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A UNIFORM DISCRETIZATION FOR SOLVING SINGULARLY PERTURBED CONVECTION-DIFFUSION BOUNDARY VALUE PROBLEMS

Abstract. In this paper, a discrete scheme is presented for solving singularly perturbed convection-diffusion equations. The stability and convergence of the proposed scheme are analyzed in the discrete maximum norm. Error estimates are carried out for both Bakhvalov (B-mesh) and Shishkin-type (S-mesh) meshes. Three numerical examples are solved to authenticate the theoretical findings.

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Key words and phrases. Bakhvalov mesh, boundary value problem, convection-diffusion equation, difference scheme, error estimate, Shishkin mesh, singular perturbation.

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1 Introduction

Singularly perturbed problems of convection-diffusion equations are crucial tools to explain mathematical modelling of many physical events and various biological phenomena. Their practices appear in electronic circuit systems [4], control theory [7], thermo-elasticity [30], direct current motors [29], chemical-reactor theory [2], population dynamics [24], spread of HIV infection [6], modelling of semiconductor devices [35] and heat transfer problems [32] (see also the references therein).

This article deals with a singularly perturbed convection-diffusion boundary value problem of the form

$$Lu = \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad 0 < x < l, \tag{1.1}$$

$$u(0) = A, \ u(l) = B, \tag{1.2}$$

where $0 < \varepsilon < 1$ is the perturbation parameter, $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$ and $f(x)$ are sufficiently smooth functions.

In general, the class of such problems involves the boundary layers. Whereas the solution of these problems changes quickly within a layer region, it behaves slowly and uniformly outside the layer region. Due to the presence of the perturbation parameter, traditional numerical approaches are not suitable when $\varepsilon \to 0$. Thus it is important to develop uniformly convergent numerical methods with respect to ε. For more details about singular perturbation theory, one can refer to [25,27,28,33].

Lots of techniques have been introduced to accomplish this complexity. Finite element and discontinuous Galerkin methods were applied in [2,21,22,32,36,39–41]. The reproducing kernel method was performed in [7]. The exponentially fitted collocation method was used in [18]. The streamline upwind Petrov–Galerkin technique was implemented in [20]. By using the fourth-order Runge–Kutta formula, the initial value method was considered in [24]. The fitted mesh method was suggested in [4,35]. The fitted non-polynomial cubic spline method was introduced in [14].

In addition to the above-mentioned techniques, various numerical schemes have been established on different meshes to obtain numerical solution of singularly perturbed convection-diffusion boundary value problems. In [19,26], hybrid type difference schemes were proposed. High-order finite difference schemes were presented in [10,17,23]. Numerov type scheme was constructed in [37]. The exponentially fitted difference scheme was established on a uniform mesh in [3]. In [38], by using Richardson extrapolation, the convergence order of the difference scheme was improved. Another schemes have been described in a series of articles [8,12,15,16,34].

Recently, different kinds of singularly perturbed convection-diffusion boundary value problems (SPCDBVPs) have been considered. This type of problems with integral boundary conditions were studied in [5, 30, 31]. Their nonlocal forms were examined in [2, 5]. Delay type SPCDBVPs were discretized on layer-adapted meshes in [6,9].

This paper aims to present uniform and robust discretization for solving SPCDBVPs and compare the obtained results for both B-mesh and S-mesh.

The structure of the study is as follows: The asymptotic behavior of the exact solution is given in Section 2. In Section 3, by using interpolating quadrature rules and linear basis functions, the finite difference scheme is constructed on a non-uniform mesh. Section 4 is devoted to the error analysis according to the node points of Shishkin and Bakhvalov meshes. In Section 5, numerical experiments are included and the obtained results are tabulated.

Notation 1. *Throughout the paper, C, C_0 and C_1 are generic positive constants independent of the perturbation parameter. For any continuous function $v(x)$, we use the maximum norm*

$$\|v\|_\infty = \max_{[0,l]} |v(x)|.$$

2 Some feautures of the analytical solution

In this section, we give the asymptotic behavior of the exact solution and its derivative that is required to analyze the stability and convergence.

Lemma 2.1. *The solution* $u(x)$ *of problem* (1.1), (1.2) *holds the following estimates:*

$$\|u\|_\infty \leq C_0 \tag{2.1}$$

and

$$|u'(x)| \leq C\Big\{1 + \frac{1}{\varepsilon}\, e^{-\frac{\alpha x}{\varepsilon}}\Big\}, \tag{2.2}$$

where

$$C_0 = |A| + |B| + \alpha^{-1}\|f\|_\infty.$$

Proof. First, we prove (2.1). For this purpose, we define the following barrier function:

$$\overline{\psi}^{\pm}(x) = \pm u(x) + |A| + |B| + \alpha^{-1}(l - x)\|f\|_\infty.$$

By applying the maximum principle to the barrier function, we obtain

$$\overline{\psi}^{\pm}(0) = |A| + |B| + \alpha^{-1}l\|f\|_\infty \pm A \geq 0,$$
$$\overline{\psi}^{\pm}(l) = |A| + |B| \pm B \geq 0.$$

Thus we can write

$$\begin{aligned} L\psi(x) &= -(|A| + |B|)b(x) + \frac{a(x)\|f\|_\infty}{\alpha} - \frac{b(x)(l - x)\|f\|_\infty}{\alpha} \pm f(x) \\ &\leq -(|A| + |B|)b(x) - \|f\|_\infty \pm f(x) \leq 0. \end{aligned}$$

According to the maximum principle, we find the following relations:

$$|A| + |B| + \frac{(l - x)\|f\|_\infty}{\alpha} \pm u(x) \geq 0,$$
$$|u(x)| \leq |A| + |B| + \frac{1}{\alpha}(l - x)\|f\|_\infty,$$

which hints at the proof of (2.1). Now, we show the proof of (2.2). Rewriting another form of equation (1.1), we have

$$\varepsilon u''(x) + a(x)u'(x) = F(x), \tag{2.3}$$

where

$$F(x) = f(x) + b(x)u(x).$$

From (2.3), we get

$$u'(x) = u'(0)\, e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)\,d\tau} + \frac{1}{\varepsilon}\int_0^x F(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau}\, d\xi. \tag{2.4}$$

By integrating (2.4) on $(0, l)$, we write

$$B - A = u'(0)\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)\,d\tau}\, dx + \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau}\, d\xi\, dx.$$

Hence we obtain

$$u'(0) = \frac{B - A - \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau}\, d\xi\, dx}{\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)\,d\tau}\, dx}. \tag{2.5}$$

Denominator of (2.5) is evaluated as

$$\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)\,d\tau}\, dx \geq \int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a^*\,d\tau}\, dx = \int_0^l e^{-\frac{a^* x}{\varepsilon}}\, dx = \frac{\varepsilon}{a^*}\,(1 - e^{\frac{-a^* l}{\varepsilon}}) = \gamma_0\varepsilon, \tag{2.6}$$

where

$$\gamma_0 \neq \gamma_0\varepsilon > 0 \ \ (a^* = \max_{(0,l]} a(x)).$$

Applying the mean-value theorem to the integral term in (2.5), we get

$$\begin{aligned}\Big| -\frac{1}{\varepsilon}\int_0^l \Big[\int_0^x F(\xi)\exp\Big(-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau\Big)\,d\xi\Big]\,dx\Big| &\le \frac{1}{\varepsilon}\int_0^l \Big[\int_0^x |F(\xi)|\exp\Big(-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau\Big)\,d\xi\Big]\,dx\\ &\le \frac{\|F\|_\infty}{\varepsilon}\int_0^l\int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}}\,d\xi\,dx = \frac{\|F\|_\infty}{\varepsilon}\,\frac{\varepsilon}{\alpha}\int_0^l [1-e^{-\frac{\alpha x}{\varepsilon}}]\,dx \le \frac{\|F\|_\infty l}{\alpha} = C_1. \quad (2.7)\end{aligned}$$

Taking into consideration (2.6) and (2.7), we find

$$|u'(0)| \le \frac{|A|+|B|+C_1}{\gamma\varepsilon} = \frac{C}{\varepsilon}\,. \tag{2.8}$$

Using relation (2.8), we obtain the following estimation:

$$\begin{aligned}|u'(x)| &= u'(0)\,e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)\,d\tau} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)\,d\tau}\,d\xi\\ &\le \frac{C}{\varepsilon}\,e^{-\frac{a^*x}{\varepsilon}} + \frac{\|F\|_\infty}{\alpha}(1-e^{-\frac{\alpha x}{\varepsilon}}) \le \frac{C}{\varepsilon}e^{-\frac{a^*x}{\varepsilon}} + \frac{\|F\|_\infty}{\alpha}\,.\end{aligned}$$

The proof of (2.2) is reached. □

3 Discretization

In this section, we construct the difference scheme on a non-uniform mesh. Let ω_N be any non-uniform mesh on $[0,l]$:

$$\omega_N = \big\{0 < x_1 < x_2 < \cdots < x_{N-1} < l,\ h_i = x_i - x_{i-1}\big\}$$

and

$$\overline{\omega}_N = \omega_N \cup \{x = 0\}.$$

Before constructing difference scheme, we define some notation for the mesh functions. For any mesh function $v(x)$ defined on $\overline{\omega}_N$, we use the following implicit difference rules:

$$\begin{aligned} v_i = v(x_i), \quad v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}\,,\\ v_{x,i} = \frac{v_{i+1}-v_i}{h_{i+1}}\,, \quad v_{\bar{x}\hat{x},i} = \frac{1}{\hbar_i}\,(v_{x,i} - v_{\bar{x},i})\,.\end{aligned}$$

The discrete maximum norm is denoted by

$$\|v\|_\infty = \|v\|_{\infty,\overline{\omega}_N} = \max_{0\le i\le N} |v_i|.$$

To design the difference method for problem (1.1), (1.2), we start by using the following integral identity:

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu\varphi_i\,dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \big(\varepsilon u'' + a(x)u' - b(x)u\big)\varphi_i\,dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i\,dx, \tag{3.1}$$

where the basis function

$$\varphi_i = \begin{cases} \varphi_i^{(1)} = \dfrac{(x - x_{i-1})}{h_i}, & x \in (x_{i-1}, x_i), \\ \varphi_i^{(2)} = \dfrac{(x_{i+1} - x)}{h_{i+1}}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}). \end{cases}$$

φ_i is a solution of the following problems:

$$\varepsilon \varphi_i^{(1)''} = 0, \quad \varphi_i^{(1)}(x_i) = 1, \quad \varphi_i^{(1)}(x_{i-1}) = 0,$$
$$\varepsilon \varphi_i^{(2)''} = 0, \quad \varphi_i^{(2)}(x_i) = 1, \quad \varphi_i^{(2)}(x_{i+1}) = 0.$$

Moreover,

$$\hbar_i = \frac{1}{2}(h_i + h_{i+1})$$

and

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i \, dx = \hbar_i^{-1} \Big(\frac{h_i}{2} + \frac{h_{i+1}}{2} \Big) = 1.$$

For the first term of (3.1), appyling interpolating quadrature rules in [1], we have

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \varepsilon u''(x) \varphi_i \, dx = -\hbar_i^{-1} \varepsilon \int_{x_{i-1}}^{x_i} u' \varphi_i^{(1)'} \, dx - \hbar_i^{-1} \varepsilon \int_{x_i}^{x_{i+1}} u' \varphi_i^{(2)'} \, dx$$
$$= -\hbar_i^{-1} \varepsilon u_{\bar{x},i} \int_{x_{i-1}}^{x_i} \varphi_i^{(1)'} \, dx - \hbar_i^{-1} \varepsilon u_{x,i} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)'} \, dx = \hbar_i^{-1} \varepsilon (u_{x,i} - u_{\bar{x},i}) = \varepsilon u_{\bar{x}\hat{x},i}. \quad (3.2)$$

For the second term of (3.1), we obtain

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} a(x) u'(x) \varphi_i \, dx = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)] u'(x) \varphi_i \, dx + \hbar_i^{-1} a_i \int_{x_{i-1}}^{x_{i+1}} u'(x) \varphi_i \, dx$$
$$= -\hbar_i^{-1} a_i \int_{x_{i-1}}^{x_{i+1}} u(x) \varphi_i' \, dx + R_{a,i} = a_i u_{x,i} + R_{a,i}, \quad (3.3)$$

where

$$R_{a,i} = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)] u'(x) \varphi_i \, dx.$$

For the third term of (3.1), we can write

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} b(x) u(x) \varphi_i \, dx = \hbar_i^{-1} \Bigg[\int_{x_{i-1}}^{x_{i+1}} [b(x) - b(x_i)] u(x) \varphi_i \, dx + b(x_i) \int_{x_{i-1}}^{x_{i+1}} u(x) \varphi_i \, dx \Bigg]$$
$$= \hbar_i^{-1} b_i \int_{x_{i-1}}^{x_{i+1}} u(x) \varphi_i \, dx + R_{b,i} = b_i \hbar_i^{-1} u_i \int_{x_{i-1}}^{x_{i+1}} \varphi_i \, dx + R_i^{(1)} + R_{b,i} = b_i u_i + R_i^{(1)} + R_{b,i}, \quad (3.4)$$

where

$$R_{b,i} = \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} [b(x) - b(x_i)] u(x) \varphi_i \, dx,$$

and the error term from the interpolating quadrature rule is indicated by

$$R_i^{(1)} = b_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} dx \varphi_i(x) \int\limits_{x_{i-1}}^{x_{i+1}} \frac{du(\xi)}{d\xi} \left[T_0(x - \xi) - \frac{(x - x_{i-1})}{\hbar_i} T_0(x_{i+1} - \xi) \right] d\xi.$$

Here, T_0 is computed by the following formula:

$$T_s(\lambda) = \begin{cases} \dfrac{\lambda^s}{s!}, & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases}$$

For the right-hand side of (3.1), we get

$$\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} f(x) \varphi_i(x) \, dx = \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} [f(x) - f(x_i)] \varphi_i(x) \, dx + \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} f(x_i) \varphi_i(x) \, dx = f_i + R_{f,i}, \quad (3.5)$$

where

$$R_{f,i} = \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} [f(x) - f(x_i)] \varphi_i(x) \, dx.$$

Combining (3.2), (3.3), (3.4) and (3.5), we find the following difference scheme:

$$\varepsilon u_{\bar{x}\hat{x},i} + a_i u_{x,i} - b_i u_i = f_i + R_i, \quad i = 1, \dots, N-1, \quad (3.6)$$

$$u_0 = A, \quad u_N = B, \quad (3.7)$$

where

$$R_i = R_{f,i} - (R_i^{(1)} + R_{a,i} + R_{b,i}). \quad (3.8)$$

Neglecting the remainder term in (3.6), for the approximate solution y we obtain the following difference problem:

$$\varepsilon y_{\bar{x}\hat{x},i} + a_i y_{x,i} - b_i y_i = f_i, \quad i = 1, \dots, N-1, \quad (3.9)$$

$$y_0 = A, \quad y_N = B. \quad (3.10)$$

4 The stability and convergence

To research the robustness of the presented scheme, let u_i be the solution of problem (3.6), (3.7) and let y_i be the solution of problem (3.9), (3.10). The error function $z_i = y_i - u_i$, $i = 0, 1, 2, \dots, N$, is the solution of following problem:

$$\varepsilon z_{\bar{x}\hat{x},i} + a_i z_{x,i} - b_i z_i = R_i, \quad i = 1, \dots, N-1, \quad (4.1)$$

$$z_0 = 0, \quad z_N = 0. \quad (4.2)$$

Lemma 4.1. *For problem* (4.1), (4.2), *the following inequality is satisfied:*

$$|z_i| \leq C h_i \sum_{i=1}^{n-1} |R_i|.$$

Proof. The proof of the lemma is similar to that given in [3]. □

Lemma 4.2. *For all remainder terms in* (3.8)*, the following estimate is valid:*

$$|R_i| \le Ch_i.$$

Proof. Using the mean-value theorem, we get

$$|a(x) - a(x_i)| = |a'(\eta_i)|\,|x - x_i|, \;\; \eta_i \in (x_i, x) \le Ch_i.$$

Thus we can write

$$|R_{a,i}| \le \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_i} Ch_i \varphi_i(x)\,dx \le Ch_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_i} \varphi_i(x)\,dx \le Ch_i.$$

Similarly, it can be shown that $|R_{b,i}| \le Ch_i$ and $|R_{f,i}| \le Ch_i$. Now, we estimate $R_i^{(1)}$:

$$\begin{aligned} |R_i^{(1)}| &\le b_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} dx |\varphi_i(x)| \int\limits_{x_{i-1}}^{x_{i+1}} \Big|\frac{du(\xi)}{d\xi}\Big| \Big| \Big[T_0(x-\xi) - \frac{(x - x_{i-1})}{\hbar_i} T_0(x_{i+1} - \xi) \Big] \Big| d\xi \\ &\le C_0 \hbar_i \int\limits_{x_{i-1}}^{x_{i+1}} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha\xi}{\varepsilon}} \Big) d\xi = C_0 \hbar_i \Big\{ 1 + \alpha^{-1} \big(e^{-\frac{\alpha x_{i+1}}{\varepsilon}} - e^{-\frac{\alpha x_{i-1}}{\varepsilon}} \big) \Big\} \le Ch_i. \end{aligned}$$ □

Lemma 4.3. *For the remainder term R_i of scheme* (3.6), (3.7)*, the following estimate is satisfied on the Bakhvalov mesh:*

$$|y_i - u_i| \le CN^{-1},$$

and for the Shishkin mesh, it is written as

$$|y_i - u_i| \le CN^{-1} \ln N.$$

Proof. First, we consider Bakhvalov-type mesh for solving problem (3.6), (3.7). For an even number N, we divide each of the subintervals $[0, \sigma]$ and $[\sigma, l]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is taken as

$$\sigma = \min_i \Big\{ \frac{l}{2}, \alpha^{-1}\varepsilon |\ln \varepsilon| \Big\}.$$

The node points x_i are specified as follows: if $\sigma < \frac{l}{2}$,

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln \Big[1 - (1-\varepsilon)\,\dfrac{2i}{N} \Big], & i = 0, 1, \ldots, \dfrac{N}{2}, \\ \sigma + \Big(i - \dfrac{N}{2} \Big) h, & i = \dfrac{N}{2} + 1, \ldots, N, \end{cases}$$

and if $\sigma = \frac{l}{2}$,

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln \Big[1 - \Big(1 - \exp\Big(\dfrac{-\alpha l}{2\varepsilon} \Big) \dfrac{2i}{N} \Big) \Big], & i = 0, 1, \ldots, \dfrac{N}{2}, \\ \sigma + \Big(i - \dfrac{N}{2} \Big) h, & i = \dfrac{N}{2} + 1, \ldots, N, \end{cases}$$

where

$$h = \frac{2(l - \sigma)}{N}.$$

According to the grade mesh's node points, we take the following estimations. First, for $x_i \in [0, \sigma]$ and $\alpha < \frac{l}{2}$, we find

$$h_i = x_i - x_{i-1} = \alpha^{-1}\varepsilon \Big\{ \ln \Big(1 - (1-\varepsilon)\,\frac{2(i-1)}{N} \Big) - \ln \Big(1 - (1-\varepsilon)\,\frac{2i}{N} \Big) \Big\}.$$

Applying the mean-value theorem according to i, we obtain

$$h_i = \alpha^{-1}\varepsilon\Big[\frac{2(1-\varepsilon)N^{-1}}{1-2i(1-\varepsilon)N^{-1}}\Big] \le 2\alpha^{-1}(1-\varepsilon)N^{-1} \le 2\alpha^{-1}N^{-1}.$$

Now, we consider $\alpha = \frac{l}{2}$ on the interval $[0,\sigma]$. Hence we have

$$h_i = x_i - x_{i-1} = \alpha^{-1}\varepsilon\Big\{\ln\Big(1-(1-e^{-\frac{\alpha l}{2\varepsilon}})\,\frac{2(i-1)}{N}\Big) - \ln\Big(1-(1-e^{-\frac{\alpha l}{2\varepsilon}})\,\frac{2i}{N}\Big)\Big\}.$$

From here, we can write the inequality

$$e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}} \le 2(l-\varepsilon)N^{-1}.$$

Thus we obtain

$$h_i = x_i - x_{i-1} \le CN^{-1}, h_{i+1} = x_i - x_{i-1} \le CN^{-1}$$

and

$$\hbar_i = \frac{(h_i + h_{i+1})}{2} \le CN^{-1}.$$

For the interval $[\sigma, l]$, taking $\sigma = \frac{l}{2}$, we find that $h = 2(l-\sigma)/N = lN^{-1}$. Consider $\sigma = -\alpha^{-1}\varepsilon\ln\varepsilon$. Using the inequalities

$$\frac{1}{\varepsilon}\,e^{-\frac{\alpha t}{\varepsilon}} \le 1$$

and

$$\hbar_i \le CN^{-1},$$

we can show that

$$|R_i| \le CN^{-1}.$$

Subsequently, we use Shishkin-type mesh for solving problem (3.6), (3.7). For an even number N, we divide each of the subintervals $[0,\sigma]$ and $[\sigma,l]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is taken as

$$\sigma = \min_i\Big\{\frac{l}{2},\ \alpha^{-1}\varepsilon\ln N\Big\}.$$

We use the notation $h^{(1)}$ for the mesh width in $[0,\sigma]$ and the notation $h^{(2)}$ for the width in $[\sigma,l]$. Hence the mesh stepsizes satisfy

$$h^{(1)} = \frac{2\sigma}{N}, \quad h^{(2)} = \frac{2(l-\sigma)}{N},$$
$$h^{(1)} \le lN^{-1}, \quad lN^{-1} \le h^{(2)} \le 2lN^{-1}, \quad h^{(1)} + h^{(2)} = 2lN^{-1}.$$

The node points x_i are specified as

$$\overline{\omega}_N = \begin{cases} x_i = ih^{(1)}, & i = 0,1,\ldots,\frac{N}{2},\ \ x_i \in [0,\sigma], \\ x_i = \sigma + \Big(i - \frac{N}{2}\Big)h^{(2)}, & i = \frac{N}{2}+1,\ldots,N,\ \ x_i \in [\sigma,l]. \end{cases}$$

We evaluate the error approximations according to the node points of Shishkin mesh. First, we consider the case $\sigma = \frac{l}{2}$. Then $\frac{l}{2} \le \alpha^{-1}\varepsilon\ln N$, $h^{(1)} = h^{(2)} = h = lN^{-1}$. Hence we can write

$$|R_i| \le C\big\{N^{-1} + \varepsilon^{-1}lN^{-1}\big\} \le C\big\{N^{-1} + \alpha^{-1}N^{-1}\ln N\big\}.$$

So, we obtain

$$|R_i| \le CN^{-1}\ln N, \ \ i = 1,2,\ldots,N.$$

Now, we deal with the case $\sigma = \alpha^{-1}\varepsilon \ln N$, so that $\alpha^{-1}\varepsilon \ln N < \frac{l}{2}$. We estimate separately R_i on $[0, \sigma]$ and $[\sigma, l]$. In the layer region $[0, \sigma]$, we write the following inequality:

$$|R_i| \le C(1+\varepsilon^{-1})h^{(1)} \le C(1+\varepsilon^{-1})\,\frac{2\alpha^{-1}\varepsilon \ln N}{N}\,.$$

Thus we find

$$|R_i| \le CN^{-1}\ln N, \quad i = 1, 2, \ldots, \frac{N}{2}\,.$$

It remains to estimate R_i for the layer region $[\sigma, l]$. In this case, we have

$$|R_i| \le C\Big\{h^{(2)} + \alpha^{-1}\big(e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}}\big)\Big\}, \quad i = \frac{N}{2}, \ldots, N. \tag{4.3}$$

Since

$$x_i = \alpha^{-1}\varepsilon \ln N + \Big(i - \frac{N}{2}\Big)\, h^{(2)},$$

it follows that

$$\begin{aligned} e^{-\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_i}{\varepsilon}} &= e^{\frac{-\alpha(\alpha^{-1}\varepsilon \ln N + (i-1-\frac{N}{2}))h^{(2)}}{\varepsilon}} - e^{\frac{-\alpha(\alpha^{-1}\varepsilon \ln N + (i-\frac{N}{2}))h^{(2)}}{\varepsilon}} \\ &= \frac{1}{N}\big(e^{\frac{-\alpha(i-1-\frac{N}{2})h^{(2)}}{\varepsilon}} - e^{\frac{-\alpha(i-\frac{N}{2})h^{(2)}}{\varepsilon}}\big) = \frac{1}{N}\, e^{\frac{-\alpha(i-1-\frac{N}{2})h^{(2)}}{\varepsilon}}(1 - e^{\frac{-\alpha h^{(2)}}{\varepsilon}}) \le N^{-1}. \end{aligned}$$

This inequality together with (4.3) give the bound

$$|R_i| \le CN^{-1}.$$

Thu, the proof of the lemma is completed. □

5 Numerical results

In this section, we test the numerical method on three examples. Toward this end, the difference problem (3.9), (3.10) is written obviously

$$\begin{gathered} (\varepsilon \hbar_i^{-1} h_i^{-1})y_{i-1} - (\varepsilon h_i^{-1} h_{i+1}^{-1} + \varepsilon h_i^{-1} h_i^{-1} + a_i h_{i+1}^{-1} + b_i)y_i + h_{i+1}^{-1}(\varepsilon \hbar_i^{-1} + a_i)y_{i+1} = f_i, \\ y_0 = A, \quad y_N = B \quad (i = 1, 2, \ldots, N-1). \end{gathered}$$

We rewrite this expression according to the following form:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \ldots, N-1,$$

where

$$A_i = \varepsilon \hbar_i^{-1} h_i^{-1}, \quad B_i = h_{i+1}^{-1}(\varepsilon \hbar_i^{-1} + a_i), \quad C_i = \varepsilon h_i^{-1} h_{i+1}^{-1} + \varepsilon h_i^{-1} h_i^{-1} + a_i h_{i+1}^{-1} + b_i, \quad F_i = -f_i.$$

Example 5.1 ([13])**.** Consider the problem

$$\begin{gathered} Lu = \varepsilon u''(x) + (1+\varepsilon)u'(x) - u(x) = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 1, \end{gathered}$$

the exact solution of which is $u(x) = \frac{e^{\frac{-x}{\varepsilon}} - e^{-x}}{e^{\frac{-1}{\varepsilon}} - e^{-1}}$. Absolute errors are defined by

$$e^N = \max_{0 \le i < N} |y_i - u_i|$$

and the convergence rates are computed as

$$p_N = \log_2\Big(\frac{e^N}{e^{2N}}\Big).$$

Error approximations and orders of convergence are given in Tables 5.1, 5.2 for different values of ε and N.

Table 5.1. Maximum pointwise errors and order of convergence on B-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.0429922095	0.0225593905	0.0115460020	0.0058356975
	e^{2N}	0.0225353745	0.0115375278	0.0058349400	0.0029327746
	p	0.9318838321	0.9673939687	0.9846036635	0.9926388976
10^{-4}	e^N	0.0423380939	0.0222654226	0.0114280413	0.0057892575
	e^{2N}	0.0222571673	0.0114228719	0.0057878366	0.0029134134
	p	0.9276863301	0.9628795803	0.9814820693	0.9906679103
10^{-6}	e^N	0.0422330787	0.0222095516	0.0114015305	0.0057768729
	e^{2N}	0.0222042150	0.0113970733	0.0057753292	0.0029077320
	p	0.9275398545	0.9625168641	0.9812524155	0.9903944328
10^{-8}	e^N	0.0422201907	0.0222022984	0.0113979575	0.0057751142
	e^{2N}	0.0221973383	0.0113935955	0.0057735942	0.0029068634
	p	0.9275464039	0.9624859339	0.9812337004	0.9903861899
10^{-10}	e^N	0.0422190013	0.0222014454	0.0113975273	0.0057749004
	e^{2N}	0.0221965294	0.0113931768	0.0057733833	0.0029067564
	p	0.9275583324	0.9624835350	0.9812319511	0.9903859112

Table 5.2. Maximum pointwise errors and order of convergence on S-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.0701539163	0.0386089625	0.0203376914	0.0104541171
	e^{2N}	0.0386089625	0.0203125711	0.0104541171	0.0053011524
	p	0.8615878562	0.9265629279	0.9600846989	0.9796933165
10^{-4}	e^N	0.1164941911	0.0696264997	0.0382962309	0.0201571709
	e^{2N}	0.0671450910	0.0382962309	0.0201420309	0.0103594904
	p	0.7949041840	0.8624340864	0.9269932512	0.9603401395
10^{-6}	e^N	0.1536545053	0.0969881244	0.0554798716	0.0298472965
	e^{2N}	0.0969881244	0.0554798716	0.0298472965	0.0155070658
	p	0.6638100553	0.8058436590	0.8943641873	0.9446745307
10^{-8}	e^N	0.1877618298	0.1197683012	0.0711238601	0.0390887200
	e^{2N}	0.1197683012	0.0689229676	0.0390887200	0.0205505616
	p	0.6486576835	0.7971893985	0.8635812830	0.9275745213
10^{-10}	e^N	0.2102571839	0.1336327415	0.0845994163	0.0478587059
	e^{2N}	0.1336327415	0.0830687954	0.0470859244	0.0254126858
	p	0.6538815671	0.6858949885	0.8453518562	0.9132325302

Example 5.2. We solve the following convection-diffusion equation:

$$Lu = \varepsilon u''(x) + \Big(\frac{1+x^2}{2}\Big)u'(x) - \frac{1}{2}u(x) = e^{\frac{-x}{\varepsilon}}, \ \ 0 < x < 1,$$

with

$$u(0) = 0, \ u(1) = 1.$$

The exact solution of this problem is unknown. Thus we use the double-mesh principle. The order of convergence is calculated as

$$p_N = \log_2\Big(\frac{e^N}{e^{2N}}\Big),$$

where the maximum pointwise errors are denoted by

$$e^N = \max_{0\leq i<N} |y_i^N - y_i^{2N}|.$$

The computed results are shown in Tables 5.3, 5.4.

Table 5.3. Maximum pointwise errors and order of convergence on B-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.0269493405	0.0134853862	0.0067297647	0.0033578900
	e^{2N}	0.0134767824	0.0067297647	0.0033578900	0.0016761978
	p	0.9997738794	1.0027688774	1.0032242063	1.0053996406
10^{-4}	e^N	0.0265537894	0.0132449498	0.0066100583	0.0033012582
	e^{2N}	0.0132449498	0.0066100583	0.0033012582	0.0016495551
	p	1.0034753869	1.0068478705	1.0106470070	1.0190390235
10^{-6}	e^N	0.0266163738	0.0132353950	0.0065942324	0.0032906622
	e^{2N}	0.0132353950	0.0065942324	0.0032906622	0.0016438256
	p	1.0079127865	1.0082260610	1.0128267837	1.0132207317
10^{-8}	e^N	0.0266797788	0.0132486354	0.0065968674	0.0032911024
	e^{2N}	0.0132486354	0.0065968674	0.0032911024	0.0016438581
	p	1.0099029347	1.0101327947	1.0132102155	1.0184851665
10^{-10}	e^N	0.0267164932	0.0132563027	0.0065984643	0.0032914253
	e^{2N}	0.0132563027	0.0065984643	0.0032914253	0.0016439217
	p	1.0110522022	1.0166418176	1.0184178559	1.0215708849

Table 5.4. Maximum pointwise errors and order of convergence on S-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.0295817240	0.0156316335	0.0080396803	0.0040779502
	e^{2N}	0.0155829893	0.0080396803	0.0040779502	0.0020539607
	p	0.9247341171	0.9592585019	0.9792939815	0.9894355928
10^{-4}	e^N	0.0529849507	0.0292186899	0.0154434378	0.0079419705
	e^{2N}	0.0292186899	0.0153875454	0.0079419705	0.0040281899
	p	0.8586911557	0.9251283817	0.9594250445	0.9793652951
10^{-6}	e^N	0.0752147791	0.0431197162	0.0232235123	0.0121049380
	e^{2N}	0.0431197162	0.0232235123	0.0120821666	0.0061791319
	p	0.8026684861	0.8927615023	0.9427069908	0.9701196240
10^{-8}	e^N	0.0931675714	0.0557940092	0.0306946111	0.0161631501
	e^{2N}	0.0537222546	0.0306946111	0.0161457073	0.0083072798
	p	0.7943080341	0.8621248304	0.9268347481	0.9602603497
10^{-10}	e^N	0.1053237522	0.0664862963	0.0377641197	0.0201298965
	e^{2N}	0.0656608505	0.0372530345	0.0200622369	0.0104088398
	p	0.6817254808	0.8356990655	0.9125336858	0.9515304917

Example 5.3. We take into account another problem

$$Lu = \varepsilon u''(x) + (1+\varepsilon)u'(x) - (1+x)u(x) = 4\sin(\pi x), \;\; 0 < x < 1,$$
$$u(0) = 0, \;\; u(1) = 1,$$

whose exact solution is unknown. Applying the double-mesh principle again, the experimental results in Tables 5.5, 5.6 are obtained.

Table 5.5. Maximum pointwise errors and order of convergence on B-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.0667522832	0.0345294730	0.0175999724	0.0088806119
	e^{2N}	0.0345294730	0.0175999724	0.0088806119	0.0044503567
	p	0.9509888676	0.9722551471	0.9868421744	0.9967381020
10^{-4}	e^N	0.0663355100	0.0349167344	0.0179308677	0.0090871973
	e^{2N}	0.0349167344	0.0179308677	0.0090871973	0.0045738304
	p	0.9258627271	0.9614733335	0.9805379956	0.9904325211
10^{-6}	e^N	0.0660621198	0.0347793257	0.0178639732	0.0090554988
	e^{2N}	0.0347793257	0.0178639732	0.0090554988	0.0045592630
	p	0.9255933015	0.9611769717	0.9801869781	0.9899934602
10^{-8}	e^N	0.0660591268	0.0347772358	0.0178628913	0.0090549544
	e^{2N}	0.0347772358	0.0178628913	0.0090549544	0.0045589905
	p	0.9256146309	0.9611776558	0.9801863380	0.9899929516
10^{-10}	e^N	0.0660597492	0.0347772386	0.0178628782	0.0090549472
	e^{2N}	0.0347772386	0.0178628782	0.0090549472	0.0045589869
	p	0.9256281109	0.9611788274	0.9801864232	0.9899929545

Table 5.6. Maximum pointwise errors and order of convergence on S-mesh

ε		N			
		128	256	512	1024
10^{-2}	e^N	0.1120071067	0.0617245242	0.0325838824	0.0167545464
	e^{2N}	0.0617245242	0.0324956302	0.0167545464	0.0084974410
	p	0.8596745574	0.9255980829	0.9596058842	0.9794522871
10^{-4}	e^N	0.1883181882	0.1130816482	0.0622338684	0.0327903298
	e^{2N}	0.1086728100	0.0622338684	0.0327417371	0.0168545876
	p	0.7931813231	0.8615929845	0.9265690608	0.9601290844
10^{-6}	e^N	0.2505570717	0.1583092810	0.0906028926	0.0487554853
	e^{2N}	0.1583092810	0.0906028926	0.0487554853	0.0253340638
	p	0.6623934195	0.8051168214	0.8939925710	0.9444860263
10^{-8}	e^N	0.3064359626	0.1956434924	0.1164230695	0.0639998541
	e^{2N}	0.1956434924	0.1126382890	0.0639998541	0.0336514918
	p	0.6473584949	0.7965297992	0.8632364376	0.9273981534
10^{-10}	e^N	0.3432540403	0.2188305892	0.1385452384	0.0784555977
	e^{2N}	0.2183350311	0.1362377247	0.0772409469	0.0416643209
	p	0.6527330779	0.6836881727	0.8429193737	0.9130639212

6 Discussion and conclusion

In this research, numerical solution of singularly perturbed convection-diffusion boundary value problems is investigated. The finite difference scheme is established on a non-uniform mesh. The stability and convergence of the method are analyzed. Three numerical examples are solved and error approximations are shown in Tables 5.1–5.6. The computational results show that the order of convergence is close to 1. Namely, the order of convergence of the scheme is $O(N^{-1})$ on B-mesh and it is $O(N^{-1}\ln N)$ for S-mesh. According to the obtained results, the proposed method gives more appropriate results on Bakhvalov mesh and it is very effective for solving these problems.

References

[1] G. M. Amiraliyev and Ya. D. Mamedov, Difference schemes on the uniform mesh for singular perturbed pseudo-parabolic equations. *Turkish J. Math.* **19** (1995), no. 3, 207–222.

[2] D. Arslan and M. Cakir, A numerical solution study on singularly perturbed convection-diffusion nonlocal boundary problem. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **68** (2019), no. 2, 1482–1491.

[3] D. Arslan, M.Çakir and Y. Masiha, A novel numerical approach for solving convection-diffusion problem with boundary layer behavior. *Gazi University Journal of Science* **33** (2020), no. 1, 152–162.

[4] P. M. Basha and V. Shanth, Fitted mesh method for a weakly coupled system of singularly perturbed reaction–convection–diffusion problems with discontinuous source term. *Ain Shams Engineering Journal* **9** (2018), no. 4, 1089–1101.

[5] M. Cakir, A numerical study on the difference solution of singularly perturbed semilinear problem with integral boundary condition. *Math. Model. Anal.* **21** (2016), no. 5, 644–658.

[6] P. P. Chakravarthy, T. Gupta and R. N. Rao, A numerical scheme for singularly perturbed delay differential equations of convection-diffusion type on an adaptive grid. *Math. Model. Anal.* **23** (2018), no. 4, 686–698.

[7] S.-B. Chen, S. Soradi-Zeid, H. Dutta, M. Mesrizadeh, H. Jahanshahi and Y.-M. Chu, Reproducing kernel Hilbert space method for nonlinear second order singularly perturbed boundary value problems with time-delay. *Chaos Solitons Fractals* **144** (2021), Paper No. 110674, 10 pp.

[8] H. Duru and B. Güneş, The finite difference method on adaptive mesh for singularly perturbed nonlinear 1D reaction diffusion boundary value problems. *J. Appl. Math. Comput. Mech.* **19** (2020), no. 4, 45–56.

[9] F. Erdogan, M. G. Sakar and O. Saldır, A finite difference method on layer-adapted mesh for singularly perturbed delay differential equations. *Appl. Math. Nonlinear Sci.* **5** (2020), no. 1, 425–436.

[10] G. Gadissa and G. File, Fitted fourth order scheme for singularly perturbed delay convection-diffusion equations. *Ethiop. J. Educ. & Sc* **14** (2019), no. 2,102–118.

[11] M. González and M. Strugaru, Adaptive solution of a singularly-perturbed convection-diffusion problem using a stabilized mixed finite element method. In: Radu, F., Kumar, K., Berre, I., Nordbotten, J., Pop, I. (Eds.) *Numerical Mathematics and Advanced Applications ENUMATH 2017. ENUMATH 2017.* Lecture Notes in Computational Science and Engineering, vol 126. *Springer, Cham.*, 2019.

[12] B. Güneş, A. B. Chianeh and M. Demirbaş, Comparison of multiple scales method and finite difference method for solving singularly perturbed convection diffusion problem. *Gümüşhane University Journal of Science and Technology* **10** (2020), no. 4, 1169–1181.

[13] P. Gupta and M. Kumar, Multiple-Scales method and numerical simulation of singularly perturbed boundary layer problems. *Appl. Math. Inf. Sci.* **10** (2016), no. 3, 1119–1127.

[14] D. A. Gurmessa, G. F. Duressa and G. G. Kiltu, Fitted non-polynomial cubic spline method for singularly perturbed delay convection-diffusion equations. *Ethiop. J. Sci.* **40** (2017), no. 1, 1–8.

[15] X. He and K. Wang, New finite difference methods for singularly perturbed convection-diffusion equations. *Taiwanese J. Math.* **22** (2018), no. 4,

[16] P.-W. Hsieh, S.-Y. Yang and Ch.-Sh. You, A robust finite difference scheme for strongly coupled systems of singularly perturbed convection-diffusion equations. *Numer. Methods Partial Differential Equations* **34** (2018), no. 1, 121–144.

[17] S. Iqbal and P. A. Zegeling, Higher order nonuniform grids for singularly perturbed convection-diffusion-reaction problems. *Comput. Math. Math. Phys.* **59** (2019), no. 12, 2057–2079.

[18] F. K. Iyanda and A. V. Iyadunni, On computational assessment of perturbation parameter ε) on third order singularly perturbed convection-diffusion boundary value problems. *Journal of Applied Mathematics and Computation (JAMC)* **3** (2019), no. 2, 583–590.

[19] A. S. V. R. Kanth and P. M. M. Kumar, Computational results and analysis for a class of linear and nonlinear singularly perturbed convection delay problems on Shishkin mesh. *Hacet. J. Math. Stat.* **49** (2020), no. 1, 221–235.

[20] B. Kaur and V. Sangwan, A posteriori error estimates for Hughes stabilized SUPG technique and adaptive refinement for a convection-diffusion problem. *Adv. Math. Phys.* **2020**, Art. ID 1361498, 9 pp.

[21] R. Lin, X. Ye, Sh. Zhang and P. Zhu, A weak Galerkin finite element method for singularly perturbed convection-diffusion-reaction problems. *SIAM J. Numer. Anal.* **56** (2018), no. 3, 1482–1497.

[22] R. Lin, X. Ye, Sh. Zhang and P. Zhu, Analysis of a DG method for singularly perturbed convection-diffusion problems. *J. Appl. Anal. Comput.* **10** (2020), no. 3, 830–841.

[23] A. Majumdar and S. Natesan, A higher-order hybrid numerical scheme for singularly perturbed convection-diffusion problem with boundary and weak interior layers. *Int. J. Math. Model. Numer. Optim.* **10** (2020, no. 1, 68–101.

[24] W. G. Melesse, A. A. Tiruneh and G. A. Derese, Solving systems of singularly perturbed convection diffusion problems via initial value method. *J. Appl. Math.* **2020**, Article ID 1062025, 8 p.

[25] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems. Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions.* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[26] K. Mukherjee, Parameter-uniform improved hybrid numerical scheme for singularly perturbed problems with interior layers. *Math. Model. Anal.* **23** (2018), no. 2, 167–189.

[27] A. H. Nayfeh, *Introduction to Perturbation Techniques.* A Wiley-Interscience Publication. Wiley-Interscience [John Wiley & Sons], New York, 1981.

[28] R. E. O'Malley, Jr., *Singular Perturbation Methods for Ordinary Differential Equations.* Applied Mathematical Sciences, 89. Springer-Verlag, New York, 1991.

[29] T. Prabha, M. Chandru and V. Shanthi, Hybrid difference scheme for singularly perturbed reaction-convection-diffusion problem with boundary and interior layers. *Appl. Math. Comput.* **314** (2017), 237–256.

[30] V. Raja and A. Tamilselvan, Numerical method for a system of singularly perturbed convection diffusion equations with integral boundary conditions. *Commun. Korean Math. Soc.* **34** (2019), no. 3, 1015–1027.

[31] V. Raja and A. Tamilselvan, A difference scheme on Shishkin mesh for convection diffusion problems with integral boundary condition. *AIP Conference Proceedings* **2277** (2020), 9 p.

[32] A. Ramesh Babu and T. Valanarasu, A weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term. *J. Appl. Math. Inform.* **37** (2019), no. 5-6, 357–382.

[33] H.-G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion-Reaction and Flow Problems.* Springer Series in Computational Mathematics, 24. Springer-Verlag, Berlin, 2008.

[34] S. Sankar Kalaiselvan, J. J. H. Miller and V. Sigamani, A parameter uniform fitted mesh method for a weakly coupled system of two singularly perturbed convection-diffusion equations. *Math. Commun.* **24** (2019), no. 2, 193–210.

[35] K. Selvakumar and G. P. Lazarus, A fitted operator and fitted mesh method for singularly perturbed convection diffusion problem. *International Journal of Recent Research Aspects, Special Issue: Conscientious Computing Technologies*, 2018, 544–550.

[36] M. K. Singh, G. Singh and S. Natesan, A unified study on superconvergence analysis of Galerkin FEM for singularly perturbed systems of multiscale nature. *J. Appl. Math. Comput.* **66** (2021), no. 1-2, 221–243.

[37] C. l. Sirisha and Y. N. Reddy, Numerical integration of singularly perturbed delay differential equations using exponential integrating factor. *Mathematical Communications* **22** (2017), no. 2, 251–264.

[38] M. M. Woldaregay and G. F. Duressa, Fitted numerical scheme for singularly perturbed differential equations having small delays. *Casp. J. Math. Sci.* (accepted) doi: 10.22080/CJMS.2020.19425.1547.

[39] S Xie, P. Zhu and X. Wang, Error analysis of weak Galerkin finite element methods for time-dependent convection-diffusion equations. *Appl. Numer. Math.* **137** (2019), 19–33.

[40] J. Zhang and X. Liu, Supercloseness of linear finite element method on Bakhvalov-type meshes for singularly perturbed convection-diffusion equation in 1D. *Appl. Math. Lett.* **111** (2021), Paper No. 106624, 7 pp.

[41] P. Zhu and S. Xie, A uniformly convergent weak Galerkin finite element method on Shishkin mesh for 1d convection-diffusion problem. *J. Sci. Comput.* **85** (2020), no. 2, Paper no. 34, 22 pp.

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