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AN EXAMPLE OF CONTRASTING COMBINATION TO STABILITY AND INSTABILITY PROPERTIES IN EVEN-DIMENSIONAL SPACES

Abstract. It was constructed examples of even-dimensional differential systems having, in a some sense, contrary stability and instability properties of different types: Lyapunov, Perron and upper-limit. Namely, all nonzero solutions of these systems tend to zero at infinity but nevertheless move away at the fixed distance from the origin at least once. In addition, these systems have a zero first approximation along the zero solution.

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Key words and phrases. Differential equations, nonlinear systems, Lyapunov stability, Perron stability, upper-limit stability.

რეზიუმე. აგებულია ლუწგანზომილებიანი დიფერენციალური სისტემების მაგალითები, რომლებსაც, გარკვეული გაგებით, სხვადასხვა ტიპის (ლიაპუნოვის, პერონისა და ზედა ზღვრის აზრით) სტაბილურობისა და არასტაბილურობის საპირისპირო თვისებები აქვს. კერძოდ, ამ სისტემების ყველა არანულოვანი ამონახსნი უსასრულობაში ნულისკენ მიისწრაფვის, მაგრამ ერთხელ მაინც შორდება სათავეს ფიქსირებულ მანძილზე. გარდა ამისა, ამ სისტემებს აქვს ნულოვანი პირველი მიახლოება ნულოვანი ამონახსნის გასწვრივ.

1 Introduction

The present paper deals with the study of such notions of qualitative theory to differential equations as *Lyapunov stability* and recently introduced *Perron* [4] and *upper-limit* [7] *stability*. This paper is a logical continuation of the papers [1–3] by the present author, where examples of differential systems are constructed that, on the one hand, give some possible relationships and connections between various types of stability and instability and, on the other hand, show that such relationships can be realized in systems with certain special properties.

In [5], I. N. Sergeev constructed an example of a system to differential equations that is completely unstable in the sense of Lyapunov and Perron and has a nonzero solution tending to zero at infinity. The first approximation of the system constructed in [5] along the zero solution is unbounded on the time semi-axis $\mathbb{R}_+$.

- The paper [1] provides an example of a system whose solutions have the same asymptotic properties as in [5], but the first approximation at zero is *bounded* on $\mathbb{R}_+$.
- This result was strengthened in [2], where it was shown that such a system may have an identically *zero* first approximation along the zero solution.
- In [3], the result of [2] was strengthened by the construction of a two-dimensional differential system having both *Perron* and *upper-limit*: on the one hand, *complete instability* (therefore and *Lyapunov global instability*) and, on the other hand, even *massive partial stability*.

The following reinforcement of the above results consists in asserting the existence of systems defined in even-dimensional spaces $\mathbb{R}^n$ of arbitrarily high dimension that simultaneously possess the following properties:

– *Lyapunov global instability* (which was also presented in examples from [1–3]);

– *Perron* and *upper-limit global stability* (unlike all the examples given above).

2 Definitions

For a number $n \in \mathbb{N}$ and a domain G in the Euclidean space $\mathbb{R}^n$ (with the norm $|\cdot|$) containing the origin, consider a system

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}_+ \equiv [0,+\infty), \ \ x \in G, \tag{2.1}$$

with a right-hand side $f : \mathbb{R}_+ \times G \to \mathbb{R}^n$ satisfying the conditions

$$f, f'_x \in C(\mathbb{R}_+ \times G), \quad f(t,0) = 0, \ \ t \in \mathbb{R}_+,$$

so that the system has a zero solution. We denote the set of all nonextendable nonzero solutions of system (2.1) by $\mathcal{S}_*(f)$ and the subset of solutions x with initial condition satisfying $|x(0)| < \delta$ by $\mathcal{S}_\delta(f) \subset \mathcal{S}_*(f)$.

Definition 2.1. We say that system (2.1) (more precisely, its *zero* solution, which we do not mention in the sequel for brevity) has the *Perron* or, respectively, the *upper-limit*

(1) *stability* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the condition

$$\underline{\lim}_{t\to+\infty} |x(t)| < \varepsilon \ \text{ or, respectively, } \ \overline{\lim}_{t\to+\infty} |x(t)| < \varepsilon; \tag{2.2}$$

(2) *instability* if there is no stability, namely, if there exists an $\varepsilon > 0$ such that for any $\delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ does not satisfy condition (2.2);

(3) *global instability* if for some $\varepsilon > 0$ no solution $x \in \mathcal{S}_*(f)$ satisfies condition (2.2);

(4) *global stability* if any solution $x \in \mathcal{S}_*(f)$ satisfies the condition

$$\underline{\lim}_{t\to+\infty} |x(t)| = 0 \ \text{ or, respectively, } \ \lim_{t\to+\infty} |x(t)| = 0. \tag{2.3}$$

Definition 2.2. Let us compare to each property introduced in Definition 1, its *Lyapunov* analogue, namely:

(1) *stability*, *instability* and *global instability* are obtained by repeating, respectively, the descriptions from steps (1)–(3) of Definition 2.1 with replacement in them condition (2.2) by the condition

$$\sup_{t\in\mathbb{R}_+} |x(t)| < \varepsilon; \tag{2.4}$$

(2) *global stability* takes place if system (2.1) has the Lyapunov stability and any solution $x \in \mathcal{S}_*(f)$ satisfies the second condition of (2.3).

We point out that in these Definitions, conditions (2.2)–(2.4) are considered not satisfied, in particular, even for the case where a solution x is not defined on the entire semi-axis $\mathbb{R}_+$, i.e., if the phase curve corresponding to this solution reaches the boundary of the phase domain G in a finite time (for the theorem on the extension of solutions, see, e.g., [6, Theorem 23]).

Here, we have not mentioned some varieties of Perron and upper-limit properties (as well as their Lyapunov analogue) — no less interesting [7, 9], but not studied in this paper – such as *asymptotic stability* or *instability, partial stability, massive partial stability, complete instability, particular instability*, and also have not indicated which of them are massive directly by definition and which ones are *dotty*, but allow the reinforcement to be massive.

3 Lemmas and theorem

The main result of this paper is that we prove the existence of even-dimensional differential systems in higher-dimensional Euclidean spaces such that, first, all nonzero solutions to these systems tend to zero in the norm as $t \to +\infty$, so that these systems have both Perron and upper-limit global stability; second, all nonzero solutions move away at the fixed distance from the origin at least once, so that these systems, nevertheless, have global Lyapunov instability.

The systems the existence of which is asserted in the following theorem are *non-autonomous* and *non-one-dimensional*, and not accidentally:

- autonomous systems with such properties do not exist [8], since the Lyapunov global instability in the autonomous case entails both Perron and upper-limit global instability;
- such one-dimensional systems do not exist [7], because in the one-dimensional case the upper-limit global stability entails Lyapunov global stability, as well.

Theorem. *For any number $n = 2, 4, 6, \ldots$ and $G = \mathbb{R}^n$, there exists system* (2.1) *with a right-hand side satisfying the condition*

$$f'_x(t, 0) = 0, \quad t \in \mathbb{R}_+,$$

that possesses the following two properties:

(1) *for each solution x of system* (2.1)*, the following equality holds:*

$$\lim_{t\to+\infty} |x(t)| = 0;$$

(2) *for each nonzero solution x of system* (2.1)*, the following inequality holds:*

$$\sup_{t\in\mathbb{R}_+} |x(t)| > 1.$$

Remark. The existence of systems with all the properties of the theorem in spaces of arbitrary odd dimensions is currently an open question.

To the proof of the theorem, we preface it with two technical lemmas.

Lemma 3.1. *The function* $\theta_1 : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *given by the conditions*

$$\theta_1(\rho, \varphi) = \begin{cases} \varphi, & 0 < \rho \leqslant 1; \\ \varphi + \dfrac{\pi}{2} \Big(\displaystyle\int_1^{\rho} \chi_1(\tau)\, d\tau \Big) \Big(\displaystyle\int_1^{2} \chi_1(\tau)\, d\tau \Big)^{-1}, & 1 < \rho < 2; \\ \varphi + \dfrac{\pi}{2}\,, & \rho \geqslant 2, \end{cases} \tag{3.1}$$

where

$$\chi_1(v) = \begin{cases} 0, & v \leqslant 1 \ \ or \ \ v \geqslant 2; \\ e^{\frac{1}{(v-1)(v-2)}}, & 1 < v < 2, \end{cases}$$

has the following properties:

(1) *for each fixed value of the variable* $\rho \in (0, +\infty)$, *the function* $\theta_1(\rho, \cdot\,)$ *is a bijection from the line* $\mathbb{R}$ *into itself and satisfies the inequality*

$$\frac{\partial \theta_1}{\partial \varphi}(\rho, \varphi) > 0, \quad \varphi \in \mathbb{R};$$

(2) *it is an infinitely differentiable function in a set of variables.*

Lemma 3.2. *The function* $\theta_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *given by the conditions*

$$\theta_2(t, u) = \begin{cases} u, & u \leqslant \dfrac{1}{3+t}\,; \\ u - \displaystyle\int_{\frac{1}{3+t}}^{u} \chi_2(t, \tau)\, d\tau, & \dfrac{1}{3+t} < u < \dfrac{1}{2+t}\,; \\ \dfrac{1}{1+t}\, u + \dfrac{1}{2+t} \Big(1 - \dfrac{1}{1+t}\Big) - \Sigma(t), & u \geqslant \dfrac{1}{2+t}\,, \end{cases} \tag{3.2}$$

where

$$\chi_2(t, v) = \begin{cases} 0, & v \leqslant \dfrac{1}{3+t}\,; \\ \sigma(t) \displaystyle\int_{\frac{1}{3+t}}^{v} e^{\frac{1}{(\tau - \frac{1}{3+t})(\tau - \frac{1}{2+t})}}\, d\tau, & \dfrac{1}{3+t} < v < \dfrac{1}{2+t}\,; \\ \dfrac{t}{1+t}\,, & v \geqslant \dfrac{1}{2+t}\,, \end{cases} \tag{3.3}$$

$$\sigma(t) = \frac{t}{1+t} \Big(\int_{\frac{1}{3+t}}^{\frac{1}{2+t}} e^{\frac{1}{(\tau - \frac{1}{3+t})(\tau - \frac{1}{2+t})}}\, d\tau \Big)^{-1}, \quad \Sigma(t) = \int_{\frac{1}{3+t}}^{\frac{1}{2+t}} \chi_2(t, \tau)\, d\tau, \ \ t \in \mathbb{R}_+, \tag{3.4}$$

has the following properties:

(1) *for each fixed value of moment* $t \in \mathbb{R}_+$, *the function* $\theta_2(t, \cdot\,)$ *is a bijection from the line* $\mathbb{R}$ *into itself and satisfies the inequality*

$$\frac{\partial \theta_2}{\partial u}(t, u) > 0, \ \ u \in \mathbb{R};$$

(2) *it is an infinitely differentiable function in a set of variables.*

4 Proofs

4.1 Lemma 3.1

Let us first show that the function $\theta_1(\,\cdot\,,\varphi)$ is a C^∞-smooth function of its argument at each fixed value of the variable $\varphi \in \mathbb{R}$. To do this, first note that there is a constant (and hence a smooth function) on the intervals $(0,1)$ and $(2,+\infty)$. Inside the interval $(1,2)$, there is also a smoothness by the construction, so it remains to check it at two points: 1 and 2.

- The validity of the equalities

$$\theta_1(1+0,\varphi) = \varphi + \frac{\pi}{2}\Bigl(\int\limits_1^{1+0} \chi_1(\tau)\,d\tau\Bigr)\Bigl(\int\limits_1^{2} \chi_1(\tau)\,d\tau\Bigr)^{-1} = \varphi = \theta_1(1-0,\varphi)$$

and

$$\theta_1(2-0,\varphi) = \varphi + \frac{\pi}{2}\Bigl(\int\limits_1^{2} \chi_1(\tau)\,d\tau\Bigr)\Bigl(\int\limits_1^{2} \chi_1(\tau)\,d\tau\Bigr)^{-1} = \varphi + \frac{\pi}{2} = \theta_1(2+0,\varphi)$$

proves a continuity at the points 1 and 2, respectively.

- Further, the validity of the equalities

$$\theta'_{1\rho}(1+0,\varphi) = \frac{\pi}{2}\,\chi_1(1+0)\Bigl(\int\limits_1^{2} \chi_1(\tau)\,d\tau\Bigr)^{-1} = 0 = \theta'_{1\rho}(1-0,\varphi)$$

and

$$\theta'_{1\rho}(2+0,\varphi) = 0 = \frac{\pi}{2}\,\chi_1(2-0)\Bigl(\int\limits_1^{2} \chi_1(\tau)\,d\tau\Bigr)^{-1} = \theta'_{1\rho}(2-0,\varphi)$$

proves a continuous differentiability at the points 1 and 2, respectively.

Now, let us note that the equalities

$$\chi_1^{(m)}(1) = \chi_1^{(m)}(2) = 0, \quad m = 0,1,2,\ldots,$$

hold, due to which and equalities (3.1), as well as a smoothness of the function $\chi_1 \in C^\infty(\mathbb{R})$, the function $\theta_1(\,\cdot\,,\varphi)$ is an infinitely differentiable function for each fixed value of variable $\varphi \in \mathbb{R}$. It follows from this and from equalities (3.1) that the function θ_1 has continuous in the set of variables (ρ,φ) derivatives of all orders on ρ and φ everywhere on the Cartesian product $(0,+\infty)\times\mathbb{R}$, so the property (2) of this lemma is valid.

Finally, we note that for every fixed value of the variable $\rho \in (0,+\infty)$, the function $\theta_1(\rho,\,\cdot\,)$ is linear everywhere on the line $\mathbb{R}$, and therefore it is a monotonic (increasing) and one-to-one function from the line $\mathbb{R}$ into itself. The property (1) follows from the above reasoning. □

4.2 Lemma 3.2

Let us first show that the function $\theta_2(t,\,\cdot\,)$ is a C^∞-smooth function of its argument at every fixed value of a moment $t \in \mathbb{R}_+$. Indeed, this is a linear (and hence smooth) function everywhere on the rays $(-\infty, \frac{1}{3+t}]$ and $[\frac{1}{2+t}, +\infty)$. In the interval $(\frac{1}{3+t}, \frac{1}{2+t})$, there is also a smoothness by the construction, so it remains to check it at the two points: $\frac{1}{3+t}$ and $\frac{1}{2+t}$.

- The validity of the equalities

$$\theta_2\Bigl(t, \frac{1}{3+t}+0\Bigr) = \frac{1}{3+t} - \int\limits_{\frac{1}{3+t}}^{\frac{1}{3+t}+0} \chi_2(t,\tau)\,d\tau = \frac{1}{3+t} = \theta_2\Bigl(t, \frac{1}{3+t}-0\Bigr)$$

and

$$\theta_2\Big(t,\frac{1}{2+t}-0\Big)=\frac{1}{2+t}-\int\limits_{\frac{1}{3+t}}^{\frac{1}{2+t}}\chi_2(t,\tau)\,d\tau=\frac{1}{2+t}-\Sigma(t)=\theta_2\Big(t,\frac{1}{2+t}+0\Big)$$

proves a continuity at the points $\frac{1}{3+t}$ and $\frac{1}{2+t}$, respectively.

- In the same way, the validity of the equalities

$$\theta'_{2u}\Big(t,\frac{1}{3+t}+0\Big)=1-\chi_2(t,\tau)\Big|_{\tau=\frac{1}{3+t}+0}=1=\theta'_{2u}\Big(t,\frac{1}{3+t}-0\Big)$$

and

$$\theta'_{2u}\Big(t,\frac{1}{2+t}-0\Big)=1-\chi_2(t,\tau)\Big|_{\tau=\frac{1}{2+t}-0}$$
$$=1-\sigma(t)\Bigg(\int\limits_{\frac{1}{3+t}}^{\frac{1}{2+t}}e^{\frac{1}{(\tau-\frac{1}{3+t})(\tau-\frac{1}{2+t})}}\,d\tau\Bigg)=1-\frac{t}{1+t}=\frac{1}{1+t}=\theta'_{2u}\Big(t,\frac{1}{2+t}+0\Big)$$

proves a continuous differentiability at the points $\frac{1}{3+t}$ and $\frac{1}{2+t}$, respectively.

Now, note that due to an infinite differentiability of the function χ_2 (which follows from the equalities (3.3) and (3.4)), as well as the following equalities

$$\frac{\partial^m\chi_2}{\partial v^m}\Big(t,\frac{1}{3+t}\Big)=\frac{\partial^m\chi_2}{\partial v^m}\Big(t,\frac{1}{2+t}\Big)=0,\ \ t\in\mathbb{R}_+,\ \ m=1,2,3,\dots,$$

the function $\theta_2(t,\cdot)$ is an infinitely differentiable function of its argument for each fixed moment $t\in\mathbb{R}_+$. Let us also show that it is strictly monotone (increasing) and one-to-one function from the line $\mathbb{R}$ into itself.

Indeed, due to the validity for the non-negative function χ_2 of the estimate

$$\chi_2(t,u)=\sigma(t)\Bigg(\int\limits_{\frac{1}{3+t}}^{u}e^{\frac{1}{(\tau-\frac{1}{3+t})(\tau-\frac{1}{2+t})}}\,d\tau\Bigg)\leqslant\frac{t}{1+t}=1-\frac{1}{1+t}<1,\ \ t\in\mathbb{R}_+,\ \ u\in\Big(\frac{1}{3+t},\frac{1}{2+t}\Big),$$

for the derivative

$$\theta'_{2u}(t,u)=\begin{cases}1, & u\leqslant\dfrac{1}{3+t};\\ 1-\chi_2(t,u), & \dfrac{1}{3+t}<u<\dfrac{1}{2+t};\\ \dfrac{1}{1+t}, & u\geqslant\dfrac{1}{2+t},\end{cases}$$

the inequality

$$\theta'_{2u}(t,u)>0,\ \ t\in\mathbb{R}_+,\ \ u\in\mathbb{R},$$

holds. It follows from this fact that the function $\theta_2(t,\cdot)$ is monotonous and injective.

Further, note that for a fixed value of a moment $t\in\mathbb{R}_+$, the relations

$$\lim_{u\to-\infty}\theta_2(t,u)=-\infty\ \text{ and }\ \lim_{u\to+\infty}\theta_2(t,u)=+\infty$$

are satisfied, by virtue of which and a continuity of the function $\theta_2(t,\cdot)\in C^\infty(\mathbb{R})$ we obtain that $\theta_2(t,\cdot)$ takes all intermediate values from $-\infty$ up to $+\infty$, i.e., it is a surjective mapping from the line $\mathbb{R}$ into itself. Hence property (1) of lemma is valid.

It remains to note that, due to the previously proved smoothness of the function $\theta_2(t,\cdot)\in C^\infty(\mathbb{R})$, as well as equalities (3.2)–(3.4), the function θ_2 has continuous in the set of variables (t,u) derivatives of all orders for t and u everywhere on the Cartesian product $\mathbb{R}_+\times\mathbb{R}$, whence the property (2) follows. □

4.3 Theorem

Let us construct a system described in the theorem in four steps.

I. On a plane with coordinates x_1, x_2, we introduce the polar coordinates ρ, φ ($\rho \geqslant 0$, $\varphi \in \mathbb{R}$):

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi, \tag{4.1}$$

and consider the autonomous two-dimensional system

$$\begin{pmatrix}\dot\rho\\ \dot\varphi\end{pmatrix} = \rho^2\begin{pmatrix}2-\rho\\ \sin^2\varphi + \chi(\varphi)\end{pmatrix}, \quad \rho^2 \equiv x_1^2 + x_2^2, \ \ x \equiv (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{4.2}$$

where

$$\chi(\varphi) = \begin{cases}\cos^2\varphi, & \cos\varphi < 0;\\ 0, & \cos\varphi \geqslant 0.\end{cases}$$

Equality (4.2) determines a vector field in the area $(0, +\infty) \times \mathbb{R}$ of the variables (ρ, φ), which can be transferred to the punctured plane $\mathbb{R}^2 \setminus \{0\}$ of the variables (x_1, x_2) with the help of the local diffeomorphism given by equalities (4.1). The resulting field can be continuously extended to the origin by zero, and the extended field is continuously differentiable at the origin and has a zero linear approximation due to a presence of the multiplier ρ^2 in a right-hand side of the system. The above-mentioned implies that system (4.2) admits a zero solution. This means that it is a system of the form (2.1).

The first equation of system (4.2) determines a dynamical system on a line (more exactly, on the half-line $\mathbb{R}_+$, since $\rho \geqslant 0$) having two fixed points 0 and 2. It is obvious that if $\rho(0)$ is not equal to 0 and 2, then $\rho(t) \to 2$ as $t \to +\infty$.

A. Let us first consider all nonzero solutions of this system satisfying the condition $\rho(0) < 2$. There are two types of such solutions.

Type 1. For solutions x satisfying the initial condition $\varphi(0) = 0$, the relations

$$\varphi(t) = 0, \ \ t \in \mathbb{R}_+, \ \ \rho(t) \to 2 \ \text{ as } \ t \to +\infty \tag{4.3}$$

are valid.

Type 2. Let us consider the solutions satisfying the condition $\varphi(0) \neq 0$. We note that the angular coordinate $\varphi(t)$ of each such solution x to system (4.2) increases as $t \to +\infty$, so at some moment $t > 0$, it necessarily enters the area (4^{th} quarter inside the circle of radius 2)

$$V^{\bullet}{}_{4;2} \equiv \left\{x \in \mathbb{R}^2 \mid \ x_1^2 + x_2^2 < 4, \ x_1 > 0, \ x_2 < 0\right\},$$

in which the function χ is zero everywhere. In this area, system (4.2) takes the following form:

$$\begin{pmatrix}\dot\rho\\ \dot\varphi\end{pmatrix} = \rho^2\begin{pmatrix}2-\rho\\ \sin^2\varphi\end{pmatrix}, \tag{4.4}$$

consequently, for the solutions under consideration the following relations are satisfied:

$$\varphi(t) \to 2\pi, \ \ \rho(t) \to 2 \ \text{ as } \ t \to +\infty. \tag{4.5}$$

B. Let us consider the solutions satisfying the condition $\rho(0) > 2$. There are two types of such solutions to system (4.2).

Type 3. For solutions x satisfying the initial condition $\varphi(0) = 0$, relations (4.3) are valid.

Type 4. Let us consider the solutions satisfying the condition $\varphi(0) \neq 0$. We note that the angular coordinate $\varphi(t)$ of each such solution x to system (4.2) increases as $t \to +\infty$, so at some moment $t > 0$, it necessarily enters the area (4^{th} quarter outside the circle of radius 2)

$$V^{\circ}{}_{4;2} \equiv \left\{x \in \mathbb{R}^2 \mid \ x_1^2 + x_2^2 > 4, \ x_1 > 0, \ x_2 < 0\right\},$$

in which the function χ is zero everywhere. In this area, system (4.2) takes form (4.4), therefore, for solutions under consideration relations, (4.5) are satisfied.

C. Let us now consider the solutions satisfying the initial condition $\rho(0) = 2$. Such solutions may also be of two types.

Type 5. The point $e_1 = (2, 0)^{\mathrm{T}}$ is singular for system under consideration (4.2) and, as is shown in items A and B, represents the interior and exterior attraction point of a circle of radius 2.

Type 6. Let us consider the solutions x satisfying the condition $\varphi(0) \neq 0$. The angular coordinate $\varphi(t)$ of each solution to this type satisfies the relation $\varphi(t) \to 2\pi$ as $t \to +\infty$, so such solutions, like all other nonzero solutions of system (4.2), are asymptotically attracted to the singular point e_1 as $t \to +\infty$.

II. Let us make in system (4.2) an autonomous replacement of the coordinates given by the equalities

$$\begin{cases} \varrho = \rho, \\ \phi = \theta_1(\rho, \varphi), \end{cases} \tag{4.6}$$

where the function θ_1 is given by equalities (3.1).

It follows from Lemma 3.1 that for every fixed value of the variable $\rho \in (0, +\infty)$ there exists an inverse function to the function $\theta_1(\rho, \cdot\,)$, which, if we consider it as a function of two arguments ρ and ϕ, is infinitely differentiable on the Cartesian product $(0, +\infty) \times \mathbb{R}$. It also should be noted that for $0 < \rho \leqslant 1$, we have

$$\theta_1(\rho, \varphi) = \varphi, \tag{4.7}$$

therefore, the vector field obtained by replacing (4.6) with a local diffeomorphism given by the equalities

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \varrho\cos\phi \\ \varrho\sin\phi \end{pmatrix}, \quad \varrho \equiv y_1^2 + y_2^2, \tag{4.8}$$

is transferred to the punctured plane $\mathbb{R}^2 \setminus \{0\}$ of the coordinates (y_1, y_2), is predetermined in a continuously differentiable way to the origin by zero, and the system thus obtained, determining this field, also has a zero linear approximation along the zero solution due to the fact that the same property, as proven earlier, system (4.2) has.

So, an autonomous differential system with zero solution is constructed as

$$\dot{y} = f_1(y), \quad y \equiv (y_1, y_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{4.9}$$

with a right-hand side $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the conditions

$$f_1, f'_{1y} \in C(\mathbb{R}^2), \;\; f_1(0) = 0, \;\; f'_{1y}(0) = 0,$$

a qualitative behaviour of whose solutions in a circular 1-circle of the origin is exactly the same as those of system (4.2) has. Let us show now that for all nonzero solutions y of the constructed system (4.9), the relations

$$\lim_{t\to+\infty} y_1(t) = 0 \;\text{ and }\; \lim_{t\to+\infty} y_2(t) = 2 \tag{4.10}$$

are valid.

Indeed, according to the above proof, the singular point e_1 is an attraction point of the whole punctured plane $\mathbb{R}^2 \setminus \{0\}$ in the variables (x_1, x_2) for system (4.2). Therefore, by virtue of the equality $\theta_1(2, 0) = \frac{\pi}{2}$ and equalities (4.6), (4.8), which determine the diffeomorphism, the point $e_2 = (0, 2)^{\mathrm{T}}$ is an attraction point of the punctured plane $\mathbb{R}^2 \setminus \{0\}$ in the variables (y_1, y_2) for system (4.9). This fact shows the validity of equality (4.10).

III. In system (4.9), let us substitute the variables by the given equalities

$$\begin{cases} z_1 = y_1, \\ z_2 = \theta_2(t, y_2), \end{cases} \tag{4.11}$$

where the function $\theta_2 \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ is given by (3.2). It follows from Lemma 3.2 that for each fixed value of a moment $t \in \mathbb{R}_+$ to the function $\theta_2(t, \cdot)$, there exists an inverse function, which, if we consider it as a function of two arguments t and z_2, is infinitely differentiable everywhere on the Cartesian product $\mathbb{R}_+ \times \mathbb{R}$. Let us show that the constructed non-autonomous two-dimensional differential system

$$\dot{z} = f_2(t, z), \quad t \in \mathbb{R}_+, \ \ z \equiv (z_1, z_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{4.12}$$

with a right-hand side $f_2 : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the condition

$$f_2, f'_{2z} \in C(\mathbb{R}_+ \times \mathbb{R}^2),$$

possesses all the properties specified in the formulation of this theorem.

To do this, let us first note that in the stripe

$$U(t) \equiv \Big\{ y \in \mathbb{R}^2 \mid \ |y_2| < \frac{1}{3+t} \Big\}$$

replacement (4.11) takes the form

$$z = y, \tag{4.13}$$

therefore, system (4.12) constructed at this stage admits a zero solution and possesses a zero linear approximation along the zero solution, since system (4.9) has the same property.

Further, in the unit semicircle

$$V^{\bullet}_{3,4;1} \equiv \big\{ z \in \mathbb{R}^2 \mid \ z_1^2 + z_2^2 \leqslant 1, \ z_2 < 0 \big\},$$

in which all nonzero solutions z to system (4.12) satisfying the initial condition $|z(0)| \leqslant 1$, enter at least once, replacement (4.11) also takes form (4.13) and hence the form $z = x$, what follows from equalities (4.6) and (4.7). Hence, just as the previously studied qualitative behaviour of solutions to system (4.2), it follows that system (4.12) possesses the property (2) of this theorem.

It remains to note that for the non-negative function $\Sigma(\cdot)$ determined by equality (3.4), the following estimates are valid:

$$\begin{aligned} \Sigma(t) = &\int\limits_{\frac{1}{3+t}}^{\frac{1}{2+t}} \chi_2(t,\tau)\, d\tau \\ &\leqslant \Big(\frac{1}{2+t} - \frac{1}{3+t}\Big) \max_{\tau \in [\frac{1}{3+t}, \frac{1}{2+t}]} \chi_2(t,\tau) \leqslant \Big(\frac{1}{2+t} - \frac{1}{3+t}\Big) \frac{t}{1+t} \to 0, \ \ t \to +\infty, \end{aligned}$$

and therefore, uniformly in $y_2 \in [1, 3]$, we have

$$\theta_2(t, y_2) = \frac{1}{1+t}\, y_2 + \frac{1}{2+t}\Big(1 - \frac{1}{1+t}\Big) - \Sigma(t) \leqslant \frac{3}{1+t} + \frac{1}{2+t} - \Sigma(t) \to 0, \ \ t \to +\infty,$$

whence it follows that system (4.12) possesses the property (1) of this theorem.

IV. Let us now consider the n-dimensional (the number n is even) differential system

$$\dot{u}^i = g_i(t, u^i), \ \ u^i \equiv (u_{2i-1}, u_{2i})^{\mathrm{T}} \in \mathbb{R}^2, \ \ g_i \equiv f_2, \ \ i = 1, 2, \ldots, \frac{n}{2}, \ \ t \in \mathbb{R}_+, \ \ u \equiv (u^1, \ldots, u^{\frac{n}{2}})^{\mathrm{T}}, \tag{4.14}$$

and show that it possesses all the properties specified in the statement of this theorem.

For this purpose, let us first note that this is a system of form (2.1) and it has a zero linear approximation along the zero solution, since, by virtue of the above, the functions $g_i : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfy the conditions

$$g_i, g'_{i u^i} \in C(\mathbb{R}_+ \times \mathbb{R}^2), \ \ g_i(t, 0) = 0, \ \ g'_{i u^i}(t, 0) = 0, \ \ t \in \mathbb{R}_+, \ \ i = 1, 2, \ldots, \frac{n}{2},$$

and hence the function $g \equiv (g_1, \dots, g_{\frac{n}{2}})^{\mathrm{T}} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions

$$g, g'_u \in C(\mathbb{R}_+ \times \mathbb{R}^n), \;\; g(t,0) = 0, \;\; g'_u(t,0) = 0, \;\; t \in \mathbb{R}_+.$$

Let us now consider an arbitrary nonzero solution $u = (u^1, \dots, u^{\frac{n}{2}})^{\mathrm{T}}$ of system (4.14).

Each of its components u^i satisfies the system

$$\dot{u}^i = f_2(t, u^i), \;\; t \in \mathbb{R}_+, \;\; u^i \in \mathbb{R}^2,$$

hence the equality $\lim\limits_{t\to+\infty} |u^i(t)| = 0$ is valid for it, since all nonzero solutions of system (4.12) have the same property, according to what was proved above. Therefore, $\lim\limits_{t\to+\infty} |u(t)| = 0$, so system (4.14) under consideration possesses the property (1) of this theorem.

This solution is $u \in \mathcal{S}_*(g)$, so at least one of its components, say u^k, satisfies the condition $u^k(0) \neq 0$. Thus it follows that

$$\sup_{t\in\mathbb{R}_+} |u(t)| \geqslant \sup_{t\in\mathbb{R}_+} |u^k(t)| > 1,$$

since system (4.12) also has the same property by virtue of what was proved above. Consequently, system (4.14) also has the property (2) of this theorem. □

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