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L^p UNCERTAINTY PRINCIPLES FOR THE WINDOWED SPHERICAL MEAN TRANSFORM

Abstract. In this work, we establish the L^p local uncertainty principle for the windowed spherical mean transform and deduce the L^p version of the Heisenberg–Pauli–Weyl uncertainty principle. Finally, using the previous uncertainty principles and the techniques of Donoho–Stark, we present uncertainty principles of concentration type in the L^p theory, when $1 < p \leqslant 2$.

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რეზიუმე. ნაშრომში დადგენილია L^p ლოკალური გაურკვევლობის პრინციპი ფანჯრიანი სფერული საშუალოს გარდაქმნისთვის და გამოყვანილია ჰეიზენბერგ-პაული-ვეილის გაურკვევლობის პრინციპის L^p ვერსია. ბოლოს, წინა გაურკვევლობის პრინციპებისა და დონოჰო-სტარკის ტექნიკის გამოყენებით, წარმოდგენილია კონცენტრაციის ტიპის გაურკვევლობის პრინციპები L^p თეორიაში, როდესაც $1 < p \leqslant 2$.

1 Introduction

The spherical mean operator $\mathscr{R}$ is defined for a function f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable [19], by

$$\mathscr{R}(f)(r,x)=\int_{S^n} f(r\eta,x+r\xi)\,d\sigma_n(\eta,\xi),\quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

where S^n is the unit sphere of $\mathbb{R}\times\mathbb{R}^n$ and $d\sigma_n$ is the surface measure on S^n, normalized to have total measure one.

The operator $\mathscr{R}$ has many important physical applications, namely, in image processing of the so-called synthetic aperture radar (SAR) data [12, 13, 23, 25], or in the linearized inverse scattering problem in acoustics [6].

The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined for every integrable function f on $[0,+\infty[\,\times\mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$ by

$$\forall\,(s,y)\in\Upsilon,\quad \mathscr{F}(f)(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\mathscr{R}\big(\cos(s\,\cdot\,)e^{-i\langle y|\,\cdot\,\rangle}\big)(r,x)\,d\nu_{n+1}(r,x),$$

where $d\nu_{n+1}$ is the measure defined on $[0,+\infty[\,\times\mathbb{R}^n$ by

$$d\nu_{n+1}(r,x)=\frac{r^n}{2^{\frac{n-1}{2}}\,\Gamma(\frac{n+1}{2})}\,dr\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}}\,,$$

$\|\cdot\|_{p,\nu_{n+1}}$ is its norm, and Υ is the set given by

$$\Upsilon=\mathbb{R}\times\mathbb{R}^n\cup\big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant|x|\big\}. \tag{1.1}$$

Many harmonic analysis results related to the Fourier transform $\mathscr{F}$ have already been proved by Jlassi, Nessibi, Rachdi and Trimèche [16, 19, 22]. Also, many uncertainty principles related to the Fourier transform $\mathscr{F}$ have been proved [14, 15, 18].

The uncertainty principles play an important role in harmonic analysis. These principles assert that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized, and there are many mathematical formulations of this general fact, one of them is the Heisenberg uncertainty principle [11] which states that if f is highly localized, then $\widehat{f}$ cannot be concentrated near a single point, but it does not preclude $\widehat{f}$ from being concentrated in a small neighborhood of two or more widely separated points.

Local uncertainty inequality states that if a function is concentrated, then not only is its Fourier transform spread out, but that it cannot be localized in a subset of finite measure. This object is proved by Faris [5] and generalized by Price [20, 21]. Other forms of the uncertainty principles can be found in [3, 4, 10].

Time frequency analysis [9] plays an important role in harmonic analysis, in particular, in signal theory. In this context and motivated by quantum mechanics, the physicist Dennis Gabor [7] has introduced the Gabor transform, in which he uses translation, convolution and modulation operators of a single Gaussian to represent one-dimensional signal. In the same context, many uncertainty principles related to the continuous Gabor and wavelet transforms have been proved in [2, 8, 26] and the references therein.

Our investigation in this work consists in defining the windowed Fourier transform $\mathscr{V}_g$ (called also the Gabor transform) associated with the integral transform $\mathscr{R}$, where g is a non-zero function. Next, based on the ideas of Faris [5] and Price [20, 21], we show the L^p local uncertainty principles for $\mathscr{V}_g$ and deduce the L^p version of the Heisenberg–Pauli–Weyl uncertainty principle. We use also the Heisenberg uncertainty principle, the properties of the windowed Fourier transform and the techniques of Donoho–Stark [4, 24] and show uncertainty principles of concentration type for the L^p theory, when $1<p\leqslant 2$.

This work is organized as follows. In Section 2, we recall some harmonic analysis results related to the spherical mean operator and its Fourier transform. In Section 3, we present some elements of harmonic analysis related to the windowed Fourier transform. In Section 4, we show local uncertainty principle. In Section 5, we deduce the L^p version of the Heisenberg–Pauli–Weyl uncertainty principle. The last section is devoted to present uncertainty principles of concentration type in the L^p theory, when $1 < p \leqslant 2$.

2 The spherical mean operator

In [19], Nessibi, Rachdi and Trimèche showed that for every $(\mu,\lambda) \in \mathbb{C}\times\mathbb{C}^n$, the function $\varphi_{(\mu,\lambda)}$ defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$\varphi_{(\mu,\lambda)}(r,x) = \mathscr{R}\big(\cos(\mu\,\cdot)e^{-i\langle\lambda|\,\cdot\,\rangle}\big)(r,x) \tag{2.1}$$

is the unique infinitely differentiable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable satisfying the following system:

$$\begin{cases} \dfrac{\partial u}{\partial x_j}(r,x_1,\dots,x_n) = -i\lambda_j u(r,x_1,\dots,x_n), & 1\leqslant j\leqslant n,\\ \ell_{\frac{n-1}{2}}u(r,x_1,\dots,x_n) - \Delta u(r,x_1,\dots,x_n) = -\mu^2 u(r,x_1,\dots,x_n), & \\ u(0,\dots,0)=1, & \\ \dfrac{\partial u}{\partial r}(0,x_1,\dots,x_n)=0, & (x_1,\dots,x_n)\in\mathbb{R}^n, \end{cases}$$

where $\ell_{\frac{n-1}{2}}$ is the Bessel operator defined by $\ell_{\frac{n-1}{2}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r}$, and Δ denotes the usual Laplacian operator defined by $\Delta = \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$. The authors also proved that the eigenfunction $\varphi_{(\mu,\lambda)}$ defined by relation (2.1) is explicitly given by

$$\forall\,(r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad \varphi_{(\mu,\lambda)}(r,x) = j_{\frac{n-1}{2}}\Big(r\sqrt{\mu^2+|\lambda|^2}\Big)e^{-i\langle\lambda|\,x\rangle}, \tag{2.2}$$

where $j_{\frac{n-1}{2}}$ is the modified Bessel function defined by

$$j_{\frac{n-1}{2}}(z) = 2^{\frac{n-1}{2}}\,\Gamma\Big(\frac{n+1}{2}\Big)\frac{J_{\frac{n-1}{2}}(z)}{z^{\frac{n-1}{2}}} = \Gamma\Big(\frac{n+1}{2}\Big)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(\frac{n+1}{2}+k)}\Big(\frac{z}{2}\Big)^{2k},\quad z\in\mathbb{C},$$

and $J_{\frac{n-1}{2}}$ is the Bessel function of the first kind and index $\frac{n-1}{2}$ (see [1,17]).

The modified Bessel function $j_{\frac{n-1}{2}}$ has the following integral representation:

$$\forall\,z\in\mathbb{C},\quad j_{\frac{n-1}{2}}(z) = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\int_0^1(1-t^2)^{\frac{n}{2}-1}\cos(zt)\,dt. \tag{2.3}$$

Relation (2.3) shows, in particular, that for every $z\in\mathbb{C}$ and for every $k\in\mathbb{N}$, we have

$$\big|j^{(k)}_{\frac{n-1}{2}}(z)\big| \leqslant e^{|\mathrm{Im}(z)|}.$$

From the properties of the modified Bessel function $j_{\frac{n-1}{2}}$, we deduce that the eigenfunction $\varphi_{(\mu,\lambda)}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$ if and only if (μ,λ) belongs to the set Υ given by relation (1.1), and in this case

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}|\varphi_{(\mu,\lambda)}(r,x)| = 1. \tag{2.4}$$

In the following, we define the translation operators, the convolution product and the Fourier transform $\mathscr{F}$ associated with the operator $\mathscr{R}$.

Definition 2.1.

(i) For every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the spherical mean operator is defined on $L^p(d\nu_{n+1})$, $p \in [1,+\infty]$, by

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})} \int_0^\pi f\Big(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\Big)\sin^{n-1}(\theta)\,d\theta.$$

(ii) The convolution product of measurable functions f and g on $[0,+\infty[\times\mathbb{R}^n$ is defined by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}^n;\quad f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} \mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)\,d\nu_{n+1}(s,y),$$

whenever the integral of the right-hand side is defined, where $\check{f}(s,y) = f(s,-y)$.

For every $(r,x) \in]0,+\infty[\times\mathbb{R}^n$, and by a standard change of variables, we have

$$\forall (s,y) \in]0,+\infty[\times\mathbb{R}^n,\quad \mathcal{T}_{(r,x)}(f)(s,y) = \frac{1}{2^{\frac{n-1}{2}}\,\Gamma(\frac{n+1}{2})}\int_0^{+\infty} f(t,x+y)\mathscr{W}_n(r,s,t)t^n\,dt,$$

where the kernel $\mathscr{W}_n$ is given by

$$\mathscr{W}_n(r,s,t) = \frac{\Gamma(\frac{n+1}{2})^2}{2^{\frac{n-3}{2}}\Gamma(\frac{n}{2})\sqrt{\pi}}\,\frac{((r+s)^2-t^2)^{\frac{n}{2}-1}(t^2-(r-s)^2)^{\frac{n}{2}-1}}{(rst)^{n-1}}\,\mathbf{1}_{]|r-s|,r+s[}(t).$$

Also, the coming properties are satisfied:

- For every $f \in L^1(d\nu_{n+1})$ and $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \mathcal{T}_{(r,x)}(f)(s,y)\,d\nu_\alpha(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(s,y)\,d\nu_\alpha(s,y).$$

- For every $f \in L^p(d\nu_{n+1})$, $p \in [1,+\infty]$, and $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_{n+1})$, and we have

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu_{n+1}} \leqslant \|f\|_{p,\nu_{n+1}}.$$

- Let $p,q,r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. Then for every $f \in L^p(d\nu_{n+1})$ and $g \in L^q(d\nu_{n+1})$, the function $f*g$ belongs to the space $L^r(d\nu_{n+1})$, and we have the following Young's inequality:

$$\|f*g\|_{r,\nu_{n+1}} \leqslant \|f\|_{p,\nu_{n+1}}\|g\|_{q,\nu_{n+1}}.$$

Definition 2.2. The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined on $L^1(d\nu_{n+1})$ by

$$\forall (\mu,\lambda) \in \Upsilon;\quad \mathscr{F}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{(\mu,\lambda)}(r,x)\,d\nu_{n+1}(r,x),$$

where $\varphi_{(\mu,\lambda)}$ is the eigenfunction given by relation (2.2), and Υ is the set defined by relation (1.1).

In the following, we give some properties of this transform.

- For every $f,g \in L^1(d\nu_{n+1})$,

$$\mathscr{F}(f*g) = \mathscr{F}(f)\mathscr{F}(g).$$

  Moreover, relation (2.4) implies that the Fourier transform $\mathscr{F}$ is a bounded linear operator from $L^1(d\nu_{n+1})$ into $L^\infty(d\gamma_{n+1})$, and that for every $f \in L^1(d\nu_{n+1})$, we have

$$\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}} \leqslant \|f\|_{1,\nu_{n+1}}.$$

- For every $f \in L^1(d\nu_{n+1})$ and $(r,x) \in [0,+\infty[\,\times\mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^1(d\nu_{n+1})$, and we have
$$\forall\,(\mu,\lambda) \in \Upsilon,\ \ \mathscr{F}(\mathcal{T}_{(r,x)}(f))(\mu,\lambda) = \overline{\varphi_{(\mu,\lambda)}(r,x)}\mathscr{F}(f)(\mu,\lambda).$$
- For every $f \in L^1(d\nu_{n+1})$, $\mathscr{F}(f)(\mu,\lambda) = \widetilde{\mathscr{F}}(f)(\sqrt{\mu^2+|\lambda|^2},\lambda)$, where $\widetilde{\mathscr{F}}$ is the mapping defined on $L^1(d\nu_{n+1})$ by
$$\widetilde{\mathscr{F}}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x) j_{\frac{n-1}{2}}(r\mu) e^{-i\langle\lambda|\,x\rangle}\, d\nu_{n+1}(r,x).$$

Theorem 2.3 (Inversion formula)**.** *Let $f \in L^1(d\nu_{n+1})$ such that $\mathscr{F}(f) \in L^1(d\gamma_{n+1})$, then for almost every $(r,x) \in \mathbb{R}\times\mathbb{R}^n$,*
$$\begin{aligned} f(r,x) &= \iint_{\Upsilon_+} \mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\, d\gamma_{n+1}(\mu,\lambda) \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} \widetilde{\mathscr{F}}(f)(\mu,\lambda) j_{\frac{n-1}{2}}(r\mu) e^{i\langle\lambda|\,x\rangle}\, d\nu_{n+1}(\mu,\lambda). \end{aligned}$$

Theorem 2.4 (Plancherel theorem)**.** *The Fourier transform $\widetilde{\mathscr{F}}$ can be extended to an isometric isomorphism from $L^2(d\nu_{n+1})$ onto itself. In particular, for every $f \in L^2(d\nu_{n+1})$,*
$$\|\widetilde{\mathscr{F}}(f)\|_{2,\nu_{n+1}} = \|f\|_{2,\nu_{n+1}}.$$

Corollary 2.5. *For all functions f and g in $L^2(d\nu_{n+1})$, we have*
$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}\, d\nu_{n+1}(r,x) \\ &\quad = \iint_{\Upsilon_+} \mathscr{F}(f)(\mu,\lambda)\overline{\mathscr{F}(g)(\mu,\lambda)}\, d\gamma_{n+1}(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} \widetilde{\mathscr{F}}(f)(\mu,\lambda)\overline{\widetilde{\mathscr{F}}(g)(\mu,\lambda)}\, d\nu_{n+1}(\mu,\lambda). \end{aligned}$$

Remark 2.6.

(i) For every $f,g \in L^2(d\nu_{n+1})$, the function $f*g$ belongs to the space $C_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ consisting of continuous functions h on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, and such that
$$\lim_{r^2+|x|^2\to+\infty} h(r,x) = 0.$$
Moreover,
$$f*g = \widetilde{\mathscr{F}}^{-1}(\widetilde{\mathscr{F}}(f)\widetilde{\mathscr{F}}(g)),$$
where $\widetilde{\mathscr{F}}^{-1}$ is the mapping defined on $L^1(d\nu_{n+1})$ by
$$\widetilde{\mathscr{F}}^{-1}(g)(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} g(\mu,\lambda) j_{\frac{n-1}{2}}(r\mu) e^{i\langle\lambda|\,x\rangle}\, d\nu_{n+1}(\mu,\lambda) = \widetilde{\mathscr{F}}(\check{g})(r,x).$$

(ii) Let $f,g \in L^2(d\nu_{n+1})$, the function $f*g$ belongs to $L^2(d\nu_{n+1})$ if and only if $\widetilde{\mathscr{F}}(f)\widetilde{\mathscr{F}}(g)$ belongs to $L^2(d\nu_{n+1})$, and we have
$$\widetilde{\mathscr{F}}(f*g) = \widetilde{\mathscr{F}}(f)\widetilde{\mathscr{F}}(g).$$

3 The windowed Fourier transform associated with the spherical mean operator

We recall some results introduced and proved in [15].

Definition 3.1. Let $g \in L^2(d\nu_{n+1})$ and $(s,y) \in [0,+\infty[\,\times\mathbb{R}^n$. The modulation of g by (s,y) is the function defined by

$$g_{(s,y)}(r,x) = \widetilde{\mathscr{F}}\Big(\sqrt{\mathcal{T}_{(s,y)}\big(|\widetilde{\mathscr{F}}(g)|^2\big)}\Big)(r,x), \quad (r,x) \in [0,+\infty[\,\times\mathbb{R}^n.$$

We denote by

- $L^p(d\mu_{n+1}) = L^p(d\nu_{n+1} \otimes d\nu_{n+1})$, $1 \leqslant p \leqslant +\infty$, the space of measurable functions f on $([0,+\infty[\,\times\mathbb{R}^n)^2$ with respect to the measure

  $$d\mu_{n+1}((r,x),(s,y)) = d\nu_{n+1}(r,x) \otimes d\nu_{n+1}(s,y)$$

  such that

  $$\|F\|^p_{p,\mu_{n+1}} = \int\int_{([0,+\infty[\,\times\mathbb{R}^n)^2} \big|F((r,x),(s,y))\big|^p \, d\mu_{n+1}((r,x),(s,y)) < \infty, \quad 1 \leqslant p < +\infty,$$

  $$\|F\|_{\infty,\mu_{n+1}} = \operatorname*{ess\,sup}_{(r,x),(s,y)\in[0,+\infty[\,\times\mathbb{R}^n} \big|F((r,x),(s,y))\big| < \infty.$$

- $\langle\,\cdot\,,\,\cdot\,\rangle_{\mu_{n+1}}$ the usual inner product in the Hilbert space $L^2(d\mu_{n+1})$.

Definition 3.2. Let g be a non-zero function in $L^2(d\nu_{n+1})$, called the window function. The windowed Fourier transform associated with the spherical mean operator is the mapping $\mathscr{V}_g$ defined for $f \in L^2(d\nu_{n+1})$ by

$$\mathscr{V}_g(f)((r,x),(s,y)) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(\mu,\lambda)\overline{\mathcal{T}_{(r,x)}(g_{(s,y)})(\mu,\lambda)}\, d\nu_{n+1}(\mu,\lambda) = \big\langle f, \mathcal{T}_{(r,x)}(g_{(s,y)})\big\rangle_{\nu_{n+1}},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\nu_{n+1}}$ is the usual inner product in the Hilbert space $L^2(d\nu_{n+1})$.

Proposition 3.3. *Let g be a window function. For every $f \in L^2(d\nu_{n+1})$, we have*

$$\|\mathscr{V}_g(f)\|_{\infty,\mu_{n+1}} \leqslant \|f\|_{2,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}. \tag{3.1}$$

Proposition 3.4. *Let g be a window function.*

(i) *Plancherel formula: for every $f \in L^2(d\nu_{n+1})$, we have*

$$\|\mathscr{V}_g(f)\|_{2,\mu_{n+1}} = \|f\|_{2,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}. \tag{3.2}$$

(ii) *Parseval formula: for all $f, h \in L^2(d\nu_{n+1})$, we have*

$$\big\langle \mathscr{V}_g(f), \mathscr{V}_g(h)\big\rangle_{\mu_{n+1}} = \|g\|_{2,\nu_{n+1}}\langle f,h\rangle_{\nu_{n+1}}.$$

(iii) *Inversion formula: for every $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ such that $\widetilde{\mathscr{F}}(f) \in L^1(d\nu_{n+1})$, we have*

$$f(\mu,\lambda) = \frac{1}{\|g\|^2_{2,\nu_{n+1}}} \iint_{([0,+\infty[\,\times\mathbb{R}^n)^2} \mathscr{V}_g(f)((r,-x),(s,y))\mathcal{T}_{(r,x)}(\check{g}_{(s,y)})(\mu,-\lambda)\, d\nu_{n+1}(r,x)\, d\nu_{n+1}(s,y).$$

By Riesz–Thorin's interpolation theorem we obtain the following

Proposition 3.5. *Let g be a window function, $f \in L^2(d\nu_{n+1})$ and $2 \leqslant p \leqslant +\infty$, then*

$$\|\mathscr{V}_g(f)\|_{p,\mu_{n+1}} \leqslant \|f\|_{2,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}. \tag{3.3}$$

4 L^p local uncertainty principle for $\mathscr{V}_g$

In this section, we establish the L^p local uncertainty principle for the windowed Fourier transform $\mathscr{V}_g$.

Theorem 4.1. *Let g be a window function and Σ be measurable subset of $([0,+\infty[\times\mathbb{R}^n)^2$ such that $0<\mu_{n+1}(\Sigma)<+\infty$. Let $p\in]1,2]$, $q=\frac{p}{p-1}$ and $0<b<\frac{2n+1}{2q}$. For every $f\in L^p(d\nu_{n+1})$,*

$$\|\chi_\Sigma\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\leqslant C_1(b,n,q)(\mu_{n+1}(\Sigma))^{\frac{2b}{2n+1}}\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)\|g\|_{2,\nu_{n+1}}. \quad (4.1)$$

Proof. It is clear that the inequality holds if $\||(r,x)|^bf\|_{2p,\nu_{n+1}}=+\infty$ or $\||(r,x)|^bf\|_{2,\nu_{n+1}}=+\infty$. Let $f\in L^p(d\nu_{n+1})$, $1<p\leqslant 2$, $q=\frac{p}{p-1}$ such that

$$\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}<+\infty.$$

For $\rho>0$, let

$$B_\rho=\big\{(r,x)\in[0,+\infty[\times\mathbb{R}^n;\ r^2+|x|^2\leqslant\rho^2\big\}.$$

Denote by χ_Σ and χ_{B_ρ} respectively, the characteristic functions associated to Σ and B_ρ. Using Minkowski's inequality, relations (3.1) and (3.3), we obtain

$$\begin{aligned}\|\chi_\Sigma\mathscr{V}_g(f)\|_{q,\mu_{n+1}}&\leqslant\|\chi_\Sigma\mathscr{V}_g(\chi_{B_\rho}f)\|_{q,\mu_{n+1}}+\|\chi_\Sigma\mathscr{V}_g(\chi_{B_\rho^c}f)\|_{q,\mu_{n+1}}\\&\leqslant(\mu_{n+1}(\Sigma))^{\frac1q}\|\mathscr{V}_g(\chi_{B_\rho}f)\|_{\infty,\mu_{n+1}}+\|\mathscr{V}_g(\chi_{B_\rho^c}f)\|_{q,\mu_{n+1}}\\&\leqslant(\mu_{n+1}(\Sigma))^{\frac1q}\|g\|_{2,\nu_{n+1}}\|\chi_{B_\rho}f\|_{2,\nu_{n+1}}+\|g\|_{2,\nu_{n+1}}\|\chi_{B_\rho^c}f\|_{2,\nu_{n+1}}.\end{aligned}$$

On the other hand, by Hölder's inequality,

$$\|\chi_{B_\rho}f\|_{2,\nu_{n+1}}\leqslant\||(r,x)|^{-b}\chi_{B_\rho}\|_{2q,\nu_{n+1}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}.$$

By simple calculus and the hypothesis $0<b<\frac{2n+1}{2q}$, we obtain

$$\|\chi_{B_\rho}f\|_{2,\nu_{n+1}}\leqslant C_{b,n,q}\rho^{\frac{2n+1}{2q}-b}\||(r,x)|^bf\|_{2p,\nu_{n+1}}. \quad (4.2)$$

Moreover,

$$\|\chi_{B_\rho^c}f\|_{2,\nu_{n+1}}\leqslant\rho^{-b}\||(r,x)|^bf\|_{2,\nu_{n+1}}. \quad (4.3)$$

From (4.2) and (4.3), we get

$$\begin{aligned}&\|\chi_\Sigma\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\\&\qquad\leqslant\rho^{-b}\||(r,x)|^bf\|_{2,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}+(\mu_{n+1}(\Sigma))^{\frac1q}C_{b,n,q}\rho^{\frac{2n+1}{2q}-b}\||(r,x)|^bf\|_{2p,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}.\end{aligned}$$

We choose

$$\rho=(C_{b,n,q})^{\frac{-2q}{2n+1}}(\mu_{n+1}(\Sigma))^{\frac{-2}{2n+1}}$$

and obtain inequality (4.1). □

Lemma 4.2. *Let g be a window function and Σ be measurable subset of $([0,+\infty[\times\mathbb{R}^n)^2$ such that $0<\mu_{n+1}(\Sigma)<+\infty$. Let $p\in]1,2]$, $q=\frac{p}{p-1}$ and $b>\frac{2n+1}{2q}$. For every $f\in L^p(d\nu_{n+1})$,*

$$\|\chi_\Sigma]\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\leqslant C_2(b,n,q)(\mu_{n+1}(\Sigma))^{\frac1q}\|f\|_{2p,\nu_{n+1}}^{1-\frac{2n+1}{2qb}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{2n+1}{2qb}}\|g\|_{2,\nu_{n+1}},$$

where

$$C_2(b,n,q)=\left(\frac{\Gamma(\frac{2n+1}{2bp})\Gamma(\frac{2qb-(2n+1)}{2bp})}{bp\,2^{n+\frac12}\Gamma(n+\frac12)\Gamma(\frac qp)}\right)^{\frac{1}{2q}}\left(\frac{2qb}{2qb-(2n+1)}\right)^{\frac{1}{2p}}\left(\frac{2qb}{2n+1}-1\right)^{\frac{2n+1}{4qbp}}.$$

Proof. We suppose naturally that $f \neq 0$. It is clear that the inequality holds if $\|f\|_{2p,\nu_{n+1}}$ or $\||(r,x)|^b f\|_{2p,\nu_{n+1}} = +\infty$. Assume that

$$\|f\|_{2p,\nu_{n+1}} + \||(r,x)|^b f\|_{2p,\nu_{n+1}} < +\infty.$$

From the hypothesis $b > \frac{2n+1}{2q}$, we deduce that the function $(r,x) \to (1+(r^2+|x|^2)^{bp})^{\frac{-1}{p}}$ belongs to $L^q(d\nu_{n+1})$ and, by Hölder's inequality, we have

$$\begin{aligned}\|f\|^{2p}_{2,\nu_{n+1}} &= \Big(\int_0^{+\infty}\int_{\mathbb{R}^n} (1+(r^2+|x|^2)^{bp})^{\frac{-1}{p}}(1+(r^2+|x|^2)^{bp})^{\frac{1}{p}}|f(r,x)|^2\,d\nu_{n+1}(r,x)\Big)^p \\ &\leqslant \Big(\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{d\nu_{n+1}(r,x)}{(1+(r^2+|x|^2)^{bp})^{\frac{q}{p}}}\Big)^{\frac{p}{q}}\Big(\|f\|^{2p}_{2p,\nu_{n+1}} + \||(r,x)|^b f\|^{2p}_{2p,\nu_{n+1}}\Big). \qquad (4.4)\end{aligned}$$

However, with a standard computation, we obtain

$$\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{d\nu_{n+1}(r,x)}{(1+(r^2+|x|^2)^{bp})^{\frac{q}{p}}}\Big)^{\frac{p}{q}} = \Big(\frac{\Gamma(\frac{2n+1}{2bp})\Gamma(\frac{2qb-(2n+1)}{2bp})}{bp\,2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})\Gamma(\frac{q}{p})}\Big)^{\frac{p}{q}}.$$

Replacing $f(r,x)$ by $f_t(r,x) = f(rt,xt)$, $t>0$, in relation (4.4), we deduce that for all $t>0$,

$$\|f\|^{2p}_{2,\nu_{n+1}} \leqslant \Big(\frac{\Gamma(\frac{2n+1}{2bp})\Gamma(\frac{2qb-(2n+1)}{2bp})}{bp\,2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})\Gamma(\frac{q}{p})}\Big)^{\frac{p}{q}}\Big(t^{(2n+1)(p-1)}\|f\|^{2p}_{2p,\nu_{n+1}} + t^{(2n+1)(p-1)-2pb}\||(r,x)|^b f\|^{2p}_{2p,\nu_{n+1}}\Big).$$

In particular, for

$$t = \Big(\frac{(2bp-(2n+1)(p-1))\||(r,x)|^b f\|^{2p}_{2p,\nu_{n+1}}}{(2n+1)(p-1)\|f\|^{2p}_{2p,\nu_{n+1}}}\Big)^{\frac{1}{2bp}},$$

we obtain

$$\|f\|_{2,\nu_{n+1}} \leqslant C_2(b,n,q)\|f\|^{1-\frac{2n+1}{2qb}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{2n+1}{2qb}}_{2p,\nu_{n+1}},$$

where

$$C_2(b,n,q) = \Big(\frac{\Gamma(\frac{2n+1}{2bp})\Gamma(\frac{2qb-(2n+1)}{2bp})}{bp\,2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})\Gamma(\frac{q}{p})}\Big)^{\frac{1}{2q}}\Big(\frac{2qb}{2qb-(2n+1)}\Big)^{\frac{1}{2p}}\Big(\frac{2qb}{2n+1}-1\Big)^{\frac{2n+1}{4qbp}}.$$

Moreover,

$$\begin{aligned}\|\chi_\Sigma \mathscr{V}_g(f)\|_{q,\mu_{n+1}} &\leqslant (\mu_{n+1}(\Sigma))^{\frac{1}{q}}\|\mathscr{V}_g(f)\|_{\infty,\mu_{n+1}} \leqslant (\mu_{n+1}(\Sigma))^{\frac{1}{q}}\|f\|_{2,\nu_{n+1}}\|g\|_{2,\nu_{n+1}} \\ &\leqslant C_2(b,n,q)(\mu_{n+1}(\Sigma))^{\frac{1}{q}}\|f\|^{1-\frac{2n+1}{2qb}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{2n+1}{2qb}}_{2p,\nu_{n+1}}\|g\|_{2,\nu_{n+1}}.\end{aligned}$$

This completes the proof. □

Lemma 4.3. *Under the same assumptions as in Lemma 4.2 and with* $b = \frac{2n+1}{2q}$ *there exists a finite constant* $C_3(b)$ *such that for all* $f \in L^p(d\nu_{n+1})$,

$$\begin{aligned}&\|\chi_\Sigma \mathscr{V}_g(f)\|_{q,\mu_{n+1}} \\ &\quad\leqslant C_3(b)(\mu_{n+1}(\Sigma))^{\frac{1}{2q}}\Big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\Big)\|g\|_{2,\nu_{n+1}}. \qquad (4.5)\end{aligned}$$

Proof. Let $s>0$, from the inequality

$$\Big(\frac{|(r,x)|}{s}\Big)^{\frac{2n+1}{4q}} \leqslant 1 + \Big(\frac{|(r,x)|}{s}\Big)^{\frac{2n+1}{2q}}$$

it follows that

$$\left\||(r,x)|^{\frac{2n+1}{4q}}f\right\|_{2p,\nu_{n+1}}\leqslant s^{\frac{2n+1}{4q}}\|f\|_{2p,\nu_{n+1}}+s^{\frac{-(2n+1)}{4q}}\||(r,x)|^{\frac{2n+1}{2q}}f\|_{2p,\nu_{n+1}}.$$

In particular, by choosing

$$s=\left\||(r,x)|^{\frac{2n+1}{2q}}f\right\|_{2p,\nu_{n+1}}^{\frac{2q}{2n+1}}\|f\|_{2p,\nu_{n+1}}^{\frac{-2q}{2n+1}},$$

we obtain

$$\left\||(r,x)|^{\frac{2n+1}{4q}}f\right\|_{2p,\nu_{n+1}}\leqslant 2\|f\|_{2p,\nu_{n+1}}^{\frac{1}{2}}\left\||(r,x)|^{\frac{2n+1}{2q}}f\right\|_{2p,\nu_{n+1}}^{\frac{1}{2}}.$$

Similarly, we prove that

$$\left\||(r,x)|^{\frac{2n+1}{4q}}f\right\|_{2,\nu_{n+1}}\leqslant 2\|f\|_{2,\nu_{n+1}}^{\frac{1}{2}}\left\||(r,x)|^{\frac{2n+1}{2q}}f\right\|_{2,\nu_{n+1}}^{\frac{1}{2}}.$$

Thus, from (4.1), we deduce that

$$\begin{aligned}
&\|\chi_\Sigma\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\\
&\quad\leqslant C_1\Big(\frac{2n+1}{4q},n,q\Big)(\mu_{n+1}(\Sigma))^{\frac{1}{2q}}\left(\left\||(r,x)|^{\frac{2n+1}{4q}}f\right\|_{2p,\nu_{n+1}}+\left\||(r,x)|^{\frac{2n+1}{4q}}f\right\|_{2,\nu_{n+1}}\right)\|g\|_{2,\nu_{n+1}}\\
&\quad\leqslant 2C_1\Big(\frac{2n+1}{4q},n,q\Big)(\mu_{n+1}(\Sigma))^{\frac{1}{2q}}\\
&\qquad\times\left(\|f\|_{2,\nu_{n+1}}^{\frac{1}{2}}\left\||(r,x)|^{\frac{2n+1}{2q}}f\right\|_{2,\nu_{n+1}}^{\frac{1}{2}}+\|f\|_{2p,\nu_{n+1}}^{\frac{1}{2}}\left\||(r,x)|^{\frac{2n+1}{2q}}f\right\|_{2p,\nu_{n+1}}^{\frac{1}{2}}\right)\|g\|_{2,\nu_{n+1}},
\end{aligned}$$

which gives the result for $b=\frac{2n+1}{2q}$. □

5 L^p Heisenberg–Pauli–Weyl uncertainty principle for $\mathscr{V}_g$

From the L^p local uncertainty principle we can find the following L^p Heisenberg–Pauli–Weyl uncertainty principle for the windowed Fourier transform $\mathscr{V}_g$.

Theorem 5.1. *Let g be a window function, $1<p\leqslant 2$, $q=\frac{p}{p-1}$ and $0<b<\frac{2n+1}{2q}$. For every $f\in L^p(d\nu_{n+1})$ and $a>0$,*

$$\begin{aligned}
&\|\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\leqslant C_1(a,b,n,q)\\
&\quad\times\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^{\frac{a}{a+4b}}\left\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\right\|_{q,\mu_{n+1}}^{\frac{4b}{a+4b}}\|g\|_{2,\nu_{n+1}}^{\frac{a}{a+4b}},
\end{aligned}\tag{5.1}$$

where

$$C_1(a,b,n,q)=\frac{(C_1(b,n,q))^{\frac{a}{a+4b}}}{(2^{2n+1}\Gamma(2n+2)))^{\frac{2ab}{(2n+1)(a+4b)}}}\left(\Big(\frac{a}{4b}\Big)^{\frac{4b}{a+4b}}+\Big(\frac{4b}{a}\Big)^{\frac{a}{a+4b}}\right)^{\frac{1}{q}}.$$

Proof. Let $0<b<\frac{2n+1}{2q}$, $a>0$. For $\rho>0$, let

$$\widetilde{B}_\rho=\Big\{((s,y),(\mu,\lambda))\in([0,+\infty[\times\mathbb{R}^n)^2;\ s^2+\mu^2+|y|^2+|\lambda|^2\leqslant\rho^2\Big\}.$$

Then

$$\|\mathscr{V}_g(f)\|_{q,\mu_{n+1}}^q=\|\chi_{\widetilde{B}_\rho}\mathscr{V}_g(f)\|_{q,\mu_{n+1}}^q+\|\chi_{\widetilde{B}_\rho^c}\mathscr{V}_g(f)\|_{q,\mu_{n+1}}^q.\tag{5.2}$$

From (4.1), we get

$$\|\chi_{\widetilde{B}_\rho}\mathscr{V}_g(f)\|_{q,\mu_\alpha}^q\leqslant C_1^q(b,n,q)(\mu_{n+1}(\widetilde{B}_\rho))^{\frac{2bq}{2n+1}}\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^q\|g\|_{2,\nu_{n+1}}^q.$$

On the other hand, by [15, Lemma 6.6], we have

$$\mu_{n+1}(\widetilde{B}_\rho)=\frac{\rho^{4n+2}}{2^{2n+1}\Gamma(2n+2)}.$$

Using the previous result, we obtain

$$\begin{aligned}&\|\chi_{\widetilde{B}_\rho}\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}\\&\quad\leqslant C_1^q(b,n,q)\Big(\frac{1}{2^{2n+1}\Gamma(2n+2)}\Big)^{\frac{2bq}{2n+1}}\rho^{4bq}\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^q\|g\|^q_{2,\nu_{n+1}}. \quad (5.3)\end{aligned}$$

Moreover,

$$\|\chi_{\widetilde{B}^c_\rho}\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}\leqslant\rho^{-aq}\Big\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\Big\|^q_{q,\mu_{n+1}}. \quad (5.4)$$

By Combining relations (5.2), (5.3) and (5.4), we get

$$\begin{aligned}&\|\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}\\&\quad\leqslant C_1^q(b,n,q)\Big(\frac{1}{2^{2n+1}\Gamma(2n+2)}\Big)^{\frac{2bq}{2n+1}}\rho^{4bq}\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^q\|g\|^q_{2,\nu_{n+1}}\\&\qquad+\rho^{-aq}\Big\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\Big\|^q_{q,\mu_{n+1}}.\end{aligned}$$

We choose

$$\rho=\left(\frac{a(2^{2n+1}\Gamma(2n+2))^{\frac{2bq}{2n+1}}}{4bC_1^q(b,n,q)}\right)^{\frac{1}{(a+4b)q}}\left(\frac{\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\|_{q,\mu_{n+1}}}{\big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\big)\|g\|_{2,\nu_{n+1}}}\right)^{\frac{1}{a+4b}}$$

and obtain inequality (5.1). □

Lemma 5.2. *Under the same assumptions as in Theorem 5.1 and with $b>\frac{2n+1}{2q}$ there exists a finite constant $C_2(a,b,n,q)$ such that for all $f\in L^p(d\nu_{n+1})$,*

$$\begin{aligned}\|\mathscr{V}_g(f)\|_{q,\mu_{n+1}}&\leqslant C_2(a,b,n,q)\|f\|_{2p,\nu_{n+1}}^{\frac{a(q-\frac{2n+1}{2b})}{4n+2+aq}}\\&\times\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{a(2n+1)}{2b(4n+2+aq)}}\|g\|_{2,\nu_{n+1}}^{\frac{aq}{4n+2+aq}}\big\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\big\|_{q,\mu_{n+1}}^{\frac{4n+2}{4n+2+aq}}, \quad (5.5)\end{aligned}$$

where

$$C_2(a,b,n,q)=\frac{(C_2(b,n,q))^{\frac{aq}{4n+2+aq}}}{(2^{2n+1}\Gamma(2n+2))^{\frac{a}{4n+2+aq}}}\left(\Big(\frac{aq}{4n+2}\Big)^{\frac{4n+2}{4n+2+aq}}+\Big(\frac{4n+2}{aq}\Big)^{\frac{aq}{4n+2+aq}}\right)^{\frac{1}{q}}.$$

Proof. Let $b>\frac{2n+1}{2q}$, $a>0$, $f\neq0$ and let $\rho>0$. From Lemma 4.2, we obtain

$$\begin{aligned}\|\chi_{\widetilde{B}_\rho}\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}&\leqslant C_2^q(b,n,q)\mu_{n+1}(\widetilde{B}_\rho)\|f\|_{2p,\nu_{n+1}}^{q-\frac{2n+1}{2b}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{2n+1}{2b}}\|g\|^q_{2,\nu_{n+1}}\\&=C_2^q(b,n,q)\,\frac{\rho^{4n+2}}{2^{2n+1}\Gamma(2n+2)}\,\|f\|_{2p,\nu_{n+1}}^{q-\frac{2n+1}{2b}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{2n+1}{2b}}\|g\|^q_{2,\nu_{n+1}}. \quad (5.6)\end{aligned}$$

Combining relations (5.2), (5.4) and (5.6), we get

$$\begin{aligned}\|\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}\leqslant C_2^q(b,n,q)\,&\frac{\rho^{4n+2}}{2^{2n+1}\Gamma(2n+2)}\,\|f\|_{2p,\nu_{n+1}}^{q-\frac{2n+1}{2b}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{2n+1}{2b}}\|g\|^q_{2,\nu_{n+1}}\\&+\rho^{-aq}\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}.\end{aligned}$$

We choose

$$\rho=\left(\frac{aq2^{2n+1}\Gamma(2n+2)\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}}}{(4n+2)C_2^q(b,n,q)\|f\|_{2p,\nu_{n+1}}^{q-\frac{2n+1}{2b}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{2n+1}{2b}}\|g\|^q_{2,\nu_{n+1}}}\right)^{\frac{1}{4n+2+aq}}$$

and obtain inequality (5.5). □

Corollary 5.3. *Under the same assumptions as in Theorem 5.1 and with* $b = \frac{2n+1}{2q}$ *there exists a finite constant* $C_3(a,b)$ *such that for all* $f \in L^p(d\nu_{n+1})$,

$$\|\mathscr{V}_g(f)\|_{q,\mu_{n+1}} \leqslant C_3(a,b)\Big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\Big)^{\frac{a}{a+2b}} \times \big\||((s,y),(\mu,\lambda))|^a \mathscr{V}_g(f)\big\|^{\frac{2b}{a+2b}}_{q,\mu_{n+1}}\|g\|^{\frac{a}{a+2b}}_{2,\nu_{n+1}},$$

where

$$C_3(a,b) = \frac{(C_3(b))^{\frac{a}{a+2b}}}{(2^{2n+1}\Gamma(2n+2))^{\frac{a}{2q(a+2b)}}}\Big(\Big(\frac{a}{2b}\Big)^{\frac{2b}{a+2b}} + \Big(\frac{2b}{a}\Big)^{\frac{a}{a+2b}}\Big)^{\frac{1}{q}}.$$

Proof. Let $b = \frac{2n+1}{2q}$, $a > 0$, $f \neq 0$ and let $\rho > 0$. From (4.5), we get

$$\begin{aligned}
&\|\chi_{\widetilde{B}_\rho}\mathscr{V}_g(f)\|^q_{q,\mu_{n+1}} \\
&\quad\leqslant (C_3(b))^q(\mu_{n+1}(\widetilde{B}_\rho))^{\frac{1}{2}}\Big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\Big)^q\|g\|^q_{2,\nu_{n+1}} \\
&\quad= (C_3(b))^q\,\frac{\rho^{2n+1}}{\sqrt{2^{2n+1}\Gamma(2n+2)}} \\
&\qquad\times\Big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\Big)^q\|g\|^q_{2,\nu_{n+1}}. \qquad (5.7)
\end{aligned}$$

Combining relations (5.2), (5.4)) and (5.7), we obtain

$$\begin{aligned}
\|\mathscr{V}_g(f)\|^q_{q,\mu_\alpha} &\leqslant (C_3(b))^q\,\frac{\rho^{2n+1}}{\sqrt{2^{2n+1}\Gamma(2n+2)}} \\
&\quad\times\Big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\Big)^q\|g\|^q_{2,\nu_{n+1}} \\
&\quad+ \rho^{-aq}\big\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\big\|^q_{q,\mu_{n+1}}.
\end{aligned}$$

We choose

$$\begin{aligned}
\rho &= \Big(\frac{aq\big(2^{2n+1}\Gamma(2n+2)\big)^{\frac{1}{2}}}{(C_3(b))^q(2n+1)}\Big)^{\frac{1}{2n+1+aq}} \\
&\quad\times\Big(\frac{\||((s,y),(\mu,\lambda))|^a\mathscr{V}_g(f)\|_{q,\mu_{n+1}}}{\big(\|f\|^{\frac{1}{2}}_{2,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2,\nu_{n+1}} + \|f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\||(r,x)|^b f\|^{\frac{1}{2}}_{2p,\nu_{n+1}}\big)\|g\|_{2,\nu_{n+1}}}\Big)^{\frac{q}{2n+1+aq}}
\end{aligned}$$

and obtain the result of this corollary. □

6 L^p-Donoho–Stark's uncertainty principle for the windowed Fourier transform $\mathscr{V}_g$

In what follows, we use the L^p Heisenberg–Pauli–Weyl uncertainty principle to obtain a concentration uncertainty principle.

Definition 6.1. Let $0 \leqslant \varepsilon < 1$ and let S be a measurable set of $[0,+\infty[\times\mathbb{R}^n$. We say that $f \in L^p(d\nu_{n+1})$, $p \in [1,2]$, is ε-concentrated on S in $L^p(d\nu_{n+1})$-norm if there is a measurable function h vanishing outside S such that

$$\|f - h\|_{p,\nu_{n+1}} \leqslant \varepsilon\|f\|_{p,\nu_{n+1}}.$$

We introduce a projection operator P_S as $P_S f(r,x) = f(r,x)$ if $(r,x) \in S$, and $P_S f(r,x) = 0$ if $(r,x) \notin S$.

Let $0 \leqslant \varepsilon_S < 1$. Then f is ε_S-concentrated on S in $L^p(d\nu_{n+1})$-norm if and only if

$$\|f - P_S f\|_{p,\nu_{n+1}} \leqslant \varepsilon_S\|f\|_{p,\nu_{n+1}}.$$

Definition 6.2. Let g be a window function and let Σ be a measurable set of $([0,+\infty[\times\mathbb{R}^n)^2$. We define a projection operator Q_Σ as

$$Q_\Sigma f = \mathscr{V}_g^{-1}\big(P_\Sigma(\mathscr{V}_g(f))\big).$$

Let $0 \leqslant \varepsilon_\Sigma < 1$. Then $\mathscr{V}_g$ is ε_Σ-concentrated on Σ in $L^q(d\mu_{n+1})$-norm, $1 \leqslant q \leqslant 2$ if and only if

$$\|\mathscr{V}_g(f) - \mathscr{V}_g(Q_\Sigma f)\|_{q,\mu_{n+1}} \leqslant \varepsilon_\Sigma \|\mathscr{V}_g(f)\|_{q,\mu_{n+1}}.$$

Proposition 6.3. *Let g be a window function and Σ be a measurable set of $([0,+\infty[\times\mathbb{R}^n)^2$. Then for every $p > 2$ and $\varepsilon > 0$, if $\mathscr{V}_g$ is ε-concentrated in Σ with respect to the norm $\|\cdot\|_{2,\mu_{n+1}}$, then*

$$\mu_{n+1}(\Sigma) \geqslant (1-\varepsilon^2)^{\frac{p}{p-2}},$$

where

$$\mu_{n+1}(\Sigma) = \int\int_\Sigma d\nu_{n+1}(r,x)\, d\nu_{n+1}(s,y).$$

Proof. Let $f \in L^2(d\nu_{n+1})$ and $p > 2$. Since $\mathscr{V}_g(f)$ is ε-concentrated in Σ with respect to the norm $\|\cdot\|_{2,\mu_{n+1}}$, we have

$$\|\chi_{\Sigma^c}\mathscr{V}_g(f)\|_{2,\mu_{n+1}} \leqslant \varepsilon \|f\|_{2,\nu_{n+1}} \|g\|_{2,\nu_{n+1}}.$$

Now, using relation (3.2), we get

$$\|\chi_\Sigma \mathscr{V}_g(f)\|^2_{2,\mu_{n+1}} \geqslant (1-\varepsilon^2)\|f\|^2_{2,\nu_{n+1}} \|g\|^2_{2,\nu_{n+1}}.$$

Applying Hölder's inequality, we obtain

$$\|\chi_\Sigma \mathscr{V}_g(f)\|^2_{2,\mu_{n+1}} \leqslant \|\mathscr{V}_g(f)\|^2_{p,\mu_{n+1}} (\mu_{n+1}(\Sigma))^{\frac{p-2}{p}}.$$

By relation (3.3), we obtain

$$\|\chi_\Sigma \mathscr{V}_g(f)\|^2_{2,\mu_{n+1}} \leqslant \|f\|^2_{2,\nu_{n+1}} \|g\|^2_{2,\nu_{n+1}} (\mu_{n+1}(\Sigma))^{\frac{p-2}{p}}.$$

Finally,

$$(\mu_{n+1}(\Sigma))^{\frac{p-2}{p}} \geqslant 1-\varepsilon^2. \qquad \square$$

Proposition 6.4. *Let g be a window function and $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ such that $\|\mathscr{V}_g(f)\|_{2,\mu_{n+1}} = 1$. If f is ε_S-concentrated on S in $L^1(d\nu_{n+1})$-norm and $\mathscr{V}_g(f)$ is ε_Σ-concentrated on Σ in $L^2(d\mu_{n+1})$-norm, then*

$$\nu_{n+1}(S) \geqslant (1-\varepsilon_S)^2 \|f\|^2_{1,\nu_{n+1}} \|g\|^2_{2,\nu_{n+1}}$$

and

$$\mu_{n+1}(\Sigma)\|f\|^2_{2,\nu_{n+1}} \|g\|^2_{2,\nu_{n+1}} \geqslant 1-\varepsilon^2_\Sigma.$$

Proof. Since $\mathscr{V}_g(f)$ is ε_Σ-concentrated on Σ in $L^2(d\mu_{n+1})$-norm, by the orthogonality of the projection operator P_Σ, it follows that

$$\|\mathscr{V}_g(f)\|^2_{2,\mu_{n+1}} - \big\|\mathscr{V}_g(f) - P_\Sigma(\mathscr{V}_g(f))\big\|^2_{2,\mu_{n+1}} = \|P_\Sigma(\mathscr{V}_g(f))\|^2_{2,\mu_{n+1}} \geqslant 1-\varepsilon^2_\Sigma.$$

Thus

$$1-\varepsilon^2_\Sigma \leqslant \|\mathscr{V}_g(f)\|^2_{\infty,\mu_{n+1}} \mu_{n+1}(\Sigma) \leqslant \mu_{n+1}(\Sigma)\|f\|^2_{2,\nu_{n+1}} \|g\|^2_{2,\nu_{n+1}}.$$

In the same way, since f is ε_S-concentrated on S in $L^1(d\nu_{n+1})$-norm, we obtain

$$(1-\varepsilon_S)\|f\|_{1,\nu_{n+1}} \leqslant \int_S |f(r,x)|\, d\nu_{n+1}(r,x).$$

Now, by the Cauchy–Schwarz inequality and the fact that $\|f\|_{2,\nu_{n+1}} = \frac{1}{\|g\|_{2,\nu_{n+1}}}$, we get

$$(1-\varepsilon_S)\|f\|_{1,\nu_{n+1}} \leqslant \frac{\nu^{\frac{1}{2}}_{n+1}(S)}{\|g\|_{2,\nu_{n+1}}}. \qquad \square$$

Definition 6.5. Let g be a window function and Σ be a measurable set of $([0,+\infty[\times\mathbb{R}^n)^2$. Let $a>0$, $f\in L^p(d\nu_{n+1})$, $p\in[1,2]$ and $0\leqslant\varepsilon_\Sigma<1$. We say that $|((s,y),(\mu,\lambda))|^a\mathscr{V}_g$ is ε_Σ-concentrated on Σ in $L^q(d\mu_{n+1})$-norm if and only if

$$\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(f)-\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}\leqslant\varepsilon_\Sigma\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(f)\Big\|_{q,\mu_{n+1}}.$$

Theorem 6.6. *Let g be a window function and Σ be a measurable set of $([0,+\infty[\times\mathbb{R}^n)^2$. Let $f\in L^p(d\nu_{n+1})$, $p\in]1,2]$, $0\leqslant\varepsilon_\Sigma<1$ and $a>0$. If $|((s,y),(\mu,\lambda))|^a\mathscr{V}_g$ is ε_Σ-concentrated on Σ in $L^q(d\mu_{n+1})$-norm, then*

$$\|\mathscr{V}_g(f)\|_{q,\mu_{n+1}}$$
$$\leqslant\begin{cases} C_1(a,b,n,q)\Big(\||(r,x)|^bf\|_{2p,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^{\frac{a}{a+4b}}\|g\|_{2,\nu_{n+1}}^{\frac{a}{a+4b}} \\ \quad\times\Big(\dfrac{1}{1-\varepsilon_\Sigma}\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}\Big)^{\frac{4b}{a+4b}} & \text{if } 0<b<\dfrac{2n+1}{2q}, \\ C_2(a,b,n,q)\|f\|_{2p,\nu_{n+1}}^{\frac{a(q-\frac{2n+1}{2b})}{4n+2+aq}}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac{a(2n+1)}{2b(4n+2+aq)}}\|g\|_{2,\nu_{n+1}}^{\frac{aq}{4n+2+aq}} \\ \quad\times\Big(\dfrac{1}{1-\varepsilon_\Sigma}\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}\Big)^{\frac{4n+2}{4n+2+aq}} & \text{if } b>\dfrac{2n+1}{2q}, \\ C_3(a,b)\Big(\|f\|_{2,\nu_{n+1}}^{\frac12}\||(r,x)|^bf\|_{2,\nu_{n+1}}^{\frac12}+\|f\|_{2p,\nu_{n+1}}^{\frac12}\||(r,x)|^bf\|_{2p,\nu_{n+1}}^{\frac12}\Big)^{\frac{a}{a+2b}} \\ \quad\times\|g\|_{2,\nu_{n+1}}^{\frac{a}{a+2b}}\Big(\dfrac{1}{1-\varepsilon_\Sigma}\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}\Big)^{\frac{2b}{a+2b}} & \text{if } b=\dfrac{2n+1}{2q}. \end{cases}$$

Proof. Let $f\in L^p(d\nu_{n+1})$, $p\in]1,2]$. Since $|((s,y),(\mu,\lambda))|^a\mathscr{V}_g$ is ε_Σ-concentrated on Σ in $L^q(d\mu_{n+1})$-norm, we have

$$\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(f)\Big\|_{q,\mu_{n+1}}$$
$$\leqslant\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}+\varepsilon_\Sigma\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(f)\Big\|_{q,\mu_{n+1}}.$$

Thus

$$\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(f)\Big\|_{q,\mu_{n+1}}\leqslant\frac{1}{1-\varepsilon_\Sigma}\Big\|\big|((s,y),(\mu,\lambda))\big|^a\mathscr{V}_g(Q_\Sigma f)\Big\|_{q,\mu_{n+1}}.$$

Then we obtain the results from Theorem 5.1, Lemma 5.2 and Corollary 5.3. □

Definition 6.7. Let Σ be a measurable subset of $([0,+\infty[\times\mathbb{R}^n)^2$ and $0\leqslant\eta<1$. Then a nonzero function $f\in L^p(d\nu_{n+1}),1\leqslant p\leqslant2$, is η-bandlimited on Σ in $L^q(d\mu_{n+1})$-norm, $q=\frac{p}{p-1}$, if

$$\|\chi_{\Sigma^c}\mathscr{V}_g(f)\|_{q,\mu_{n+1}}\leqslant\eta\|f\|_{p,\nu_{n+1}}.$$

Corollary 6.8. *Let g be a window function such that $\|g\|_{2,\nu_{n+1}}=1$.*

(i) *If $0<b<\frac{2n+1}{4}$, then there exists a positive constant C such that for every function f which is η-bandlimited on Σ,*

$$(\mu_{n+1}(\Sigma))^{\frac{4b}{2n+1}}\Big(\||(r,x)|^bf\|_{4,\nu_{n+1}}+\||(r,x)|^bf\|_{2,\nu_{n+1}}\Big)^2\geqslant C(1-\eta^2)\|f\|_{2,\nu_{n+1}}^2.$$

(ii) *If $b>\frac{2n+1}{4}$, then there exists a positive constant C such that for every function f which is η-bandlimited on Σ,*

$$\mu_{n+1}(\Sigma)\|f\|_{4,\nu_{n+1}}^{2-\frac{2n+1}{2b}}\||(r,x)|^bf\|_{4,\nu_{n+1}}^{\frac{2n+1}{2b}}\geqslant C(1-\eta^2)\|f\|_{2,\nu_{n+1}}^2.$$

Proof. Since $f\in L^2(d\nu_{n+1})$ is η-bandlimited on Σ, we have

$$\|\chi_\Sigma\mathscr{V}_g(f)\|_{2,\mu_{n+1}}^2=\|f\|_{2,\nu_{n+1}}^2-\|\chi_{\Sigma^c}\mathscr{V}_g(f)\|_{2,\mu_{n+1}}^2\geqslant(1-\eta^2)\|f\|_{2,\nu_{n+1}}^2.$$

For (i) and (ii), we use the local inequalities given, respectively, by Theorem 4.1 and Lemma 4.2. □

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