Memoirs on Differential Equations and Mathematical Physics

Volume 86, 2022, 1–14

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CONVERGENCE OF NÖRLUND MEANS WITH RESPECT TO VILENKIN SYSTEMS OF INTEGRABLE FUNCTIONS

Abstract. In this paper, we derive the convergence of Nörlund means of Vilenkin–Fourier series with monotone coefficients of integrable functions at Lebesgue and Vilinkin–Lebesgue points. Moreover, we discuss pointwise convergence and convergence in L_p norms of such N rlund means.

2010 Mathematics Subject Classification. 42C10, 42B25.

Key words and phrases. Vilenkin systems, Vilenkin groups, Nörlund means, a.e. convergence, Lebesgue points, Vilenkin–Lebesgue points.

რეზიუმე. სტატიაში ვღებულობთ ინტეგრებადი ფუნქციიებისთვის მონოტონური მიმდევრობით წარმოქმნილი ვილენკინ-ფურიეს მწკრივის ნორლუნდის საშუალოების ვილენკინ-ლებეგის წერტილებში კრებადობას. უფრო მეტიც, განვიხილავთ ასეთივე ნორლუნდის საშუალოების წერტილოვან კრებადობას და კრებადობას L_p ნორმით.

1 Introduction

It is well-known (see, e.g., the books [38] and [42]) that there exists an absolute constant c_p depending only on p such that

$$\|S_n f\|_p \leq c_p \|f\|_p, \quad \text{when} \quad p > 1.$$

On the other hand (see [12, 37, 48–51, 57] for details), the boundedness does not hold for $p = 1$. The analogue of Carleson's theorem for the Walsh system was proved by Billard [3] for $p = 2$ and by Sj lin [44] for $1 < p < \infty$, while for the bounded Vilenkin systems by Gosselin [20]. For the Walsh–Fourier series, Schipp [42] gave a proof by using the methods of martingale theory. A similar proof for th Vilenkin–Fourier series can be found in Schipp and Weisz [43, 58] (see also [31] and [38]). In each proof they show that the maximal operator of partial sums is bounded in L_p, i.e., there exists an absolute constant c_p such that

$$\|S^* f\|_p \leq c_p \|f\|_p, \quad \text{when} \quad f \in L_p, \quad p > 1.$$

Hence, if $f \in L_p(G_m)$, where $p > 1$, then $S_n f \to f$ a.e. on G_m. Stein [45] constructed an integrable function whose Vilenkin–Fourier (Walsh–Fourier) series diverges almost everywhere. In [42], it was proved that there exists an integrable function whose Walsh–Fourier series diverges everywhere. The a.e convergence of subsequences of Vilenkin–Fourier series of integrable functions was considered in [10], where the methods of martingale Hardy spaces was used. Considering the following restricted maximal operator $\widetilde{S}^*_{\#} f := \sup_{n\in\mathbb{N}} \|S_{M_n} f\|$, we get a weak $(1, 1)$ type inequality for $f \in L_1(G_m)$. That is,

$$\lambda\mu\left\{\widetilde{S}^*_{\#} f > \lambda\right\} \leq c\|f\|_1, \quad f \in L_1(G_m), \quad \lambda > 0.$$

Hence, if $f \in L_1(G_m)$, then $S_{M_n} f \to f$ a.e. on G_m. Moreover, for any integrable function, it is known that a.e. point is Lebesgue point and for any such point x of integrable function f we have

$$S_{M_n} f(x) \to f(x) \ \text{ as } n \to \infty \text{ for any such point } x \text{ of } f \in L_1(G_m). \tag{1.1}$$

In the one-dimensional case, Yano [61] proved that

$$\|\sigma_n f - f\|_p \to 0 \ \text{ as } \ n \to \infty \ \left(f \in L_p(G_m), \ \ 1 \leq p \leq \infty \right).$$

If we consider the maximal operator of Féjer means

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|,$$

then

$$\lambda\mu\left\{\sigma^* f > \lambda\right\} \leq c\|f\|_1, \quad f \in L_1(G_m), \quad \lambda > 0.$$

This result can be found in Zygmund [62] for the trigonometric series, in Schipp [41] (see also [39, 40] and [4, 17, 32–34, 46, 52–55]) for Walsh series and in Pál, Simon [30] for bounded Vilenkin series. The boundedness does not hold from the Lebesgue space $L_1(G_m)$ to the space $L_1(G_m)$. From the weak $(1, 1)$ type inequality it follows that for any $f \in L_1(G_m)$, $\sigma_n f(x) \to f(x)$, a.e. as $n \to \infty$. Moreover (for details see [16]), for any integrable function, it is known that a.e. point is the Vilenkin–Lebesgue point and for any such point x of integrable function f we have $\sigma_n f(x) \to f(x)$, as $n \to \infty$.

It is also well-known (see [42] for details) that the maximal operator $\sigma^{\alpha,*}$ of Cesŕo means is bounded from the Lebesgue space L_1 to the weak-L_1 space. It follows that $\sigma^{\alpha}_n f(x) \to f(x)$ a.e. as $n \to \infty$ for any $f \in L_1(G_m)$. The maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ with respect to Vilenkin systems was also investigated by Weisz [58] (see also [5, 7] and [59, 60]).

In [14], Gt and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the Lebesgue space L_1. In particular (see also [11, 36, 47, 56]), they proved that there exists a function in the space L_1 such that $\sup_{n\in\mathbb{N}} \|L_n f\|_1 = \infty$. However, Goginava [15] proved that $\|L_{2^n} f\|_1 \leq c\|f\|_1$, $f \in L_1$, $n \in \mathbb{N}$. Moreover, considering a restricted maximal operator

$$\widetilde{L}^*_{\#} f := \sup_{n\in\mathbb{N}} |L_{M_n} f|$$

of Nörlund means, Goginava [15] proved that

$$\lambda\mu\left\{\widetilde{L}_{\#}^{*}f>\lambda\right\}\leq c\|f\|_1,\quad f\in L_1(G_m),\quad \lambda>0.$$

It follows that for any $f\in L_1(G_m)$, $L_{M_n}f(x)\to f(x)$, a.e. as $n\to\infty$.

Móricz and Siddiqi [23] investigated properties of approximation by norm of some special Nörlund means of Walsh–Fourier series of L_p functions Similar results for the two-dimensional case can be found in Nagy [24, 25], Nagy and Tephnadze [26–29]. Approximation properties of some general summability methods can be found in [2,6,8,9,18,19,22]. Fridli, Manchanda and Siddiqi [13] improved and extended the results of Móricz and Siddiqi [23] to the Martingale Hardy spaces. The almost everywhere convergence of Nörlund means of Vilenkin–Fourier series with monotone coefficients of integrable functions was proved in [35].

In this paper, we derive the convergence of Nörlund means of Vilenkin–Fourier series with monotone coefficients of integrable functions at the Lebesgue and Vilinkin–Lebesgue points.

The paper is organized as follows. In Section 3, we present and prove some auxiliary lemmas. In Section 4, we present and prove our main results. Moreover, in order not to disturb our discussions in these sections, some preliminaries are given in Section 2.

2 Preliminaries

Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $m:=(m_0,m_1,\dots)$ be a sequence of positive integers not less than 2. Denote by $Z_{m_k}:=\{0,1,\dots,m_k-1\}$ the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}'s. In this paper, we discuss the bounded Vilenkin groups, i.e., the case $\sup_n m_n<\infty$. The direct product μ of the measures

$$\mu_k(\{j\}):=\frac{1}{m_k}\quad (j\in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m)=1$.

The elements of G_m are represented by the sequences $x:=(x_0,x_1,\dots,x_j,\dots)$, $x_j\in Z_{m_j})$.

It is easy to give a base for the neighborhood of G_m:

$$I_0(x):=G_m,\quad I_n(x):=\left\{y\in G_m \mid\ y_0=x_0,\dots,y_{n-1}=x_{n-1}\right\},$$

where $x\in G_m$, $n\in\mathbb{N}$. Denote $I_n:=I_n(0)$ for $n\in\mathbb{N}_+$, and $\overline{I_n}:=G_m\setminus I_n$.

If we define the so-called generalized number system based on m in the following way $M_0:=1$, $M_{k+1}:=m_kM_k$ $(k\in\mathbb{N})$, then every $n\in\mathbb{N}$ can be uniquely expressed as $n=\sum_{j=0}^{\infty}n_jM_j$, where $n_j\in Z_{m_j}$ $(j\in\mathbb{N}_+)$, and only a finite number of n_j's differs from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First, we define the complex-valued functions $r_k(x):G_m\to\mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x):=\exp\left(\frac{2\pi i x_k}{m_k}\right)\quad (i^2=-1,\quad x\in G_m,\quad k\in\mathbb{N}).$$

Now, define the Vilenkin system $\psi:=(\psi_n:n\in\mathbb{N})$ on G_m as follows:

$$\psi_n(x):=\prod_{k=0}^{\infty}r_k^{n_k}(x)\quad (n\in\mathbb{N}).$$

Specifically, we call this system the Walsh–Paley system when $m\equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and weak-$L_p(G_m)$ $(0 < p < \infty)$ are defined, respectively, by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu \quad \text{and} \quad \|f\|_{\text{weak-}L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [1]).

If $f \in L_1(G_m)$, we can define the Fourier coefficients, partial sums of the Fourier series and the Dirichlet kernels in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n \, d\mu, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k \ \ (n \in \mathbb{N}_+),$$

respectively. Recall that

$$\int_{G_m} D_n(x)\, dx = 1, \tag{2.1}$$

$$D_{M_n - j}(x) = D_{M_n}(x) - \psi_{M_n-1}(x)\overline{D}_j(x), \ \ j < M_n. \tag{2.2}$$

The convolution of two functions $f, g \in L_1(G_m)$ is defined by

$$(f * g)(x) := \int_{G_m} f(x-t)g(t)\, dt \ \ (x \in G_m).$$

It is easy to see that if $f \in L_p(G_m)$, $g \in L_1(G_m)$ and $1 \le p < \infty$, then $f * g \in L_p(G_m)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1. \tag{2.3}$$

Let $\{q_k : k \ge 0\}$ be a sequence of non-negative numbers. The Nörlund and T means for a Fourier series of f are defined, respectively, by

$$t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f \ \text{ and } \ T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \ \text{ where } \ Q_n := \sum_{k=0}^{n-1} q_k.$$

It is obvious that

$$t_n f(x) = \int_{G_m} f(t) F_n(x-t)\, d\mu(t) \ \text{ and } \ T_n f(x) = \int_{G_m} f(t) F_n^{-1}(x-t)\, d\mu(t),$$

where F_n and F_n^{-1} are, respectively, the Nörlund kernels and T kernels:

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k \ \text{ and } \ F_n^{-1} := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k.$$

We always assume that $\{q_k : \ k \ge 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then Nörlund means generated by $\{q_k : k \ge 0\}$ are regular if and only if $\frac{q_{n-1}}{Q_n} \to 0$ as $n \to \infty$. Analogical regularity condition for T means is the condition $\lim\limits_{n\to\infty} Q_n = \infty$.

If we invoke Abel's transformation, we get the following identities, which are very important for the investigations of Nörlund summability:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1} n q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n, \tag{2.4}$$

$$F_n = \frac{1}{Q_n} \Big(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \Big) \tag{2.5}$$

and

$$t_n=\frac{1}{Q_n}\Big(\sum_{j=1}^{n-1}(q_{n-j}-q_{n-j-1})j\sigma_j+q_0n\sigma_n\Big). \tag{2.6}$$

Let us consider some class of Nörlund means with monotone and bounded sequence $\{q_k:\ k\in\mathbb{N}\}$ such that $q:=\lim_{n\to\infty}q_n>c>0$. It easy to check that

$$\frac{q_{n-1}}{Q_n}=O\Big(\frac{1}{n}\Big)\ \text{ as }\ n\to\infty.$$

The well-known example of Nörlund summability are the (C,α) -means (Cesàro means), where $0<\alpha<1$, which are defined by

$$\sigma_n^{\alpha}f:=\frac{1}{A_n^{\alpha}}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}S_kf,\ \text{ where }\ A_0^{\alpha}:=0,\ \ A_n^{\alpha}:=\frac{(\alpha+1)\cdots(\alpha+n)}{n!}.$$

Let V_n^{α} denote the Nörlund mean, where $\{q_0=1,\ q_k=k^{\alpha-1}:\ k\in\mathbb{N}_+\}$, that is,

$$V_n^{\alpha}f:=\frac{1}{Q_n}\sum_{k=0}^{n-1}(n-k)^{\alpha-1}S_kf,\ \ 0<\alpha<1.$$

It is easy to show that

$$\frac{q_{n-1}}{Q_n}=O\Big(\frac{1}{n}\Big)\to 0\ \text{ as }\ n\to\infty.$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_nf:=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_kf}{k},\quad L_nf:=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_kf}{n-k},\ \text{ where }\ l_n:=\sum_{k=1}^{n-1}\frac{1}{k}.$$

Up to now we have considered T mean in the case where the sequence $\{q_k:\ k\in\mathbb{N}\}$ is bounded, but now we consider T summabilities with an unbounded sequence $\{q_k:k\in\mathbb{N}\}$.

Let us define the class of Nörlund means with non-decreasing coefficients:

$$\beta_n^{\alpha}f:=\frac{1}{Q_n}\sum_{k=1}^{n}\log^{\alpha}(n-k-1)S_kf.$$

It is obvious that

$$\frac{n}{2}\ \log^{\alpha}\Big(\frac{n}{2}\Big)\le Q_n\le n\log^{\alpha}n.$$

It follows that

$$\frac{1}{Q_n}=O\Big(\frac{1}{n}\Big)\to 0\ \text{ as }\ n\to\infty\ \text{ and }\ \frac{q_{n-1}}{Q_n}=O\Big(\frac{1}{n}\Big)\to 0\ \text{ as }\ n\to\infty.$$

A point $x\in G_m$ is called a Lebesgue point of integrable function f if

$$\lim_{n\to 0}\frac{1}{|I_n(x)|}\int\limits_{I_n(x)}|f(t)-f(x)|\,d\mu(t)=0.$$

It is known that a.e. point $x\in G_m$ is a Lebesgue point of the function f, and the Fejér means σ_nf of trigonometric Fourier series of $f\in L_1(G_m)$ converge to f at each Lebesgue point. It is also known that if $x\in G_m$ is a point of continuity of the function f, then it is Lebesgue point.

Let us introduced the operator

$$W_nf(x):=\sum_{s=0}^{n-1}M_s\sum_{r_s=1}^{m_s-1}\int\limits_{I_n(x-r_se_s)}|f(t)-f(x)|\,d\mu(t).$$

A point $x\in G_m$ is a Vilenkin–Lebesgue point of $f\in L_1(G_m)$ if

$$\lim_{n\to\infty}W_nf(x)=0.$$

3 Auxiliary lemmas

The next two lemmas can be found in [1].

Lemma 3.1. *Let $n \in \mathbb{N}$. Then for some constant c, we have*

$$n|K_n| \leq c \sum_{l=\langle n \rangle}^{|n|} M_l |K_{M_l}| \leq c \sum_{l=0}^{|n|} M_l |K_{M_l}|. \tag{3.1}$$

Lemma 3.2. *Let $n \in \mathbb{N}$. Then for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} K_n(x)\, d\mu(x) = 1,$$

$$\sup_{n \in \mathbb{N}} \int_{G_m} |K_n(x)|\, d\mu(x) < \infty, \tag{3.2}$$

$$\int_{G_m \setminus I_N} |K_n(x)|\, d\mu(x) \to 0 \ \text{ as } \ n \to \infty. \tag{3.3}$$

First, we consider the Nörlund kernels with non-decreasing sequences.

Lemma 3.3. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers satisfying the condition*

$$\frac{q_{n-1}}{Q_n} = O\Big(\frac{1}{n}\Big) \ \text{ as } \ n \to \infty. \tag{3.4}$$

Then for some constant c, we have

$$|F_n| \leq \frac{c}{n} \Big\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \Big\}.$$

Proof. Let the sequence $\{q_k : \ k \in \mathbb{N}\}$ be non-decreasing. Then, by using (3.4), we get

$$\frac{1}{Q_n} \Big(\sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \Big) \leq \frac{1}{Q_n} \Big(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \Big) \leq \frac{q_{n-1}}{Q_n} \leq \frac{c}{n}.$$

Hence, in view of (3.4), if we apply Lemma 3.1 and use equalities (2.4) and (2.5), we obtain

$$|F_n| \leq \Big(\frac{1}{Q_n} \Big(\sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \Big) \Big) \sum_{i=0}^{|n|} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{|n|} M_i |K_{M_i}|.$$

The proof is complete. □

Corollary 3.1. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers. Then for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_n(x)\ d\mu(x) = 1, \tag{3.5}$$

$$\sup_{n \in \mathbb{N}} \int_{G_m} |F_n(x)|\, d\mu(x) < \infty,$$

$$\int_{G_m \setminus I_N} |F_n(x)|\, d\mu(x) \to 0 \ \text{ as } \ n \to \infty.$$

Proof. According to (2.1), we readily obtain (3.5). By using (3.2) in Lemma 3.2, combined with (2.4) and (2.5), we get

$$\begin{aligned}\int_{G_m} |F_n(x)|\,d\mu(x) &\leq \frac{1}{Q_n}\sum_{j=1}^{n-1}(q_{n-j}-q_{n-j-1})j\int_{G_m}|K_j|\,d\mu + \frac{q_0 n}{Q_n}\int_{G_m}|K_n|\,d\mu \\ &\leq \frac{c}{Q_n}\sum_{j=1}^{n-1}(q_{n-j}-q_{n-j-1})j + \frac{cq_0 n}{Q_n} < c < \infty.\end{aligned}$$

By using (3.3) in Lemma 3.2 and inequalities (2.4) and (2.5), we can conclude that

$$\begin{aligned}\int_{G_m\backslash I_N} |F_n|\,d\mu &\leq \frac{1}{Q_n}\sum_{j=0}^{n-1}(q_{n-j}-q_{n-j-1})j\int_{G_m\backslash I_N}|K_j|\,d\mu + \frac{q_0 n}{Q_n}\int_{G_m\backslash I_N}|K_n|\,d\mu \\ &\leq \frac{1}{Q_n}\sum_{j=0}^{n-1}(q_{n-j}-q_{n-j-1})j\alpha_j + \frac{q_0 n\alpha_n}{Q_n} := \mathrm{I}+\mathrm{II}, \ \text{ where } \ \alpha_n\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Since the sequence is non-decreasing, we can conclude that $\mathrm{II}\leq\alpha_n\to 0$ as $n\to\infty$.

On the other hand, since α_n converges to 0, we find that there exists an absolute constant A such that $\alpha_n\leq A$ for any $n\in\mathbb{N}$, and for any $\varepsilon>0$, there exists $N_0\in\mathbb{N}$ such that $\alpha_n<\varepsilon$ when $n>N_0$. Hence

$$\mathrm{I} = \frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{n-j}-q_{n-j-1})j\alpha_j + \frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j}-q_{n-j-1})j\alpha_j := \mathrm{I}_1+\mathrm{I}_2.$$

Since $|q_{n-j}-q_{n-j-1}|<2q_{n-1}$ and $\alpha_n<A$, we obtain

$$\begin{aligned}\mathrm{I}_1 &= \frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{n-j}-q_{n-j-1})j\alpha_j \leq \frac{2AN_0q_{n-1}}{Q_n}\to 0 \ \text{ as } \ n\to\infty, \\ \mathrm{I}_2 &= \frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j}-q_{n-j-1})j\alpha_j \\ &\leq \frac{\varepsilon}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j}-q_{n-j-1})j \leq \frac{\varepsilon}{Q_n}\sum_{j=0}^{n-1}(q_{n-j}-q_{n-j-1})j < \varepsilon.\end{aligned}$$

We conclude that $\mathrm{I}_2\to 0$ as well, so the proof is complete. □

Lemma 3.4. *Let $\{q_k:\ k\in\mathbb{N}\}$ be a sequence of non-increasing numbers satisfying the condition*

$$\frac{1}{Q_n} = O\Big(\frac{1}{n}\Big) \ \text{ as } \ n\to\infty. \tag{3.6}$$

Then for some constant c, we have

$$|F_n| \leq \frac{c}{n}\Big\{\sum_{j=0}^{|n|}M_j|K_{M_j}|\Big\}.$$

Proof. Let the sequence $\{q_k:k\in\mathbb{N}\}$ be non-increasing satisfying condition (3.6). Then

$$\frac{1}{Q_n}\Big(\sum_{j=1}^{n-1}|q_{n-j}-q_{n-j-1}|+q_0\Big) \leq \frac{1}{Q_n}\Big(\sum_{j=1}^{n-1}-(q_{n-j}-q_{n-j-1})+q_0\Big) \leq \frac{2q_0-q_{n-1}}{Q_n} \leq \frac{2q_0}{Q_n} \leq \frac{c}{n}.$$

If we apply Lemma 3.1 and invoke equalities (2.4) and (2.5), we immediately get

$$|F_n| \leq \frac{1}{Q_n}\Big(\sum_{j=1}^{n-1}|q_{n-j}-q_{n-j-1}|+q_0\Big)\sum_{i=0}^{|n|}M_i|K_{M_i}| \leq \frac{c}{n}\sum_{i=0}^{|n|}M_i|K_{M_i}|.$$

The proof is complete. □

Corollary 3.2. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying condition* (3.6)*. Then for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_n(x)\,d\mu(x) = 1, \quad \sup_{n\in\mathbb{N}} \int_{G_m} |F_n(x)|\,d\mu(x) < \infty,$$

$$\int_{G_m \setminus I_N} |F_n(x)|\,d\mu(x) \to 0 \ \text{ as } \ n \to \infty \ \text{ for any } \ N \in \mathbb{N}_+.$$

Proof. If we compare the estimation of F_n in Corollary 3.4 with the estimation of F_n in Lemma 3.3, we find that they are quite the same. Hence the proof is analogous to that of Corollary 3.1, so, we leave out the details. □

Finally, we study special subsequences of Nörlund kernels and T means.

Lemma 3.5. *Let $n \in \mathbb{N}$. Then*

$$F_{M_n}(x) = D_{M_n}(x) - \psi_{M_n-1}(x)\overline{F^{-1}}_{M_n}(x).$$

Proof. By using (2.2), we get

$$\begin{aligned} F_{M_n}(x) &= \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n-k} D_k(x) = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k D_{M_n-k}(x) \\ &= \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k \big(D_{M_n}(x) - \psi_{M_n-1}(x)\overline{D}_j(x)\big) = D_{M_n}(x) - \psi_{M_n-1}(x)\overline{F^{-1}}_{M_n}(x). \end{aligned}$$ □

Lemma 3.6. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_n^{-1}(x)\,d\mu(x) = 1, \quad \sup_{n\in\mathbb{N}} \int_{G_m} |F_n^{-1}(x)|\,d\mu(x) < \infty,$$

$$\int_{G_m \setminus I_N} |F_n^{-1}(x)|\,d\mu(x) \to 0 \ \text{ as } \ n \to \infty.$$

Proof. If we follow analogous steps of Corollaries 3.1 and 3.2, we immediately get the proof. So, we leave out the details. □

Corollary 3.3. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_{M_n}(x)\,d\mu(x) = 1, \quad \sup_{n\in\mathbb{N}} \int_{G_m} |F_{M_n}(x)|\,d\mu(x) \le c < \infty,$$

$$\int_{G_m \setminus I_N} |F_{M_n}(x)|\,d\mu(x) \to 0 \ \text{ as } \ n \to \infty.$$

Proof. The proof immediately follows from Lemmas 3.5 and 3.6. □

4 Proofs of the theorems

Theorem 4.1.

(a) *Let $1 \le p < \infty$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers. Then for all $f \in L_p(G_m)$, $\|t_n f - f\|_p \to 0$ as $n \to \infty$. Let the function $f \in L_1(G_m)$ be continuous at a point x. Then $t_n f(x) \to f(x)$ as $n \to \infty$. Moreover, for all Vilenkin–Lebesgue points of $f \in L_p(G_m)$,*

$$\lim_{n\to\infty} t_n f(x) = f(x).$$

(b) *Let $1 \le p < \infty$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying condition (3.6). Then for all $f \in L_p(G_m)$, $\|t_n f - f\|_p \to 0$ as $n \to \infty$. Let the function $f \in L_1(G_m)$ be continuous at a point x. Then $t_n f(x) \to f(x)$ as $n \to \infty$. Moreover, for all Vilenkin–Lebesgue points of $f \in L_p(G_m)$,*

$$\lim_{n\to\infty} t_n f(x) = f(x).$$

Proof. Let $\{q_k : k \in \mathbb{N}\}$ be a non-decreasing sequence. Corollary 3.1 immediately implies the stated norm and pointwise convergence. Suppose that x is either a point of continuity or a Vilenkin–Lebesgue point of the function $f \in L_p(G_m)$. Then

$$\lim_{n\to\infty} |\sigma_n f(x) - f(x)| = 0.$$

Hence, by combining (2.4) and (2.6), we can conclude that

$$\begin{aligned}|t_n f(x) - f(x)| &\le \frac{1}{Q_n}\Big(\sum_{j=1}^{n-2}(q_{n-j} - q_{n-j-1})j|\sigma_j f(x) - f(x)| + q_0 n|\sigma_n f(x) - f(x)|\Big)\\ &\le \frac{1}{Q_n}\sum_{j=0}^{n-2}(q_{n-j} - q_{n-j-1})j\alpha_j + \frac{q_0 n\alpha_n}{Q_n} := \mathrm{I} + \mathrm{II}, \quad \text{where } \alpha_n \to 0 \text{ as } n \to \infty.\end{aligned}$$

To prove $\mathrm{I} \to 0$ as $n \to \infty$ and $\mathrm{II} \to 0$ as $n \to \infty$, we just have to use analogous steps of Corollary 3.1. It follows that part (a) is proved.

Let the sequence be non-increasing satisfying condition (3.6). According to Corollary 3.2, we get the norm convergence and the pointwise convergence. To prove the convergence at Vilenkin–Lebesgue points, we use estimations (2.4) and (2.6) to obtain

$$|t_n f - f(x)| \le \frac{1}{Q_n}\sum_{j=0}^{n-2}(q_{n-j-1} - q_{n-j})j\alpha_j + \frac{q_0 n\alpha_n}{Q_n} = \mathrm{III} + \mathrm{IV}, \quad \text{where } \alpha_n \to 0 \text{ as } n \to \infty.$$

It is evident that $\mathrm{IV} \le \frac{q_0 n\alpha_n}{Q_n} \le C\alpha_n \to 0$ as $n \to \infty$. Moreover, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\alpha_n < \varepsilon$ when $n > N_0$. It follows that

$$\begin{aligned}&\frac{1}{Q_n}\sum_{j=1}^{n-2}(q_{n-j-1} - q_{n-j})j\alpha_j\\ &\quad= \frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{n-j-1} - q_{n-j})j\alpha_j + \frac{1}{Q_n}\sum_{j=N_0+1}^{n-2}(q_{n-j-1} - q_{n-j})j\alpha_j = \mathrm{III}_1 + \mathrm{III}_2.\end{aligned}$$

Since the sequence is non-increasing, we can conclude that $|q_{n-j} - q_{n-j-1}| < 2q_0$. Hence

$$\mathrm{III}_1 \le \frac{2q_0 N_0}{Q_n} \to 0 \text{ as } n \to \infty$$

and

$$\begin{aligned}\mathrm{III}_2 &\le \frac{1}{Q_n}\sum_{j=N_0+1}^{n-2}(q_{n-j-1} - q_{n-j})j\alpha_j\\ &\le \frac{\varepsilon(n-1)}{Q_n}\sum_{j=N_0+1}^{n-2}(q_{n-j} - q_{n-j-1}) \le \frac{\varepsilon(n-1)}{Q_n}(q_0 - q_{n-N_0}) \le \frac{2q_0\varepsilon(n-1)}{Q_n} < C\varepsilon.\end{aligned}$$

The proof of part (b) is also complete. □

Corollary 4.1. *Let $f \in L_1(G_m)$. Then*

$$\sigma_n f \to f \ \text{ a.e. as } n \to \infty, \quad \sigma_n^{\alpha} f \to f \ \text{ a.e. as } n \to \infty,$$
$$V_n f \to f \ \text{ a.e. as } n \to \infty, \quad B_n^{\alpha,\beta} f \to f \ \text{ a.e. as } n \to \infty.$$

Theorem 4.2. *Let $1 \le p < \infty$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then for all $f \in L_p(G_m)$,*

$$\|t_{M_n} f - f\|_p \to 0 \ \text{ as } n \to \infty$$

Moreover, for all Lebesgue points of $f \in L_p(G_m)$,

$$\lim_{n\to\infty} t_{M_n} f(x) = f(x).$$

Proof. From Corollary 3.6 we immediately get the norm convergence. To prove a.e. convergence, we use Lemma 3.5 to write that

$$\begin{aligned} t_{M_n} f(x) &= \int_{G_m} f(t) F_n(x-t)\, d\mu(t) \\ &= \int_{G_m} f(t) D_{M_n}(x-t)\, d\mu(t) - \int_{G_m} f(t)\psi_{M_n-1}(x-t)\overline{F^{-1}}_{M_n}(x-t) = \mathrm{I} - \mathrm{II}. \end{aligned}$$

Applying (1.1), we get that $\mathrm{I} = S_{M_n} f(x) \to f(x)$ for all Lebesgue points of $f \in L_p(G_m)$.

According to $\psi_{M_n-1}(x-t) = \psi_{M_n-1}(x)\overline{\psi}_{M_n-1}(t)$, we can conclude that

$$\mathrm{II} = \psi_{M_n-1}(x) \int_{G_m} f(t)\overline{F^{-1}}_{M_n}(x-t)\overline{\psi}_{M_n-1}(t)\, d(t).$$

By combining (2.3) and Lemma 3.6, we find that

$$f(t)\overline{F^{-1}}_{M_n}(x-t) \in L_p, \ \text{ where } \ p \ge 1 \ \text{ for any } \ x \in G_m,$$

and II are Fourier coefficients of integrable function. According to the Riemann–Lebesgue Lemma, we get that $\mathrm{II} \to 0$ for any $x \in G_m$. The proof is complete. ☐

Corollary 4.2. *Let $f \in L_1(G_m)$ be continuous at a point x. Then $t_n f(x) \to f(x)$ as $n \to \infty$.*

Corollary 4.3. *Let $f \in L_1(G_m)$. Then for all Lebesgue points of f,*

$$L_{M_n} f(x) \to f(x) \ \text{ as } n \to \infty,$$

Corollary 4.4. *Let $f \in L_1(G_m)$ be continuous at a point x. Then $L_{M_n} f(x) \to f(x)$ as $n \to \infty$.*

Acknowledgements

The research was supported by Shota Rustaveli National Science Foundation grant no. PHDF-21-1702.

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(Received 01.07.2021; revised 03.11.2021; accepted 05.11.2021)

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