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ON SOLVABILITY CONDITIONS FOR THE CAUCHY PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NON-VOLTERRA OPERATORS AND COMPOSITE POINTWISE RESTRICTIONS

Dedicated to the blessed memory of Professor Nikolay Viktorovich Azbelev

Abstract. The Cauchy boundary value problem for the second order linear functional differential equations with non-Volterra operators is considered. Unimprovable sufficient solvability conditions are obtained for various composite pointwise restrictions on functional operators.

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1 Introduction

We consider the Cauchy boundary value problem for the second order functional differential equation

$$\begin{gathered}\ddot{x}(t)=(T^+x)(t)-(T^-x)(t)+f(t),\ \ t\in[0,1],\\ x(0)=c_0,\quad \dot{x}(0)=c_1,\end{gathered}\tag{1.1}$$

with, generally speaking, non-Volterra positive linear operators $T^+, T^- : \mathbf{C}[0,1]\to\mathbf{L}[0,1]$, $f\in\mathbf{L}[0,1]$, $c_0, c_1\in\mathbb{R}$. Here and further, we regard equalities (and inequalities) with integrable functions as equalities (and inequalities) fulfilled almost everywhere on the appropriated interval. An operator is called *positive* if it maps each non-negative function into almost everywhere non-negative one. By $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ denote the spaces of real continuous and integrable functions on $[0,1]$ with the standard norms.

Many known solvability conditions for problem (1.1) are obtained in terms of integrals or least upper bounds of the functions $T^+\mathbf{1}, T^-\mathbf{1}\in\mathbf{L}[0,1]$, where $\mathbf{1}(t)=1$, $t\in[0,1]$, is the unit function (see, e.g., [6–11]). This choice has important reasons. However, obviously, this is not the only possibility. In this work, we get some non-improvable solvability conditions for problem (1.1) in new terms of integrable functions p^+ and p^-, when

$$p^+=T^+v,\quad p^-=T^-u\tag{1.2}$$

for some given continuous functions $v,u:[0,1]\to(0,+\infty)$. These functions may differ from the identity function that gives us some advantages. The new solvability conditions cannot be deduced from the previously obtained for the unit functions.

The solvability conditions obtained here are non-improvable for the given functions p^+, p^- in the following sense. If they are not fulfilled, then there exist linear positive operators $T^+, T^- : \mathbf{C}[0,1]\to\mathbf{L}[0,1]$ such that equalities (1.2) hold and problem (1.1) is not uniquely solvable. So, these conditions are necessary and sufficient for problem (1.1) to be uniquely solvable for all linear positive operators T^+, T^- satisfying (1.2). Obviously, some results of previous works can be obtained if one takes $v=u=\mathbf{1}$.

Definition 1.1. Let functions $p^+, p^-\in\mathbf{L}[0,1]$ be non-negative and functions $v,u\in\mathbf{C}[0,1]$ be positive.

By $\mathcal{S}_{v,u}^{p^+,p^-}$ denote the set of all operators $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ having the representation $T=T^+-T^-$, where linear positive operators $T^+, T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ satisfy conditions (1.2).

We say that the 4-vector (p^+,p^-,v,u) belongs *the set of unique solvability* $\mathcal{U}$ if problem (1.1) is uniquely solvable for all linear positive operators $T^+, T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ satisfying conditions (1.2).

Boundary value problem (1.1) possesses the Fredholm property if (see, e.g., [3]), in particular, problem (1.1) is uniquely solvable if and only if the corresponding homogeneous boundary value problem has only the trivial solution.

Many papers [4,12,17,18,23–25,27] are devoted to the conditions for the solvability of the Cauchy problem for functional differential equations with Volterra and non-Volterra functional operators.

Various boundary value problems for functional differential equations with non-Volterra operators, when there exist both leads and lags, are also being studied more often (see, e.g., [1–3,5,13,14,19,26]). Such kind of functional dependencies arise in many fields from physics to economics and optimal control [16,22].

Usually, in the non-Volterra case, criteria for the solvability are obtained by using the variants of contraction mapping principle or the authors use numerical methods in their study.

We apply an approach of the papers [9,10,15,20,21] where unimprovable solvability conditions are found for integral restrictions on functions p^+ and p^- for $v=u=\mathbf{1}$ in conditions (1.2). However, we consider much narrower and more flexible classes of *operator uncertainties*, which are given by pointwise restrictions with different functions v and u. Thus the obtained solvability conditions can be more effectively applied in practice.

In Section 2, we obtain general necessary and sufficient conditions for the solvability of the Cauchy problem (1.1) for all positive operators satisfying conditions (1.2). Further, in Sections 3 and 4, effective solvability conditions will be obtained in cases where v and u are proportional (Section 3), the functions p^+, p^- are proportional (Section 4). The effective solvability conditions (3.18)–(3.23), (4.6)–(4.8) obtained here for the simplest problem from the class under consideration (with concentrated measurable deviating argument) are unimprovable, do not follow from the contraction mapping principle, do not follow from each other, explicitly include deviating arguments.

2 Necessary and sufficient solvability conditions

Suppose problem (1.1) is not uniquely solvable. Then the homogeneous problem

$$\begin{gathered} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t), \quad t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \end{gathered} \tag{2.1}$$

has a non-trivial solution (by the Fredholm property [3]).

Lemma 2.1. *If* $(p^+, p^-, v, u) \in \mathcal{U}$ *and*

$$\widetilde{p}^+, \widetilde{p}^- \in \mathbf{L}[0,1], \quad 0 \le \widetilde{p}^+(t) \le p^+(t), \quad 0 \le \widetilde{p}^-(t) \le p^-(t), \quad t \in [0,1], \tag{2.2}$$

then $(\widetilde{p}^+, \widetilde{p}^-, v, u) \in \mathcal{U}$.

Proof. Suppose $(\widetilde{p}^+, \widetilde{p}^-, v, u) \notin \mathcal{U}$ and conditions (2.2) hold. Then there exists $T \in \mathcal{S}_{v,u}^{\widetilde{p}^+,\widetilde{p}^-}$ such that problem (2.1) has a non-trivial solution. Therefore, the problem

$$\begin{gathered} \ddot{x}(t) = (T^+x)(t) + (p^+(t) - \widetilde{p}^+(t))\,\frac{x(0)}{v(0)} - (T^-x)(t) - (p^-(t) - \widetilde{p}^-(t))\,\frac{x(0)}{u(0)}\,, \quad t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \end{gathered}$$

has the same non-trivial solution. Since

$$\begin{gathered} (T^+v)(t) + (p^+(t) - \widetilde{p}^+(t))\,\frac{v(0)}{v(0)} = p^+(t), \quad t \in [0,1], \\ (T^-u)(t) + (p^-(t) - \widetilde{p}^-(t))\,\frac{u(0)}{u(0)} = p^-(t), \quad t \in [0,1], \end{gathered}$$

we get $(p^+, p^-, v, u) \notin \mathcal{U}$. □

Let $v(t) > 0$, $u(t) > 0$ for all $t \in [0,1]$. Define the points t_1, t_2, s_1, s_2 as follows:

$$\frac{x(t_1)}{v(t_1)} = \min_{s \in [0,1]} \frac{x(s)}{v(s)}\,, \quad \frac{x(t_2)}{v(t_2)} = \max_{s \in [0,1]} \frac{x(s)}{v(s)}\,, \tag{2.3}$$

$$\frac{x(s_1)}{u(s_1)} = \min_{s \in [0,1]} \frac{x(s)}{u(s)}\,, \quad \frac{x(s_2)}{u(s_2)} = \max_{s \in [0,1]} \frac{x(s)}{u(s)}\,. \tag{2.4}$$

Then we have

$$\frac{x(s_1)}{u(s_1)}\,u(t) \le \frac{x(t)}{u(t)}\,u(t) = x(t) \le \frac{x(s_2)}{u(s_2)}\,u(t), \quad t \in [0,1], \tag{2.5}$$

$$\frac{x(t_1)}{v(t_1)}\,v(t) \le \frac{x(t)}{v(t)}\,v(t) = x(t) \le \frac{x(t_2)}{u(t_2)}\,v(t), \quad t \in [0,1]. \tag{2.6}$$

Suppose that equalities (1.2) hold. Applying positive operators T^+ and T^- to inequalities (2.5), (2.6), we get the following inequalities:

$$\begin{gathered} \frac{x(t_1)}{v(t_1)}\,p^+(t) = \frac{x(t_1)}{v(t_1)}\,(T^+v)(t) \le (T^+x)(t) \le \frac{x(t_2)}{v(t_2)}\,(T^+v)(t) = \frac{x(t_2)}{v(t_2)}\,p^+(t), \quad t \in [0,1], \\ \frac{x(s_1)}{u(s_1)}\,p^-(t) = \frac{x(s_1)}{u(s_1)}\,(T^-u)(t) \le (T^-x)(t) \le \frac{x(s_2)}{u(s_2)}\,(T^-u)(t) = \frac{x(s_2)}{u(s_2)}\,p^-(t), \quad t \in [0,1]. \end{gathered}$$

So, the function $Tx = T^+x - T^-x$ satisfies the inequalities

$$\frac{x(t_1)}{v(t_1)}\,p^+(t) - \frac{x(s_2)}{u(s_2)}\,p^-(t) \le (Tx)(t) \le \frac{x(t_2)}{v(t_2)}\,p^+(t) - \frac{x(s_1)}{u(s_1)}\,p^-(t), \quad t \in [0,1].$$

We can rewrite them as an equality

$$(Tx)(t) = \xi(t)\Big(\frac{x(t_1)}{v(t_1)}\,p^+(t) - \frac{x(s_2)}{u(s_2)}\,p^-(t)\Big) + (1-\xi(t))\Big(\frac{x(t_2)}{v(t_2)}\,p^+(t) - \frac{x(s_1)}{u(s_1)}\,p^-(t)\Big),$$

where $\xi : [0,1] \to [0,1]$ is a measurable function. So, we show that x satisfies the Cauchy problem

$$\begin{aligned} \ddot{x}(t) = \xi(t)p^+(t)\,\frac{x(t_1)}{v(t_1)} &+ (1-\xi(t))p^+(t)\,\frac{x(t_2)}{v(t_2)} \\ &- \xi(t)p^-(t)\,\frac{x(s_2)}{u(s_2)} - (1-\xi(t))p^-(t)\,\frac{x(s_1)}{u(s_1)}\,, \quad t \in [0,1], \\ &x(0) = 0, \quad \dot{x}(0) = 0. \end{aligned} \tag{2.7}$$

Every solution of this Cauchy problem satisfies the equation

$$\begin{aligned} x(t) = (\mathcal{C}\xi p^+)(t)\,\frac{x(t_1)}{v(t_1)} \\ + (\mathcal{C}(1-\xi)p^+)(t)\,\frac{x(t_2)}{v(t_2)} - (\mathcal{C}\xi(t)p^-)(t)\,\frac{x(s_2)}{u(s_2)} - (\mathcal{C}(1-\xi)p^-)(t)\,\frac{x(s_1)}{u(s_1)}\,, \quad t \in [0,1], \end{aligned} \tag{2.8}$$

where $(\mathcal{C}z)(t) = \int\limits_0^t (t-s)z(s)\,ds$, $t \in [0,1]$, for every $z \in \mathbf{L}[0,1]$.

Substituting $t = t_1$, $t = t_2$, $t = s_1$, $t = s_2$ into equation (2.8), we obtain the algebraic system with respect to $x(t_1)$, $x(t_2)$, $x(s_1)$, $x(s_2)$:

$$\begin{aligned} x(t_1)\Big(1 - \frac{(\mathcal{C}\xi p^+)(t_1)}{v(t_1)}\Big) - x(t_2)\,\frac{(\mathcal{C}(1-\xi)p^+)(t_1)}{v(t_2)} \\ + x(s_1)\,\frac{(\mathcal{C}(1-\xi)p^-)(t_1)}{u(s_1)} + x(s_2)\,\frac{(\mathcal{C}\xi p^-)(t_1)}{u(s_2)} = 0, \\ -x(t_1)\,\frac{(\mathcal{C}\xi p^+)(t_2)}{v(t_1)} + x(t_2)\Big(1 - \frac{(\mathcal{C}(1-\xi)p^+)(t_2)}{v(t_2)}\Big) \\ + x(s_1)\,\frac{(\mathcal{C}(1-\xi)p^-)(t_2)}{u(s_1)} + x(s_2)\,\frac{(\mathcal{C}\xi p^-)(t_2)}{u(s_2)} = 0, \\ -x(t_1)\,\frac{(\mathcal{C}\xi p^+)(s_1)}{v(t_1)} - x(t_2)\,\frac{(\mathcal{C}(1-\xi)p^+)(s_1)}{v(t_2)} \\ + x(s_1)\Big(1 + \frac{(\mathcal{C}(1-\xi)p^-)(s_1)}{u(s_1)}\Big) + x(s_2)\,\frac{(\mathcal{C}\xi p^-)(s_1)}{u(s_2)} = 0, \\ -x(t_1)\,\frac{(\mathcal{C}\xi p^+)(s_2)}{v(t_1)} - x(t_2)\,\frac{(\mathcal{C}(1-\xi)p^+)(s_2)}{v(t_2)} \\ + x(s_1)\,\frac{(\mathcal{C}(1-\xi)p^-)(s_2)}{u(s_1)} + x(s_2)\,\Big(1 + \frac{(\mathcal{C}\xi p^-)(s_2)}{u(s_2)}\Big) = 0. \end{aligned} \tag{2.9}$$

Equation (2.8) with respect to x has a non-trivial solution if and only if system (2.9) has a non-trivial solution with respect to $(x(t_1), x(t_2), x(s_1), x(s_2))$, that is, if

$$\Delta = \begin{vmatrix} 1-\dfrac{(\mathcal{C}\xi p^+)(t_1)}{v(t_1)} & -\dfrac{(\mathcal{C}(1-\xi)p^+)(t_1)}{v(t_2)} & \dfrac{(\mathcal{C}(1-\xi)p^-)(t_1)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(t_1)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(t_2)}{v(t_1)} & 1-\dfrac{(\mathcal{C}(1-\xi)p^+)(t_2)}{v(t_2)} & \dfrac{(\mathcal{C}(1-\xi)p^-)(t_2)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(t_2)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(s_1)}{v(t_1)} & -\dfrac{(\mathcal{C}(1-\xi)p^+)(s_1)}{v(t_2)} & 1+\dfrac{(\mathcal{C}(1-\xi)p^-)(s_1)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(s_1)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(s_2)}{v(t_1)} & -\dfrac{(\mathcal{C}(1-\xi)p^+)(s_2)}{v(t_2)} & \dfrac{(\mathcal{C}(1-\xi)p^-)(s_2)}{u(s_1)} & 1+\dfrac{(\mathcal{C}\xi p^-)(s_2)}{u(s_2)} \end{vmatrix} \neq 0. \quad (2.10)$$

In determinant (2.10), we can add to the second column the first column multiplied by $v(t_1)/v(t_2)$ and add to the third column the forth column multiplied by $u(s_2)/u(s_1)$. Then we get

$$\Delta = \begin{vmatrix} 1-\dfrac{(\mathcal{C}\xi p^+)(t_1)}{v(t_1)} & \dfrac{v(t_1)}{v(t_2)}-\dfrac{(\mathcal{C}p^+)(t_1)}{v(t_2)} & \dfrac{(\mathcal{C}p^-)(t_1)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(t_1)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(t_2)}{v(t_1)} & 1-\dfrac{(\mathcal{C}p^+)(t_2)}{v(t_2)} & \dfrac{(\mathcal{C}p^-)(t_2)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(t_2)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(s_1)}{v(t_1)} & -\dfrac{(\mathcal{C}p^+)(s_1)}{v(t_2)} & 1+\dfrac{(\mathcal{C}p^-)(s_1)}{u(s_1)} & \dfrac{(\mathcal{C}\xi p^-)(s_1)}{u(s_2)} \\ -\dfrac{(\mathcal{C}\xi p^+)(s_2)}{v(t_1)} & -\dfrac{(\mathcal{C}p^+)(s_2)}{v(t_2)} & \dfrac{u(s_2)}{u(s_1)}+\dfrac{(\mathcal{C}p^-)(s_2)}{u(s_1)} & 1+\dfrac{(\mathcal{C}\xi p^-)(s_2)}{u(s_2)} \end{vmatrix}. \quad (2.11)$$

So, if the corresponding determinant Δ is different from zero for all $t_1, t_2, s_1, s_2 \in [0,1]$ and for all measurable function ξ, then $(p^+, p^-, v, u) \in \mathcal{U}$. Conversely, if system (2.9) has a non-trivial solution for some $t_2, s_1, s_2 \in [0,1]$ and ξ, then there exists a non-trivial solution of equation (2.8) and of problem (2.7). Therefore, problem (2.1) has a non-trivial solution if positive linear operators T^+, T^- are defined by the equalities

$$(T^+y)(t) \equiv \xi(t)p^+(t)\,\frac{y(t_1)}{v(t_1)} + (1-\xi(t))p^+(t)\,\frac{y(t_2)}{v(t_2)}\,, \ \ t \in [0,1],$$
$$(T^-y)(t) \equiv \xi(t)p^-(t)\,\frac{y(s_2)}{u(s_2)} + (1-\xi(t))p^-(t)\,\frac{y(s_1)}{u(s_1)}\,, \ \ t \in [0,1],$$

for every $y \in \mathbf{C}[0,1]$. It is clear that $T^+v = p^+$, $T^-u = p^-$. Therefore, $(p^+, p^-, v, u) \notin \mathcal{U}$. We prove the following assertion.

Theorem 2.1. *$(p^+, p^-, v, u) \in \mathcal{U}$ if and only if in (2.11) $\Delta > 0$ for all $t_1, t_2, s_1, s_2 \in [0,1]$ and for all measurable functions $\xi : [0,1] \to [0,1]$.*

Proof. It remains to prove that if Δ from (2.11) does not equal zero for all for all $t_1, t_2, s_1, s_2 \in [0,1]$ and for all measurable function $\xi : [0,1] \to [0,1]$, then $\Delta > 0$ for all these variables. □

Unfortunately, we do not have simple efficient means to check the condition of Theorem 2.1, because the dependence on the functional variable ξ is too complicated.

There are two cases when we are able to find effective solvable conditions. The first case, where the functions v and u are proportional, is essentially simpler than the second case, where the functions p^+ and p^- are proportional.

3 The case of identical functions v and u

If the functions v and u are proportional, then $s_1 = t_1$ and $s_2 = t_2$, moreover, without loss of generality, we can put $v = u$.

Now, a non-trivial solution x of problem (1.1) satisfies the following problem:

$$\begin{gathered}\ddot{x}(t) = \big(-p^-(t) + \xi(t)(p^+(t) + p^-(t))\big)\,\frac{x(t_1)}{v(t_1)} + \big(p^+(t) - \xi(t)(p^+(t) + p^-(t))\big)\,\frac{x(t_2)}{v(t_2)}\,,\ \ t \in [0,1],\\ x(0) = 0, \quad \dot{x}(0) = 0.\end{gathered} \tag{3.1}$$

Every solution of this Cauchy problem satisfies the equation

$$x(t) = \mathcal{C}(-p^- + \xi(p^+ + p^-))(t)\,\frac{x(t_1)}{v(t_1)} + \mathcal{C}(p^+ - \xi(p^+ + p^-))(t)\,\frac{x(t_2)}{v(t_2)}\,,\ \ t \in [0,1], \tag{3.2}$$

where $(\mathcal{C}z)(t) = \int\limits_0^t (t-s)z(s)\,ds$, $t \in [0,1]$, for every $z \in \mathbf{L}[0,1]$.

Substituting $t = t_1$ and $t = t_2$ into equation (3.2), we obtain the algebraic system with respect to $x(t_1)$, $x(t_2)$:

$$\begin{aligned}x(t_1)\Big(1 - \frac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_1)}{v(t_1)}\Big) - x(t_2)\,\frac{\mathcal{C}(p^+ - \xi(p^+ + p^-))(t_1)}{v(t_2)} &= 0,\\ -x(t_1)\,\frac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_2)}{v(t_1)} + x(t_2)\Big(1 - \frac{\mathcal{C}(p^+ - \xi(p^+ + p^-))(t_2)}{v(t_2)}\Big) &= 0.\end{aligned} \tag{3.3}$$

Equation (3.2) with respect to x has a non-trivial solution if and only if system (3.3) has a non-trivial solution with respect to $(x(t_1), x(t_2))$, that is, if

$$\Delta = \begin{vmatrix} 1 - \dfrac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_1)}{v(t_1)} & -\dfrac{\mathcal{C}(p^+ - \xi(p^+ + p^-))(t_1)}{v(t_2)} \\ -\dfrac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_2)}{v(t_1)} & 1 - \dfrac{\mathcal{C}(p^+ - \xi(p^+ + p^-))(t_2)}{v(t_2)} \end{vmatrix} \neq 0. \tag{3.4}$$

Adding in determinant (3.4) the first column multiplied by $v(t_1)/v(t_2)$ to the second column, we get

$$\begin{aligned}\Delta &= \begin{vmatrix} 1 - \dfrac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_1)}{v(t_1)} & \dfrac{v(t_1)}{v(t_2)} - \dfrac{\mathcal{C}(p^+ - p^-)(t_1)}{v(t_2)} \\ -\dfrac{\mathcal{C}(-p^- + \xi(p^+ + p^-))(t_2)}{v(t_1)} & 1 - \dfrac{\mathcal{C}(p^+ - p^-)(t_2)}{v(t_2)} \end{vmatrix}\\ &= 1 - \frac{\mathcal{C}(p^+)(t_2)}{v(t_2)} + \frac{\mathcal{C}(p^-)(t_2)}{v(t_2)}\\ &\qquad + \frac{1}{v(t_1)v(t_2)}\Big((\mathcal{C}p_1)(t_1)(v(t_2) - (\mathcal{C}p)(t_2)) - (\mathcal{C}p_1)(t_2)(v(t_1) - (\mathcal{C}p)(t_1))\Big)\\ &= \psi(t_2) + \int\limits_0^1 p_1(s)\phi(t)\,ds \neq 0,\end{aligned} \tag{3.5}$$

where

$$p_1(t) = -p^-(t) + \xi(t)(p^+(t) + p^-(t)) \in [-p^-(t), p^+(t)],\ \ t \in [0,1],$$

is an arbitrary measurable function from this cone interval,

$$\phi(s) = \frac{(t_2 - s)_+}{v(t_2)}\,\psi(t_1) - \frac{(t_1 - s)_+}{v(t_1)}\,\psi(t_2),\ \ s \in [0,1], \tag{3.6}$$

$$\psi(t) = 1 - (\mathcal{C}_v p)(t), \quad (\mathcal{C}_v z)(t) \equiv \frac{(\mathcal{C}z)(t)}{v(t)}\,,\ \ z \in \mathbf{L}[0,1],\ \ p(t) = p^+(t) - p^-(t),\ \ t \in [0,1],$$

for every $\alpha \in \mathbb{R}$ let $a_+ = \max\{0, a\}$.

We always can choose a solution of (2.1) x such that $1 \geq t_2 > t_1 \geq 0$. So, we have proved the following assertion.

Lemma 3.1. *Let $u(t) = v(t) > 0$ for all $t \in [0,1]$. Then $(v,u,p^+,p^-) \in \mathcal{U}$ if and only if*

$$\Delta = \psi(t_2) + \int_0^1 p_1(s)\Big[\frac{(t_2-s)_+}{v(t_2)}\,\psi(t_1) - \frac{(t_1-s)_+}{v(t_1)}\,\psi(t_2)\Big]\,ds \neq 0 \tag{3.7}$$

for all $0 \le t_1 \le t_2 \le 1$ and all measurable functions p_1 such that

$$-p^-(s) \le p_1(s) \le p^+(s), \quad s \in [0,1]. \tag{3.8}$$

It is obvious that if $\Delta \neq 0$ for all t_1, t_2, ξ, then $\psi(t) > 0$ for all $t \in (0,1]$ (this follows from (3.5) for $p_1 \equiv 0$ and for $p_1 \equiv\!= 0$, $t_2 = 0$). Therefore, $\Delta > 0$ for all t_1, t_2, ξ.

If $(v,u,p^+,p^-) \in \mathcal{U}$, the inequality $\psi(t) > 0$, $t \in [0,1]$, can be improved. Indeed, by Lemma 2.1, we have $(v,u,p^+,0) \in \mathcal{U}$. Therefore, the inequality

$$1 - (\mathcal{C}_v p^+)(t) > 0, \quad t \in [0,1], \tag{3.9}$$

is a necessary condition for the unique solvability of (1.1).

Let us find $p_1 \in \mathbf{L}[0,1]$, $p_1(t) \in [-p^-(t), p^+(t)]$, $t \in [0,1]$, such that Δ be minimal for the given $t_1 \le t_2$. We have

$$\phi(s) \ge 0, \quad s \in [t_1, 1].$$

Therefore, $p_{1,\min}(s) = -p^-(s)$ for $s \in [t_1, 1]$.

Since ϕ is linear on $[0, t_1]$, then either

$$\frac{t_2\psi(t_1)}{v(t_2)} - \frac{t_1\psi(t_2)}{v(t_1)} \ge 0,$$

then $\phi(s) \ge 0$, $s \in [0,t_1]$, $p_{1,\min}(s) = -p^-(s)$ for all $s \in [0,1]$, or

$$\frac{t_2\psi(t_1)}{v(t_2)} - \frac{t_1\psi(t_2)}{v(t_1)} < 0, \tag{3.10}$$

then there exists

$$t_3 = \frac{t_1\psi(t_2)v(t_2) - t_2\psi(t_1)v(t_1)}{\psi(t_2)v(t_2) - \psi(t_1)v(t_1)} \in (0, t_1) \tag{3.11}$$

such that

$$\phi(s) \le 0, \quad s \in [0, t_3]; \qquad \phi(s) \ge 0, \quad s \in [t_3, 1].$$

Therefore,

$$p_{1,\min}(s) = p^+(s), \quad s \in [0, t_3]; \qquad p_{1,\min}(s) = -p^-(s), \quad s \in [t_3, 1]. \tag{3.12}$$

In the former case, we have

$$\begin{aligned}\Delta_{\min} &= \big(1 - (\mathcal{C}_v p^+)(t_2) + (\mathcal{C}_v p^-)(t_2)\big)\big(1 + (\mathcal{C}_v p^-)(t_1)\big) - (\mathcal{C}_v p^-)(t_2)\big(1 - (\mathcal{C}_v p^+)(t_1) + (\mathcal{C}_v p^-)(t_1)\big)\\ &= (1 - (\mathcal{C}_v p^+)(t_2))(\mathcal{C}_v p^-)(t_1) + (\mathcal{C}_v p^-)(t_2)(\mathcal{C}_v p^+)(t_1) > 0\end{aligned}$$

by the necessary condition (3.9).

In the latter case, where (3.10) holds and t_3 is defined by (3.11), we have the following solvability condition:

$$\begin{aligned}\Delta_{\min} = \psi(t_2) &+ \int_0^{t_3} p^+(s)\Big(\frac{t_2-s}{v(t_2)}\,\psi(t_1) - \frac{t_1-s}{v(t_1)}\,\psi(t_2)\Big)\,ds\\ &- \int_{t_3}^{t_2} p^-(s)\,\frac{t_2-s}{v(t_2)}\,\psi(t_1)\,ds + \int_{t_3}^{t_1} p^-(s)\,\frac{t_1-s}{v(t_1)}\,\psi(t_2)\,ds > 0.\end{aligned} \tag{3.13}$$

Lemma 3.2. *If a function* $t \mapsto \frac{t}{v(t)} : [0,1] \to \mathbb{R}$ *does not decrease, then in order to verify inequality* (3.13) *for all* $0 \le t_3 \le t_1 \le t_2 \le 1$ *it suffices to check it for all* $0 \le t_3 \le t_1 \le 1 = t_2$.

Proof. In fact, we need to check inequality (3.7) only for the solutions x of (3.1) when p_1 from (3.6) is defined by equalities (3.12). In this case, we have

$$\ddot{x}(t) = \begin{cases} \dfrac{p^+(t)x(t_1)}{v(t_1)} - \dfrac{p^-(t)x(t_2)}{v(t_2)}, & t \in [0,t_3], \\ \dfrac{p^+(t)x(t_2)}{v(t_2)} - \dfrac{p^-(t)x(t_1)}{v(t_1)}, & t \in (t_3,t_2], \\ 0, & t \in (t_2,1], \end{cases}$$

where the points $t_1 \le t_2$ are defined by (2.3). If $x(t_1)$ and $x(t_2)$ have the same sign, then $x(t_1) = 0$. Therefore, in this case, the solution x of (3.1) is a solution of the equation

$$x(t) = (\mathcal{C}_v p_1)(t_2)x(t_2), \quad t \in [0,1],$$

with some p_1 satisfying (3.8). It is easy to check that this equation has only the trivial solution if the necessary condition (3.9) holds. Therefore, $x(t_1) < 0 < x(t_2)$. It implies that $\ddot{x}(t) \le 0$ for $t \in [0,t_3]$; $\ddot{x}(t) \ge 0$ for $t \in [t_3,1]$. Let $x(\theta) = 0$, $\theta \in (t_3,t_2)$. Then $\dot{x}(\theta) > 0$ and

$$x(t) = \dot{x}(\theta)(t-\theta) + \int_\theta^t (t-s)\ddot{x}(s)\,ds, \quad t \in [\theta,1].$$

The function $t \mapsto x(t)/t$ increases on $[\theta,1]$, since

$$\left(\frac{x(t)}{t}\right)'_t = \frac{\dot{x}(\theta)\theta + \int_\theta^t s\ddot{x}(s)\,ds}{t^2} > 0, \quad t \in [\theta,1].$$

Therefore, the function $t \mapsto \frac{x(t)}{v(t)} = \frac{x(t)}{t}\frac{t}{v(t)}$ increases on $(\theta,1]$. So,

$$\max_{t\in[0,1]} \frac{x(t)}{v(t)} = \frac{x(1)}{v(1)},$$

that implies $t_2 = 1$. □

Remark 3.1. Obviously, if the function v is positive for $t \in (0,1]$ and $\lim\limits_{t\to 0} \frac{x(t)}{v(t)} = 0$ for every solution x of problem (2.1), then all previous statements of this section remain valid.

Every solution of (2.1) is bounded, therefore, for some $K > 0$, the following estimate is fulfilled:

$$|x(t)| \le K \int_0^t (t-s)\big((T^+\mathbf{1})(s) + (T^-\mathbf{1})(s)\big)\,ds, \quad t \in [0,1]. \tag{3.14}$$

Further, suppose that for $w = v$ the function

$$t \mapsto \frac{\int_0^t (t-s)((T^+\mathbf{1})(s) + (T^-\mathbf{1})(s))\,ds}{w(t)}, \quad t \in (0,1], \tag{3.15}$$

is bounded, and

$$\lim_{t\to 0} \frac{\int_0^t (t-s)(p^+(s) + p^-(s))\,ds}{w(t)} = 0. \tag{3.16}$$

Then it follows from (3.14)–(3.16) that if problem (2.1) has a non-trivial solution x and equalities (1.2) are fulfilled, then $\lim\limits_{t\to 0}\frac{x(t)}{v(t)}=0$, and the functions $(\mathcal{C}_v p^+)(t)$, $(\mathcal{C}_v p^-)(t)$ can be extended at $t=0$ by zero.

The singular case, where function (3.15) is bounded, but equality (3.16) does not hold, will be consider later.

Let us formulate all the results obtained in this item. Here, either v is a positive function, or $v(0)=0$ and $v(t)>0$, $t\in(0,1]$. In the last case, we suppose that function (3.15) is bounded and equality (3.16) holds.

Theorem 3.1 (Necessary condition). *If $(p^+,p^-,v,v)\in\mathcal{U}$, then inequality* (3.9) *holds.*

Theorem 3.2 (Rough sufficient condition). *Suppose inequality* (3.9) *holds and the function*

$$t\mapsto \frac{v(t)-\int\limits_0^t (t-s)(p^+(s)-p^-(s))\,ds}{t}$$

does not increase on $(0,1]$*, then* $(p^+,p^-,v,v)\in\mathcal{U}$.

Corollary. *If the function $t\mapsto\frac{v(t)}{t}$ does not increase on $(0,1]$ and $p^+(t)\ge p^-(t)$, $t\in[0,1]$, then $(p^+,p^-,v,v)\in\mathcal{U}$.*

Theorem 3.3 (Necessary and sufficient condition). *$(p^+,p^-,v,v)\in\mathcal{U}$ if and only if inequality* (3.13)*, where t_3 is defined by* (3.11)*, is fulfilled for every $0<t_1\le t_2\le 1$ such that inequality* (3.10) *holds.*

The necessary and sufficient condition of Theorem 3.3 can be formulated in perhaps a more easily verifiable way.

Theorem 3.4 (Necessary and sufficient condition). *$(p^+,p^-,v,v)\in\mathcal{U}$ if and only if inequality* (3.13) *is fulfilled for every $0<t_3\le t_1\le t_2\le 1$.*

Remark 3.2. By Lemma 3.2, if the function $t\mapsto\frac{v(t)}{t}$ does not decrease on $t\in(0,1]$, then we have to verify the conditions of Theorems 3.2 and 3.3 only for $t_2=1$.

As far as we know, all these results are new.

Let us give some examples.

Example 3.1. Let $v(t)=u(t)=t$, $p^+=p$, $p^-=m$ be the constants. Then function (3.15) is bounded and condition (3.16) holds. By Theorem 2.1, the inequality $p^+<2$ is necessary for $(p,m,v,v)\in\mathcal{U}$. The function $t\mapsto v(t)/t$ does not increase, therefore, the equality $p\ge m$ is a sufficient condition from Theorem 3.2 and $t_2=1$ in Theorems 3.3, 3.4.

If $m>p$ in Theorem 3.3, we have

$$t_3=\frac{t_1}{1+t_1+2/(m-p)}\,.$$

So, by Theorem 3.3, we obtain the following necessary and sufficient conditions.

Theorem 3.5. *Let $v(t)=t$, $t\in[0,1]$, $p^+=p$, $p^-=m$ be the non-negative constants. Then $(p,m,v,v)\in\mathcal{U}$ if and only if*

(1) *$p<2$;*

(2) *$m\le p$, or $p<m$, and the inequality*

$$(m-p)(m^2-p^2+2m)t^2+((m-p)m^2-(2m-p)(m-p+4))t+2(2-p)(m-p+2)>0$$

holds for all $t\in[0,1]$.

Example 3.2. Let $v(t) = u(t) = 1 - t/2$, $p^+ = p$, $p^- = m$ be the constants. Then function (3.15) is bounded and condition (3.16) holds. By Theorem 3.1, the inequality $p < 1$ is necessary for $(p, m, v, v) \in \mathcal{U}$. The function $t \mapsto v(t)/t$ does not increase, therefore, the equality $p \geq m - 2$ is a sufficient condition from Theorem 3.2 and $t_2 = 1$ in Theorems 3.3, 3.4.

If $m > p + 2$ in Theorem 3.3, we have

$$t_3 = \frac{t_1 - 2/(m-p)}{t_1 + 1 - 1/(m-p)} \text{ if } t_1 > \frac{2}{m-p}.$$

So, by Theorem 3.3, we obtain the following necessary and sufficient conditions.

Theorem 3.6. *Let $v(t) = 1 - t/2$, $t \in [0, 1]$, $p^+ = p$, $p^- = m$ be the non-negative constants. Then $(p, m, v, v) \in \mathcal{U}$ if and only if*

(1) $p < 1$;

(2) $m \leq p + 2$, *or* $p < m - 2$, *and the inequality*

$$(m-p)(m^2 - p^2 + m)t^3 + (m^2p + mp^2 - p^3 - 3m^2 + 3p^2 - 2m + p)t^2 \\ + (4m + 3p + 1)(m - p + 1)t - 2mp + 2p^2 - 2m - 4p - 2 > 0$$

holds for all $t \in [2/(m-p), 1]$.

Example 3.3. Let $v(t) = u(t) = 1$, $p^+ = p$, $p^- = m$ be the constants. Then function (3.15) is bounded and condition (3.16) holds. By Theorem 3.1, the inequality $p < 2$ is necessary for $(p, m, v, v) \in \mathcal{U}$. The function $t \mapsto v(t)/t$ does not increase, therefore, the equality $m \leq p + 2$ is a sufficient condition from Theorem 3.2 and $t_2 = 1$ in Theorems 3.3, 3.4.

If $m < p + 2$ in Theorem 3.3, we have

$$t_3 = \frac{t_1 - 2/(m-p)}{t_1 + 1 - 1/(m-p)} \text{ if } t_1 > \frac{2}{m-p}.$$

So, by Theorem 3.3, we obtain the following necessary and sufficient conditions.

Theorem 3.7. *Let $v(t) = 1$, $t \in [0, 1]$, $p^+ = p$, $p^- = m$ be the non-negative constants. Then $(p, m, v, v) \in \mathcal{U}$ if and only if*

(1) $p < 2$;

(2) $m \leq p + 2$, *or* $m > p + 2$, *and the inequality*

$$t_3(1 - t_1)(2t_1 - t_3(1 + t_1))\,\frac{m^2 - p^2}{4} + m\Big(\frac{t_1^2}{2} + t_3(1 - t_1)\Big) + p\Big(-\frac{1}{2} + t_3(1 - t_1)\Big) + 1 > 0$$

holds for all $0 \leq t_3 \leq t_1 \leq 1$.

Example 3.4. We can compare the sufficient conditions of Theorems 3.5–3.7 for the Cauchy problem

$$\begin{gathered} \ddot{x}(t) = p(t)x(h(t)) - m(t)x(g(t)) + f(t), \quad t \in [0, 1], \\ x(0) = c_1, \quad \dot{x}(0) = c_2, \end{gathered} \tag{3.17}$$

where $p, m \in \mathbf{L}[0, 1]$ are non-negative functions, $h, g : [0, 1] \to [0, 1]$ are measurable functions.

We use Theorems 3.5, 3.6, 3.7 when the constant p equals zero or 1/2. Everywhere below in this item, P and M are some non-negative constants. Theorem 3.5 gives the following solvability conditions:

$$p(t)h(t) \leq P < \frac{1}{2}, \quad m(t)g(t) \leq M < m_1 \approx 7.435, \quad t \in [0, 1], \tag{3.18}$$

where m_1 is the largest root of the equation

$$64m^6 - 320m^5 - 1168m^4 - 96m^3 + 1212m^2 + 44m + 25 = 0,$$

and

$$p(t) \equiv 0, \quad m(t)g(t) \le M < 8, \quad t \in [0,1]. \tag{3.19}$$

Theorem 3.6 gives the conditions

$$p(t)\Big(1 - \frac{h(t)}{2}\Big) \le P < \frac{1}{2}, \quad m(t)\Big(1 - \frac{g(t)}{2}\Big) \le M < m_2 \approx 10.2, \;\; t \in [0,1], \tag{3.20}$$

where m_2 is the largest root of the equation

$$y(m,t_1) \equiv (2m-1)(4m^2+4m-1)t_1^3 \\ + (-8m^3 - 20m^2 - 14m + 9)t_1^2 + 2(8m+5)(2m+1)t_1 - 4(6m+7) = 0$$

when t_1 is the largest root of the equation $y(t,m)'_t = 0$;

$$p(t) \equiv 0, \quad m(t)\Big(1 - \frac{g(t)}{2}\Big) \le M < \frac{a}{12} + \frac{893}{6a} + \frac{11}{3} \approx 10.7, \;\; t \in [0,1], \tag{3.21}$$

where $a = \sqrt[3]{75716 + \sqrt{330}^3}$.

Thus Theorem 3.7 gives

$$p(t) \le P < \frac{1}{2}, \quad m(t) \le M < m_3 \approx 15.5, \;\; t \in [0,1], \tag{3.22}$$

where $m_3 = \min\limits_{0 < t_3 \le t_1 < 1} r(t_3,t_1)$, $r(t_3,t_1)$ is the largest root of the equation

$$-t_3(1-t_1)(2t_1 - t_3(1+t_1))\Big(r^2 - \frac{1}{4}\Big) + (4t_3(1-t_1) + 2t_1^2)r + 2(1-t_1)t_3 + 3 = 0,$$

and

$$p(t) \equiv 0, \quad m(t) \le M < 16, \;\; t \in [0,1]. \tag{3.23}$$

Each of these conditions (3.18)–(3.23) is sufficient for the unique solvability of problem (3.17), none of them follow from the others, the constants on the right-hand sides of all these inequalities cannot be increased. These conditions do not follow from the contraction mapping principle.

4 The case of proportional functions p^+, p^-

When the functions p^+, p^- are proportional, we can take $p^+ = p^-$ without loss of generality. But it is more convenient to consider

$$p^+(t) = P^+ p_0(t), \quad p^-(t) = P^- p_0(t),$$

where $p_0 \in \mathbf{L}[0,1]$ is a non-negative function, P^+, P^- are the non-negative constants. In fact, we have to consider only $P^+ > 0$, $P^- > 0$, because if one of the numbers P^+, P^- equals zero, then this case can be considered for $v = v$, as in the previous section.

Add to the first column in (2.11) the forth column multiplied by $\frac{u_2 P^+}{v_1 P^-}$, add to the second column the third column multiplied by $\frac{u_1 P^+}{v_2 P^-}$. Then add to the second column the first column multiplied by $-\frac{v_1}{v_2}$. So, we obtain

$$\Delta = \begin{vmatrix} 1 & 0 & \dfrac{P^-(\mathcal{C}p_0)(t_1)}{u(s_1)} & \dfrac{P^-(\mathcal{C}\xi p_0)(t_1)}{u(s_2)} \\ 0 & 1 & \dfrac{P^-(\mathcal{C}p_0)(t_2)}{u(s_1)} & \dfrac{P^-(\mathcal{C}\xi p_0)(t_2)}{u(s_2)} \\ 0 & \dfrac{u(s_1)P^+}{v(t_2)P^-} & 1 + \dfrac{P^-(\mathcal{C}p_0)(s_1)}{u(s_1)} & \dfrac{P^-(\mathcal{C}\xi p_0)(s_1)}{u(s_2)} \\ \dfrac{u(s_2)P^+}{v(t_1)P^-} & 0 & \dfrac{u(s_2)}{u(s_1)} + \dfrac{P^-(\mathcal{C}p_0)(s_2)}{u(s_1)} & 1 + \dfrac{P^-(\mathcal{C}\xi p_0)(s_2)}{u(s_2)} \end{vmatrix}$$
$$= \alpha - (\mathcal{C}_v \xi p_0)(t_1)\alpha P^+ + (\mathcal{C}_v \xi p_0)(t_2)\beta P^+ - (\mathcal{C}_u \xi p_0)(s_1)\beta P^- + (\mathcal{C}_u \xi p_0)(s_2)\alpha P^-,$$

where

$$\alpha = 1 + P^-(G_u p_0)(s_1) - P^+(G_v p_0)(t_2), \quad \beta = 1 + P^-(G_u p_0)(s_2) - P^+(G_v p_0)(t_1), \tag{4.1}$$

$$(\mathcal{C}_w z)(t) = \frac{(\mathcal{C} z)(t)}{w(t)} = \frac{\int_0^t (t-s) z(s)\, ds}{w(t)}, \ \ t \in [0,1], \ \text{ for any } z \in \mathbf{L}[0,1] \text{ and for } w = u \text{ or } w = v.$$

If $\xi \equiv 0$ and $s_1 = 0$, we have $\Delta = 1 - P^+(G_v p_0)(t_2)$. Therefore, the inequality

$$P^+(G_v p_0)(t) < 1, \ \ t \in [0,1], \tag{4.2}$$

is necessary for $(P^+ p_0, P^- p_0, v, u) \in \mathcal{U}$. Let further inequality (4.2) be fulfilled. Then $\alpha > 0$, $\beta > 0$ for all points t_1, t_2, s_1, s_2.

Further, we have

$$\Delta = \alpha + \int_0^1 \xi(s) p_0(s) \phi(s)\, ds,$$

where

$$\phi(s) = -\alpha P^+ \frac{(t_1 - s)_+}{v(t_1)} + \beta P^+ \frac{(t_2 - s)_+}{v(t_2)} - \beta P^- \frac{(s_1 - s)_+}{u(s_1)} + \alpha P^- \frac{(s_2 - s)_+}{u(s_2)}. \tag{4.3}$$

Obviously, for the given t_1, t_2, s_1, s_2, we need to verify the inequality $\Delta > 0$ only for

$$\xi(s) = \begin{cases} 0 & \text{if } \ s \in E^+, \\ 1 & \text{if } \ s \in E^-, \end{cases} \tag{4.4}$$

where

$$E^- = \big\{ s \in [0,1] : \ \phi(s) < 0 \big\}, \quad E^+ = \big\{ s \in [0,1] : \ \phi(s) \geq 0 \big\}. \tag{4.5}$$

Since ϕ is a piecewise linear function, each of the sets E^+, E^- is a union of not more than two intervals ends of which can be easily found for all given t_1, t_2, s_1, s_2. Thus we have proved the main assertion of this section.

Theorem 4.1. *Suppose that functions $v, u \in \mathbf{C}[0,1]$ are positive on $(0,1]$ and if $v(0) = 0$ or $u(0) = 0$, then for this function, conditions* (3.15) *and* (3.16) *are satisfied. Let $p_0 \in \mathbf{L}[0,1]$ be a non-negative function, $P^+ \geq 0$, $P^- \geq 0$. Let for every t_1, t_2, s_1, s_2, the parameters ϕ, α, E^- be defined by equalities* (4.1), (4.3), (4.5), *respectively.*

Then $(P^+ p_0, P^- p_0, v, u) \in \mathcal{U}$ if and only if

(1) *inequality* (4.2) *holds;*

(2) *for all t_1, t_2, s_1, $s_2 \in (0,1]$, the inequality*

$$\alpha + \int_{E^-} p_0(s) \phi(s)\, ds > 0$$

is fulfilled.

Every solution of problem (2.7) corresponding to the function ξ (see (4.4)) satisfies the equation

$$\ddot{x}(t) = p_0(t) \begin{cases} \dfrac{x(t_2) P^+}{v(t_2)} - \dfrac{x(s_1) P^+}{u(s_1) P^-} & \text{if } \ s \in E^+, \\[2ex] \dfrac{-x(s_2) P^-}{u(s_2)} + \dfrac{x(t_1) P^+}{v(t_1) P^+} & \text{if } \ s \in E^-. \end{cases}$$

We are interested only in solutions of (2.7) for which equalities (2.3), (2.4) are fulfilled, therefore, for such solutions $x(t_2) \geq 0$, $x(s_2) \geq 0$, $x(t_1) \leq 0$, $x(s_1) \leq 0$. Thus, all solutions of (2.7) corresponding to function (4.4) have the same sign of the second derivative on each of the sets E^+ and E^-: $\ddot{x}(t) \geq 0$, $t \in E^+$, $\ddot{x}(t) \leq 0$, $t \in E^-$. Such considerations make it possible to reduce the enumeration of mutual dispositions of points t_1, t_2, s_1 when applying Theorem 4.1.

Example 4.1. Let $v(t) \equiv \mathbf{1}$, $u(t) = t$, $t \in [0,1]$, $p_0(t) \equiv \mathbf{1}$. Then conditions (3.15), (3.16) are fulfilled, the functions $t \mapsto t/v(t)$ and $t \mapsto t/u(t)$ do not decrease. Arguments similar to those in the proof of Lemma 3.2 allow to consider in Theorem 4.1 only the points t_1, t_2, s_1, s_2, and the set E^- that satisfy the conditions

$$E^- = [0,r], \quad r \le \min(t_1, s_1), \;\; t_2 = s_2 = 1.$$

Thus, taking in Theorem 4.1 $P^+ = 1/2$, $P^+ = 1$, $P^+ = 2$, we obtain some unimprovable solvability conditions for problem (3.17) (here, P and M are some non-negative constants):

$$p(t) \le P < \frac{1}{2}, \quad m(t)g(t) \le M < m_4 \approx 7.537, \;\; t \in [0,1], \tag{4.6}$$

$$p(t) \le P < 1, \quad m(t)g(t) \le M < m_5 \approx 7, \;\; t \in [0,1], \tag{4.7}$$

where m_4 and m_5 equal $\min\limits_{0<r\le s_1 \le 1} M(r,s_1)$, $M(r,s_1)$ is the largest root of the equation

$$(2 + M\,s1 - P^+)(2 + M\,r(2-r) - P^+\,r^2) + (2 + M - P^+r^2)\Big(\frac{Mr(r-2s_1)}{s_1} + P^+r(2-r)\Big) = 0$$

for $P^+ = 1/2$ and $P^+ = 1$, respectively;

$$p(t) \le P < 2, \quad m(t)g(t) \le M < m_6 \approx 4.5, \;\; t \in [0,1], \tag{4.8}$$

where m_6 is the largest root of the equation $m^3 - 2m^2 - 8m + 8 = 0$.

Note that conditions (3.19) complement these results for $P^+ = 0$.

Each of conditions (4.6)–(4.8) is sufficient for the unique solvability of problem (3.17), none of them follow from the others and from conditions (3.18)–(3.23), the constants in the right-hand sides of all these inequalities cannot be increased. All these conditions are not a consequence of the contraction mapping principle.

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