Memoirs on Differential Equations and Mathematical Physics

Volume 83, 2021, 13–29

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TWO EXPLICIT NON-ALGEBRAIC CROSSING LIMIT CYCLES FOR A FAMILY OF PIECEWISE LINEAR SYSTEMS

Abstract. For a given family of planar piecewise linear differential systems, it is a very difficult problem to determine an upper bound for the number of its limit cycles and its explicit expressions. In this paper, we give a family of planar discontinuous piecewise linear differential systems formed by two regions separated by a straight line and having only one focus whose limit cycles can be explicitly described by using the first integrals. We show that these systems may have at most two explicit non-algebraic limit cycles.

2010 Mathematics Subject Classification. 34C25, 34A36, 34C07.

Key words and phrases. Discontinuous piecewise differential systems, first integrals, non-algebraic limit cycles.

რეზიუმე. ბრტყელი უბან-უბან წრფივი დიფერენციალური განტოლებების მოცემული ოჯახისთვის ძალზე რთულია მისი ზღვრული ციკლების რიცხვის ზედა საზღვრის განსაზღვრა და ცხადი სახით მოცემა. ნაშრომში მოცემულია წრფით გაყოფილი ორი არით განსაზღვრული ბრტყელი წყვეტილი უბან-უბან წრფივი დიფერენციალური სისტემები, რომლებსაც აქვს მხოლოდ ერთი ფოკუსი და რომელთა ზღვრული ციკლები შეიძლება ცხადად აღიწეროს პირველი ინტეგრალების საშუალებით. ნაჩვენებია, რომ ამ სისტემებს შეიძლება ჰქონდეს არა უმეტეს ორი ცხადი არაალგებრული ზღვრული ციკლი.

1 Introduction

The study of piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1] and still continues to receive attention by researchers. Piecewise linear systems often appear in the descriptions of many real processes such as dry friction in mechanical systems or switches in electronic circuits (see, e.g., [5, 15, 18, 19]). This kind of systems is generally modeled by ordinary differential equations with discontinuous right-hand sides which can exhibit very complicated dynamics and rich bifurcation phenomena.

A limit cycle is a periodic orbit of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system. There are two types of limit cycles in the planar discontinuous piecewise linear differential systems, the crossing and sliding ones. The "sliding limit cycles" contain some arc of the lines of discontinuity that separate the different linear differential systems (more precise definition can be found in [17]). The "crossing limit cycles" contain only isolated points of the lines of discontinuity. In this paper, we consider only the crossing limit cycles of some planar discontinuous piecewise linear differential systems separated by one straight line.

Limit cycles of discontinuous piecewise linear differential systems separated by a straight line have been studied by many authors (see, e.g., [2, 7, 8, 10, 11, 13] and the references therein). There are examples of such systems exhibiting three limit cycles (see [3, 4, 9, 12, 14]), but at present moment we do not know whether discontinuous piecewise linear differential systems separated by a straight line may have more than three limit cycles.

On the other hand, it seems intuitively clear that "most" limit cycles of discontinuous piecewise linear differential systems have to be non-algebraic. Nevertheless, in all these papers devoted to the study of the crossing limit cycles of piecewise linear differential systems, explicit non-algebraic limit cycles do not appear, their existence is proved by using different methods as the first integrals, the averaging theory, the Poincaré map, the Newton–Kantorovich Theorem, the Melnikov function.

The goal of this paper is to give a discontinuous piecewise linear differential systems separated by a straight line for which we can get two explicit limit cycles which are not algebraic. As far as we know, there are no examples of this situation in the literature.

We consider planar piecewise linear systems with two linearity regions separated by a straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : \ x = 0\}$, where we assume that the two linearity regions in the phase plane are the left and right half-planes

$$\Sigma_- = \big\{(x, y) \in \mathbb{R}^2 : \ x < 0\big\}, \quad \Sigma_+ = \big\{(x, y) \in \mathbb{R}^2 : \ x > 0\big\}.$$

We suppose that one of the two linear differential systems has no equilibria, neither real nor virtual, and the other one has a focus at the origin. We prove that these two systems are integrable. Moreover, we determine sufficient conditions for a discontinuous piecewise linear differential systems to possess two or one explicit non-algebraic limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

2 Preliminaries

The following normal form for the linear differential system in $\mathbb{R}^2$ and its first integral will help us to prove our main result.

Lemma 2.1. *A linear differential system having a focus at the origin can be written as*

$$\dot{x} = (2\lambda - \delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\left((\lambda - \delta)^2 + \omega^2\right)x + \delta y \tag{2.1}$$

with $\omega > 0$. Moreover, this system has the first integral

$$H_1(x, y) = \Big(\big((\lambda - \delta)^2 + \omega^2\big)x^2 + 2\beta(\lambda - \delta)xy + \beta^2 y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x+\beta y}\right)}.$$

Proof. Consider a general linear differential system

$$\dot{x} = \alpha x + \beta y, \quad \dot{y} = \eta x + \delta y. \tag{2.2}$$

The eigenvalues of this system are

$$\lambda_{1,2} = \frac{1}{2}\left(\alpha + \delta \pm \sqrt{(\alpha-\delta)^2 + 4\beta\eta}\right).$$

We know that system (2.2) has a real focus if $\frac{1}{2}(\alpha+\delta) = \lambda$, and $(\alpha-\delta)^2 + 4\beta\eta = -4\omega^2$, for some $\omega > 0$, $\beta\eta < 0$ and $\lambda \in \mathbb{R}$, then

$$\alpha = 2\lambda - \delta, \quad \eta = -\frac{1}{\beta}\left((\lambda-\delta)^2 + \omega^2\right).$$

Therefore, we obtain system (2.1).

Since the unique equilibrium is located at the origin $O(0,0)$ and is of focus type, any orbit of system (2.1) crosses the straight line $x = 0$ at least at one point, namely, $(0, C)$, $C \in \mathbb{R}$, thus the general solution of (2.1) is given by

$$x(t) = \frac{\beta}{\omega} C e^{t\lambda} \sin t\omega, \quad y(t) = \frac{1}{\omega} C e^{t\lambda} v\left(\omega \cos t\omega + (\delta - \lambda)\sin t\omega\right), \tag{2.3}$$

where $C \in \mathbb{R}$. So, from the first equation of (2.3), we obtain

$$e^{t\lambda} \sin \omega t = \frac{\omega}{\beta C}\, x.$$

Substituting this last expression into the second equation, we get

$$e^{t\lambda} \cos \omega t = \frac{1}{C\beta}\left((\lambda-\delta)x + \beta y\right).$$

Therefore,

$$\tan \omega t = \frac{\omega x}{(\lambda-\delta)x + \beta y}\,.$$

From the last equation, we obtain

$$t = \frac{1}{\omega}\,\arctan\left(\frac{\omega x}{(\lambda-\delta)x + \beta y}\right).$$

Substituting the previous expressions in the first equation of (2.3) and simplifying, we obtain

$$\left(\left((\lambda-\delta)^2 + \omega^2\right)x^2 + 2\beta(\lambda-\delta)xy + \beta^2 y^2\right) e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x + y\beta}\right)} = h,$$

where $h = (\beta C)^2 \in \mathbb{R}$. □

It is known that if the vector field has no equilibrium points, it can be written as

$$\dot{x} = ax + by + c, \quad \dot{y} = \mu a x + \mu b y + d, \tag{2.4}$$

where a, b, c, μ and d are real constants such that $d \neq \mu c$ and $\mu \neq 0$.

The following Lemma provides a first integral for an arbitrary linear differential system without equilibrium points.

Lemma 2.2. *For system* (2.4)*, the following statements hold.*

(i) *If $a + b\mu = 0$, then system* (2.4) *is Hamiltonian and all its solutions are algebraic and given by parabolas. Moreover, this system has the first integral*

$$H_2(x,y) = b\mu^2 x^2 - 2b\mu xy - 2\,dx + by^2 + 2cy.$$

(ii) *If $a + b\mu \neq 0$, the only algebraic invariant curve of* (2.4) *is an invariant line. Moreover, this system has the first integral*

$$H_3(x,y) = \left((a+b\mu)(ax+by) + ac + bd\right) e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)}.$$

Proof.

(i) Via the change of variables $x = v$, $u = \frac{1}{d-c\mu}(y - \mu x)$, where $d - c\mu \neq 0$, system (2.4) is transformed into

$$\dot{v} = (a + b\mu)v + b(d - c\mu)u + c, \quad \dot{u} = 1. \tag{2.5}$$

If $a + b\mu = 0$, the last system is Hamiltonian and it has the first integral

$$H_2(v, u) = v - \frac{b(d - c\mu)}{2}\, u^2 - cu,$$

and statement (i) follows.

(ii) If $a + b\mu \neq 0$, the general solution of (2.5) is

$$\begin{aligned} v(t) &= \frac{1}{(a + b\mu)^2}\Big((a + b\mu)^2(C_2 + e^{at+bt\mu}C_1) - ac - bd + b(c\mu - d)(a + b\mu)t\Big), \\ u(t) &= -\frac{1}{b(d - c\mu)}\big((a + b\mu)C_2 + b(c\mu - d)t\big), \end{aligned} \tag{2.6}$$

where C_1 and C_2 are real constants. So, from the second equation of (2.6), we obtain

$$t = \frac{(a + b\mu)C_2 + bu(d - c\mu)}{b(d - c\mu)}.$$

Substituting the expression of t into the first equation of (2.6), we get

$$\Big(b(d - c\mu)(a + b\mu)u + (a + b\mu)^2 v + ac + bd\Big)e^{-(a+b\mu)u} = C_1(a + b\mu)^2 e^{\frac{C_2(a+b\mu)^2}{bd-bc\mu}}.$$

Going back through the changes of variables, we obtain

$$\big((a + b\mu)(ax + by) + ac + bd\big)e^{\frac{(a+b\mu)}{d-c\mu}(\mu x - y)} = h, \tag{2.7}$$

where $h = C_1(a + b\mu)^2 e^{\frac{C_2(a+b\mu)^2}{bd-bc\mu}} \in \mathbb{R}$. From (2.7), we define a first integral of (2.4) as follows:

$$H_3(x, y) = \big((a + b\mu)(ax + by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)},$$

statement (ii) holds. □

Suppose that we have a discontinuous piecewise linear differential system separated by Σ. We assume, without loss of generality, that the left half-system has no equilibria, neither real nor virtual, and the right half-system is of focus type at the origin. By Lemma 2.1, and using the normal form (2.4), we can write such a discontinuous piecewise linear differential system as

$$\begin{aligned} &\dot{x} = (2\lambda - \delta)x + \beta y, \qquad \dot{y} = -\frac{1}{\beta}\big((\lambda - \delta)^2 + \omega^2\big)x + \delta y \ \text{ in } \Sigma_+, \\ &\dot{x} = ax + by + c, \qquad \dot{y} = \mu a x + \mu b y + d \ \text{ in } \Sigma_-. \end{aligned} \tag{2.8}$$

In order to state precisely our results, we introduce first some notations and definitions. Consider the piecewise differential system (2.8) defined in $\Sigma_\pm$. We use the techniques and approaches presented by Filippov in [6] and by di Bernardo et al. in [5] to establish these notations. An equilibrium point is called a real (resp. virtual) singular point of the right system of (2.8) if this point locates in the region Σ_+ (resp. Σ_-). A similar definition can be done for the left system of (2.8). Otherwise it is called a virtual equilibrium point. In order to extend the definition of a trajectory to $\Sigma = \{(x, y) \in \mathbb{R}^2 :\ x = 0\}$, we split Σ into three parts depending on whether or not the vector field points towards it:

1. Crossing region:
$$\Sigma_c = \big\{(0, y) \in \Sigma :\ \ \beta(by + c)y > 0\big\},$$

2. Attractive sliding region:

$$\Sigma_{as} = \Big\{(0,y) \in \Sigma : \quad \beta y < 0, \quad by + c > 0\Big\},$$

3. Repulsive sliding region:

$$\Sigma_{rs} = \Big\{(0,y) \in \Sigma : \quad \beta y > 0, \quad by + c < 0\Big\}.$$

These three regions are relatively open in Σ and may have several connected components. Therefore, their definitions exclude the so-called tangency points, that is, points where one of the two vector fields is tangent to Σ, which can be characterized by

$$\Big\{(0,y) \in \Sigma : \quad y = 0 \ \text{ or } \ by + c = 0\Big\}.$$

These points are on the boundary of the regions Σ_c, Σ_{as} and Σ_{rs}.

Periodic orbits that have neither sliding part nor tangent points are called crossing periodic orbits, otherwise, they are called sliding periodic orbits. We say that an isolated periodic orbit Γ is an algebraic limit cycle if all its points are contained in the level sets of polynomials. Otherwise, they are called non-algebraic limit cycles.

3 Main result

Our main result is contained in the following

Theorem 3.1. *The discontinuous piecewise linear differential system* (2.8) *may have at most two non-algebraic crossing limit cycles. Moreover, there are the systems in this class having one or two non-algebraic crossing limit cycles.*

Theorem 3.1 is proved in Section 4.

The next Propositions show that there are discontinuous piecewise linear differential systems of the form (2.8) (in case the left half-linear system of (2.8) is non-Hamiltonian) with two, or one (respectively) non-algebraic crossing limit cycles.

Proposition 3.1. *For $a = \mu+1$, $c = -1$, $d = -\mu-3$, $b = -1$, $\mu \neq 0$ and $\lambda = -\frac{1}{2}\,\omega$, the discontinuous piecewise linear differential system* (2.8) *defined by*

$$\begin{aligned} \dot{x} &= -(\omega+\delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= (\mu+1)x - y - 1, \quad \dot{y} = \mu(\mu+1)x - \mu y - (\mu+3) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.1}$$

when $\omega > 1.7525$, $\mu \neq 0$ and $\beta < 0$, has exactly two nested crossing limit cycles. Moreover, these limit cycles are hyperbolic, non-algebraic and given by

$$\begin{aligned} \Gamma_1 = \Big\{(x,y) \in \Sigma_+ : \quad &\Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{-\arctan\left(\frac{2\omega x}{(2\delta+\omega)x - 2\beta y}\right)} = 50.971\beta^2\Big\} \\ &\cup \Big\{(x,y) \in \Sigma_- : \quad ((1+\mu)x - y + 2)e^{\frac{1}{3}\,y - \frac{\mu}{3}\,x} = 0.846\,03\Big\}, \\ \Gamma_2 = \Big\{(x,y) \in \Sigma_+ : \quad &\Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\,\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{-\arctan\left(\frac{2\omega x}{(2\delta+\omega)x - 2\beta y}\right)} = 19.825\beta^2\Big\} \\ &\cup \Big\{(x,y) \in \Sigma_- : \quad ((1+\mu)x - y + 2)e^{\frac{1}{3}\,y - \frac{\mu}{3}\,x} = 1.462\,7\Big\}. \end{aligned}$$

This proposition will be proved in Section 5.

Proposition 3.2. *For $a = \mu - 1$, $c = -3$, $d = -(3\mu + 10)$, $b = -1$, $\lambda = -\omega$ and $\mu \neq 0$, the discontinuous piecewise linear differential system* (2.8) *defined by*

$$\begin{aligned} \dot{x} &= -(2\omega + \delta)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\left((\omega+\delta)^2 + \omega^2\right)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= (\mu - 1)x - y - 3, \qquad \dot{y} = \mu(\mu-1)x - \mu y - (3\mu + 10) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.2}$$

when $\omega > 5.315$, $\mu \neq 0$ and $\beta < 0$, has exactly one explicit hyperbolic non-algebraic crossing limit cycle given by

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \Big(\big((\omega+\delta)^2 + \omega^2\big)x^2 - 2\beta(\omega+\delta)xy + \beta^2 y^2\Big)e^{-2\arctan\left(\frac{\omega x}{(\delta+\omega)x - \beta y}\right)} = 32.1\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ (x + y - \mu x + 13)e^{\frac{\mu}{10}x - \frac{1}{10}y} = 12.925\Big\}.$$

This proposition will be proved in Section 6.

The next proposition shows that there are discontinuous piecewise linear differential systems of the form (2.8) (in case the left half-linear system of (2.8) is Hamiltonian) with one crossing non-algebraic limit cycle.

Proposition 3.3. *For $a = \mu$, $\lambda = -\frac{\omega}{2}$, $c = -3$, $b = -1$, $d = -(1 + 3\mu)$ and $\mu \neq 0$, the discontinuous piecewise linear differential system defined by*

$$\begin{aligned} \dot{x} &= -(\delta + \omega)x + \beta y, \quad \dot{y} = -\frac{1}{\beta}\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x + \delta y \ \text{ in } \Sigma_+, \\ \dot{x} &= \mu x - y - 3, \quad \dot{y} = \mu^2 x - \mu y - (1 + 3\mu) \ \text{ in } \Sigma_-, \end{aligned} \tag{3.3}$$

when $\omega > 0.34337$, $\mu \neq 0$ and $\beta < 0$, has exactly one explicit hyperbolic non-algebraic crossing limit cycle given by

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{\arctan\left(\frac{-2x\omega}{(2\delta+\omega)x - 2\beta y}\right)} = 57.375\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ -\mu^2x^2 + 2\mu xy + 2(1+3\mu)x - y^2 - 6y = -11.927\Big\}.$$

This proposition will be proved in Section 7.

Remark 3.1. The assumption $\beta < 0$ in Propositions 3.1, 3.2 and 3.3 is a necessary condition for the existence of crossing limit cycles of system (3.1) (resp. (3.2) and (3.3)). Effectively, if the crossing region of (3.1) (resp. (3.2) and (3.3)) exists with $\beta > 0$, then the inequality $y(-y-1) > 0$ (resp. $y(-y-3) > 0$) implies that the crossing region is an open interval $(-1, 0)$ (resp. $(-3, 0)$) of the line Σ. Since the right half-system is of focus type at the origin, any orbit starting at the point $(0, y_0)$ with $y_0 < 0$ goes into the left zone Σ_- under the flow of the left linear differential systems. If these orbits can reach Σ again at some point $(0, y_1)$ after some time $t > 0$, must be $y_1 > 0$ and so, the condition $\beta > 0$ precludes the existence of crossing limit cycles.

4 Proof of Theorem 3.1

Suppose that we have a discontinuous piecewise linear differential system (2.8). In order to investigate the crossing limit cycles of this system, we use the first integrals for the right and the left side systems of (2.8). Due to Lemmas 2.1 and 2.2, these first integrals are

$$H_1(x,y) = \Big(\big((\lambda-\delta)^2 + \omega^2\big)x^2 + 2\beta(\lambda-\delta)xy + \beta^2y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan\left(\frac{\omega x}{(\lambda-\delta)x + \beta y}\right)},$$
$$H_2(x,y) = \begin{cases} \big((a+b\mu)(ax+by) + ac + bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)} & \text{if } a + b\mu \neq 0, \\ b\mu^2x^2 - 2b\mu xy - 2\,dx + by^2 + 2cy & \text{if } a + b\mu = 0 \end{cases}$$

in Σ_+ and Σ_-, respectively. Suppose that this discontinuous piecewise differential system has some limit cycles intersecting Σ at two points, namely, $(0,y_0)$ with $y_0<0$, and $(0,y_1)$ with $y_1>0$. Then the first integrals H_1 and H_2 must satisfy the following two equations:

$$\begin{aligned} H_1(0,y_0)-H_1(0,y_1)&=0,\\ H_2(0,y_0)-H_2(0,y_1)&=0, \end{aligned} \tag{4.1}$$

it is easy to see that the implicit form of the orbit arc of (2.8) in Σ_+ which starting at the point $(0,y_0)$, where $y_0<0$ when $t=0$, is given by $H_1(x,y)-\beta^2y_0^2=0$, this last orbit can be given also by the analytic curves $(x_+(t),y_+(t))$, where

$$\begin{aligned} x_+(t)&=\frac{\beta}{\omega}\,y_0e^{\lambda t}\sin\omega t,\\ y_+(t)&=\frac{1}{\omega}\,y_0e^{\lambda t}(\omega\cos\omega t+(\delta-\lambda)\sin\omega t). \end{aligned}$$

Denote by t_+ the minimum positive time such that $x(t_+)=x(0)=0$, then $t_+=\frac{\pi}{\omega}$. Since the orbits starting at the point $(0,y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0,y_1)$ after the time $t_+=\frac{\pi}{\omega}$, we have

$$y_1=y(t_+)=-y_0e^{\frac{\lambda\pi}{\omega}},$$

which is proves that $H_1(0,y_0)-H_1(0,y_1)=0$. Now, it is easy to see that the existence of crossing periodic solutions of discontinuous piecewise linear differential system (2.8) is equivalent to the existence of negative values of y_0 satisfying

$$H_2(0,y_0)=H_2(0,-y_0e^{\frac{\lambda\pi}{\omega}}). \tag{4.2}$$

Here, we have to separate the proof of Theorem 3.1 in two cases.

Case 1. $a+\mu b=0$.

In this case (4.2) becomes

$$y_0\big(b(1-e^{\frac{2\lambda\pi}{\omega}})y_0+2c(1+e^{\frac{\lambda\pi}{\omega}})\big)=0. \tag{4.3}$$

It is easy to see that when $b=0$ or $c=0$, the unique solution of (4.3) is $y_0=0$. So, in this case, the discontinuous piecewise linear differential system (2.8) has no limit cycles.

When $b\neq 0$ and $c\neq 0$, equation (4.3) has two roots: $y_{01}=0$, which cannot contribute a limit cycle and $y_0=\frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)}\neq 0$. Moreover, we can choose the appropriate parameters b, c, λ and ω in such a way that (4.3) has exactly one real negative root $y_0=\frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)}$, thus obtaining at most one limit cycle for the discontinuous piecewise linear differential system (2.8). Using the first integrals of both linear differential systems and knowing that the non-algebraic crossing periodic orbit passes through the point $(0,y_0)$ when $t=0$ and through the point $(0,-y_0e^{\frac{\lambda\pi}{\omega}})$ when $t=\frac{\pi}{\omega}$, where $y_0=\frac{2c(e^{\frac{\lambda\pi}{\omega}}+1)}{b(e^{\frac{2\lambda\pi}{\omega}}-1)}<0$, we get the expression

$$\begin{aligned}\Gamma=\Big\{(x,y)\in\Sigma_+:\ \Big(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan(\frac{\omega x}{(\lambda-\delta)x+\beta y})}=\beta^2y_0^2\Big\}\\ \cup\Big\{(x,y)\in\Sigma_-:\ b\mu^2x^2-2b\mu xy-2\,dx+by^2+2cy=(2c+by_0)y_0\Big\}\end{aligned}$$

So, Theorem 3.1 is proved in Case 1.

Case 2. $a+\mu b\neq 0$.

In this case (4.2) becomes

$$\big(ac+bd-b(a+b\mu)e^{\pi\frac{\lambda}{\omega}}y_0\big)e^{\frac{a+b\mu}{d-c\mu}y_0e^{\frac{\lambda\pi}{\omega}}}=(b(a+b\mu)y_0+ac+bd)e^{-\frac{a+b\mu}{d-c\mu}y_0}. \tag{4.4}$$

Then the existence of crossing periodic solutions of discontinuous piecewise linear differential system (2.8) is equivalent to the existence of zeros for equation (4.4) with respect to the variable y_0. On the other hand, this equation can be rewritten as

$$\big(ac+bd-b(a+b\mu)e^{\frac{\lambda\pi}{\omega}}y_0\big)e^{\frac{(a+b\mu)(e^{\frac{\lambda\pi}{\omega}}+1)}{d-c\mu}y_0}-b(a+b\mu)y_0-ac-bd=0.$$

For convenience, we use the notation

$$f(y)=\big(ac+bd-b(a+b\mu)e^{\frac{\lambda\pi}{\omega}}y\big)e^{\frac{(a+b\mu)(e^{\frac{\lambda\pi}{\omega}}+1)}{d-c\mu}y}-b(a+b\mu)y-ac-bd. \tag{4.5}$$

Now, solving (4.4) is equivalent to finding the solutions y_{0j} of the equation $f(y)=0$. In order to investigate a number of solutions of $f(y)=0$, and since f is a differentiable function in $\mathbb{R}$, we use the first two derivatives of the function f. Simple calculations yield

$$\begin{aligned} f'(y)={}&\frac{a+b\mu}{c\mu-d}\Big(be^{\frac{\lambda\pi}{\omega}}(1+e^{\frac{\lambda\pi}{\omega}})(a+b\mu)y-ac-bd-c(a+b\mu)e^{\frac{\lambda\pi}{\omega}}\Big)e^{\frac{(e^{\frac{\lambda\pi}{\omega}}+1)(a+b\mu)}{d-c\mu}y}\\ &-\frac{b(a+b\mu)(d-c\mu)}{d-c\mu},\\ f''(y)={}&-\Big(be^{\pi\frac{\lambda}{\omega}}(1+e^{\pi\frac{\lambda}{\omega}})(a+b\mu)y-e^{\pi\frac{\lambda}{\omega}}(ac-bd+2bc\mu)-bd-ac\Big)(e^{\pi\frac{\lambda}{\omega}}+1)\\ &\times\frac{(a+b\mu)^2}{(d-c\mu)^2}\,e^{\frac{(e^{\frac{\lambda}{\omega}\pi}+1)(a+b\mu)}{d-c\mu}y}. \end{aligned}$$

It is easy to see that f' and f'' are continuous functions in $\mathbb{R}$.

It is obvious that $f^{''}(y)=0$ has at most one root y_0, thus the equation $f'(y)=0$ has at most two zeros y_{0j}, $j=1,2$, and the equation $f(y)=0$ has at most three roots y_{0i}, $i=1,2,3$.

Note that the equation $f(y)=0$ has the solution $y_0=0$, which cannot contribute a limit cycle. So, in this case, the equation $f(y)=0$ may have eventually two real solutions, $y_{0j}\neq 0$ for $j=1,2$ that can provide at most 2 limit cycles for the discontinuous piecewise linear differential system (2.8). Moreover, we can choose the appropriate parameters a, b, c, d, λ, δ, μ and ω in such a way that $f(y)=0$ has exactly 2 real negative roots y_{0i}, $i=1,2$, that can provide 2 limit cycles for the discontinuous piecewise linear differential system (2.8).

Using the first integrals of both linear differential systems and knowing that the non-algebraic crossing periodic orbits pass through the points $(0,y_{0i})$ when $t=0$, and through the point $(0,-y_{0i}e^{\frac{\lambda\pi}{\omega}})$ when $t=\frac{\pi}{\omega}$, where y_{0i}, $i=1,2$, are the zeros of $f(y)=0$. Thus the expressions for these orbits are:

$$\begin{aligned} \Gamma_i={}&\Big\{(x,y)\in\Sigma_+:\ \ \Big(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\Big)e^{-\frac{2\lambda}{\omega}\arctan(\frac{\omega x}{(\lambda-\delta)x+\beta y})}=\beta^2y_{0i}^2\Big\}\\ &\cup\Big\{(x,y)\in\Sigma_-:\ \big((a+b\mu)(ax+by)+ac+bd\big)e^{\frac{a+b\mu}{d-c\mu}(\mu x-y)}=\big(b(a+b\mu)y_{0i}+ac+bd\big)e^{\frac{a+b\mu}{c\mu-d}y_{0i}}\Big\}. \end{aligned}$$

This completes the proof of Theorem 3.1 in Case 2.

Remark 4.1. The orbit arc passing through the crossing point $(0,-y_0e^{\frac{\lambda\pi}{\omega}})$ is $H_1(x,y)-\beta^2(y_0e^{\frac{\lambda\pi}{\omega}})^2=0$, this orbit, when $(\lambda-\delta)x+\beta y\neq 0$ and $((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2\neq 0$, can be rewritten as

$$\tan\Big(\frac{-\omega}{2\lambda}\ln\frac{\beta^2y_0^2}{(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2)}-\pi\Big)=\frac{\omega x}{(\lambda-\delta)x+\beta y},$$

thus

$$\tan\Big(\frac{-\omega}{2\lambda}\ln\frac{\beta^2y_0^2}{(((\lambda-\delta)^2+\omega^2)x^2+2\beta(\lambda-\delta)xy+\beta^2y^2)}\Big)=\frac{\omega x}{(\lambda-\delta)x+\beta y},$$

this last equation is equivalent to

$$H_1(x,y)-\beta^2y_0^2=0,$$

and shows that $H_1(0,y_0)-H_1(0,-y_0e^{\frac{\lambda\pi}{\omega}})=0$.

5 Proof of Proposition 3.1

We prove that the discontinuous piecewise linear differential system (3.1) has exactly two hyperbolic non-algebraic limit cycles. It is easy to see that the left half-system has no equilibria, neither real nor virtual, and since $-\frac{1}{2} \pm i\omega$, $\omega > 0$ are the eigenvalues of the matrices of the right half-system of (3.1), this system has its equilibria as focus type at the origin.

The two linear differential systems of (3.1) have the following first integrals:

$$H_1(x,y) = \Big(\Big(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\Big)e^{-\arctan(\frac{2\omega x}{(2\delta+\omega)x-2\beta y})},$$
$$H_2(x,y) = ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x}$$

in Σ_+ and Σ_-, respectively. The parametric solution of the right half-system of (2.8) starting at the point $(0, y_0)$ with $y_0 < 0$ when $t = 0$, is

$$x_+(t) = \frac{\beta}{\omega}y_0 e^{-\frac{\omega}{2}t}\sin\omega t,$$
$$y_+(t) = \frac{1}{\omega}y_0 e^{-\frac{\omega}{2}t}\Big(\omega\cos\omega t + \Big(\delta + \frac{\omega}{2}\Big)\sin\omega t\Big).$$

Let t_+ denote the minimum positive time such that $x(t_+) = x(0) = 0$, then $t_+ = \frac{\pi}{\omega}$. Since the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\omega}$, we have

$$y_1 = y(t_+) = -y_0 e^{-\frac{\pi}{2}}.$$

Then, for the discontinuous piecewise linear differential system (3.1), the function (4.5) becomes

$$f(y) = (ye^{\frac{-\pi}{2}} + 2)e^{-\frac{1}{3}(e^{\frac{-\pi}{2}}+1)y} + y - 2.$$

The graphic of this function is given in Figure 5.1.

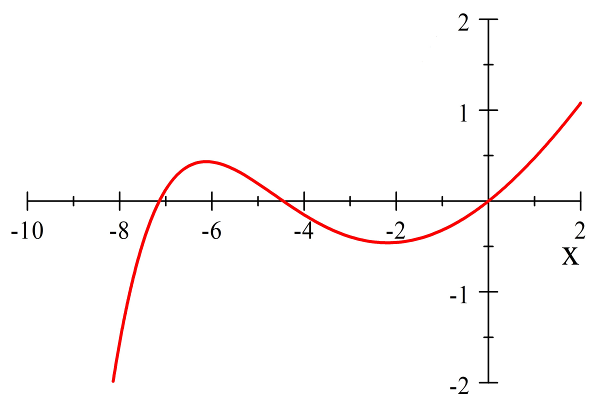

Figure 5.1. The graphic of the function $f(y)$.

The equation $f(y) = 0$ has exactly three zeros $y_{00} = 0$, $y_{01} = -4.4522$ and $y_{02} = -7.1392$. From these values of y_{0i}, $i = 0, 1, 2$, we get the values $y_{10} = 0$, $y_{11} = 0.92558$ and $y_{12} = 1.4841$.

Straightforward computations show that the solution passing through the crossing points $(0, y_{01})$ and $(0, y_{11})$ corresponds to

$$\Gamma_1 = \Big\{(x,y) \in \Sigma_+ : \ \Big(\Big(\delta^2+\delta\omega+\frac{5}{4}\omega^2\Big)x^2 - \beta(2\delta+\omega)xy + \beta^2y^2\Big)e^{-\arctan\frac{2\omega x}{(2\delta+\omega)x-2\beta y}} = 19.825\beta^2\Big\}$$
$$\cup \Big\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} = 1.4627\Big\},$$

and the solution passing through the crossing points $(0, y_{02})$ and $(0, y_{12})$ corresponds to

$$\Gamma_2 = \left\{(x,y) \in \Sigma_+ : \ \left(\left(\delta^2 + \delta\omega + \frac{5}{4}\omega^2\right)x^2 - \beta(2\delta+\omega)xy + \beta^2 y^2\right)e^{-\arctan\frac{2\omega x}{(2\delta+\omega)x-2\beta y}} = 50.971\beta^2\right\}$$
$$\cup \left\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} = 0.84603\right\}.$$

Moreover, Γ_1 and Γ_2 are non-algebraic and travel in a counterclockwise sense around the sliding segment $\Sigma_{rs} = \{(0,y) \in \Sigma : \ -1 \le y \le 0\}$. Clearly, Γ_1 and Γ_2 are nested, and Γ_1 is the inner one and Γ_2 is the outer one. Now we prove that these non-algebraic crossing periodic orbits are the hyperbolic limit cycles.

Let T be the period of the periodic solution

$$\Gamma : \ \{(x(t), y(t)), \ t \in [0,T]\}.$$

To see that Γ is, in fact, a limit cycle, we recall a classic result characterizing limit cycles among other periodic orbits for a smooth differential system in the plane (see, e.g., Perko [16] for more details), which means that $\Gamma(t)$ is a hyperbolic limit cycle when

$$\int_0^T \operatorname{div}(\Gamma(t)\, dt \neq 0, \tag{5.1}$$

stable if $\int_0^T \operatorname{div}(\Gamma(t))\, dt < 0$, and instable if $\int_0^T \operatorname{div}(\Gamma(t))\, dt > 0$.

Using the form parametric $(x_{-i}(t), y_{-i}(t))$ of the curve $H_2(x,y) = (-y_{1i} + 2)e^{\frac{1}{3}y_{1i}}$ starting at the point $(0, y_{1i})$ in the half-plane Σ_-

$$\begin{aligned} x_{-i}(t) &= y_{1i} - 3t + (2 - y_{1i})e^t - 2, \\ y_{-i}(t) &= y_{1i} - 2\mu - (3\mu+3)t + (2\mu - \mu y_{1i})e^t + \mu y_{1i}, \end{aligned}$$

where $i = 1, 2$ and $y_{1i} = -y_{0i}e^{-\frac{\pi}{2}}$, it is easy to check that the periodic orbits Γ_1 and Γ_2 have periods $T_1 = 1.7926$ and $T_2 = 2.8745$, respectively.

Formula (5.1) can be extended to the discontinuous piecewise linear differential systems considered here, then for the discontinuous piecewise linear differential system, we have

$$\begin{aligned} \Gamma_1 &: \left\{(x_{+1}(t), y_{+1}(t)), \ t \in \left[0, \frac{\pi}{\omega}\right]\right\} \cup \left\{(x_{-1}(t), y_{-1}(t)), \ t \in \left[\frac{\pi}{\omega}, T\right]\right\}, \\ \Gamma_2 &: \left\{(x_{+2}(t), y_{+2}(t)), \ t \in \left[0, \frac{\pi}{\omega}\right]\right\} \cup \left\{(x_{-2}(t), y_{-2}(t)), \ t \in \left[\frac{\pi}{\omega}, T\right]\right\}, \end{aligned}$$

where

$$\begin{aligned} x_{+i}(t) &= \frac{\beta}{\omega}\, y_{0i}e^{-\frac{1}{2}\omega t}\sin\omega t, \\ y_{+i}(t) &= \frac{1}{\omega}\, y_{0i}e^{-\frac{1}{2}\omega t}\left(\omega\cos\omega t + \left(\delta + \frac{1}{2}\omega\right)\sin\omega t\right). \end{aligned}$$

Thus

$$\begin{aligned} \int_0^{T_1} \operatorname{div}(\Gamma_1(t))\, dt &= \int_0^{\frac{\pi}{\omega}} -\omega\, dt + \int_{\frac{\pi}{\omega}}^{1.7926} dt = 1.7926 - \frac{\pi}{\omega} - \pi, \\ \int_0^{T_2} \operatorname{div}(\Gamma_2(t))\, dt &= \int_0^{\frac{\pi}{\omega}} -\omega\, dt + \int_{\frac{\pi}{\omega}}^{2.8745} dt = 2.8745 - \frac{\pi}{\omega} - \pi. \end{aligned}$$

Since $\omega > 1.7525$, we have $\frac{\pi}{\omega} < 1.7926$, thus $\int_0^{T_1} \operatorname{div}(\Gamma_1(t))\, dt \neq 0$ and $\int_0^{T_2} \operatorname{div}(\Gamma_2(t))\, dt \neq 0$, so we obtain two hyperbolic non-algebraic crossing limit cycles.

Example 5.1. When $\mu = 2$, $\beta = -1$, $\omega = 2$ and $\delta = 1$, system (3.1) reads as

$$\begin{aligned} \dot{x} &= -3x - y, \quad \dot{y} = 8x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= 3x - y - 1, \quad \dot{y} = 6x - 2y - 5 \ \text{ in } \Sigma_-. \end{aligned} \tag{5.2}$$

This system has exactly two explicit hyperbolic and non-algebraic crossing limit cycles Γ_i, $i = 1, 2$. The smallest one Γ_1 intersects the switching line Σ at two points

$$y_{01} = -4.4522, \quad y_{11} = 0.92558$$

and is given by

$$\Gamma_1 = \left\{(x,y) \in \Sigma_+ : \ (8x^2 + 4xy + y^2)e^{-\arctan \frac{2x}{2x+y}} = 19.825\right\} \cup \left\{(x,y) \in \Sigma_- : \ (3x - y + 2)e^{\frac{1}{3}y - \frac{2}{3}x} = 1.4627\right\}.$$

The biggest limit cycle Γ_2 intersects the switching line Σ at two points

$$y_{02} = -7.1392, \quad y_{12} = 1.4841.$$

and the expression of this limit cycle is given by

$$\Gamma_2 = \left\{(x,y) \in \Sigma_+ : \ (8x^2 + 4xy + y^2)e^{-\arctan \frac{2x}{2x+y}} = 50.971\right\} \cup \left\{(x,y) \in \Sigma_- : \ ((1+\mu)x - y + 2)e^{\frac{1}{3}y - \frac{\mu}{3}x} = 0.84603\right\}$$

(see Figure 5.2).

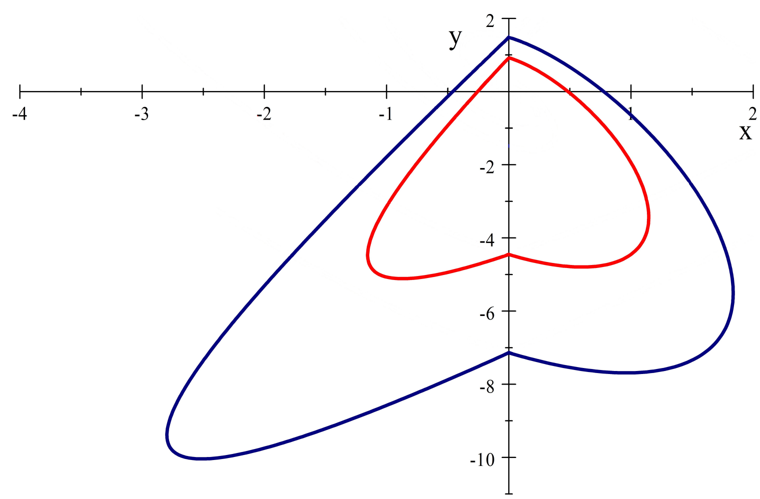

Figure 5.2. The two crossing non-algebraic limit cycles of the discontinuous piecewise linear differential systems (5.2).

6 Proof of Proposition 3.2

We consider the planar piecewise linear system (3.2), for this system it is easy to check that the left linear differential system has neither real nor virtual equilibria and the right linear differential system is a focus with eigenvalues $-1 \pm \omega i$, $\omega > 0$. In order to prove that the discontinuous piecewise linear differential system (3.2) has exactly one hyperbolic non-algebraic limit cycle, we use the first integrals for the right and the left side systems of (3.2).

The first integrals of the two linear differential systems of (3.2) are

$$H_1(x,y)=\Big((\omega^2+(\delta+\omega)^2)x^2-2\beta(\delta+\omega)xy+\beta^2y^2\Big)e^{2\arctan\left(\frac{\omega x}{(\delta+\omega)x-\beta y}\right)},$$
$$H_2(x,y)=(x+y-x\mu+13)e^{\frac{\mu}{10}x-\frac{1}{10}y}$$

in Σ_+ and Σ_-, respectively. The solution $(x_+(t),y_+(t))$ of right half-system of (3.2) such that $(x_+(0),y_+(0))=(0,y_0)$ with $y_0<0$ is

$$x_+(t)=\frac{\beta}{\omega}y_0e^{-\omega t}\sin\omega t,$$
$$y_+(t)=\frac{1}{\omega}y_0e^{-\omega t}\big(\omega\cos\omega t+(\delta-\lambda)\sin\omega t\big).$$

The time t_+ that the solution $(x_+(t),y_+(t))$ contained in Σ_+ needs to reach the point $(0,y_1)$ is $t_+=\frac{\pi}{\omega}$. Therefore,

$$y_1=y(t_+)=-y_0e^{-\pi}.$$

Then, for the discontinuous piecewise linear differential system (3.2), the function (4.5) becomes

$$f(y)=-(e^{-\pi}y-13)e^{\frac{1}{10}(e^{-\pi}+1)y}-y-13.$$

The graphic of this function is given in Figure 6.1.

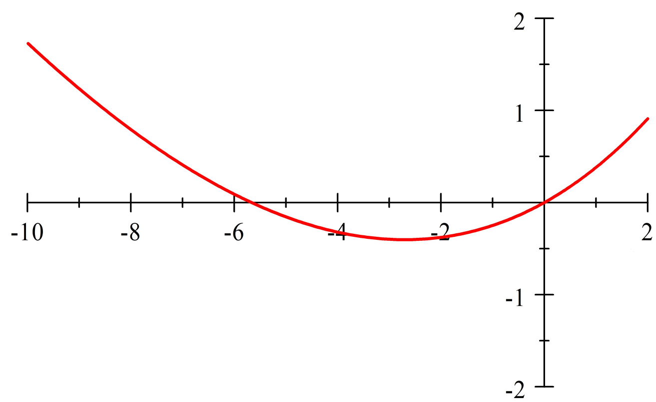

Figure 6.1. The graphic of the function $f(y)$.

The unique solution $y_0\neq 0$ of the equation $f(y)=0$ is $y_0=-5.6657$. From this value of y_0, we get the value of $y_1=0.24484$.

Thus, the solution passing through the crossing points $(0,y_0)$ and $(0,y_1)$ corresponds to

$$\Gamma=\Big\{(x,y)\in\Sigma_+:\ \Big(((\omega+\delta)^2+\omega^2)x^2-2\beta(\omega+\delta)xy+\beta^2y^2\Big)e^{-2\arctan\left(\frac{\omega x}{(\delta+\omega)x-\beta y}\right)}=32.1\beta^2\Big\}$$
$$\cup\Big\{(x,y)\in\Sigma_-:\ (x+y-\mu x+13)e^{\frac{\mu}{10}x-\frac{1}{10}y}=12.925\Big\}.$$

Moreover, Γ is non-algebraic and travels in a counterclockwise sense around the sliding segment $\Sigma_{rs}=\{(0,y)\in\Sigma:\ -3\leq y\leq 0\}$.

Using the form parametric $(x_-(t),y_-(t))$ of the curve $H_2(x,y)=(y_1+2)e^{\frac{-1}{10}y_1}$ starting at the point $(0,y_1)$ in the half-plane Σ_-

$$x_-(t)=10t-y_1+e^{-t}(y_1+13)-13,$$
$$y_-(t)=y_1-13\mu+10(\mu-1)t-\mu y_1+\mu(13+y_1)e^{-t},$$

where $y_1 = -y_0 e^{-\pi}$, it is easy to check that the periodic orbit Γ has period $T = 0.59108$.
Then, for the discontinuous piecewise linear differential system (3.2), we have

$$\Gamma : \left\{(x_+(t), y_+(t)),\ t \in \left[0, \frac{\pi}{\omega}\right]\right\} \cup \left\{(x_-(t), y_-(t)),\ t \in \left[\frac{\pi}{\omega}, T\right]\right\},$$

and

$$\int_0^T \operatorname{div}(\Gamma(t))\, dt = \int_0^{\frac{\pi}{\omega}} -2\omega\, dt - \int_{\frac{\pi}{\omega}}^{0.59108} dt = \frac{\pi}{\omega} - 2\pi - 0.59108.$$

Since $\omega > 5.315$, $\frac{\pi}{\omega} < 0.59108$ which leads to $\int_0^T \operatorname{div}(\Gamma(t))\, dt < 0$, hence the non-algebraic crossing periodic orbit Γ is a stable and hyperbolic limit cycle. This completes the proof of Proposition 3.2.

Example 6.1. When $\mu = -2$, $\beta = -1$, $\delta = 1$ and $\omega = 8$, system (3.2) reads as

$$\begin{aligned} \dot{x} &= -17x - y, \quad \dot{y} = 145x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= -3x - y - 3, \quad \dot{y} = 6x + 2y - 4 \ \text{ in } \Sigma_-. \end{aligned} \tag{6.1}$$

Then, this system has exactly one explicit hyperbolic and non-algebraic crossing limit cycle Γ. This limit cycle intersects the switching line Σ at two points

$$y_0 = -5.6657, \quad y_1 = 0.24484$$

and is given by

$$\Gamma = \left\{(x, y) \in \Sigma_+ : \ (145x^2 + 18xy + y^2)e^{-2\arctan\left(\frac{8x}{9x+y}\right)} = 32.1\right\}$$
$$\cup \left\{(x, y) \in \Sigma_- : \ (x + y + 2x + 13)e^{\frac{-1}{5}x - \frac{1}{10}y} = 12.925\right\}.$$

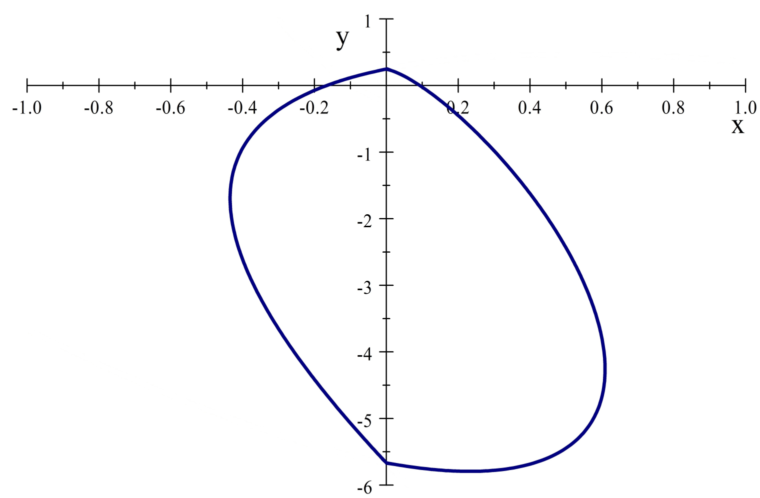

Figure 6.2. The unique crossing non-algebraic limit cycle of system (6.1).

7 Proof of Proposition 3.3

Suppose that we have a discontinuous piecewise linear differential system (3.3). It is easy to see that the left half-system is Hamiltonian without equilibrium points and, since $-\frac{1}{2} \pm i\omega$, $\omega > 0$ are the eigenvalues of the matrices of the right half-system, this system has its equilibria as focus type at the origin. In order for the piecewise linear differential system (3.3) to have exactly one hyperbolic non-algebraic limit cycle, it must intersect the discontinuous curve Σ at two points. Let $(0, y_0)$ with $y_0 < 0$, and $(0, y_1)$ with $y_1 > 0$ be two intersecting points. Then, taking into account that

$$\begin{aligned} H_1(x,y) &= \Big(\Big(\delta^2+\delta\omega+\frac{5}{4}\,\omega^2\Big)x^2-\beta(2\delta+\omega\Big)xy+\beta^2y^2)e^{\arctan(\frac{-2x\omega}{(2\delta+\omega)x-2\beta y})},\\ H_2(x,y) &= -\mu^2x^2+2\mu xy+2(1+3\mu)x-y^2-6y \end{aligned}$$

are first integrals of the two linear differential systems of (3.3) in Σ_+ and Σ_-, respectively, these two points satisfy equations (4.1).

The solution of the right half-system of (3.3) starting at the point $(0, y_0)$, $y_0 < 0$ when $t = 0$, is

$$\begin{aligned} x_+(t) &= \frac{\beta}{\omega}\,y_0e^{-\frac{\omega}{2}t}(\sin\omega t),\\ y_+(t) &= \frac{1}{\omega}\,y_0e^{-\frac{\omega}{2}t}\Big(\omega\cos\omega t+\Big(\delta+\frac{\omega}{2}\Big)\sin\omega t\Big). \end{aligned}$$

The time t_+ that the solution $(x_+(t), y_+(t))$ contained in Σ_+ needs to reach the point $(0, y_1)$ is $t_+ = \frac{\pi}{\omega}$. Since the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left linear differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\omega}$, we have

$$y_1 = y(t_+) = -y_0e^{-\frac{\pi}{2}}.$$

This proves that $H_1(0, y_0) - H_1(0, y_1) = 0$. Then, for the discontinuous piecewise linear differential system (3.3), equation (4.2) becomes

$$\big((e^{-\pi}-1)y_0-6(1+e^{\frac{-\pi}{2}})\big)y_0 = 0.$$

The unique solution $y_0 \neq 0$ of this last equation is

$$y_0 = \frac{6(e^{-\frac{\pi}{2}}+1)}{e^{-\pi}-1} = -7.5746.$$

From this value of y_0, we get the value of $y_1 = 1.5746$.
Therefore, the solution passing through the crossing points $(0, y_0)$ and $(0, y_1)$ is written as

$$\begin{aligned} \Gamma = \Big\{(x,y)\in\Sigma_+ :\ \Big(\Big(\delta^2+\delta\omega+\frac{5}{4}\,\omega^2\Big)x^2-\beta(2\delta+\omega)xy+\beta^2y^2\Big)e^{\arctan(\frac{-2x\omega}{(2\delta+\omega)x-2\beta y})} = 57.375\beta^2\Big\}\\ \cup\Big\{(x,y)\in\Sigma_- :\ -\mu^2x^2+2\mu xy+2(1+3\mu)x-y^2-6y = -11.927\Big\}. \end{aligned}$$

Moreover, Γ is non-algebraic and travels in a counterclockwise sense around the sliding segment $\Sigma_{rs} = \{(0, y) \in \Sigma :\ -3 \le y \le 0\}$.

Now, we prove that this non-algebraic crossing periodic orbit is a hyperbolic limit cycle. From the analytical form $(x_-(t), y_-(t))$ of the curve $H_2(x, y) = -(6 + y_1)y_1$ starting at the point $(0, y_1)$ in the half-plane Σ_-, we have

$$\begin{aligned} x_-(t) &= -\frac{1}{2}\,t^2-t(y_1+3),\\ y_-(t) &= \frac{1}{2}\,\mu t^2-(3\mu+1)t+y_1, \end{aligned}$$

where $y_1 = -y_0e^{-\pi}$, it is easy to check that the periodic orbit Γ has period $T = 9.1492$.

Then, for the discontinuous piecewise linear differential system (3.3), we have

$$\Gamma :\ \Big\{(x_+(t), y_+(t)),\ t\in\Big[0,\frac{\pi}{\omega}\Big]\Big\}\cup\Big\{(x_-(t), y_-(t)),\ t\in\Big[\frac{\pi}{\omega}, T\Big]\Big\}$$

and

$$\int_0^T \operatorname{div}(\Gamma(t))\,dt = \int_0^{\frac{\pi}{\omega}} -\omega\,dt = -\pi < 0,$$

hence, the non-algebraic crossing periodic orbit Γ is a stable and hyperbolic limit cycle. This completes the proof of Proposition 3.3.

Example 7.1. When $\beta = -1$, $\mu = -2$, $\delta = 1$ and $\omega = 1$, system (3.3) reads as

$$\begin{aligned} \dot{x} &= -2x - y, \quad \dot{y} = \frac{13}{4}x + y \ \text{ in } \Sigma_+, \\ \dot{x} &= -2x - y - 3, \quad \dot{y} = 4x + 2y + 6 \ \text{ in } \Sigma_-. \end{aligned} \tag{7.1}$$

Then, this system has exactly one explicit hyperbolic, non-algebraic crossing limit cycle Γ. This limit cycle intersects the switching line Σ at two points

$$y_0 = -7.5746, \quad y_1 = 1.5746$$

and is given by

$$\Gamma = \Big\{(x,y) \in \Sigma_+ : \ \frac{1}{4}(13x^2 + 12xy + 4y^2)e^{-\arctan(\frac{2x}{3x+2y})} = 57.375\Big\} \\ \cup \Big\{(x,y) \in \Sigma_- : \ -4x^2 - 4xy - y^2 - 10x - 6y = -11.927\Big\}$$

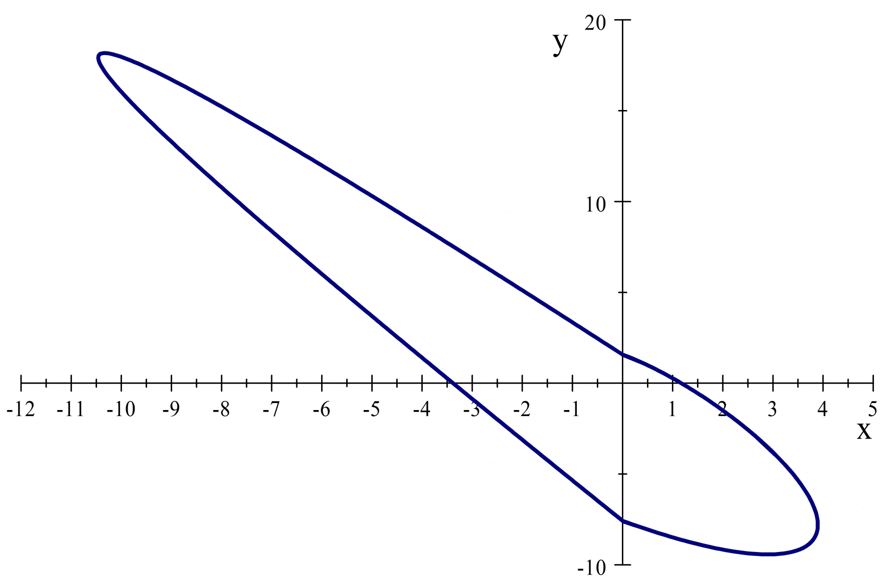

Figure 7.1. The unique crossing non-algebraic limit cycle of system (7.1).

Acknowledgments

We thank the reviewers for their good comments and suggestions that help us to improve the presentation of our results.

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(Received 06.09.2020)

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