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STEKLOV EIGENVALUES PROBLEMS FOR GENERALIZED (p, q)-LAPLACIAN TYPE OPERATORS

Abstract. In this paper, we study the following class of (p,q) elliptic problems under Steklov-type boundary conditions

$$\begin{cases} -\operatorname{div}\big(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\big) + a(|u|^p)|u|^{p-2}u = 0 \ \text{ in } \Omega, \\ a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nu = \lambda |u|^{m-2}u \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, ν is the outward unit normal vector on $\partial\Omega$, $2 \leq p < N$, $m \in \mathbb{R}$ with $m > 1$ in suitable ranges listed later and a is a C^1 real function and $\lambda > 0$ is a real parameter. Using variational methods, we establish the existence of a continuous and unbounded set of positive generalized eigenvalues.

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Key words and phrases. (p,q)-Laplacian, nonlinear boundary conditions, Steklov eigenvalue problem, mountain pass theorem, Ekeland variational.

რეზიუმე. სტატიაში შესწავლილია (p,q) ელიფსური ოპერატორების შემდეგი კლასი სტეკლოვის ტიპის სასაზღვრო პირობებით

$$\begin{cases} -\operatorname{div}\big(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\big) + a(|u|^p)|u|^{p-2}u = 0 \ \text{ in } \Omega, \\ a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nu = \lambda |u|^{m-2}u \ \text{ on } \partial\Omega, \end{cases}$$

სადაც Ω არის გლუვი შემოსაზღვრული არე $\mathbb{R}^N$-ში $(N \geq 2)$, ν გარე ერთეულოვანი ნორმალია $\partial\Omega$-ზე, $2 \leq p < N$, $m \in \mathbb{R}$, $m > 1$, იცვლება შესაბამის დიაპაზონში, რომელიც ქვემოთ იქნება ჩამოთვლილი, a არის C^1 კლასის ნამდვილი ფუნქცია და $\lambda > 0$ არის ნამდვილი პარამეტრი. ვარიაციული მეთოდების გამოყენებით დადგენილია დადებითი განზოგადებული საკუთრივი მნიშვნელობების უწყვეტი და შემოუსაზღვრელი სიმრავლის არსებობა.

1 Introduction

The aim of this paper is to present a preliminary study on a more general class of (p,q)-type eigenvalues problems under Steklov-type boundary conditions given by

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) + a(|u|^p)|u|^{p-2}u = 0 & \text{in } \Omega, \\ \left\langle a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u, \nu\right\rangle = \lambda |u|^{m-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\langle\cdot\,,\,\cdot\rangle$ is the scalar product of $\mathbb{R}^N$, $1 \leq p < N, m \in \mathbb{R}$, with $m > 1, \lambda > 0$ is a real parameter, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 real function satisfying

(a_1) There exist constants $\xi_i > 0$, $i = 0,1,2,3$, $1 < p \leq q < N$ such that

$$\xi_0 + \xi_1 t^{(q-p)/p} \leq a(t) \leq \xi_2 + \xi_3 t^{(q-p)/p} \ \text{ for all } \ t \geq 0.$$

Moreover, additional hypotheses on a which are useful in some specific cases are listed here:

(a_2) There exists a positive real constant α with $\frac{q}{p} \leq \alpha < \frac{m}{p}$ such that

$$\frac{1}{\alpha(t)t} \leq A(t) = \int_0^t a(s)\,ds \ \text{ for all } \ t \geq 0;$$

(a_3) The map $t \to a(t)t^{(p-2)/p}$ is increasing for $t \geq 0$;

(a_4) The map a and its derivative a' satisfy

$$a'(t)t < \left(\frac{q-p}{p}\right)a(t) \ \text{ for all } \ t > 0.$$

Now, we present some examples of functions a in order to illustrate the degree of generality of the kind of problems studied here.

Example 1.1. If $a \equiv 1$, our operator is the p-Laplacian, so problem (1.1) becomes

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \dfrac{\partial u}{\partial \upsilon} = \lambda |u|^{m-2}u & \text{on } \partial\Omega, \end{cases}$$

with $q = p$ and $\xi_0 + \xi_1 = \xi_2 + \xi_3$.

Example 1.2. If $a(t) = 1 + t^{\frac{q-p}{p}}$, we get

$$\begin{cases} \Delta_p u + \Delta_q u = |u|^{p-2}u + |u|^{q-2}u & \text{in } \Omega, \\ \left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\dfrac{\partial u}{\partial \upsilon} = \lambda |u|^{m-2}u & \text{on } \partial\Omega, \end{cases}$$

with $q = p$, $\xi_0 = \xi_1 = \xi_2 = \xi_3 = 1$.

Example 1.3. Taking

$$a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}},$$

we get

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \dfrac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) = -|u|^{p-2}u - \dfrac{|u|^{p-2}u}{(1+|u|^p)^{\frac{p-2}{p}}} & \text{in } \Omega, \\ \left(|\nabla u|^{p-2}\nabla u + \dfrac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right)\dfrac{\partial u}{\partial \upsilon} = \lambda |u|^{m-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\xi_0 = \xi_1 = \xi_2 = \xi_3 = 1$ and $\xi_2 = 2$.

Example 1.4. If we consider
$$a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}},$$
we obtain
$$\begin{cases} -\Delta_p u - \Delta_q u - \operatorname{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\Big) = -|u|^{p-2}u - |u|^{q-2}u - \dfrac{|u|^{p-2}u}{(1+|u|^p)^{\frac{p-2}{p}}} & \text{in } \Omega, \\ \big(|\nabla u|^{p-2} + |\nabla u|^{q-2}\big)\nabla u \dfrac{\partial u}{\partial \upsilon} + \dfrac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}} u \dfrac{\partial u}{\partial \upsilon} = \lambda |u|^{m-2}u & \text{on } \partial\Omega, \end{cases}$$
with $q = p$, $\xi_0 = \xi_1 = \xi_2 = \xi_3 = 1$.

This class of problems comes, for example, from a general reaction-diffusion system
$$u_t = \operatorname{div}(H(u)\nabla u) + c(x,u), \tag{1.2}$$
where
$$H(u) = \big(|\nabla u|^{p-2} + \mu |\nabla u|^{q-2}\big).$$
This system has a wide range of applications in physics and related sciences like chemical reaction design [4], biophysics [13] and plasma physics [17]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.2) corresponds to the diffusion with a diffusion coefficient $H(u)$; whereas the second one is the reaction and relates to the source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x;u)$ has a polynomial form with respect to the concentration.

This class of problems has received special attention past years. For example, the particular case $a(t) = 1 + t^{\frac{q-p}{p}}$ (see Example 1.2) was studied in [7,8,20,21].

The problem on $\mathbb{R}^N$ was studied in [5,6,11,12]. A result of the existence of solutions for a problem with critical growth can be found in [11], and a result of multiplicity of solutions for a problem with subcritical growth vis category theory was studied in [12]. In [5,6], the authors established the existence of the principal eigenvalue and of a continuous family of eigenvalues.

A further interesting and more general study of the Steklov eigenvalue problems for the (p,q)-Laplacian, being in a close relationship with both the above results and our problem, is provided in [21] (see also some references there, in particular, [20]), where the authors consider the following Steklov problem involving (p,q)-Laplacian:
$$\begin{cases} \Delta_p u + \mu \Delta_q u = |u|^{p-2}u + \mu |u|^{q-2}u & \text{in } \Omega, \\ \big(|\nabla u|^{p-2}\nabla u + \mu |\nabla u|^{q-2}\nabla u\big) \dfrac{\partial u}{\partial \upsilon} = \lambda\big[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u\big] & \text{on } \partial\Omega, \end{cases}$$
where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\mu > 0$, $\lambda \in \mathbb{R}$, and m_p, m_q are the bounded weights on $\partial\Omega$.

In the light of the specificity of the results cited above and for the sake of completeness within the study of generalized p,q-Laplacian type operators, we are therefore motivated in this paper to deal with the more general problem (1.1), thus covering the case of eigenvalues problems, including the ones with p- or q-Laplacian operators.

2 Variational framework

First, let us observe that by the boundedness of Ω and $p \leq q$, $W^{1,p}(\Omega) \cap W^{1,q}(\Omega) = W^{1,q}(\Omega)$. Therefore, as functional space, we consider the space $W^{1,q}(\Omega)$ endowed with the norm
$$\|u\|_{1,q} := \Big(\int_\Omega |\nabla u|^q \, dx + \int_\Omega |u|^q \, dx\Big)^{\frac{1}{q}}.$$

We denote by $W^{1,p}(\Omega)^*$ a dual space, and the duality pairing between $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)^*$ is written as $\langle\cdot,\cdot\rangle$. It is well known that the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact for each $r \in [1, p^*)$, where $p^* = p(N-1)/(N-p)$, and there exists $C_p > 0$ such that

$$\|u\|_{L^r(\partial\Omega)} \leq C\|u\|_{1,r}.$$

Recall that $W^{1,p}(\Omega)$ is continuously embedded in $L^r(\partial\Omega)$ for all $r \in [1, p^*)$ and

$$W^{1,q}(\Omega) \text{ is compactly embedded in } L^r(\partial\Omega) \text{ for all } r \in [1, p^*). \tag{2.1}$$

In the following, we denote by $\|\cdot\|_r$ the classical norm on $L^r(\partial\Omega)$. Recall also that a weak solution of problem (1.1) is a function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega a(|u|^p)|u|^{p-2}uv\,dx - \lambda\int_{\partial\Omega}|u|^{m-2}uv\,d\sigma \text{ for all } v \in W^{1,q}(\Omega). \tag{2.2}$$

In order to use variational methods, it is needed to observe that by assumption (a_1), we have

$$a(t)t^{(p-1)/p} \leq \xi_2 t^{(p-1)/p} + \xi_3 2t^{(q-1)/q} \text{ for all } t \geq 0. \tag{2.3}$$

By these subcritical growth conditions the energy functional $J_\lambda : W^{1,q}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx + \frac{1}{p}\int_\Omega A(|u|^p)\,dx - \frac{\lambda}{m}\int_{\partial\Omega}|u|^m\,d\sigma$$

and expression (2.3) are well defined, where $A(t) = \int_0^t a(s)\,ds$. Moreover, by the standard arguments, $J_\lambda \in C^1(W^{1,q}(\Omega), \mathbb{R})$ with the following Frechet derivative:

$$\langle J'_\lambda(u), v\rangle = \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega a(|u|^p)|u|^{p-2}uv\,dx - \lambda\int_{\partial\Omega}|u|^{m-2}uv\,d\sigma$$

for every $u, v \in W^{1,q}(\Omega)$ and, thanks to the variational structure of the problem, any weak solution of (1.1) is a critical point of J_λ and vice versa.

Let us recall that $\lambda > 0$ is a generalized eigenvalue of problem (1.1) if there exists $u \in W^{1,q}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla v\,dx + a(|u|^p)|u|^{p-2}uv\,dx = \lambda\int_{\partial\Omega}|u|^{m-2}uv\,d\sigma = 0$$

for every $v \in W^{1,q}(\Omega)$. We point out that if λ is a generalized eigenvalue to (1.1), then the corresponding eigenfunction $u \in W^{1,q}(\Omega)\setminus\{0\}$ is a nontrivial weak solution to problem (1.1).

In what follows, we denote by $\lambda_1(q)$ (resp., $\psi_1(q)$) the principal eigenvalue (resp., eigenfunction) to the following Steklov (eigenvalue) problem involving q-Laplacian:

$$\begin{cases} \Delta_q u = |u|^{q-2}u & \text{in } \Omega, \\ |\nabla u|^{q-2}\nabla u\,\dfrac{\partial u}{\partial v} = \lambda(q)|u|^{q-2}u & \text{on } \partial\Omega \end{cases}$$

(with $1 < p \leq q < N$).

From now on, specifically, we will distinguish the following cases:

Case I: $1 < m \leq 2 \leq p \leq q < q^*$;

Case II: $1 < p < q < q^*$; or $1 < p < m = q < m^* = q^*$;

Case III: $1 < p \leq q < mq^*$.

3 Case I: $1 < m \leq 2 \leq p \leq q < q^*$

We analyze the following subcases.

3.1 Case 1: $1 < m \leq 2 < p = q < p^* = q^*$

The main result of this subsection is proved by a generalized version of Weierstrass Theorem as follows. Steklov eigenvalues problems involving p-Laplacian operator and its generalizations (see Examples 1.1 and 1.3) are included.

Theorem 3.1. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $1 < m \leq 2 < p = q < p^* = q^*$. Then for every $\lambda > 0$ problem* (1.1) *possesses at least one nontrivial weak solution or, equivalently, any $\lambda > 0$ is a generalized eigenvalue to* (1.1).

Proof. First, let us observe that by (a_1) and for every $u \in W^{1,p}(\Omega)$:

$$(\xi_0 + \xi_1)|\nabla u|^p \leq A(|\nabla u|^p) \text{ and } a(|\nabla u|^p)|\nabla u|^p \leq (\xi_2 + \xi_3)|\nabla u|^p,$$
$$(\xi_0 + \xi_1)|u|^p \leq A(|u|^p) \text{ and } a(|u|^p)|u|^p \leq (\xi_2 + \xi_3)|u|^p,$$

so that, together with Sobolev continuous embedding we find that

$$J_\lambda(u) = \frac{1}{p}\int_\Omega A(|\nabla u|^p)\, dx + \frac{1}{p}\int_\Omega A(|u|^p)\, dx - \frac{\lambda}{m}\int_{\partial\Omega} |u|^m\, d\sigma \geq \frac{(\xi_0 + \xi_1)}{p}\|u\|_{1,p}^p - \frac{\lambda}{m}\|u\|_{1,p}^m.$$

Now, denote by $\varphi(\|u\|_{1,p})$ the right-hand side of the above inequality and $\|u\|_{1,p} = t \geq 0$, it is clear that $\varphi : [0, +\infty) \to \mathbb{R}$ is a continuous function such that, by $m < p$, $\varphi/t \to +\infty$ as $t \to +\infty$ and $J_\lambda(u) \geq \varphi(\|u\|_{1,p})$ for every $u \in W^{1,p}(\Omega)$. So, for every $K > 0$, there exists $t_0 > 0$ such that $\varphi(t)/t \geq K$ for any $t > t_0$. Therefore,

$$J_\lambda(u) \geq \varphi(\|u\|_{1,p}) \geq K\|u\|_{1,p} > Kt_0 \text{ if } \|u\|_{1,p} > t_0.$$

If $t = \|u\|_{1,p} \leq t_0$, $J_\lambda(u) \geq \overline{C}$, where $\overline{C} = \min\{\varphi(t) : 0 \leq t \leq t_0\}$. This implies that for all $\lambda > 0$, J_λ is bounded from below. Moreover, J_λ is also coercive and this is due to the fact that the stronger condition $J_\lambda(u)/\|u\|_{1,p} \to +\infty$ holds.

Now, we prove that J_λ is weakly lower semicontinuous in $W^{1,p}(\Omega)$. Indeed, for every sequence $(u_n)_{n\geq 0} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, by the compact embedding (2.1) we have $u_n \to u$ in $L^r(\partial\Omega)$ for all $r \in [1, p^*)$ and, since $m < p < p^*$, it follows that

$$\int_{\partial\Omega} |u_n|^m\, d\sigma \longrightarrow \int_{\partial\Omega} |u|^m\, d\sigma \text{ as } n \to +\infty.$$

Furthermore, by a subcritical growth in (2.3) for $p < p^*$, the functional $u \longmapsto \int_\Omega (A(|\nabla u|^p) + A(|u|^p))\, dx$ is weakly lower semicontinuous, i.e.,

$$\int_{\partial\Omega} |u_n|^m\, dx \leq \liminf_{n\to+\infty} \int_{\partial\Omega} |u_n|^m\, d\sigma,$$

thus we obtain

$$\begin{aligned}
\liminf_{n\to+\infty} J_\lambda(u_n) &= \frac{1}{p}\liminf_{n\to+\infty}\int_\Omega \big(A(|\nabla u|^p) + A(|u|^p)\big)\, dx - \frac{\lambda}{m}\lim_{n\to+\infty}\int_{\partial\Omega} |u_n|^m\, dx \\
&\geq \frac{1}{p}\int_\Omega \big(A(|\nabla u|^p) + A(|u|^p)\big)\, dx - \frac{\lambda}{m}\int_{\partial\Omega} |u_n|^m\, d\sigma = J_\lambda(u).
\end{aligned}$$

Consequently, by a generalized version of Weierstrass type theorem, we get the existence of a global minimum point $w \in W^{1,p}(\Omega)$ such that

$$J_\lambda(w) = \min_{u \in W^{1,p}(\Omega)} J_\lambda(u)$$

and then w is a solution for our problem. In order to show that this solution is nontrivial, we consider $v \in W^{1,p}(\Omega)$ and $t^* > 0$ sufficiently small such that

$$J_\lambda(w) \leq J_\lambda(t^* v) \leq \frac{(\xi_2 + \xi_3)(t^*)^p}{p} \|v\|_{1,p}^p - \frac{\lambda (t^*)^m}{m} \|v\|_{L^m(\partial\Omega)}^m < 0$$

for all $\lambda > 0$. Then we conclude that $w \neq 0$. $\square$

To complete the paper, let us show that the existence of a non-trivial global minimum for J_λ proved above follows also by means of a direct application of Ekeland's variational principle [18] under the additional assumption (a_3).

Theorem 3.2. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) and (a_3) with $1 < m \leq 2 < p = q < p^* = q^*$. Then for every $\lambda > 0$ problem* (1.1)*, possesses at least one nontrivial weak solution or, equivalently, any $\lambda > 0$ is a generalized eigenvalue to* (1.1)*.*

Proof. Indeed, since $J_\lambda \in C^1(W^{1,p}(\Omega), \mathbb{R}^+)$ is bounded from below (and coercive) on $W^{1,p}(\Omega)$, by adding (a_3) we are able to show that J_λ satisfies the Palais–Smale condition at level $c = \inf\limits_{u \in W^{1,p}(\Omega)} J_\lambda(u)$. This means that any sequence $(u_n)_{n\geq 0} \subset W^{1,p}(\Omega)$ satisfying $J_\lambda(u) \to c$ and $\|J_\lambda^{'}(u)\|_{(W^{1,p}(\Omega))'} \to 0$ as $n \to +\infty$ (briefly, $\{u_n\}$ is a $(PS)_c$ sequence) has a strongly convergent subsequence. Further, we make use and adapt the arguments that can be found in [11, Lemma 2.4]. First, it is not difficult to observe that taking a $(PS)_c$ sequence $\{u_n\}_n$, by the coerciveness of J_λ as proved before, $(u_n)_{n\geq 0}$ is bounded in $W^{1,p}(\Omega)$. So, there exists $u \in W^{1,p}(\Omega)$ such that, up to subsequences, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, and our aim is to prove the strong convergence of u_n to u in $W^{1,p}(\Omega)$. By $a(t) \geq \xi_0$ for every $t \geq 0$, that follows by the left-hand side inequality in (a_1) and the monotonicity assumption (a_3), we have

$$C|x-y|^p \leq \left\langle a(|x|^P)|x|^{p-2}x - a(|y|^P)|x|^{p-2}y, x-y \right\rangle \tag{3.1}$$

for all $x, y \in \mathbb{R}^{\mathbb{N}}$ with $N \geq 1$ and $\langle \cdot\,, \cdot \rangle$ being the scalar product in $\mathbb{R}^{\mathbb{N}}$ (for the proof, see [11, Lemma 2.4]). Now, we can write

$$\begin{aligned}\left\langle J_\lambda'(u_n) - J_\lambda^{'}(u), u_n - u \right\rangle &= \int\limits_\Omega \Big(a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n - a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \Big) \cdot (\nabla u_n - \nabla u)\, dx \\ &\quad + \int\limits_\Omega \big(a(|u_n|^p)|u_n|^{p-2}u_n - a(|u|^p)|u|^{p-2}u \big)(u_n - u)\, dx \\ &\quad - \lambda \int\limits_{\partial\Omega} \big(|u_n|^{m-2}u_n - |u|^{m-2}u \big)(u_n - u)\, d\sigma.\end{aligned}$$

Since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $J_\lambda^{'}(u) \in (W^{1,p}(\Omega))'$ and $\{u_n\}$ is a $(PS)_c$ sequence implies that $J_\lambda^{'}(u_n) \to 0$ as $n \to +\infty$ in $(W^{1,p}(\Omega))'$, we get

$$\left\langle J_\lambda^{'}(u_n) - J_\lambda^{'}(u), u_n - u \right\rangle = o_n(1)$$

and, by the compact embedding (2.3),

$$\int\limits_{\partial\Omega} \big(|u_n|^{m-2}u_n - |u|^{m-2}u \big)(u_n - u)\, d\sigma = o_n(1).$$

Therefore, by (3.1),

$$0 \leq C\|u_n - u\|_{1,p}^p \leq \int\limits_\Omega \Big(a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n - a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \Big) \cdot (\nabla u_n - \nabla u)\, dx$$

$$+ \int\limits_\Omega \big(a(|u_n|^p)|u_n|^{p-2}u_n - a(|u|^p)|u|^{p-2}u \big)(u_n - u)\, dx = o_n(1),$$

which is the desired conclusion

$$\|u_n - u\|_{1,p}^p = o_n(1).$$

□

3.2 Case 1: $1 < m \leq 2 < p = q < p^*$

Unlike the previous case, here we treat the more general cases involving p- and q-Laplacians and their perturbations as in Examples 1.2 and 1.4). Nevertheless, also in this case, first by a generalized Weierstrass Theorem and then for completeness by Ekeland's Variational Principle, we get the existence of a non-trivial weak solution to problem (1.1) as is stated in the following

Theorem 3.3. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $1 < m \leq 2 < p = q < p^*$. Then, for every $\lambda > 0$ there is the generalized eigenvalue to problem (1.1), namely, for every $\lambda > 0$, problem (1.1) possesses at least one nontrivial weak solution.*

Proof. Using hypothesis (a_1) and Sobolev embedding, for every $u \in W^{1,p}(\Omega)$ we get

$$J_\lambda(u) \geq \frac{\xi_0}{p}\,\|u\|_{1,p}^p + \frac{\xi_1}{q}\,\|u\|_{1,q}^q - \frac{\lambda}{m}\,\|u\|_{L^m(\partial\Omega)}^m \geq \frac{\xi_1}{q}\,\|u\|_{1,q}^q\,\frac{\lambda}{m}\,C\|u\|_{1,q}^m.$$

By adapting the reasoning used in the previous section and denoting by $\varphi(\|u\|_{1,q})$ the right-hand side of the above inequality, it follows that J_λ is bounded from below in $W^{1,q}(\Omega)$ for all $\lambda > 0$. Moreover, J_λ is also coercive and by the compact embedding (2.2) and subcritical growth (2.3), it is weakly lower semicontinuous on $W^{1,q}(\Omega)$. Therefore, there exists a global minimum point $w \in W^{1,q}(\Omega)$ such that

$$J_\lambda(w) = \int\limits_{u \in w \in W^{1,q}(\Omega)} J_\lambda(u).$$

Thus w is a solution to (1.1). Now, taking $v \in W^{1,q}(\Omega)$ and t^* sufficiently small, by hypothesis (a_1), again we have

$$J_\lambda(w) \leq J_\lambda(t^* v) \leq \frac{\xi_2 (t^*)^p}{p}\,\|v\|_{1,p}^p + \frac{\xi_3 (t^*)^q}{q}\,\|v\|_{1,q}^q - \frac{\lambda}{m}\,\|u\|_{L^m(\partial\Omega)}^m < 0$$

for all $\lambda > 0$. Hence the above-found solution w is non-trivial. □

Also, in this section, if (a_3) also holds, we have to prove the existence of a non-trivial global minimum for J_λ by applying Ekeland's variational principle [18].

Theorem 3.4. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) and (a_3) with $1 < m \leq 2 < p < q < q*$. Then for every $\lambda > 0$, problem (1.1) possesses at least one nontrivial weak solution or, equivalently, any $\lambda > 0$ is generalized a eigenvalue to (1.1).*

Proof. The desired follows by adapting the arguments to the proof of Theorem 3.2. □

3.3 Case 1: $2 \leq m = p = q < m^* = p^* = q^*$

In this situation, we require a little bit more than in Subsection 3.1, namely, $m = p = q$, and we deal with

$$\begin{cases} -\operatorname{div}\big(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\big) + a(|u|^p)|u|^{p-2}u = 0 & \text{in } \Omega, \\ a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\,\dfrac{\partial u}{\partial v} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases}$$

Now, since more specific mathematical techniques are required to study (1.1) due to the presence of a non-homogeneous operator, and since the aim of this paper is to give a preliminary overview on the search of eigenvalues to (1.1), we state the following result.

Theorem 3.5. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le m = p = q < m^* = p^* = q^*$. Then there exists $\lambda^* > 0$ such that every eigenvalue $\lambda > 0$ to problem (1.1) is bounded from below by λ^*, i.e., $\lambda \ge \lambda^*$.*

Proof. Indeed, if $u \neq 0$ is a solution of problem (1.1), then

$$\int_\Omega a(|\nabla u|^p)\nabla u\, dx + \int_\Omega a(|u|^p)u\, dx = \lambda \int_{\partial\Omega} |u|^p\, d\sigma.$$

Using hypothesis (a_1) and Sobolev embedding, we get

$$(\xi_0 + \xi_1)\|u\|_{1,p}^p \le \frac{\lambda}{\lambda_1(p)}\, \|u\|_{1,p}^p$$

and hence there exists only a trivial solution for $0 < \lambda < (\xi_0 + \xi_1)\lambda_1(p) = \lambda^*$. □

3.4 Case 2: $2 \le m = p < q < q^*$

First, let us prove the following result.

Theorem 3.6. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le m = p < q < q^*$. Then there exists $\lambda^* > 0$ such that problem (1.1) does not possess any nontrivial weak solution for all $\lambda \in (0, \lambda^*]$, i.e., the interval $(0, \lambda^*]$ does not contain any generalized eigenvalue.*

Proof. Reasoning as in the proof of Theorem 3.5, the desired follows by (a_1) and the Sobolev embedding by choosing $\lambda^* = \xi_0\lambda_1(p)$. □

Let us add that, as we prove in the next theorem, for λ greater than a suitable positive constant, J_λ is obviously weakly lower semicontinuous but also bounded from below and coercive on $W^{1,q}(\Omega)$ for J_λ belonging to a neighborhood of the origin, then u_0 is the unique solution of the equation (nonnegative or not) and 0 is the global minimum of the functional J_λ. On the other hand, problem (1.1) admits a non-trivial solution for J_λ in a neighborhood of $+\infty$ as stated below.

Theorem 3.7. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le m = p < q < q^*$. Then for every $\lambda > 0$ bounded from below by a positive constant, problem (1.1) possess at least one nontrivial weak solution or, equivalently, there exists $\lambda^{**} > 0$ such that every λ belonging to the set $(\lambda^{**}, +\infty)$ is a generalized eigenvalue to (1.1).*

Proof. By hypothesis (a_1), Hölder's inequality and $m = p < q$, we note that for every $u \in W^{1,q}(\Omega)$,

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\int_\Omega A(|\nabla u|^p)\, dx + \frac{1}{p}\int_\Omega A(|u|^p)\, dx - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p\, d\sigma \\ &\ge \frac{\xi_0}{p}\, \|u\|_{1,p}^p + \frac{\xi_1}{q}\, \|u\|_{1,q}^q - \frac{\lambda}{p}\, \|u\|_{L^q(\partial\Omega)}^p \ge \frac{\xi_1}{q}\, \|u\|_{1,q}^q - \frac{\lambda}{p}\, C^{'}\, \|u\|_{1,p}^p, \end{aligned}$$

where $C^{'} = S_p(\lambda_1(q))^{-1}$. Then, by adapting slightly the arguments used in the proof of Theorem 3.3, J_λ is bounded from below and coercive on $W^{1,q}(\Omega)$ for all $\lambda > 0$. So, by (2.2), there exists $w \in W^{1,q}(\Omega)$ such that

$$J_\lambda(w) = \min_{u \in W^{1,q}(\Omega)} J_\lambda(u).$$

Consequently, w is a solution to problem (1.1). In order to conclude that $w \neq 0$, namely, that w is a nontrivial solution, let $v \in W^{1,q}(\Omega)$ and $t^* > 0$ be sufficiently small so that hypothesis (a_1) again implies

$$J_\lambda(w) \le J_\lambda(t^* v) \le \frac{\xi_2 (t^*)^p}{p}\, \|v\|_{1,p}^p + \frac{\xi_3 (t^*)^q}{q}\, \|v\|_{1,q}^q - \frac{\lambda (t^*)^p}{p}\, \|v\|_{L^p(\partial\Omega)}^p < 0$$

and this is possible if $\lambda > 0$ is in a neighborhood of $+\infty$. More precisely, we can choose $\lambda > \xi_2\lambda_1(p)$. Indeed, taking $\psi_1(p)$, the regular and positive eigenfunction corresponding to $\lambda_1(p)$ such that $\|\psi_1(p)\|^p_{L^p(\partial\Omega)} = 1$, by (a_1), we get

$$\begin{aligned} J_\lambda(w) \le J_\lambda(t^*\psi_1(p)) &\le \frac{\xi_2(t^*)^p}{p}\|\psi_1(p)\|^p_{1,p} + \frac{\xi_3(t^*)^q}{q}\|\psi_1(p)\|^q_{1,q} - \frac{\lambda(t^*)^p}{p}\|\psi_1(p)\|^p_{L^p(\partial\Omega)} \\ &= (t^*)^p\Big(\frac{\xi_2\lambda_1(p)-\lambda}{p}\Big) + \frac{\xi_3(t^*)^q}{q}\|\psi_1(p)\|^q_{1,q} < 0 \end{aligned}$$

for t sufficiently small and $p < q$. $\square$

Furthermore, if we assume that (a_3) holds, then in this case a non-trivial global minimum for J_λ also exists as a consequence of Ekeland's variational principle [18].

Theorem 3.8. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le m < q < p < q^*$. Then problem (1.1) possesses at least one nontrivial weak solution for every $\lambda > 0$ bounded from below by a positive constant or, equivalently, there exists $\lambda^{**} > 0$ such that every λ belonging to the set $(\lambda^{**}, +\infty)$ is a generalized eigenvalue to (1.1).*

Proof. We refer to the proof of Theorem 3.4 with some minor changes. $\square$

4 Case II: $2 \le p < m < q < q^*$ or $2 \le p < m = q < m^* = q^*$

In the following subsection let us first consider the case $2 \le p < m < q$ and then the case $2 \le p < m = q$.

4.1 Case 2: $2 \le p < m < q < q^*$

Here, we prove that for λ bounded from above by a suitable positive constant, J_λ does not possess nontrivial critical points as is stated in the following

Theorem 4.1. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le p < m < q < q^*$. Then there exists $\lambda^* > 0$ such that problem (1.1) does not possess any nontrivial weak solution for all $\lambda \in (0, \lambda^*)$; that is, any $\lambda \in (0, \lambda^*)$ is not a generalized eigenvalue to (1.1).*

Proof. Assume to the contrary that there exists a nontrivial weak solution u of problem (1.1). So, multiplying the equation by u and integrating over Ω we obtain

$$\int_\Omega a(|\nabla u|^p)|\nabla|^p\,dx + \int_\Omega a(|u|^p)|u|^p\,dx = \lambda\int_{\partial\Omega}|u|^m\,d\sigma.$$

By the left-hand side of (a_1) and the Poincaré inequality, it follows that

$$\xi_0 C\|u\|^p_p + \xi_1 C^{'}\|u\|^q_q \le \lambda\|u\|^m_{L^m(\partial\Omega)},$$

where C and $C^{'}$ denote the Poincaré constants. Since $p < m < q$, by the interpolation inequality, we have

$$\|u\|^m_{L^m(\partial\Omega)} \le \|u\|^{vp}_{L^p(\partial\Omega)}\|u\|^{1-v}_{L^q(\partial\Omega)}$$

with $v \in (0,1)$ and $\frac{1}{m} = \frac{v}{p} + \frac{(1-v)}{q}$ so, by the Young inequality, we obtain

$$\|u\|^m_{L^m(\partial\Omega)} \le v\|u\|^p_{L^p(\partial\Omega)} + (1-v)\|u\|^q_{L^q(\partial\Omega)}.$$

Consequently,

$$(\lambda v - \xi_0 C)\|u\|^p_p + \big(\lambda(1-v) - \xi_1 C^{'}\big)\|u\|^q_q \ge 0$$

and if we choose λ such that

$$\lambda < \min\Big\{\frac{\xi_0 C}{v}, \frac{\xi_0 C^{'}}{(1-v)}\Big\} = \lambda^*.$$ $\square$

Theorem 4.2. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \le p < m < q < q^*$. Then there exists $\lambda^{**} > 0$ such that problem (1.1) possesses at least one nontrivial weak solution for all $\lambda \in (\lambda^{**}, +\infty)$; equivalently, any $\lambda \in (0, \lambda^{**})$ is a generalized eigenvalue to problem (1.1) with $\lambda^{**} > 0$.*

Proof. Since the boundedness of the level sets of J_λ assures that the functional is coercive, we begin by proving that $J_\lambda^b = \{u \in W^{1,q}(\Omega) : \ J_{\lambda(v) \le b}\}$ are bounded. If $u \in J_\lambda^b$, then

$$\frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx + \frac{1}{p}\int_\Omega A(|u|^p)\,dx - \frac{\lambda}{m}\int_{\partial\Omega} |u|^m\,d\sigma \le b.$$

By (a_1), the Hölder and Poincaré inequalities applied to the left-hand side of the previous expression, we have

$$\frac{\xi_0}{p}\,\|u\|_{1,p}^p + \|u\|_{1,p}^m\Big(\frac{\xi_1}{q}\|u\|_{1,p}^{q-m} - \frac{\lambda}{m}\big(\operatorname{meas}(\partial\Omega)\big)^{\frac{q-m}{q}}(C^{''})^{\frac{m}{q}}\Big) \le b, \tag{4.1}$$

where $C^{''}$ is a Poincaré constant. If

$$\|u\|_{1,q} \le 2\Big(\frac{\lambda q}{\xi_1 m}\Big)^{\frac{1}{q-m}}\big(\operatorname{meas}(\partial\Omega)\big)^{\frac{1}{q}}(C^{''})^{\frac{m}{q(q-m)}},$$

then $\|u\|_{1,q}$ is bounded and, by (4.1), we obtain

$$\frac{\xi_0}{p}\,\|u\|_{1,p}^p \le b + D$$

with $D = D(q, \lambda, m, \operatorname{meas}(\partial\Omega), C^{''})$. Therefore, $\|u\|$ is also bounded. On the other side, if

$$\|u\|_{1,p}^p \ge 2\Big(\frac{\lambda q}{\xi_1 m}\Big)^{\frac{1}{q-m}}\big(\operatorname{meas}(\partial\Omega)\big)^{\frac{1}{q}}(C^{''})^{\frac{m}{q(q-m)}},$$

from (4.1) it follows that

$$\frac{\xi_0}{p}\,\|u\|_{1,p}^p \le b, \quad \text{and} \quad \|u\|_{1,p}^m\Big(\frac{\xi_1}{q}\,\|u\|_{1,p}^{q-m} - \frac{\lambda}{m}\big(\operatorname{meas}(\partial\Omega)\big)^{\frac{q-m}{q}}(C^{''})^{\frac{m}{q}}\Big) \le b.$$

If $\|u\|_{1,q} \to +\infty$, for $m < q$ we would obtain a contradiction, thus the proof of coercivity is complete. Moreover, following the proof of Theorem 3.3, J_λ is bounded from below and weakly lower semicontinuous on $W^{1,q}(\Omega)$ and, by a generalized version of Weierstrass type theorem, we obtain the existence of a global minimum. As a conclusion, taking $w \in W^{1,q}(\Omega)$, $w \ne 0$, for λ sufficiently large, we have $J_\lambda(w) < 0$ and this ensures that the found minimum is nontrivial. Under the addition of hypothesis (a_3) and taking into account the reasoning used in the proof of Theorem 3.4, we can here apply a consequence of Ekeland's variational principle to obtain what follows. $\square$

Theorem 4.3. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) and (a_3) with $2 \le p < m < q < q^*$. Then there exists $\lambda^{**} > 0$ such that problem (1.1) possesses at least one non-trivial weak solution for all $\lambda \in (0, \lambda^*]$, namely, there exists $\lambda^* > 0$ such that $(0, \lambda^*]$ does not contain a generalized eigenvalue to problem (1.1) with $\lambda^{**} > 0$.*

4.2 Case 2: $2 \le m = p < m^* = q^*$

At this point, we study problem (1.1) in the case $2 \le p < m = q < m^* = q^*$, namely,

$$\begin{cases} -\operatorname{div}\big(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\big) + a(|u|^p)|u|^{p-2}u = 0 & \text{in } \Omega,\\ a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\,\dfrac{\partial u}{\partial v} = \lambda|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

We first claim and prove that for λ belonging to a suitable neighborhood of the origin, problem (1.1) possesses only a trivial solution.

Theorem 4.4. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \leq p < m = q < m^* = q^*$. Then there exists $\lambda^* > 0$ such that problem (1.1) does not possess any nontrivial weak solution for all $\lambda \in (0, \lambda^*]$, namely, there exists $\lambda^* > 0$ such that $(0, \lambda^*]$ does not contain generalized eigenvalues.*

Proof. Following the reasoning of the proof of Theorem 3.6, by (a_1) and the Sobolev embedding, the result is proved with $\lambda^* = \xi_1 \lambda_1(q)$. □

Also, in this case, J_λ is bounded from below and coercive for $\lambda \leq \lambda^*$, therefore 0 is the global minimum of the functional and $u \equiv 0$ is the unique solution of the equation (nonnegative or not). Moreover, the result just proved above is confirmed by the proof of the proposition, which we show below.

First, recall an interpolation inequality which is a well-known consequence of the Hölder inequality and the Sobolev continuous embedding.

Lemma 4.5. *For every $u \in W^{1,q}(\Omega)$, we have*

$$\|u\|^q_{L^q(\partial\Omega)} \leq S_q \|u\|^{qt}_{1,p} \|u\|^{q(1-t)}_{L^q(\partial\Omega)},$$

where $t \in (0, 1)$ and C is a suitable positive real constant.

Now we prove the following

Proposition 4.6.

(1) *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) with $2 \leq p < m = q < m^* = q^*$. Then there exists $\wedge^* \in \mathbb{R}$ with $\xi_1\lambda_1(q) \leq \wedge^* \leq \xi_3\lambda_1(q)$ such that there is no (positive) eigenvalue $\lambda < \wedge^*$.*

(2) *If $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1) and, in addition, (a_4) and $2 \leq p < m = q < m^* = q^*$, then $\wedge^*$ cannot be an eigenvalue to problem (1.1); consequently, every (positive) eigenvalue λ to problem (1.1) satisfies $\lambda > \wedge^*$.*

Proof. First, let us prove (1). Indeed, let us introduce the functional $\wedge^* : W^{1,q}(\Omega) \backslash \{0\} \to \mathbb{R}$ defined by

$$\wedge(u) = \frac{\int\limits_\Omega a(|\nabla u|^p)|\nabla|^p\,dx + \int\limits_\Omega a(|u|^p)|u|^p\,dx}{\int\limits_{\partial\Omega} |u|^q\,d\sigma}.$$

For any $u \in W^{1,q}(\Omega) \setminus \{0\}$, by (a_1), we have

$$\begin{aligned} \wedge(u) &\geq \int\limits_{u \in W^{1,p}(\Omega),\, u \neq 0} \frac{\xi_0 \int\limits_\Omega (|\nabla u|^p + |u|^p)\,dx + \xi_1 \int\limits_\Omega (|\nabla u|^p + |u|^p)\,dx}{\int\limits_{\partial\Omega} |u|^q\,d\sigma} \\ &\geq \int\limits_{u \in W^{1,p}(\Omega),\, u \neq 0} \frac{\xi_1 \int\limits_\Omega (|\nabla u|^p + |u|^p)\,dx}{\int\limits_{\partial\Omega} |u|^q\,d\sigma} = \xi_1\lambda_1(q) > 0 \end{aligned}$$

from which it follows that $\wedge$ is bounded from below in $W^{1,q}(\Omega) \setminus \{0\}$ and $\wedge^* = \inf\limits_{u \in W^{1,p}(\Omega),\, u \neq 0} \wedge(u)$ is a positive real number with $\wedge^* \geq \xi_1\lambda_1(q)$.

Now we prove that $\wedge^* \leq \xi_3\lambda_1(q)$. First, let $\psi_1(q)$ be the positive eigenfunction corresponding to $\lambda_1(q)$ such that $\int\limits_{\partial\Omega} |\psi_1(q)|^q\,d\sigma = 1$, so $\psi_1(q)$ satisfies $\|\psi_1(q)\|^q_{1,q} = \lambda_1(q)$. Thus since (a_1) holds, we obtain

$$\wedge^* \leq \wedge(t\psi_1(q)) \leq \xi_2 t^{p-q} \|\psi_1(q)\|^q_{1,q} + \xi_3\lambda_1(q),$$

and by $q - p > 0$ and letting $t \to +\infty$, we get the desired upper bound to $\wedge^*$. Consequently, $\xi_1\lambda_1(q) \leq \wedge^* \leq \xi_3\lambda_1(q)$.

Suppose that there exists an eigenvalue $\lambda > 0$ and then a non-trivial weak solution to problem (1.1) in $W^{1,q}(\Omega)$ satisfies $\lambda < \wedge^*$. So, we can find $v \in W^{1,q}(\Omega)$, $v \neq 0$, such that

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^p\,dx + \int_\Omega a(|u|^p)|u|^p\,dx - \lambda\int_{\partial\Omega}|v|^q\,d\sigma = 0.$$

Then we get

$$\begin{aligned}\wedge > \lambda &= \frac{\int_\Omega a(|\nabla u|^p)|\nabla u|^p\,dx + \int_\Omega a(|u|^p)|u|^p\,dx}{\lambda\int_{\partial\Omega}|v|^q\,d\sigma}\\ &\geq \int_{u\in W^{1,p}(\Omega),\,u\neq 0}\frac{\int_\Omega a(|\nabla u|^p)|\nabla u|^p\,dx + \int_\Omega a(|u|^p)|u|^p\,dx}{\lambda\int_{\partial\Omega}|v|^q\,d\sigma} = \wedge^*,\end{aligned}$$

which is a contradiction. So, there is no eigenvalue less than $\wedge^*$.

At this point, if (a_4) also holds, we prove (2), namely, that $\wedge^*$ cannot be an eigenvalue to problem (1.1).

Indeed, observe that $\wedge$ is a C^1 weakly lower semicontinuous functional bounded from below as just we have proved.

So let $(u_n)_{n\geq 0}$ be a minimizing sequence in $W^{1,q}(\Omega)\backslash\{0\}$ such that $\wedge^* = \lim_{n,\to+\infty}\wedge(u_n)$. This implies that $(u_n)_{n\geq 0}$ is bounded. Indeed, since $\wedge(u_n)$ is a convergent real sequence, it is also bounded, i.e., $\wedge \leq C$. On the other hand, by Lemma 4.5 and (a_1) we obtain

$$\wedge(u_n) \geq \frac{\xi_0\|u_n\|_{1,p}^p + \xi_1\|u_n\|_{1,q}^q}{C\|u\|_{1,p}^{qt}\|u\|_{1,p}^{q(1-t)}}.$$

Consequently, $(u_n)_{n\geq 0}$ is bounded, otherwise $\wedge(u_n) \to +\infty$ as $n \to +\infty$ which contradictions $\wedge(u_n) \leq C$. Then there exists $u \in W^{1,q}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$ and by a weakly lower semicontinuity of $\wedge$, we have $\wedge(u) \leq \liminf_{n\to+\infty}\wedge(u_n)$, thus $\wedge(u) = \wedge^*$ implies that $u \neq 0$. Therefore, u is a critical point to $\wedge$, i.e., $\langle\wedge'(u), v\rangle = 0$ for every $v \in W^{1,q}(\Omega)$. This is equivalent to

$$\begin{aligned}&p\int_\Omega\big(a'(|\nabla u|^p)|\nabla u|^p + a(|\nabla u|^p)\big)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx\\ &\qquad + p\int_\Omega\big(a'(|u|^p)|u|^p + a(|u|^p)\big)u|^{p-2}uv\,dx = \wedge^* q\int_{\partial\Omega}|u|^{q-2}uv\,d\sigma,\end{aligned}$$

and, since we have supposed by the contradiction that $\wedge^*$ is an eigenvalue with an associated eigenfunction $u \neq 0$, we have

$$\begin{aligned}&\int_\Omega\big(pa'(|\nabla^p|^p)|\nabla^p|^p - (q-p)a(|\nabla^p|^p)\big)|\nabla^p|^{p-2}\nabla u\cdot\nabla v\,dx\\ &\qquad + \int_\Omega\big(pa'(|u|^p)|u|^p - (q-p)a(|u|^p)\big)u|^{p-2}uv\,dx = 0\end{aligned}$$

for every $v \in W^{1,q}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$. By assumption (a_4), we conclude that $u = 0$. Thus we have a contradiction. Therefore, $\wedge^*$ cannot be an eigenvalue to problem (1.1), and any positive eigenvalue λ to problem (1.1) satisfies $\lambda > \wedge^*$. $\square$

Lemma 4.7. *Assume that (a_1) holds and $2 \leq p < m = q < m^* = q^*$. Then for each $\lambda > \xi_3\lambda_1(q)$, the functional J_λ satisfies the following conditions:*

(J_α) *There exist* $\rho, \alpha > 0$ *such that*

$$J_\lambda(u) \geq \alpha, \;\; \textit{for every} \;\; u \in W^{1,q}(\Omega): \;\; \|u\| = \rho;$$

(J_e) *There exists* $e \in B^c_\rho(0)$ *verifying* $J_\lambda(e) < 0$.

Proof.
(J_α) Let $u \in W^{1,q}(\Omega)$ and recall that $\|u\| := \|u\|_{1,p} + \|u\|_{1,q}$. By (a_1) and Lemma 4.5,

$$\begin{aligned} J_\lambda(u) &\geq \frac{\xi_1}{q}\,\|u\|^q_{1,q} - \frac{\lambda}{q}\,\|u\|^q_{L^q(\partial\Omega)} \geq \frac{\xi_1}{q}\,\|u\|^q_{1,q} - \frac{\lambda}{q}\,S_q\|u||y^{q(1-t)}_{1,q}\|u\|^{qt}_{1,q}. \\ &\geq \frac{\xi_1}{q}\,\|u\|^{q(1-t)}_{1,q}\big(\|u\|^{qt}_{1,q} - \lambda C'\|u||y^{qt}_{1,q}\big). \end{aligned}$$

We can choose $\|u\|_{1,p} = \epsilon$ and $\|u\|_{1,q} = (1 + \lambda C' \epsilon^{qt})^{\frac{1}{qt}}$ for $\epsilon > 0$ sufficiently small. Consequently,

$$J_\lambda(u) \geq \frac{\xi_1}{q}\,(1 + \lambda C' \epsilon^{qt})^{\frac{1-t}{t}}$$

for any $u \in W^{1,q}(\Omega)$ such that $\|u\| = \rho = (1 + \lambda C' \epsilon^{qt})^{\frac{1}{qt}} + \epsilon$.
(J_e) We denote by $\psi_1(q)$ the normalized eigenfunction associated to $\lambda_1(q)$, namely,

$$\begin{cases} -\operatorname{div}\big(|\nabla\psi_1|^{q-2}\nabla\psi_1\big) + |\psi_1|^{q-2}\psi_1 = 0 & \text{in } \Omega, \\ |\nabla\psi_1|^p)\nabla\psi_1\,\dfrac{\partial\psi_1}{\partial\upsilon} = \lambda|\psi_1|^{q-2}\psi_1 & \text{on } \partial\Omega \end{cases}$$

and

$$\int_{\partial\Omega} |\nabla\psi_1(q)|^{q-1}\,d\sigma + |\psi_1|^{q-1} = 1.$$

Hence, taking $t > 0$, by

$$\int_{\partial\Omega} |\psi_1(q)|^q\,d\sigma = \frac{1}{\lambda_1(q)}\,,$$

we have

$$\begin{aligned} J_\lambda(t_1\psi_1(p)) &\leq \frac{t^p}{p}\,\xi_2\|\psi_1(p)\|^p_{1,p} + \frac{t^q}{q}\,\xi_3\|\psi_1(p)\|^q_{1,q} - \lambda\,\frac{t^q}{q}\,\|\psi_1(q)\|^q_{L^q(\partial\Omega)} \\ &\leq \frac{t^p}{p}\,\xi_2\|\psi_1(p)\|^p_{1,p} + \frac{t^q}{q}\,\Big(\xi_3 - \frac{\lambda}{\lambda_1(q)}\Big). \end{aligned}$$

If $\lambda > \xi_3\lambda_1(q)$, then as $t \to +\infty$, we get $J_\lambda(t\psi_1(q)) \to -\infty$. Consequently, there exists $t^* > 0$ such that $J_\lambda(t\psi_1(q)) < 0$ and we put $e = t^*\psi_1(q)$. □

By the Mountain Pass Theorem (see [1]), there exists a Palais-Smale sequence at level

$$\widetilde{C} = \int_{\eta\in\Gamma} \max_{t\in[0,1]} J_\lambda(\eta(t))$$

with $\Gamma = \{\eta \in [0,1], W^{1,q}(\Omega): \;\; \eta(0) = 0 \text{ and } J_\lambda(\eta(t)) \leq 0\}$. However, this condition together with (a_2) is not useful because we cannot show that such a sequence is bounded in X. Indeed, since $m = q$, condition (a_2) becomes $q/p \leq \alpha < q/p$, then $q/p < q/p$ which is impossible.

This enables us to affirm that we cannot obtain a critical point for J_λ by using this method involving the Mountain Pass Theorem. But if the Palais-Smale sequence is bounded under suitable assumptions, there exists $u \in W^{1,q}(\Omega)$ such that, up to the subsequences, $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$, and it remains to prove the strong convergence ofu_n to u in $W^{1,q}(\Omega)$. Since $p < q$, we can refer to the last part of the proof of Theorem 3.4 with due changes to make and we have done. Therefore, defining $\lambda^{**} := \max\{\wedge^*, \xi_3\lambda_1(q)\} = \xi_3\lambda_1(q)$, we can conclude that for every $\lambda > \lambda^{**}$, problem (1.1) admits a non-trivial Mountain Pass solution and we have proved the following result.

Theorem 4.8. *Suppose that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (a_1), (a_3) and (a_4) with $2 \le p < m = q < m^* = q^*$ and every Palais–Smale sequence at any level $\widetilde{c} \in \mathbb{R}$ is bounded in X. Then there exists $\lambda^{**} > 0$ such that problem (1.1) admits both a trivial and a nontrivial weak solution for all $\lambda \in (\lambda^{**}, +\infty)$. In particular, $(\lambda^{**}, +\infty)$ with $\lambda^{**} > 0$ contains generalized eigenvalues to problem (1.1).*

5 Case III: $2 \le m = p < m^* = q^*$

Let us consider together the cases 2: $2 \le p = q < m < p^* = q^*$ and $2 \le p < q < q^*$.

Theorem 5.1. *Assume that (a_1)–(a_3) hold and $2 \le m = p < m^* = q^*$. Then for every $\lambda > 0$, there exist both a trivial and a nontrivial weak solution to problem (1.1). In particular, there exists a continuous set of positive generalized eigenvalues to problem (1.1).*

The above theorem will be proved once we show that the functional J_λ has the geometry of Mountain Pass Theorem as shown below.

Lemma 5.2. *Assume that (a_1) holds. Then for each $\lambda > 0$, the functional J_λ satisfies conditions (J_α) and (J_e) (as in Lemma 4.7).*

Proof.
(J_α) By (J_e), we have

$$\begin{aligned} J_\lambda(u) &\ge \frac{\xi_0}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) dx + \frac{\xi_1}{q}\int_\Omega \left(|\nabla u|^q + |u|^q\right) dx - \frac{\lambda}{m}\int_{\partial\Omega} |u|^m \, dx \\ &\ge C_1\left(\|u\|_{1,q}^q + \|u\|_{1,q}^q\right) - \lambda \|u\|_{L^m(\partial\Omega)}^m. \end{aligned}$$

Choosing $0 < \|u\| = \rho < 1$, since $p < q$, we get $\|u\|_{1,q}^q \le \|u\|_{1,q}^p$, hence

$$J_\lambda(u) \ge C_2\left(\|u\|_{1,q}^q + \|u\|_{1,q}^q\right) - C_m\lambda\|u\|_{L^m(\partial\Omega)}^m \ge C_2\|u\|^q - C_s\lambda\|u\|_{L^m(\partial\Omega)}^m,$$

where in the last inequality we have exploited the Sobolev embedding. Now, since $q \le p < m$, the condition (J_α) follows easily.
(J_e). Fixing $v \in C^\infty(\overline{\Omega})$ with $v > 0$ on $\overline{\Omega}$, from (a_1) we derive

$$J_\lambda(tv) \le \frac{t^p\xi_2}{p}\int_\Omega \left(|\nabla v|^p + |u|^p\right) dx + \frac{t^q\xi_3}{q}\int_\Omega \left(|\nabla v|^q + |u|^q\right) dx - \frac{\lambda t^m}{m}\int_{\partial\Omega} |v|^m \, d\sigma.$$

Since $p \le q < m$, there exists $\bar{t} > 1$ such that $e = \bar{t}v$ satisfies $J_\lambda(e) < 0$ and $\|e\| \ge \rho$. □

By exploiting and adapting the proofs of Theorems 3.2 and 3.4, by (a_2), (a_3) it follows that J_λ satisfies the Palais–Smale condition at the Mountain Pass level

$$\widetilde{C} = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} J_\lambda(\eta(t))$$

with $\Gamma = \{\eta \in [0,1], W^{1,q}(\Omega) : \ \eta(0) = 0 \text{ and } J_\lambda(\eta(t)) \le 0\}$ and then Theorem 5.1 will be proved.
In fact, also in this case, we use the arguments that can be found in [11, Lemma 2.4] and in the proof of Theorems 3.2 and 3.4. So, let $(u_n)_{n\ge 0}$ be a $(PS)_c$ sequence. Therefore,

$$\begin{aligned} C(1 + \|u_n\|) &\ge J_\lambda(u_n) - \frac{1}{m}\left\langle J_\lambda^{'}(u_n), u_n\right\rangle \\ &\ge \frac{1}{p}\int_\Omega A(|\nabla(u_n)|^p)\, dx + \frac{1}{p}\int_\Omega A(|u_n|^p)\, dx \\ &\qquad - \frac{1}{m}\int_\Omega a(|\nabla(u_n)|^p)|\nabla(u_n)|^p\, dx - \frac{1}{m}\int_\Omega a(u_n|^p)|u_n|^p\, dx, \end{aligned}$$

and (a_2) and then (a_1) imply

$$C(1+\|u_n\|) \geq \Big(\frac{1}{p\alpha}-\frac{1}{m}\Big)\int\limits_{\Omega}\Big(a(|\nabla(u_n)|^p)|\nabla(u_n)|^p + a(|u_n|^p)|u_n|^p\Big)\,dx$$
$$\geq C_1\|u_n\|_{1,p}^p + C_2\|u_n\|_{1,p}^q. \tag{5.1}$$

Now, if $q=p$, the boundedness of $(u_n)_{n\geq 0}$ in $W^{1,p}(\Omega)$ follows easily. If $q>p$, suppose by contradiction that, up to subsequence, $\|u_n\|\to+\infty$. If $\|u_n\|_{1,q}$ is bounded and $\|u_n\|_{1,q}\to+\infty$, by (5.1), we obtain an absurd. If $\|u_n\|_{1,p}\to+\infty$ and $\|u_n\|_{1,q}\to+\infty$, since $\|u_n\|_{1,q}^{q-p}\geq 1$ implies $\|u_n\|_{1,q}^q\geq\|u_n\|_{1,q}^p$, (5.1) involves

$$C\big(1+\|u_n\|\big)\geq C_1\|u_n||y_{1,q}^p + C_2\|u_n\|_{1,q}^q \geq C_3\big(\|u_n\|_{1,q}+\|u_n\|_{1,q}\big)^p = C_3\|u_n\|^p,$$

which is also an absurd. Consequently, by the boundedness of u_n in $W^{1,p}(\Omega)$ a function exists such that, up to subsequence, $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and it remains to prove the strong convergence of u_n to u in $W^{1,p}(\Omega)$. In the case $q=p$, we can refer to the last part of the proof of Theorem 3.2, while when $p<q$, to the one of the proof of Theorem 3.4 with the due changes.

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