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THE KOPLATADZE–CHANTURIYA TYPE THEOREM FOR LINEAR FIRST-ORDER DELAY DIFFERENTIAL EQUATION OF GENERAL FORM

Abstract. We present effective oscillation conditions for all solutions of a linear first-order delay differential equation written in terms of the Riemann–Stiltjes integral and including equations with concentrated and distributed delays as special cases. The main result of the paper is a generalization of the so-called low-limit oscillation test for the equation with several concentrated delays.

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1 Introduction

We say that a function $f : [0, \infty) \to \mathbb{R}$ *oscillates* if it has zeros to the right of any point $T > 0$.

It is well known that all solutions to an autonomous linear delay differential equation $\dot{x}(t) + ax(t-r) = 0$ oscillate if and only if $ar > 1/e$. Oscillation of solutions of nonautonomous equations was first studied by Myshkis [9] in the middle of the XXth century. In particular, it was Myshkis who first proposed to express the conditions for the oscillation of solutions in terms of the estimate of the lower limit of some functional defined on equation parameters. However, the interest to oscillation conditions for delay equations of the first order, in contrary to equations of the second order, appeared in the international literature only in the early 1970s. Estimating the upper and lower limits of a function

$$v(t) = \int_{h(t)}^{t} a(s)\,ds$$

determined by parameters of an equation

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \ge 0, \tag{1.1}$$

where $h(t) \le t$, gave rise to two main areas of obtaining the oscillation conditions for delay differential equations of the first order. These areas are still developing, influencing each other. In the 1990s, the idea arose of "filling the gap" between the results obtained in two directions. This idea in various forms continues to be, explicitly or implicitly, the main point of application of the efforts of modern researchers. Modern lines of research are characterized, for example, by the following recent works: in [3] and [4], a search is underway for equations that are similar in properties to autonomous equations; in [10], an attempt is made to take into account the interaction between retarded terms; in [11], a refinement of estimating $\limsup v(t)$ is proposed.

This paper is devoted to another line of research coming from Myshkis and generalizing the well-known Koplatadze–Chanturia theorem on an estimate of $\liminf v(t)$ [5]. Our main result generalizes the theorem from paper [1] on conditions for the oscillation of solutions to a linear differential equation with several concentrated delays. The main advantage of these conditions is that all delays are taken into account equally. We show that the key idea of the proof of the theorem can be transferred to a linear equation written in the most general form using the Stieltjes integral. For such an equation, the result takes on a clear and complete form.

2 The Koplatadze–Chanturiya theorem and its generalizations

Equation (1.1) is called an *equation of stable type* if $a(t) \ge 0$ and $h(t) \to \infty$ as $t \to \infty$. Consider such an equation assuming that the functions a and h are continuous and solutions are continuously differentiable functions.

Theorem 2.1 (see [5])**.** *If equation* (1.1) *is an equation of stable type and*

$$\liminf_{t \to +\infty} \int_{h(t)}^{t} a(s)\,ds > \frac{1}{e}\,,$$

then all solutions to (1.1) *oscillate.*

The author of this article knows at least three fundamentally different proofs of this theorem. The first one was proposed by the authors of [5], and this proof, apparently, underlies all generalizations of Theorem 2.1 published over the past 40 years. The second proof was proposed in the work by Koplatadze and Kvinikadze [6], where the iterative approach in estimating parameters of (1.1) was first applied. The third method of proof, proposed by V. V. Malygina, consists in deriving Theorem 2.1

directly from the results by Myshkis using a change of variables. This approach is very illustrative and effective, but is hardly applicable to more general equations.

Let us pass to the case of several delays. Suppose $a_k, h_k : [0,\infty) \to \mathbb{R}$, $k = \overline{1,m}$, are integrable functions, and consider an equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \tag{2.1}$$

with respect to a locally absolutely continuous function x. Define m families of sets

$$E_k(t) = \{s \ge t \mid \; h_k(s) < t\}.$$

If $a_k(t) \ge 0$ and $h_k(t) \le t$ almost everywhere and $h_k(t) \to \infty$ as $t \to \infty$, then equation (2.1) is an *equation of stable type*.

Theorem 2.2 (see [1]). *If* (2.1) *is an equation of stable type and*

$$\liminf_{t\to+\infty} \sum_{k=1}^{m} \int\limits_{E_k(t)} a_k(s)\,ds > \frac{1}{e}\,,$$

then all solutions to (2.1) *oscillate.*

Note that this theorem is an essential generalization of Theorem 2.1 even in the case $m = 1$. Indeed, the meaning of Theorem 2.1 does not change when the function $x(t)$ is replaced by the function $\delta(t) = \max\{h(s) \mid s \in [0,t]\}$, while Theorem 2.2 takes into account the case where the nonincrease of the function h is essential.

However, the main advantage of Theorem 2.2 is that it overcomes the main imperfection of the known iterative oscillation tests, which is using an integral along the segment $[\max\{h(s) \mid s \in [0,t]\}, t]$, where $t - h(t)$ is the least delay at the point t. This segment may be arbitrarily small and even equal to zero.

The proof of Theorem 2.2 inherits the scheme of the proof of Theorem 2.1 proposed in [5], although the method of taking into account the after-effect essentially changes. The idea of passing from Theorem 2.1 to Theorem 2.2 and some related known results are discussed in [1], and there is an attempt to compare different approaches in applying low-limit oscillation conditions in [2].

3 Oscillation conditions for equation of general form

In this section, we formulate and prove our main result.

Consider an equation

$$\dot{x}(t) + \int\limits_0^t x(s)\,d_s r(t,s) = 0, \;\; t \ge 0. \tag{3.1}$$

Here, the integral is understood in the Riemann–Stieltjes sence, the function $r(t,\cdot\,)$ has bounded variation for each t, the variation function $\rho(t) = \bigvee\limits_{s=0}^{t} r(t,s)$ is locally integrable, $r(t,0) = 0$, and the function $r(\,\cdot\,,s)$ is measurable for each s.

We say that a solution to equation (3.1) is a locally continuous function $x : [0,+\infty) \to \mathbb{R}$ that satisfies equality (3.1) almost everywhere. Given an initial condition $x(0) = x_0$, equation (3.1) is uniquely solvable.

This general form (3.1) of a linear delay differential equation was first studied in [8] and [7]. It contains, as special cases, equations without delay, equations with one and several concentrated delays, integro-differential equations with delay, and equations containing delay terms corresponding to different mentioned types. To determine the solution of equations (1.1) and (2.1), an initial function is needed, which is absent in equation (3.1). This does not limit the generality of (3.1), since equations

(1.1) and (2.1) with a nonzero initial function can be written in the form of the inhomogeneous equation corresponding to (3.1).

We say that equation (3.1) is an equation of *stable type* if for each t, the function $r(t,\cdot)$ does not decrease and for each $s \geq 0$, there exists $T(s) > s$ such that for each $t \geq T(s)$, we have $r(t,s) = 0$.

The main results of this work is the following

Theorem 3.1. *If equation* (3.1) *is an equation of stable type and*

$$\liminf_{t\to+\infty} \int_t^{+\infty} \int_0^t d_\tau\, r(s,\tau)\, ds > \frac{1}{e}\,, \tag{3.2}$$

then all solutions to (3.1) *oscillate.*

Note that in Theorem 3.1 and everywhere below, the outer integral in double integrals is understood in the sense of Lebesgue. Under the assumed conditions, this integral is always defined.

Proof.

1. Denote

$$C(t) = \int_t^{+\infty} \int_0^t d_\tau\, r(s,\tau)\, ds, \quad y(t) = \int_t^{+\infty} \int_0^t x(\tau)\, d_\tau\, r(s,\tau)\, ds.$$

Note that, in general,

$$\int_0^t d_\tau\, r(s,\tau) \leq r(s,t).$$

Consider an arbitrary solution $x : \mathbb{R}_+ \to \mathbb{R}$ of equation (3.1). Suppose it does not oscillate, while condition (3.2) is fulfilled. Without loss of generality, we suppose the solution to be positive from some point. In virtue of the equation and the theorem conditions, there are $t_0 \in \mathbb{R}_+$ and $K \in (1/e, 1)$ such that for all $t \geq t_0$, we have $x(t) > 0$ and

$$C(t) \geq K. \tag{3.3}$$

It follows from the theorem conditions that $x(t) \to 0$ for $t \to +\infty$. Indeed, there is a strongly increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $r(t,t_n) = 0$ for all $t > t_{n+1}$, hence

$$\begin{aligned} x(t_{n+1}) = x(t_n) - \int_{t_n}^{t_{n+1}} \int_0^s x(\tau)\, d_\tau\, r(s,\tau)\, ds \\ \leq x(t_n) - \int_{t_n}^{+\infty} \int_0^{t_n} x(\tau)\, d_\tau\, r(s,\tau)\, ds \leq x(t_n)(1 - C(t_n)) < x(t_n)\, \frac{e-1}{e}\,. \end{aligned}$$

Thus, in virtue of equation (3.1), for $t \geq 0$ we have

$$x(t) = -(0 - x(t)) = \int_t^{+\infty} \int_0^s x(\tau)\, d_\tau\, r(s,\tau)\, ds,$$

therefore, for some $t_1 \geq t_0$ and for all $t \geq t_1$, we have $y(t) < x(t)$.

Denote $\varepsilon(t) = x(t) - y(t)$. In the subsequent items 2–5 we show that $\varepsilon(t)/x(t) \to 0$ for $t \to +\infty$. For this, we show that for any $c > 0$, if there is $t_2 \geq t_1$ such that for all $t \geq t_2$ an inequality $x(t)/\varepsilon(t) \geq c$ holds, then there is $t_3 \geq t_2$ such that for all $t \geq t_3$, the inequality $x(t)/\varepsilon(t) \geq K \cdot e \cdot c > c$ is satisfied. We show in item 6 that the deduced fact contradicts the conditions of the theorem.

2. Suppose $c > 0$, $t_1 \geq t_0$, and for all $t \geq t_1$, we have $x(t) \geq c\,\varepsilon(t)$.

Fix $t_2 \geq t_1$ such that for all $t \geq t_2$ and $s \leq t_1$, we have $r(t,s) = 0$, hence

$$\int_t^\infty \int_0^{t_1} d_\tau\, r(s,\tau)\, ds = 0.$$

For convenience, define

$$T(t) = \inf\Big\{ s \geq t \mid \int_s^\infty d_\tau\, r(t,\tau) = 0 \Big\}.$$

So,

$$\int_{t_2}^\infty \int_0^{t_2} d_s\, r(t,s)\, dt = \int_{t_2}^{T(t_2)} \int_0^{t_2} d_s\, r(t,s)\, dt = C(t_2).$$

Divide the double integral $\int_{t_2}^{T(t_2)} \int_0^{t_2}$ into h terms, each is equal to $C(t_2)/h$.

Decompose the segment $[t_1, t_2]$ by a sequence of points s_i, $i = \overline{0,h}$:

$$t_1 = s_0 < s_1 \leq \cdots \leq s_{h-1} \leq s_h = t_2. \tag{3.4}$$

Define functions $p : [t_2, \infty) \times (t_1, t_2) \to \mathbb{R}$ and $Q : (t_1, t_2) \to \mathbb{R}$ by the equalities

$$p(t,s) = \lim_{\sigma \to s+0} \int_{t_1}^{\sigma} d_\tau\, r(t,\tau),$$

$$Q(s) = \int_{t_2}^{T(t_2)} p(t,s)\, dt = \int_{t_2}^{T(t_2)} \lim_{\sigma \to s+0} \int_{t_1}^{\sigma} d_\tau\, r(t,\tau)\, dt,$$

and complete the definition by

$$p(t,t_1) \equiv 0, \quad p(t,t_2) = \int_{t_1}^{t_2} d_\tau\, r(t,\tau), \quad Q(t_1) = 0, \quad Q(t_2) = C(t_2).$$

For each t, the function $r(t,\cdot)$ is not decreasing, hence the function $p(t,\cdot)$, as well as the function $Q(s)$, is not decreasing on the segment $[t_1,t_2]$.

For brevity, denote

$$q_i = \frac{i}{h} C(t_2), \quad i = \overline{1, h-1}\,.$$

Choose arbitrarily

$$s_i \in \Big\{ \inf\{s \mid Q(s) \geq q_i\}, \sup\{s \mid Q(s) \leq q_i\} \Big\}.$$

If the function Q has the inverse and q_i is in the range of values of Q, then $s_i \in Q^{-1}(q_i)$. So, we obtain decomposition (3.4).

Note that one can describe the division by the other way: put s_i, $i = \overline{1, h-1}$ such that

$$q_i \in \Big[\lim_{s \to s_i - 0} Q(s), Q(s_i) \Big].$$

3. Define $u_i \in [t_2, T(t_2)]$, $i = \overline{1,h}$, in the following way.

If $q_1 = Q(s_1)$, then put $u_1 = T(t_2)$. If, conversely,

$$q_1 < Q(s_1) = \int_{t_2}^{T(t_2)} p(s,s_1)\, ds = \int_{t_2}^{T(t_2)} (p(s,s_1) - p(s,s_0))\, ds,$$

then in virtue of the boundedness of the function p, there is a number u_1 such that

$$S_1 = \int_{t_2}^{u_1} p(s, s_1)\, ds = \int_{t_2}^{u_1} \lim_{\sigma \to s_1+0} \int_{s_0}^{\sigma} d_\tau\, r(s, \tau)\, ds = q_1 = \frac{C(t_2)}{h}\,.$$

Further, choose $u_i \in [t_2, T(t_2)]$, $i = \overline{2, h}$, and compose the terms S_i, $i = \overline{2, h}$, expressed in the form of integrals of the function p. Each of such terms is equal to $C(t_2)/h$.

If $s_{i-1} < s_i$, then $Q(s_{i-1}) < q_i$. Put u_i and S_i so that

$$Q(s_{i-1}) + \int_{t_2}^{u_i} (p(s, s_i) - p(s, s_{i-1}))\, ds = \frac{i\, C(t_2)}{h} = \sum_{j=1}^{i} S_j;$$

However, if $s_i = \cdots = s_{i-k+1}$, $s_{i-k} < s_i$, then put

$$\int_{u_{i-1}}^{u_i} (p(s, s_i) - p(s, s_{i-k}))\, ds = \frac{C(t_2)}{h} = S_i$$

or, which is the same,

$$Q(s_{i-k}) + \int_{t_2}^{u_i} (p(s, s_i) - p(s, s_{i-k}))\, ds = \frac{i\, C(t_2)}{h} = \sum_{j=1}^{i} S_j.$$

Note that $u_h = T(t_2)$, $S_h = C(t_2)/h$,

$$\sum_{j=1}^{h} S_j = \int_{t_2}^{T(t_2)} \int_{t_1}^{t_2} d_\tau\, r(s, \tau)\, ds = C(t_2).$$

4. Define a function

$$P(t, s) = \lim_{\sigma \to s+0} \int_{t_1}^{\sigma} x(\tau)\, d_\tau\, r(t, \tau)$$

and a sequence $\{X_i\}_{i=1}^{h}$, each term of which X_i is obtained from the expression for the corresponding term S_i by replacing the function p with the function P. So, for example,

$$X_1 = \int_{t_2}^{s_1} P(s, s_1)\, ds.$$

Define a sequence $\{\xi_i\}$, $i = \overline{0, h}$, as follows:

$$\xi_0 = x(t_2); \quad \xi_1 = \xi_0 - \sum_{j=1}^{i} X_j, \ \ i = \overline{2, h}\,.$$

We obtain

$$\xi_h = \xi_0 - \int_{t_2}^{T(t_2)} \int_{t_1}^{t_2} x(\tau)\, d_\tau\, r(s, \tau)\, ds = x(t_2) - y(t_2) = \varepsilon(t_2),$$

that is,

$$\frac{x(t_2)}{\varepsilon(t_2)} = \frac{\xi_0}{\xi_h}\,.$$

Show that $x(s_i) \geq c\,\xi_i$, $i = \overline{0,h}$. We have

$$x(s_i) \geq c\,\varepsilon(s_i) = c(x(s_i) - y(s_i)) = c\Big(x(s_i) - \int_{s_i}^{T(t_2)} \int_0^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds\Big)$$

$$= c\Big(x(s_i) - \int_{s_i}^{t_2} \int_0^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds - \int_{t_2}^{T(t_2)} \int_{t_1}^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds\Big).$$

Compare the expression in braces with ξ_i. If $s_i = \cdots = s_{i-k+1}$, $s_{i-k} < s_i$, then

$$\xi_i = \xi_0 - \sum_{j=1}^{i} X_j = x(s_i) - (x(s_i) - x(t_2)) - \sum_{j=1}^{i} X_j$$

$$\leq x(s_i) - \int_{s_i}^{t_2} \int_0^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds - \int_{t_2}^{T(t_2)} P(s, s_{i-k})\, ds - \int_{t_2}^{u_i} [P(s,s_i) - P(s,s_{i-k})]\, ds$$

$$\leq x(s_i) - \int_{s_i}^{t_2} \int_0^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds - \int_{t_2}^{T(t_2)} P(s,s_i)\, ds$$

$$= x(s_i) - \int_{s_i}^{t_2} \int_0^{s} x(\tau)\, d_\tau\, r(s,\tau)\, ds - \int_{t_2}^{T(t_2)} \int_{s_0}^{s_i} x(\tau)\, d_\tau\, r(s,\tau)\, ds.$$

Thus $x(s_i) \geq c\,\xi_i$.

5. Show that $x(t_2)/\varepsilon(t_2) \geq K \cdot e \cdot c$. For $i = \overline{1,h}$, we have

$$\xi_i = \xi_{i-1} - X_i \leq \xi_{i-1} - x(s_i) S_i \leq \xi_{i-1} - c\,\xi_i\, \frac{C(t_2)}{h}\,,$$

that is,

$$\xi_i\Big(1 + \frac{c\,C(t_2)}{h}\Big) \leq \xi_{i-1},$$

and so,

$$\xi_0 \geq \xi_h\Big(1 + \frac{c\,C(t_2)}{h}\Big)^h.$$

Since one can take h arbitrarily large, taking account the inequalities $C(t_2) \geq K$ and $e^x \geq ex$ for $x \geq 0$, we obtain

$$\varepsilon(t_2) = \xi_h \leq \xi_0 \exp(-c\,C(t_2)) \leq \frac{x(t_2)}{K \cdot e \cdot c}\,.$$

Thus, in items 2–5, we have fulfilled our program given at the end of item 1, and it has been shown that $\varepsilon(t)/x(t) \to 0$ for $t \to +\infty$.

It remains to show that the obtained fact contradicts the conditions of the theorem.

6. We have (see item 1) $x(t) \to 0$ for $t \to \infty$ and $K \in (1/e, 1)$. For all $t \geq t_0$, define $f(t) \geq t_0$ so that

$$x(f(t)) \in \Big(\frac{Kx(t)}{e}\,, \frac{x(t)}{e}\Big). \tag{3.5}$$

For $t \geq t_0$, we have

$$\varepsilon(t) = x(t) - y(t) = \int_t^{+\infty} \int_t^{s} x(\tau)\, d_\tau\, r(s,\tau)\, ds$$

$$\geq \int_{f(t)}^{\infty} \int_t^{f(t)} x(\tau)\, d_\tau\, r(s,\tau)\, ds \geq x(f(t)) \int_{f(t)}^{\infty} \int_t^{f(t)} d_\tau\, r(s,\tau)\, ds > \frac{K}{e}\, x(t) \int_{f(t)}^{\infty} \int_t^{f(t)} d_\tau\, r(s,\tau)\, ds.$$

Hence the assertion that

$$\int_{f(t)}^{\infty}\int_{t}^{f(t)} d_\tau\, r(s,\tau)\, ds \not\to 0 \ \text{ for } \ t\to\infty$$

contradicts the established above fact that $\varepsilon(t)/x(t)\to 0$ for $t\to\infty$. So,

$$\int_{f(t)}^{\infty}\int_{t}^{f(t)} d_\tau\, r(s,\tau)\, ds \to 0 \ \text{ for } \ t\to\infty.$$

Taking into account inequality (3.3), it follows that for t large enough we have

$$\int_{f(t)}^{\infty}\int_{0}^{t} d_\tau\, r(s,\tau)\, ds \ge K > \frac{1}{e}\,.$$

Therefore,

$$x(f(t)) \ge \int_{f(t)}^{\infty}\int_{0}^{t} x(\tau)\, d_\tau\, r(s,\tau)\, ds \ge x(t)\int_{f(t)}^{\infty}\int_{0}^{t} d_\tau\, r(s,\tau)\, ds > \frac{x(t)}{e}\,.$$

This contradicts definition (3.5).

So, the assertion that a solution x of equation (3.1) does not oscillate leads to contradiction. □

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