Memoirs on Differential Equations and Mathematical Physics

Volume 87, 2022, 17–24

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ON ASYMPTOTIC EQUIVALENCE OF n-TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. This paper is devoted to the problem of asymptotic equivalence of n-th order differential equations with exponentially equivalent right-hand sides. With the help of the obtained result asymptotic behavior of solutions to perturbed differential equations is described.

2010 Mathematics Subject Classification. 34C41.

Key words and phrases. Higher-order nonlinear equation, asymptotic equivalence.

რეზიუმე. ნაშრომი ეძღვნება ექსპონენციალურად ეკვივალენტური მარჯვენა მხარეების მქონე n-ური რიგის დიფერენციალური განტოლებების ასიმპტოტური ეკვივალენტობის პრობლემას. მიღებული შედეგის დახმარებით აღწერილია შეშფოთებული დიფერენციალური განტოლებების ამონახსნების ასიმპტომური ყოფაქცევა.

1 Introduction

We study the problem of asymptotic equivalence of the equations

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)|y(x)|^k \operatorname{sgn} y(x) = f(x) \tag{1.1}$$

and

$$z^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)|z(x)|^k \operatorname{sgn} z(x) = 0 \tag{1.2}$$

with $n \geq 2$, $k > 1$, and continuous functions $p(x)$, $f(x)$ and $a_j(x)$. Equation (1.2) is a so-called Emden–Fowler type differential equation. It was considered from different points of view (see, e.g., [3,12] and the references there). In particular, the asymptotic behavior of its solutions vanishing at infinity is described (see also [2,4,13]). So, if an asymptotic equivalence of equations (1.1) and (1.2) exists, it is possible to describe the asymptotic behavior of vanishing at infinity solutions to equation (1.1), too. Previous results are formulated in [1,5–7,11]. The asymptotic equivalence of ordinary differential equations and their systems can be useful to investigate some problems for partial differential equations (see, e.g., [10]). Note that the notion of asymptotic equivalence can be used in different senses (cf. [8,9,14–19]).

Hereafter we denote $|y|^k \operatorname{sgn} y$ by $[y]^k_{\pm}$.

2 Asymptotic equivalence of nonlinear perturbed differential equations

Theorem 2.1. *Let $a_0, \dots, a_{n-1}$, p, f, and g be continuous functions defined in a neighborhood of ∞. Suppose $p(x)$, $f(x)$ and $g(x)$ are bounded while $a_0, \dots, a_{n-1}$ satisfy the inequalities*

$$\int_{x_0}^{\infty} x^{n-j-1} |a_j(x)| \, dx < \infty, \quad j \in \{0, \dots, n-1\}. \tag{2.1}$$

If y is a solution to the equation

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x) \, [y(x)]^k_{\pm} = f(x) \, e^{-\gamma x} \tag{2.2}$$

with $n \geq 2$, $k > 1$, $\gamma > 0$ and $y(x) \to 0$ as $x \to +\infty$, then there exists a unique solution z to the equation

$$z^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) z^{(j)}(x) + p(x) \, [z(x)]^k_{\pm} = g(x) \, e^{-\gamma x} \tag{2.3}$$

such that $|z(x) - y(x)| = O(e^{-\gamma x})$ as $x \to +\infty$.

Lemma 2.1. *Any linear differential operator*

$$L : y \mapsto y^{(n)} + \sum_{j=0}^{n-1} a_j y^{(j)} \tag{2.4}$$

with all continuous functions $a_j(x)$ satisfying (2.1) can be represented in a neighbourhood of $+\infty$ as the composition operator

$$L = D_b = b_0 B_1 \circ \dots \circ B_n,$$

where $b = (b_0, b_1, \dots, b_n)$, all B_j, $j = 1, \dots, n$, are the first-order operators $u \mapsto (b_j u)'$ and each b_j, $j = 0, \dots, n$, is a $\mathcal{C}^j$ function satisfying at infinity the following conditions:

(i) $b_j(x) \to 1$,

(ii) $x^i b_j^{(i)}(x) \to 0$ *for all* $i \in \{1, \dots, j-1\}$,

(iii) $\int\limits_{x_0}^{\infty} x^{i-1} |b_j^{(i)}(x)|\, dx < \infty$ *for all* $i \in \{1, \dots, j\}$ *and some* $x_0 \in \mathbb{R}$.

Now, for $b = (b_0, b_1, \dots, b_n)$ and $j \in \{0, \dots, n\}$, put

$$b - j = (b_j, \dots, b_n).$$

Note that if a tuple b satisfies the conditions from Lemma 2.1, then so does the tuple $b - j$.

Lemma 2.2. *Let $b = (b_0, b_1, \dots, b_n)$ satisfy the conditions from Lemma 2.1. If a function y satisfies at infinity both $y \to 0$ and $D_b(y) \to 0$, then the same is true for all functions $D_{b-j}(y)$, $0 < j < n$.*

Proof. Suppose the contrary, i.e., that for some $j \in \{1, \dots, n-1\}$, the function $D_{b-j}(y)$ does not tend to zero. Consider the greatest of those j.

Since $b_j \to 1$ as $x \to \infty$ for all $j \in \{0, \dots, n\}$, we can assume the inequality $\beta < b_j < \beta^{-1}$ to hold for all those j and for some common $\beta \in (0; 1)$. Without loss of generality, we can also assume that for some $\varepsilon > 0$ there exists a sequence of points $x_i \to \infty$ such that $D_{b-j}(y)(x_i) > \varepsilon$. Let x_i' be the closest point to the right of x_i such that $D_{b-j}(y)(x_i') = \beta\varepsilon$. Such a point exists. Indeed, otherwise $D_{b-j}(y) = b_j (D_{b-(j+1)}(y))' > \beta\varepsilon$ on $[x_i; \infty)$, whence

$$\begin{aligned} D_{b-(j+1)}(y)(x) &= D_{b-(j+1)}(y)(x_i) + \int\limits_{x_i}^{x} \frac{D_{b-j}(y)(s)\, ds}{b_j(s)} \\ &\qquad > D_{b-(j+1)}(y)(x_i) + \beta^2 \varepsilon (x - x_i) \to \infty \ \text{ as } \ x \to \infty, \\ D_{b-(j+2)}(y)(x) &= D_{b-(j+2)}(y)(x_i) + \int\limits_{x_i}^{x} \frac{D_{b-(j+1)}(y)(s)\, ds}{b_{j+1}(s)} \to \infty, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_n(x) y(x) = D_{b-n}(y)(x) &= D_{b-n}(y)(x_i) + \int\limits_{x_i}^{x} \frac{D_{b-(n-1)}(y)(s)\, ds}{b_{n-1}(s)} \to \infty, \end{aligned}$$

which contradicts the assumption of Lemma 2.2 that $y \to 0$. So, $D_{b-j}(y) \ge \beta\varepsilon$ on $[x_i; x_i']$. To complete the proof we need the following

Lemma 2.3. *Suppose a tuple $b = (b_0, b_1, \dots, b_n)$ satisfies the conditions from Lemma 2.1 and a function y satisfies, on a segment I of length Δ, the inequality $|D_{b-j}(y)| \ge W$ with some $j \in \{1, \dots, n\}$ and a constant $W > 0$. Then there exists a segment $I' \subset I$ of length $4^{j-n}\Delta$ such that $|y| \ge (2^{j-n}\beta)^{n+1-j} W \Delta^{n-j}$ on I'.*

Proof. Still assuming $\beta < b_j < \beta^{-1}$ to hold for all $j \in \{0, \dots, n\}$ and for some common $\beta \in (0; 1)$, we prove the lemma by reverse induction on j. For $j = n$, the statement is trivial since if $|D_{b-n}(y)| = |b_n y| \ge W$, then $|y| \ge \beta W$.

Suppose it is proved for some $j > 1$ and on a segment I of length Δ the inequality $|D_{b-(j-1)}(y)| \ge W > 0$ holds.

Since the derivative of the function $D_{b-j}(y)$ equals $D_{b-(j-1)}(y)/b_{j-1}$ and hence does not vanish on I, the function itself is monotone there and therefore can vanish at most at a single point.

If both $D_{b-j}(y)$ and $D_{b-(j-1)}(y)$ are non-negative at the middle point c of the segment I, then on the last quarter of I we have

$$D_{b-j}(y)(x) \ge D_{b-j}(y)(c) + \beta W \cdot (x - c) \ge \frac{\beta W \Delta}{4} > 0.$$

For other sign combinations of $D_{b-j}(y)(c)$ and $D_{b-(j-1)}(y)(c)$ we can prove by the same way the inequality

$$|D_{b-j}(y)| \geq W' = \frac{\beta W \Delta}{4} > 0$$

to hold on either the first or last quarter $I' \subset I$ of length $\Delta' = \Delta/4$.

Now, according to the induction hypothesis, there exists a segment $I'' \subset I'$ of length $4^{j-n}\,\Delta' = 4^{(j-1)-n}\,\Delta$, where the function y satisfies

$$\begin{aligned} |y| &\geq (2^{j-n}\,\beta)^{n+1-j} W'(\Delta')^{n-j} = (2^{j-n}\,\beta)^{n+1-j} \cdot \frac{\beta W \Delta^{1+n-j}}{4^{1+n-j}} \\ &= \beta(2^{j-n-2}\,\beta)^{n+1-j} \cdot W\Delta^{n-(j-1)} = (2^{(j-1)-n}\,\beta)^{n+1-(j-1)} W\Delta^{n-(j-1)}. \end{aligned}$$

So, the statement for $(j-1)$ and Lemma 2.3 are proved. □

Now we continue proving Lemma 2.2.

We have a sequence of segments $[x_i; x_i']$ such that $D_{b-j}(y) \geq \beta\varepsilon$ on each of them as well as $D_{b-j}(y)(x_i) \geq \varepsilon$ and $D_{b-j}(y)(x_i') = \beta\varepsilon$ on their ends.

By Lemma 2.3, there exist the segments $[x_i''; x_i'''] \subset [x_i; x_i']$ with the inequality

$$|y| \geq (2^{j-n}\,\beta)^{n+1-j} \beta\varepsilon (x_i' - x_i)^{n-j}$$

holding on each of them.

Since by assumption $y \to 0$, the length of the segments $[x_i; x_i']$ must also tend to zero. Now we can choose a sequence of points $c_i \in [x_i; x_i']$ with

$$|D_{b-(j-1)}(y)(c_i)| = b_{j-1}(c_i)\left|\frac{D_{b-j}(y)(x_i') - D_{b-j}(y)(x_i)}{x_i' - x_i}\right| \geq \frac{\varepsilon - \beta\varepsilon}{x_i' - x_i} \to \infty.$$

This contradicts the choice of j as the smallest of those with $D_{b-j}(y)$ non-tending to zero. So, Lemma 2.2 is proved. □

Corollary 2.1. *Under the conditions of Theorem 2.1, a function y is a solution to equation* (2.2) *tending to zero as $x \to +\infty$ if and only if*

$$b_n y = (J_{n-1} \circ \cdots \circ J_0)\Big[e^{-\gamma x} f(x) - p(x)\,[y(x)]_{\pm}^{k}\ \Big], \tag{2.5}$$

where the operators J_j take each sufficiently rapidly decreasing continuous function φ to the vanishing at infinity primitive function of φ/b_j:

$$J_j[\varphi](x) = -\int_x^\infty \frac{\varphi(\xi)}{b_j(\xi)}\, d\xi.$$

Proof. Under the conditions of Theorem 2.1, equation (2.2) can be written, in a neighborhood of $+\infty$, as

$$D_b(y)(x) = e^{-\gamma x} f(x) - p(x)\,[y(x)]_{\pm}^{k}\,. \tag{2.6}$$

So, if a solution y to (2.6) tends to 0 as $x \to \infty$, then so does $D_b(y)$. By Lemma 2.2, the same is true for all functions $D_{b-j}(y)$, $0 < j < n$, which ensures that we can obtain (2.5) from (2.6) by successively (for $j = 0, \ldots, n-1$) applying the formula

$$D_{b-(j+1)}(y) = J_j[D_{b-j}(y)], \tag{2.7}$$

which is true whenever its left-hand side tends to zero at infinity.

For the converse statement, first, note that any function satisfying (2.5) tends to 0 due to the definition of the operators J_j. To prove that such a function satisfies (2.6), we also successively (for $j = n-1, \ldots, 0$) apply the same formula (2.7) to equation (2.5) with its left-hand side treated as $D_{b-n}(y)$, whereafter take into account that functions having equal images under J_j must be equal to each other. □

Proof of Theorem 2.1. Suppose that y is a vanishing at infinity solution to equation (2.2). Let $M > 0$ be a common upper bound for $|f|$, $|g|$, and $|p|$ on their domains and

$$H = \frac{3M}{\beta^{n+1}\,\gamma^n}\,. \tag{2.8}$$

Consider the space $\mathcal{H}$ of all continuous functions $\eta : [x_*, +\infty) \to [-H;\, H]$, where x_* is a sufficiently large positive constant such that all the functions $y(x)$, $f(x)$, $g(x)$, and $p(x)$ are defined on $[x_*, +\infty)$ and, moreover, the values $e^{-\gamma x_*}$ and $Y = \sup\{|y(x)| :\ x \ge x_*\}$ are sufficiently small to ensure

$$k(Y + He^{-\gamma x_*})^{k-1} \le H^{-1}. \tag{2.9}$$

Now we define an operator $R : \mathcal{H} \to C[x_*; \infty)$ by the formula

$$R(\eta)(x) = p(x)\Big(\,[y(x)]_{\pm}^k - \big[y(x) + \eta(x)e^{-\gamma x}\big]_{\pm}^k\Big) + e^{-\gamma x}(g(x) - f(x)).$$

Taking into account the inequality

$$\big|\,[a]_{\pm}^k - [b]_{\pm}^k\,\big| \le k \max\big\{|a|, |b|\big\}^{k-1}|a - b|$$

as well as (2.8) and (2.9), we obtain, for $\eta \in \mathcal{H}$, that

$$|R(\eta)(x)| \le Mk(Y + He^{-\gamma x_*})^{k-1}He^{-\gamma x} + 2Me^{-\gamma x} \le MH^{-1}He^{-\gamma x} + 2Me^{-\gamma x} = 3Me^{-\gamma x}.$$

This allows us to define an operator $F : \mathcal{H} \to C[x_*; \infty)$ by

$$F(\eta)(x) = \frac{e^{\gamma x}(J_{n-1} \circ \cdots \circ J_0 \circ R)[\eta](x)}{b_n(x)} \tag{2.10}$$

and to note that $|F(\eta)(x)| \le e^{\gamma x}\gamma^{-n}\beta^{-n-1}\,3Me^{-\gamma x} = H$ for all $\eta \in \mathcal{H}$, i.e., $F(\mathcal{H}) \subset \mathcal{H}$. Similar estimates show that F is a contraction. Indeed, suppose $\eta_1, \eta_2 \in \mathcal{H}$ and

$$\delta = \sup\big\{|\eta_1(x) - \eta_2(x)| :\ x \ge x_*\big\}.$$

Then

$$\big|R(\eta_1)(x) - R(\eta_2)(x)\big| \le Mk(Y + He^{-\gamma x})^{k-1}\delta\, e^{-\gamma x_*} \le \frac{M\delta e^{-\gamma x}}{H}$$

for all $x \ge x_*$, and therefore

$$\big|F(\eta_1)(x) - F(\eta_2)(x)\big| \le \frac{e^{\gamma x}M\delta e^{-\gamma x}}{H\beta^{n+1}\gamma^n} = \frac{\delta}{3}\,.$$

So, F is a contraction and there exists a unique $\eta \in \mathcal{H}$ such that $F(\eta) = \eta$. Taking into account (2.10), this can be written as

$$e^{\gamma x}(J_{n-1} \circ \cdots \circ J_0 \circ R)[\eta](x) = b_n(x)\,\eta(x)$$

or, taking into account the definition of R and putting $z = y + \eta\, e^{-\gamma x}$, as

$$(J_{n-1} \circ \cdots \circ J_0)\Big[p \cdot \big(\,[y]_{\pm}^k - [z]_{\pm}^k\,\big) + e^{-\gamma x}(g - f)\Big] = b_n{\cdot}(z - y).$$

Since y is a vanishing at infinity solution to equation (2.2), we can use Corollary 2.1 to remove in the last equality all terms with y and f:

$$(J_{n-1} \circ \cdots \circ J_0)\big[e^{-\gamma x}g - p\,[z]_{\pm}^k\,\big] = b_n z.$$

Now the same Corollary 2.1 ensures z to be a solution to equation (2.3). By definition, z also satisfies $|z(x) - y(x)| = O(e^{-\gamma x})$ as $x \to \infty$. Suppose there exist two functions $z_1(x)$ and $z_2(x)$ defined on some half-line $[c; \infty)$, $c \ge x_*$, and satisfying the statement of Theorem 2.1.

Then $D = \sup\{\, e^{\gamma x}|z_1(x) - z_2(x)| : \ x \geq c\} < \infty$. Moreover, both $z_1(x)$ and $z_2(x)$ tend to zero as $x \to +\infty$ and therefore satisfy

$$b_n z_j = (J_{n-1} \circ \cdots \circ J_0)\big[e^{-\gamma x} g - p\,[z_j]_{\pm}^{k}\,\big], \ \ j = 1, 2.$$

So, putting

$$Z_c = \sup\Big\{\max\big\{|z_1(x)|, |z_2(x)|\big\} : \ x \geq c\Big\},$$

we obtain

$$e^{\gamma x}|z_1(x) - z_2(x)| \leq e^{\gamma x} \cdot \frac{MkZ_c^{k-1} D e^{-\gamma x}}{\beta^{n+1}\gamma^n}\,, \ \ \text{whence} \ \ D \leq \frac{MkZ_c^{k-1}}{\beta^{n+1}\gamma^n} \cdot D.$$

Now, choosing c large enough, we can make Z_c to become sufficiently small so that the last inequality holds only if $D = 0$. So, the uniqueness is proved. □

Corollary 2.2. *Suppose that the function $f(x)$ in equation* (1.1) *is continuous and satisfies the condition*

$$|f(x)| \leq Ce^{-\gamma x}, \ \ C > 0, \ \ \gamma > 0, \tag{2.11}$$

$p(x)$ is a bounded continuous function, and $a_0, \dots, a_{n-1}$ are continuous functions satisfying (2.1).

Then for any solution $y(x)$ to equation (1.1) *tending to zero as $x \to \infty$, there exists a solution $z(x)$ to equation* (1.2) *such that*

$$|y(x) - z(x)| = O(e^{-\gamma x}), \ \ x \to \infty. \tag{2.12}$$

Similarly, for any solution $z(x)$ to equation (1.2) *tending to zero as $x \to \infty$, there exists a solution $y(x)$ to equation* (1.1) *satisfying* (2.12).

Acknowledgments

The research was partially supported by Russian Science Foundation (scientific project # 20-11-20272).

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(Received 15.08.2022; accepted 08.09.2022)

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