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Roland Gachechiladze

DYNAMICAL CONTACT PROBLEMS WITH REGARD TO FRICTION OF COUPLE-STRESS VISCOELASTICITY FOR INHOMOGENEOUS ANISOTROPIC BODIES

Abstract. The paper deals with the three-dimensional boundary-contact problems of couple-stress viscoelasticity for inhomogeneous anisotropic bodies with friction. The uniqueness theorem is proved by using the corresponding Green's formulas and positive definiteness of the potential energy. To analyze the existence of solutions, the problem under consideration is reduced equivalently to a spatial variational inequality. A special parameter-dependent regularization of this variational inequality is considered, which is equivalent to the relevant regularized variational equation depending on a real parameter, and its solvability is studied by the Faedo–Galerkin method. Some a priori estimates for solutions of the regularized variational equation are established and with the help of an appropriate limiting procedure the existence theorem for the original contact problem with friction is proved.

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Key words and phrases. Couple-stress elasticity theory, viscoelasticity, contact problem with friction, variational inequality, variational equation, Faedo–Galerkin method.

რეზიუმე. ნაშრომში განხილულია ბლანტი დრეკადობის მომენტური თეორიის დინამიკის სამგანზომლებიანი სასაზღვრო-საკონტაქტო ამოცანა არაერთგვაროვანი, ანიზოტროპული სხეულისთვის ხახუნის ეფექტის გათვალისწინებით. შესწავლილია ამოცანის სუსტი ამონახსნის არსებობისა და ერთადერთობის საკითხი. ამონახსნის ერთადერთობის დადგენა ეფუძნება გრინის ფორმულებს და პოტენციალური ენერგიის დადებითად განსაზღვრულობას. ამონახსნის არსებობის შესწავლისთვის სასაზღვრო-საკონტაქტო ამოცანა ეკვივალენტურად დაიყვანება სივრცით ვარიაციულ უტოლობაზე, რომელიც, თავის მხრივ, ეკვივალენტურია მცირე პარამეტრზე დამოკიდებული რეგულარიზებული განტოლების. ამ განტოლების ამონახსნის არსებობა შესწავლილია ფაედო-გალიორკინის მეთოდის მეშვეობით და მიღებულია ამონახსნის გარკვეული აპრიორული შეფასებები. ეს შეფასებები იძლევა ზღვარზე გადასვლის საშუალებას ჯერ განზომილების, ხოლო შემდეგ მცირე პარამეტრის მიმართ. და ბოლოს, ნაჩვენებია, რომ ზღვარითი ფუნქცია წარმოადგენს დასმული სასაზღვრო-საკონტაქტო ამოცანის ამონახსნს.

1 Introduction

The general and widespread use of the linear theory of viscoelasticity has been observed since the early seventies of the past century. Activity in this area is associated with a wide application of polymeric materials with properties that can obviously be described neither by elastic nor by viscous models, but combine the features of both models. Mathematical strictly grounded theory of linear viscoelasticity with numerous practical applications is contained in the monographs of D. R. Bland and R. M. Christensen (see [1,2] and the references therein).

Viscoelastic materials are those supplied with the "memory" in the sense that the state at time t depends on all the deformations that the material undergoes. A particularly important class of "viscoelastic equations of state" is associated with materials for which there is a linear relationship between the time derivatives of the stress and strain tensors. We will consider viscoelastic materials with short-term memory, i.e., when the stress of the moment at time t depends only on the deformations, the moment at time t and the nearest previous moments of time. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed in terms of the displacement and rotation vectors. In their work [4], E. Cosserat and F. Cosserat created and presented the model of a solid medium in which every material point has six degrees of freedom, three of which are defined by the displacement components and the other three by the components of rotation (for the history of the model of elasticity see [6, 24, 27, 31] and the references therein). The main equations of that model are interrelated and generate a matrix second order differential operator of dimension 6×6. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity for smooth and non-smooth Lipschitz domains were studied in [28]. The one-sided contact problems of statics of the hemitropic theory of elasticity, free from friction, were investigated in [11, 12, 16, 18, 21], and the contact problems of statics and dynamics with a friction were considered in [9, 10, 13–15, 17, 19, 20]. Analogous, one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [5, 7, 8, 22, 23] and the references therein). Particular problems of the viscoelasticity theory are considered in [1, 2]. As for the dynamical and quasistatical boundary-contact problems of viscoelasticity with friction, we have considered them in [5].

The paper is organized as follows. First, we present general field equations of the linear theory of couple-stress viscoelasticity and formulate the boundary-contact problem of dynamics with regard to the friction. We prove the uniqueness theorem by using Green's formulas and positive definiteness of the potential energy. Afterwards, the contact problem is equivalently reduced to a spacial variational inequality. The latter is in its turn replaced by the relevant regularized equation depending on a real positive parameter ε, and its solvability is studied by the Faedo–Galerkin method in appropriate approximate function spaces of dimension m. Furthermore, some a priori estimates are established, which allow us to pass to the limit with respect to dimension m as $m \to \infty$ and to parameter ε as $\varepsilon \to 0$. As a result, we prove that the limiting function is a solution of the variational inequality and, consequently, the limiting function solves the original contact problem.

2 Field equations and Green's formulas

2.1 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with C^∞ smooth boundary $S := \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at a point $x \in S$.

The basic equilibrium equations of dynamics of couple-stress viscoelasticity for inhomogeneous anisotropic bodies read as

$$\begin{aligned} \partial_i \sigma_{ij}(x,t) + \varrho F_j(x,t) &= \varrho \, \frac{\partial^2 u_j(x,t)}{\partial t^2}\,, \\ \partial_i \mu_{ij}(x,t) + \varepsilon_{ikj}\sigma_{ik}(x,t) + \varrho G_j(x,t) &= \mathcal{J} \, \frac{\partial^2 \omega_j(x,t)}{\partial t^2}\,, \end{aligned} \tag{2.1}$$

where t is the time variable, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_i = \frac{\partial}{\partial x_i}$, ϱ is the mass density of the elastic material, $\mathcal{J}$ is the moment of inertia per unit volume, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are, respectively, the body force and body couple vectors per unit mass, $u = (u_1, u_2, u_3)^\top$ is the *displacement vector*, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the *micro-rotation vector*, ε_{ikj} is the permutation (Levi–Civita) symbol;

Here and in what follows, the symbol $(\,\cdot\,)^\top$ denotes transposition and the repetition of the index means summation over this index from 1 to 3. For the *force stress tensor* $\{\sigma_{ij}\}$ and the *couple-stress tensor* $\{\mu_{ij}\}$, we have

$$\begin{aligned}
\sigma_{ij}(x,t) &:= \sigma_{ij}(U(t)) \\
&= a^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + b^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + a^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),\\
\mu_{ij}(x,t) &:= \mu_{ij}(U(t)) \\
&= b^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + c^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t)) + c^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),
\end{aligned}$$

where $U(t) := U(x,t) = (u(x,t), \omega(x,t))^\top$, $\zeta_{lk}(U(t)) = \partial_l u_k(x,t) - \varepsilon_{lkm}\omega_m(x,t)$ and $\eta_{lk}(U(t)) = \partial_l\omega_k(x,t)$ are the so-called strain and torsion (curvature) tensors; the real-valued functions $a^{(0)}_{ijlk}$, $b^{(0)}_{ijlk}$, $c^{(0)}_{ijlk}$ (respectively, $a^{(1)}_{ijlk}$, $b^{(1)}_{ijlk}$, $c^{(1)}_{ijlk}$), called the elastic constants (respectively, viscosity constants), satisfy certain smoothness and symmetry conditions

(i) $a^{(q)}_{ijlk}, b^{(q)}_{ijlk}, c^{(q)}_{ijlk} \in C^1(\overline{\Omega})$,

(ii) $a^{(q)}_{ijlk} = a^{(q)}_{lkij}$, $c^{(q)}_{ijlk} = c^{(q)}_{lkij}$,

(iii) there exists $\alpha_0 > 0$ such that $\forall\, x \in \overline{\Omega}$ and $\forall\, \xi_{ij}, \eta_{ij} \in R$:

$$a^{(q)}_{ijlk}(x)\xi_{ij}\xi_{lk} + 2b^{(q)}_{ijlk}(x)\xi_{ij}\eta_{lk} + c^{(q)}_{ijlk}(x)\eta_{ij}\eta_{lk} \geq \alpha_0(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij}) \ \ (q = 0,1).$$

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$\mathcal{M}(x,\partial) = \begin{bmatrix} \mathcal{M}^{(1)}(x,\partial) & \mathcal{M}^{(2)}(x,\partial) \\ \mathcal{M}^{(3)}(x,\partial) & \mathcal{M}^{(4)}(x,\partial) \end{bmatrix}_{6\times 6}, \quad \mathcal{M}^{(p)}(x,\partial) = \Big[\mathcal{M}^{(p)}_{jk}(x,\partial)\Big]_{3\times 3}, \ \ p = \overline{1,4},$$

where

$$\begin{aligned}
\mathcal{M}^{(1)}_{jk}(x,\partial) &= \partial_i\big(\big[a^{(0)}_{ijlk}(x) + a^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big),\\
\mathcal{M}^{(2)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{ijlk}(x) + b^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[a^{(0)}_{ijlr}(x) + a^{(1)}_{ijlr}(x)\partial_t\big];\\
\mathcal{M}^{(3)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{lkij}(x) + b^{(1)}_{lkij}(x)\partial_t\big]\partial_l\big) + \varepsilon_{irj}\big[a^{(0)}_{irlk}(x) + a^{(1)}_{irlk}(x)\partial_t\big]\partial_l;\\
\mathcal{M}^{(4)}_{jk}(x,\partial) &= \partial_i\big(\big[c^{(0)}_{ijlk}(x) + c^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[b^{(0)}_{lrij}(x) + b^{(1)}_{lrij}(x)\partial_t\big]\\
&\qquad + \varepsilon_{irj}\big[b^{(0)}_{irlk}(x) + b^{(1)}_{irlk}(x)\partial_t\big]\partial_l - \varepsilon_{ipj}\varepsilon_{lrk}\big[a^{(0)}_{iplr}(x) + a^{(1)}_{iplr}(x)\partial_t\big].
\end{aligned}$$

Denote by $\mathcal{N}(\partial,n)$ the generalized 6×6 matrix differential stress operator

$$\mathcal{N}(\partial,n) = \begin{bmatrix} \mathcal{N}^{(1)}(\partial,n) & \mathcal{N}^{(2)}(\partial,n) \\ \mathcal{N}^{(3)}(\partial,n) & \mathcal{N}^{(4)}(\partial,n) \end{bmatrix}_{6\times 6}, \quad \mathcal{N}^{(p)}(\partial,n) = \Big[\mathcal{N}^{(p)}_{jk}(\partial,n)\Big]_{3\times 3}, \ \ p = \overline{1,4},$$

where

$$\begin{aligned}
\mathcal{N}^{(1)}_{jk}(\partial,n) &= \big[a^{(0)}_{ijlk} + a^{(1)}_{ijlk}\partial_t\big]n_i\partial_l;\\
\mathcal{N}^{(2)}_{jk}(\partial,n) &= \big[b^{(0)}_{ijlk} + b^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[a^{(0)}_{ijlr} + a^{(1)}_{ijlr}\partial_t\big]n_i;\\
\mathcal{N}^{(3)}_{jk}(\partial,n) &= \big[b^{(0)}_{lkij} + b^{(1)}_{lkij}\partial_t\big]n_i\partial_l;\\
\mathcal{N}^{(4)}_{jk}(\partial,n) &= \big[c^{(0)}_{ijlk} + c^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[b^{(0)}_{lrij} + b^{(1)}_{lrij}\partial_t\big]n_i.
\end{aligned} \tag{2.2}$$

Here $\partial_n = \partial/\partial n$ denotes the directional derivative along the vector n (normal derivative). In the sequel, for the force stress and couple-stress vectors we use the following notation:

$$\mathcal{T}U = \mathcal{N}^{(1)}u + \mathcal{N}^{(2)}\omega, \quad MU = \mathcal{N}^{(3)}u + \mathcal{N}^{(4)}\omega,$$

where $\mathcal{N}^{(p)}$, $p = 1,2,3,4$, is defined by formula (2.2).

The system of equations (2.1) can be rewritten in the matrix form

$$\mathcal{M}(x,\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,, \;\; x \in \Omega, \;\; 0 < t < T, \tag{2.3}$$

where T is an arbitrary positive number, $U = (u,\omega)^\top$, $\mathcal{G} = (\varrho F, \varrho G)^\top$, $P = [p_{ij}]_{6\times 6}$, $p_{ii} = \varrho$, when $i = 1,2,3$, $p_{ii} = \mathcal{J}$, when $i = 4,5,6$, and $p_{ij} = 0$, when $i \neq j$.

Throughout the paper, $L_p(\Omega)$ $(1 \leq p \leq \infty)$, $L_2(\Omega) = H^0(\Omega)$ and $H^s(\Omega) = H^s_2(\Omega)$, $s \in \mathbb{R}$, denote the Lebesgue and Bessel potential spaces (see, e.g., [25, 32]). We denote the corresponding norms by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$, respectively. Denote by $D(\Omega)$ the class of $C^\infty(\Omega)$ functions with a support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M \subset \partial\Omega$, $M \neq \partial\Omega$: then we denote by $H^s(M)$ the restriction of the space $H^s(\partial\Omega)$ on M,

$$H^s(M) := \big\{ r_{_M}\varphi : \; \varphi \in H^s(\partial\Omega) \big\},$$

where $r_{_M}$ stands for the restriction operator on the set M. Further, let

$$\widetilde{H}^s(M) := \big\{ \varphi \in H^s(\partial\Omega) : \; \operatorname{supp}\varphi \subset \overline{M} \big\}.$$

The total strain energy of the respective media has the form

$$\mathcal{B}^{(q)}(U,V) = \int\limits_\Omega \Big\{ a^{(q)}_{ijlk}(x)\zeta_{ij}(U)\zeta_{lk}(V) + b^{(q)}_{ijlk}(x)\zeta_{ij}(U)\eta_{lk}(V) + b^{(q)}_{ijlk}(x)\zeta_{ij}(V)\eta_{lk}(U) + c^{(q)}_{ijlk}(x)\eta_{ij}(U)\eta_{lk}(V) \Big\}\, dx,$$

where $q = 1,2$, $U = (u,\omega)^\top$, $V = (v,w)^\top$ and $\zeta_{ij}(U) = \partial_i u_j - \varepsilon_{ijr}\omega_r$, $\eta_{ij}(U) = \partial_i \omega_j$.

From properties (ii) and (iii), it is clear that $\mathcal{B}^{(q)}(U,V) = \mathcal{B}^{(q)}(V,U)$ and $\mathcal{B}^{(q)}(U,U) \geq 0$. Moreover, there exist positive constants C_1 and C_2, depending only on the material parameters, such that Korn's type inequality (cf., [8, Part I, § 12], [3, § 6.3])

$$\mathcal{B}^{(q)}(U,U) \geq C_1 \|U\|^2_{[H^1(\Omega)]^6} - C_2 \|U\|^2_{[L_2(\Omega)]^6}, \;\; q = 1,2, \tag{2.4}$$

holds for an arbitrary real-valued vector function $U \in [H^1(\Omega)]^6$.

Remark 2.1. If $U \in [H^1(\Omega)]^6$ and on some open part $S^* \subset \partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+ = 0$, then we have the strict Korn's inequality

$$\mathcal{B}^{(q)}(U,U) \geq c\|U\|^2_{[H^1(\Omega)]^6}$$

with some positive constant $c > 0$ which does not depend on the vector U. This follows from (2.4) and the fact that in this case $\mathcal{B}^{(q)}(U,U) > 0$ for $U \neq 0$ (see [29], [26, Ch. 2, Exercise 2.17]).

2.2 Green's formulas

For the real-valued vector functions $U(t) = (u(t),\omega(t))^\top$ and $\widetilde{U}(t) = (\widetilde{u}(t),\widetilde{\omega}(t))^\top$ of the class $[C^2(\overline{\Omega})]^6$ and for an arbitrary $t \in [0;T]$, the following Green's formula (see [13])

$$\int\limits_\Omega \mathcal{M}(x,\partial)U(t)\cdot\widetilde{U}(t)\,dx = \int\limits_S \big\{\mathcal{N}(\partial,n)U(t)\big\}^+ \cdot \{\widetilde{U}(t)\}^+\,dS - \Big\{ \mathcal{B}^{(0)}(U(t),\widetilde{U}(t)) + \partial_t \mathcal{B}^{(1)}(U(t),\widetilde{U}(t)) \Big\} \tag{2.5}$$

holds, where $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω.

By the standard limiting arguments, Green's formula (2.5) can be extended to the Lipschitz domains and to vector functions $U, \widetilde{U} \in [H^1(\Omega)]^6$ with $\mathcal{M}(x,\partial)U(t) \in [L_2(\Omega)]^6$ (see [25, 29]),

$$\int\limits_{\Omega} \mathcal{M}(x,\partial)U(t)\cdot\widetilde{U}(t)\,dx = \Big\langle \big\{\mathcal{N}(\partial,n)U(t)\big\}^+ \cdot \{\widetilde{U}(t)\}^+ \Big\rangle_s\, dS \\ - \Big\{\mathcal{B}^{(0)}(U(t),\widetilde{U}(t)) + \partial_t \mathcal{B}^{(1)}(U(t),\widetilde{U}(t))\Big\}, \;\; t \in (0;T), \quad (2.6)$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality between the spaces $[H^{-1/2}(S)]^6$ and $[H^{1/2}(S)]^6$, which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By this relation, the generalized trace of the stress operator $\{\mathcal{N}(\partial,n)U\}^+ \in [H^{-1/2}(S)]^6$ is well defined.

The following assertion describes the null space of the energy quadratic form $\mathcal{B}^{(q)}(U(t),U(t))$ (see [13]).

Lemma 2.2. *Let for an arbitrary $t \in (0;T)$, $U(t) = (u(t),\omega(t))^\top \in [C^1(\overline{\Omega})]^6$ and $\mathcal{B}^{(q)}(U(t),U(t)) = 0$ in Ω. Then*

$$u(t) = [a^{(q)} \times x] + b^{(q)}, \;\; \omega(t) = a^{(q)}, \;\; x \in \Omega,$$

where $a^{(q)}$ and $b^{(q)}$ are arbitrary three-dimensional constant vectors and the symbol $[\,\cdot\times\cdot\,]$ denotes the cross product of two vectors.

The vectors of type $([a^{(q)}\times x]+b^{(q)}, a^{(q)})$ are called *generalized rigid displacement vectors*. Observe that a generalized rigid displacement vector vanishes, i.e., $a^{(q)} = b^{(q)} = 0$, if it is zero at a single point.

3 Contact problems with friction

3.1 Coulomb's law

Let the boundary S of the domain Ω be divided into two open, connected and non-overlapping parts S_1 and S_2 of positive measure, $S = \overline{S_1} \cup \overline{S_2}$, $S_1 \cap S_2 = \varnothing$. Assume that the viscoelastic body occupying the domain Ω is in a contact with another rigid body along the subsurface S_2. Denote by $F(x,t)$ the force stress vector by which the hemitropic body acts upon the rigid body at the point $x \in S_2$. Throughout the paper, F_n and F_s stand for the normal and tangential components of the vector F, respectively: $F_n = F\cdot n$ and $F_s = F - (F\cdot n)\,n$. Further, let $\mathcal{F}(x)$ be the *friction coefficient* at the point $x \in S_2$. It is a nonnegative scalar function which depends on the geometry of the contacting surfaces and also on the physical properties of the interacting materials.

Coulomb's law describing the contact interaction of materials with friction reads as follows (for details see [5]):

If the contact of two bodies is described by the force vector F, then

$$|F_s(x,t)| \le \mathcal{F}(x)|F_n(x,t)|\,.$$

Moreover, if

$$|F_s(x,t)| < \mathcal{F}(x)|F_n(x,t)|,$$

then

$$\frac{\partial u_s(x,t)}{\partial t} = 0,$$

and if

$$|F_s(x,t)| = \mathcal{F}(x)|F_n(x,t)|,$$

then there exist nonnegative functions λ_1 and λ_2 not vanish simultaneously such that

$$\lambda_1(x,t)\,\frac{\partial u_s(x,t)}{\partial t} = -\lambda_2(x,t)F_s(x,t).$$

3.2 Pointwise and variational formulation of the contact problem

Let X be a Banach space with the norm $\|\cdot\|_X$. We denote by $L_p(0,T;X)$ $(1\le p\le\infty)$ the space of measurable functions $t\mapsto f(t)$ defined on the interval $(0;T)$ with values in the space X such that

$$\|f\|_{L_p(0,T;X)} := \bigg\{ \int\limits_0^T \|f(t)\|_X^p\,dt \bigg\}^{1/p} < \infty \ \text{ for } \ 1\le p<\infty$$

and

$$\|f\|_{L_\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0;T)} \big\{ \|f(t)\|_X \big\} < \infty \ \text{ for } \ p=\infty.$$

Definition 3.1. The vector-function $U:(0;T)\to[H^1(\Omega)]^6$ is said to be a weak solution of equation (2.3) for $\mathcal{G}:(0;T)\to[L_2(\Omega)]^6$ if

$$U(t), U'(t)\in L_\infty(0,T;[H^1(\Omega)]^6),\ \ U''(t)\in L_\infty(0,T;[L_2(\Omega)]^6),$$

and for every $\Phi\in[\mathcal{D}(\Omega)]^6$,

$$(PU''(t),\Phi)+\mathcal{B}^{(0)}(U(t),\Phi)+\mathcal{B}^{(1)}(U'(t),\Phi)=(\mathcal{G}(t),\Phi).$$

Here and in what follows, the symbol $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product in the space $L_2(\Omega)$.
Further, let

$$\mathcal{G}:(0,T)\to[L_2(\Omega)]^6,\quad \varphi:(0;T)\to[H^{-1/2}(S_2)]^3,\quad f:(0;T)\to L_\infty(S_2),$$

and set

$$g:=\mathcal{F}|f|\ge 0. \tag{3.1}$$

Consider the following contact problem of dynamics with friction.

Problem (A_0)**.** Find a weak solution $U:(0;T)\to[H^1(\Omega)]^6$ of the equation

$$\mathcal{M}(x,\partial)U(x,t)+\mathcal{G}(x,t)=P\,\frac{\partial^2U(x,t)}{\partial t^2}\,,\ \ x\in\Omega,\ \ t\in(0;T), \tag{3.2}$$

satisfying the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+\in[L_\infty(S_2\times(0;T))]^3$, the initial conditions

$$U(x,0)=0,\ \ x\in\Omega, \tag{3.3}$$

$$U'(x,0)=0,\ \ x\in\Omega, \tag{3.4}$$

and the boundary contact conditions

$$r_{S_1}\{U\}^+=0\ \text{ on }\ S_1\times(0;T), \tag{3.5}$$

$$r_{S_2}\{(\mathcal{T}U)_n\}^+=f\ \text{ on }\ S_2\times(0;T), \tag{3.6}$$

$$r_{S_2}\{MU\}^+=\varphi\ \text{ on }\ S_2\times(0;T), \tag{3.7}$$

$$r_{S_2}\Big\{\frac{\partial u_s}{\partial t}\Big\}^+=0\ \text{ if }\ \big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big|<g\ \text{ on }\ S_2\times(0;T), \tag{3.8}$$

and if $|r_{S_2}\{(\mathcal{T}U)_s\}^+|=g$, then there exist nonnegative functions λ_1 and λ_2 do not vanishing simultaneously, such that

$$\lambda_1(x,t)r_{S_2}\Big\{\frac{\partial u_s}{\partial t}\Big\}^+=-\lambda_2(x,t)\,r_{S_2}\{(\mathcal{T}U)_s\}^+\ \text{ on }\ S_2\times(0;T). \tag{3.9}$$

This problem can be reformulated in terms of a variational inequality. To this end, on the space $[H^1(\Omega)]^6$ we introduce the continuous convex functional

$$j(V)=\int\limits_{S_2} g|\{v_s\}^+|\,dS,\quad V=(v,w)^\top:(0;T)\to[H^1(\Omega)]^6 \tag{3.10}$$

and the closed convex sets $\mathcal{K}$ and $\mathcal{K}_0$:

$$\mathcal{K} := \Big\{ V \mid \ V(t), V'(t) \in L_\infty(0,T;[H^1(\Omega)]^6), \\ V''(t) \in L_\infty(0,T;[L_2(\Omega)]^6),\ r_{_{S_1}}\{V\}^+ = 0,\ V(0) = V'(0) = 0 \Big\};$$

$$\mathcal{K}_0 := \Big\{ V \mid \ V \in [H^1(\Omega)]^6,\ r_{_{S_1}}\{V\}^+ = 0 \Big\}.$$

Consider the following variational inequality: Find a $(u,\omega)^\top \in \mathcal{K}$ such that the variational inequality

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) + j(V) - j(U'(t)) \\ \geq (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ \, dS + \big\langle \varphi(t), r_{_{S_2}}\{w - \omega'(t)\}^+ \big\rangle_{S_2} \tag{3.11}$$

holds for all $V = (v,w)^\top \in \mathcal{K}_0$.

Here and in what follows, the symbol $\langle \cdot\, , \cdot \rangle$ denotes the duality relation between the corresponding dual pairs $X^*(M)$ and $X(M)$. In particular, $\langle \cdot\, , \cdot \rangle_{S_2}$ in (3.11) denotes the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\widetilde{H}^{1/2}(S_2)]^3$.

4 Equivalence theorem

Here we prove the following equivalence result.

Theorem 4.1. *If $U : (0;T) \to [H^1(\Omega)]^6$ is a solution of problem (A_0), then U is a solution of the variational inequality* (3.11)*, and vice versa.*

Proof. Let $U = (u,\omega)^\top : (0;T) \to [H^1(\Omega)]^6$ be a solution of problem (A_0), and $V = (v,w)^\top \in \mathcal{K}_0$. By virtue of the interior regularity theorems (see [8]), we have $U(t) \in [H^2(\Omega')]^6$ for every domain $\overline{\Omega'} \subset \Omega$. Hence the equation

$$\mathcal{M}(x,\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,, \ \ x \in \Omega, \ \ t \in (0;T)$$

holds almost everywhere in the domain Ω. By virtue of Green's formula (2.6), we get

$$(PU''(t), V - U'(t)) - \big\langle \{\mathcal{T}U\}^+, \{v - u'(t)\}^+ \big\rangle_S - \big\langle \{MU\}^+, \{w - \omega'(t)\}^+ \big\rangle_S \\ + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) = (\mathcal{G}(t), V - U'(t)). \tag{4.1}$$

Taking into account the boundary conditions (3.5), (3.6), (3.7) and the form of the functional (3.10), we deduce that for all $V = (v,w)^\top \in \mathcal{K}_0$ from (4.1), we have

$$(PU''(t), V - U'(t)) + \mathcal{B}^{(0)}(U(t), V - U'(t)) + \mathcal{B}^{(1)}(U'(t), V - U'(t)) + j(V) - j(U'(t)) \\ = (\mathcal{G}(t), V - U'(t)) + \int\limits_{S_2} f(t)\{v_n - u_n'(t)\}^+ \, dS + \big\langle \varphi(t), r_{_{S_2}}\{w - \omega'(t)\}^+ \big\rangle_{S_2} \\ + \int\limits_{S_2} \Big[\{(\mathcal{T}U)_s\}^+ \cdot \{v_s - u_s'(t)\}^+ + g\big(|\{v_s\}^+| - |\{u_s'(t)\}^+|\big) \Big] \, dS.$$

It is easy to see that if conditions (3.8) and (3.9) hold, then

$$r_{_{S_2}}\{(\mathcal{T}U)_s\}^+ \cdot r_{_{S_2}}\{v_s - u_s'(t)\}^+ + g\big(|r_{_{S_2}}\{v_s\}^+| - |r_{_{S_2}}\{u_s'(t)\}^+|\big) \geq 0.$$

Hence we have

$$
\begin{aligned}
(PU''(t), V-U'(t)) + \mathcal{B}^{(0)}(U(t), V-U'(t)) + \mathcal{B}^{(1)}(U'(t), V-U'(t)) + j(V) - j(U'(t)) \\
\geq (\mathcal{G}(t), V-U'(t)) + \int\limits_{S_2} f(t)\{v_n - u'_n(t)\}^+ \, dS + \big\langle \varphi(t), r_{S_2}\{w-\omega'(t)\}^+ \big\rangle_{S_2}
\end{aligned}
$$

for all $V=(v,w)^\top \in \mathcal{K}_0$. Thus $U=(u,\omega)^\top : (0;T) \to [H^1(\Omega)]^6$ is a solution of the variational inequality (3.11).

Let now $U=(u,\omega)^\top \in \mathcal{K}$ be a solution of the variational inequality (3.11). Substituting $U'(t) \pm \Phi$ instead of V in (3.11) with an arbitrary $\Phi \in [\mathcal{D}(\Omega)]^6$, we obtain

$$
(PU''(t), \Phi) + \mathcal{B}^{(0)}(U(t), \Phi) + \mathcal{B}^{(1)}(U'(t), \Phi) = (\mathcal{G}(t), \Phi) \quad \forall\, \Phi \in [\mathcal{D}(\Omega)]^6,
$$

which implies that U is a weak solution of equation (3.2). Again, by virtue of the interior regularity theorem (see [8]), equation (3.2) is satisfied almost everywhere in the domain Ω. Thus, taking into account the fact that $r_{S_1}\{V-U'(t)\}^+ = 0$ for all $V=(v,w)^\top \in \mathcal{K}_0$, Green's formula (2.6) yields

$$
\begin{aligned}
(PU''(t), V-U'(t)) + \mathcal{B}^{(0)}(U(t), V-U'(t)) + \mathcal{B}^{(1)}(U'(t), V-U'(t)) \\
= (\mathcal{G}(t), V-U'(t)) + \Big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+, r_{S_2}\{v_n - u'_n(t)\}^+ \Big\rangle_{S_2} \\
+ \Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u'_s(t)\}^+ \Big\rangle_{S_2} + \Big\langle r_{S_2}\{MU\}^+, r_{S_2}\{w-\omega'(t)\}^+ \Big\rangle_{S_2} \quad \forall\, V \in \mathcal{K}_0.
\end{aligned}
$$

Subtracting the above equality from (3.11), we obtain

$$
\begin{aligned}
\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u'_s(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u'_s(t)\}^+|\big)\, dS \\
+ \Big\langle r_{S_2}\{(\mathcal{T}U)_n\}^+ - f(t), r_{S_2}\{v_n - u'_n(t)\}^+ \Big\rangle_{S_2} + \Big\langle r_{S_2}\{MU\}^+ - \varphi(t), r_{S_2}\{w-\omega'(t)\}^+ \Big\rangle_{S_2} \geq 0 \quad (4.2)
\end{aligned}
$$

for all $V=(v,w)^\top \in \mathcal{K}_0$. For an arbitrary t from the interval $(0;T)$, we choose $V=(v,w)^\top \in \mathcal{K}_0$ such that $r_{S_2}\{w\}^+ = r_{S_2}\{\omega'(t)\}^+$, $r_{S_2}\{v_s\}^+ = r_{S_2}\{u'_s(t)\}^+$, and $r_{S_2}\{v_n\}^+ = r_{S_2}[\{u'_n(t)\}^+ \pm \psi]$, where $\psi \in \widetilde{H}^{1/2}(S_2)$ is an arbitrary scalar function. Then from (4.2) we infer

$$
r_{S_2}\{(\mathcal{T}U)_n\}^+ = f(t), \tag{4.3}
$$

i.e., condition (3.6) is fulfilled. Taking into account (4.3), from (4.2) we find that

$$
\begin{aligned}
\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u'_s(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u'_s(t)\}^+|\big)\, dS \\
+ \Big\langle r_{S_2}\{MU\}^+ - \varphi(t), r_{S_2}\{w-\omega'(t)\}^+ \Big\rangle_{S_2} \geq 0 \quad \forall\, V=(v,w)^\top \in \mathcal{K}_0. \quad (4.4)
\end{aligned}
$$

Let now the vector-function $V=(v,w)^\top \in \mathcal{K}_0$ be such that $r_{S_2}\{v_s\}^+ = r_{S_2}\{u'_s(t)\}^+$ and $r_{S_2}\{w\}^+ = r_{S_2}[\{\omega'(t)\}^+ \pm \psi]$, where $\psi \in [\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector-function. Then (4.4) yields

$$
r_{S_2}\{MU\}^+ = \varphi(t)\,. \tag{4.5}
$$

Consequently, condition (3.7) is satisfied. Note that conditions (3.5), (3.3) and (3.4) are automatically fulfilled, since $U=(u,\omega)^\top \in \mathcal{K}$. Taking into account condition (4.5), from (4.4) we obtain

$$
\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u'_s(t)\}^+ \Big\rangle_{S_2} + \int\limits_{S_2} g\big(|\{v_s\}^+| - |\{u'_s(t)\}^+|\big)\, dS \geq 0 \quad \forall\, V=(v,w)^\top \in \mathcal{K}_0, \tag{4.6}
$$

whence

$$\Big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s-u_s'(t)\}^+\Big\rangle_{S_2}+\int\limits_{S_2} g\big|\{v_s\}^+-\{u_s'(t)\}^+\big|\,dS\geq 0\quad \forall\, V=(v,w)^\top\in\mathcal{K}_0. \tag{4.7}$$

Further, let us choose the vector-function $V=(v,w)^\top\in\mathcal{K}_0$ such that $r_{S_2}\{w\}^+=r_{S_2}\{\omega'(t)\}^+$, $r_{S_2}\{v_n\}^+=r_{S_2}\{u_n'(t)\}^+$, and $r_{S_2}\{v_s\}^+=r_{S_2}\{u_s'(t)\}^+\pm r_{S_2}\psi_s$, where $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector-function. Then from (4.7) we obtain

$$\pm\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\big\rangle_{S_2}+\int\limits_{S_2} g|\psi_s|\,dS\geq 0. \tag{4.8}$$

For an arbitrary $\psi\in[\widetilde{H}^{1/2}(S_2)]^3$, we have $|r_{S_2}\psi_s|\leq|r_{S_2}\psi|$ and

$$\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\big\rangle_{S_2}=\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2}.$$

Therefore, from (4.8) we derive

$$\Big|\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2}\Big|\leq\int\limits_{S_2} g|\psi|\,dS\quad \forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3. \tag{4.9}$$

Let $t\in(0;T)$ and consider in the space $[\widetilde{H}^{1/2}(S_2)]^3$ the linear functional

$$\Phi_t(\psi)=\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\big\rangle_{S_2},\ \ \psi\in[\widetilde{H}^{1/2}(S_2)]^3.$$

Due to inequality (4.9), this functional is continuous on the space $[\widetilde{H}^{1/2}(S_2)]^3$ with respect to the topology induced by the space $[L_1(S_2)]^3$. Since the space $[\widetilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ_t can be continuously extended to the whole space $[L_1(S_2)]^3$ preserving the norm. Since the dual of $[L_1(S_2)]^3$ is isomorphic to $[L_\infty(S_2)]^3$, there exists a function $\Phi_t^*\in[L_\infty(S_2)]^3$ such that

$$\Phi_t(\psi)=\int\limits_{S_2}\Phi_t^*\cdot\psi\,dS\quad\forall\,\psi\in[L_1(S_2)]^3.$$

Hence

$$r_{S_2}\{(\mathcal{T}U)_s\}^+=\Phi_t^*\in[L_\infty(S_2)]^3.$$

Using again inequality (4.9) we derive

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}U)_s\}^+\cdot\psi-g|\psi|\big]\,dS\leq 0\quad\forall\,\psi\in[\widetilde{H}^{1/2}(S_2)]^3, \tag{4.10}$$

whence the inequality

$$\big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big|\leq g\ \text{ almost everywhere on }\ S_2\times(0;T)$$

follows. Indeed, it is well known that for an arbitrary essentially bounded function $\widetilde{\psi}\in L_\infty(S_2)$ there is a sequence $\widetilde{\varphi}_l\in C^\infty(S_2)$ with supports in S_2 for which (see [30, Lemma 1.4.2])

$$\lim_{l\to\infty}\widetilde{\varphi}_l(x)=\widetilde{\psi}(x)\ \text{ for almost all }\ x\in S_2\ \text{ and }\ |\widetilde{\varphi}_l(x)|\leq\operatorname*{ess\,sup}_{y\in S_2}|\widetilde{\psi}(y)|$$

for almost all $x\in S_2$. Therefore, from inequality (4.10), by the Lebesgue dominated convergence theorem, it follows that

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}U)_s\}^+\cdot\psi-g|\psi|\big]\,dS\leq 0\quad\forall\,\psi\in[L_\infty(S_2)]^3,$$

whence we get

$$\pm r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi - g|\psi| \le 0$$

on S_2 for every $\psi\in[L_\infty(S_2)]^3$. Substituting $\psi = r_{S_2}\{(\mathcal{T}U)_s\}^+$ in the above inequality, we finally get the inequality

$$\big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big| \le g. \tag{4.11}$$

Now let us set

$$\vartheta_s := r_{S_2}\{v_s\}^+, \quad \vartheta_{0s} := r_{S_2}\{u_s'(t)\}^+. \tag{4.12}$$

Clearly, $\vartheta_s, \vartheta_{0s}\in[H^{1/2}(S_2)]^3$. Due to the inclusion

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ \in \big[L_2(S_2\times(0;T))\big]^3,$$

from (4.6) we get

$$\big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+,\vartheta_s\big\rangle_{S_2} + \int\limits_{S_2} g|\vartheta_s|\,dS - \big\langle r_{S_2}\{(\mathcal{T}U)_s\}^+,\vartheta_{0s}\big\rangle_{S_2} - \int\limits_{S_2} g|\vartheta_{0s}|\,dS \ge 0. \tag{4.13}$$

Let $\psi\in[H^{1/2}(S_2)]^3$ be an arbitrary vector-function. Substitute in (4.13) $\vartheta_s = q\psi$ for a nonnegative number $q\ge 0$, and take into consideration that $|\psi_s|\le|\psi|$ and $r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi_s = r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot\psi$ to obtain

$$q\int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\psi + g|\psi|\,\big]\,dS - \int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\vartheta_{0s} + g|\vartheta_{0s}|\,\big]\,dS \ge 0.$$

Sending q to 0, we arrive at the inequality

$$\int\limits_{S_2}\big[\{(\mathcal{T}U)_s\}^+\cdot\vartheta_{0s} + g|\vartheta_{0s}|\,\big]\,dS \le 0,$$

whence by (4.11) and (4.12) we arrive at the equation

$$r_{S_2}\{(\mathcal{T}U)_s\}^+\cdot r_{S_2}\{u_s'(t)\}^+ + g\big|r_{S_2}\{u_s'(t)\}^+\big| = 0. \tag{4.14}$$

Clearly, if $|r_{S_2}\{(\mathcal{T}U)_s\}^+|<g$, then it follows from (4.14) that $r_{S_2}\{u_s'(t)\}^+=0$. But if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then (4.14) can be rewritten in the form

$$g\big|r_{S_2}\{u_s'(t)\}^+\big|(\cos\alpha+1) = 0 \ \text{ on } \ S_2\times(0;T),$$

where α is the angle lying between the vectors $r_{S_2}\{u_s'(t)\}^+$ and $r_{S_2}\{(\mathcal{T}U)_s\}^+$ at the point $x\in S_2$. Consequently, there exist the functions λ_1 and λ_2 such that $\lambda_1(x,t)+\lambda_2(x,t)>0$ and

$$\lambda_1(x,t)\,r_{S_2}\{u_s'(t)\}^+ = -\lambda_2(x,t)r_{S_2}\{(\mathcal{T}U)_s\}^+ \ \text{ on } \ S_2\times(0;T).$$

Moreover, we may assume that λ_1 belongs to the same class as $\{(\mathcal{T}U)_s\}^+$, while λ_2 belongs to the same class as $\{u_s'(t)\}^+$. This completes the proof. □

5 The uniqueness theorem

We start the investigation of the variational inequality (3.11) with the following uniqueness result.

Theorem 5.1. *The variational inequality* (3.11) *and hence Problem* (A_0) *have at most one weak solution.*

Proof. Let $U=(u,\omega)^\top\in\mathcal{K}$ and $\widetilde{U}=(\widetilde{u},\widetilde{\omega})^\top\in\mathcal{K}$ be two solutions of inequality (3.11). Substituting in (3.11) $\widetilde{U}'(t)$ instead of V, we obtain

$$\begin{aligned}\big(PU''(t),\widetilde{U}'(t)-U'(t)\big)+\mathcal{B}^{(0)}\big(U(t),\widetilde{U}'(t)-U'(t)\big)+\mathcal{B}^{(1)}\big(U'(t),\widetilde{U}'(t)-U'(t)\big)+j(\widetilde{U}'(t))-j(U'(t))\\ \geq(\mathcal{G}(t),\widetilde{U}'(t)-U'(t))+\int\limits_{S_2}f(t)\big\{\widetilde{u}_n'(t)-u_n'(t)\big\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{\widetilde{\omega}'(t)-\omega'(t)\}^+\big\rangle_{S_2}.\end{aligned}\tag{5.1}$$

Analogously, substituting $U(t)=\widetilde{U}(t)$ and $V=U'(t)$ in (3.11), we get

$$\begin{aligned}\big(P\widetilde{U}''(t),U'(t)-\widetilde{U}'(t)\big)+\mathcal{B}^{(0)}\big(\widetilde{U}(t),U'(t)-\widetilde{U}'(t)\big)+\mathcal{B}^{(1)}\big(\widetilde{U}'(t),U'(t)-\widetilde{U}'(t)\big)+j(U'(t))-j(\widetilde{U}'(t))\\ \geq(\mathcal{G}(t),U'(t)-\widetilde{U}'(t))+\int\limits_{S_2}f(t)\big\{u_n'(t)-\widetilde{u}_n'(t)\big\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{\omega'(t)-\widetilde{\omega}'(t)\}^+\big\rangle_{S_2}.\end{aligned}\tag{5.2}$$

Combining (5.1) and (5.2) and denoting the difference $U(t)-\widetilde{U}(t)$ by $W(t)$, we obtain

$$-(PW''(t),W'(t))-\mathcal{B}^{(0)}(W(t),W'(t))-\mathcal{B}^{(1)}(W'(t),W'(t))\geq 0,\tag{5.3}$$

Note that

$$(PW''(t),W'(t))=\frac{1}{2}\,\frac{d}{dt}\Big(\sqrt{P}\,W'(t),\sqrt{P}\,W'(t)\Big)=\frac{1}{2}\,\frac{d}{dt}\Big[\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}\Big]$$

and

$$\mathcal{B}^{(0)}(W(t),W'(t))=\frac{1}{2}\,\frac{d}{dt}\,\mathcal{B}^{(0)}(W(t),W(t)),$$

where $\sqrt{P}=[\sqrt{p_{ij}}]_{6\times6}$ with $\sqrt{p_{ii}}=\sqrt{\varrho}$ for $i=1,2,3$, $\sqrt{p_{ii}}=\sqrt{\mathcal{J}}$ for $i=4,5,6$, and $p_{ij}=0$ if $i\neq j$.

Then, from (5.3) we get

$$\frac{1}{2}\,\frac{d}{dt}\Big\{\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))\Big\}+\mathcal{B}^{(1)}(W'(t),W'(t))\leq 0.\tag{5.4}$$

Since $\mathcal{B}^{(1)}(W'(t),W'(t))$ is nonnegative, (5.4) can be rewritten as

$$\frac{1}{2}\,\frac{d}{dt}\Big\{\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))\Big\}\leq 0.\tag{5.5}$$

On the basis of (5.5), we can conclude that the scalar function

$$\big\|\sqrt{P}\,W'(t)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(W(t),W(t))$$

decreases on the interval $(0;T)$. Since $\mathcal{B}^{(0)}(W(t),W(t))\geq 0$ $\forall\,t\in(0;T)$ and $W(0)=W'(0)=0$, we see that $\mathcal{B}^{(0)}(W(t),W(t))=0$. Hence, by virtue of Lemma 2.2, we conclude that $W(t)=0$, which completes the proof. □

6 The existence results

The existence of a solution to the variational inequality (3.11) is obtained by the following scheme. First, we reduce the variational inequality (3.11) to an equivalent regularized variational equation depending on a small parameter ε whose solvability is studied by the Faedo–Galerkin approximation method. Then we establish some a priori estimates which allow us to pass to the limit with respect to the dimension m of the approximation space of test functions as $m\to+\infty$ and with respect to the parameter as $\varepsilon\to0$. We will show that the limiting function solves the variational inequality (3.11) and, consequently, by virtue of Theorem 4.1, it will be a solution of problem (A_0), as well. The assumptions which are to be satisfied by the data of problem (A_0) will be given below in the course of discussions and, finally, we will formulate the basic existence theorem.

6.1 Reduction to regularized variational equation

To reduce the variational inequality (3.11) to the regularized variational equation, we consider on the space $\mathcal{K}_0$ the convex differentiable functional

$$j_\varepsilon(V) = \int\limits_{S_2} g(x)\varphi_\varepsilon(|\{v_s\}^+|)\,dS, \quad V = (v,w)^\top \in \mathcal{K}_0, \tag{6.1}$$

where ε is an arbitrary positive number, $\varphi_\varepsilon : \mathbb{R} \to (0;\infty)$ is defined by

$$\varphi_\varepsilon(\lambda) = \sqrt{\lambda^2 + \varepsilon^2}\,,$$

g is defined by (3.1) and, in what follows, we assume that it does not depend on the time variable t. Denote by $\mathcal{K}_0'$ the dual space to $\mathcal{K}_0$ and by j_ε' the Gâteaux derivative of the functional (6.1). It is easy to show that for almost all t from the interval $(0;T)$,

$$j_\varepsilon' : \mathcal{K}_0 \to \mathcal{K}_0'$$

is given by

$$\big\langle j_\varepsilon'(V), U\big\rangle_{S_2} = \int\limits_{S_2} g(x)\,\frac{\{v_s\}^+ \cdot \{u_s\}^+}{\sqrt{|\{v_s\}^+|^2 + \varepsilon^2}}\,dS \quad \forall\, V = (v,w)^\top \in \mathcal{K}_0, \ \ \forall\, U = (u,\omega)^\top \in \mathcal{K}_0. \tag{6.2}$$

Consider the following regularized variational equation: Find $U_\varepsilon \in \mathcal{K}$ satisfying for almost all t from the interval $(0;T)$, the equation

$$(PU_\varepsilon''(t), V) + \mathcal{B}^{(0)}(U_\varepsilon(t), V) + \mathcal{B}^{(1)}(U_\varepsilon'(t), V) + \big\langle j_\varepsilon'(U_\varepsilon'(t)), V\big\rangle_{S_2} = \langle \Psi(t), V\rangle_{\mathcal{K}_0}, \tag{6.3}$$

where $V = (v,w)^\top \in \mathcal{K}_0$ and the linear functional $\Psi(t)$ is defined as

$$\langle \Psi(t), V\rangle_{\mathcal{K}_0} := (\mathcal{G}(t), V) + \int\limits_{S_2} f(t)\{v_n\}^+\,dS + \big\langle \varphi(t), r_{S_2}\{w\}^+\big\rangle_{S_2} \tag{6.4}$$

with $\mathcal{G}$, f, and φ involved in the formulation of Problem (A_0).

It can be easily shown that the variational inequality (3.11), in which U and j are replaced, respectively, by U_ε and j_ε, is equivalent to the regularized variational equation (6.3). Therefore, we investigate the regularized variational equation (6.3).

Since the space $\mathcal{K}_0$ is separable, there exists a countable basis $W_1, W_2, \dots, W_m, \dots$ in the sense that for every m the system of vectors $W_1, W_2, \dots, W_m$ is linearly independent and the space of all finite linear combinations is dense in $\mathcal{K}_0$. We denote by $\mathbf{W}_m := [W_1, W_2, \dots, W_m]$ the linear span of elements $W_1, W_2, \dots, W_m$.

Consider the auxiliary problem: Find a vector-function $U_{\varepsilon m} : (0;T) \to \mathbf{W}_m$ such that $U_{\varepsilon m}, U_{\varepsilon m}'$, $U_{\varepsilon m}'' \in L_\infty(0,T;\mathbf{W}_m)$ and the variational equation

$$(PU_{\varepsilon m}''(t), V) + \mathcal{B}^{(0)}(U_{\varepsilon m}(t), V) + \mathcal{B}^{(1)}(U_{\varepsilon m}'(t), V) + \big\langle j_\varepsilon'(U_{\varepsilon m}'(t)), V\big\rangle_{S_2} = \langle \Psi(t), V\rangle_{\mathcal{K}_0} \tag{6.5}$$

and the initial conditions

$$U_{\varepsilon m}(0) = 0, \tag{6.6}$$

$$U_{\varepsilon m}'(0) = 0 \tag{6.7}$$

are satisfied for almost all t from the interval $(0;T)$ and $\forall\, V \in \mathbf{W}_m$.

Let us look for a solution of the above problem in the form of a linear combination with unknown coefficients $C_{j\varepsilon m}(t)$:

$$U_{\varepsilon m}(t) = \sum_{j=1}^{m} C_{j\varepsilon m}(t) W_j. \tag{6.8}$$

Replace in (6.5) the test vector-function V by W_k and instead of $U_{\varepsilon m}$ substitute the above linear combination to obtain

$$\sum_{j=1}^{m}(PW_j,W_k)C''_{j\varepsilon m}(t)+\sum_{j=1}^{m}\mathcal{B}^{(0)}(W_j,W_k)C_{j\varepsilon m}(t)+\sum_{j=1}^{m}\mathcal{B}^{(1)}(W_j,W_k)C'_{j\varepsilon m}(t)$$
$$+\Big\langle j'_\varepsilon\Big(\sum_{j=1}^{m}C'_{j\varepsilon m}(t)W_j\Big),W_k\Big\rangle_{S_2}=\langle\Psi(t),W_k\rangle_{\mathcal{K}_0},\ \ k=1,2,\dots,m. \tag{6.9}$$

Introduce the notation:

$$\Phi_k(C'_{1\varepsilon m},\dots,C'_{m\varepsilon m}):=\Big\langle j'_\varepsilon\Big(\sum_{j=1}^{m}C'_{j\varepsilon m}(t)W_j\Big),W_k\Big\rangle_{S_2},\quad \Phi:=(\Phi_1,\dots,\Phi_m)^\top,$$
$$\mathcal{P}_k(t):=\big\langle\Psi(t),W_k\big\rangle_{\mathcal{K}_0},\ \ k=\overline{1,m},\quad \mathcal{P}:=(\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_m)^\top,$$
$$\mathcal{B}:=\big[(PW_j,W_k)\big]_{m\times m},\quad D^{(0)}:=\big[\mathcal{B}^{(0)}(W_j,W_k)\big]_{m\times m},$$
$$D^{(1)}:=\big[\mathcal{B}^{(1)}(W_j,W_k)\big]_{m\times m},\quad C_{\varepsilon m}(t):=\big(C_{1\varepsilon m}(t),C_{2\varepsilon m}(t),\dots,C_{m\varepsilon m}(t)\big)^\top.$$

System (6.9) can be then rewritten as

$$\mathcal{B}C''_{\varepsilon m}(t)+D^{(1)}\,C'_{\varepsilon m}(t)+D^{(0)}\,C_{\varepsilon m}(t)+\Phi(C'_{\varepsilon m}(t))=\mathcal{P}(t). \tag{6.10}$$

The initial conditions (6.6) and (6.7) result in

$$C_{\varepsilon m}(0)=C'_{\varepsilon m}(0)=0. \tag{6.11}$$

Note that $\det\mathcal{B}\neq 0$, since the system of vectors $W_1,W_2,\dots,W_m$ is linearly independent, and hence from (6.10) we get

$$C''_{\varepsilon m}(t)+\mathcal{B}^{-1}D^{(1)}C'_{\varepsilon m}(t)+\mathcal{B}^{-1}D^{(0)}\,C_{\varepsilon m}(t)+\mathcal{B}^{-1}\Phi(C'_{\varepsilon m}(t))=\mathcal{B}^{-1}\mathcal{P}(t). \tag{6.12}$$

To reduce system (6.12) to the normal type, we introduce the notation

$$S_{\varepsilon m}(t):=C'_{\varepsilon m}(t),\quad Y_{\varepsilon m}(t):=\big(S_{\varepsilon m}(t),C_{\varepsilon m}(t)\big)^\top$$

and

$$\mathcal{L}(t,Y_{\varepsilon m}):=\begin{bmatrix}\mathcal{B}^{-1}\mathcal{P}(t)-\mathcal{B}^{-1}\Phi(S_{\varepsilon m})-\mathcal{B}^{-1}D^{(1)}C'_{\varepsilon m}-\mathcal{B}^{-1}D^{(0)}C_{\varepsilon m}\\ S_{\varepsilon m}\end{bmatrix}_{2m\times 1}.$$

Then equation (6.12) and the initial conditions (6.11) take the form

$$Y'_{\varepsilon m}(t)=\mathcal{L}(t,Y_{\varepsilon m}),\quad Y_{\varepsilon m}(0)=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}_{2m\times 1}. \tag{6.13}$$

Let us show that the matrix function $\mathcal{L}$ is continuous with respect to the first argument t. To this end, we estimate the difference

$$\big|\mathcal{P}_k(t+\Delta t)-\mathcal{P}_k(t)\big|=\big|\big\langle\Psi(t+\Delta t)-\Psi(t),W_k\big\rangle_{\mathcal{K}_0}\big|$$
$$=\bigg|\big(\mathcal{G}(t+\Delta t)-\mathcal{G}(t),W_k\big)+\int_{S_2}\big(f(t+\Delta t)-f(t)\big)\{(\xi_k)_n\}^+\,dS+\big\langle\varphi(t+\Delta t)-\varphi(t),r_{S_2}\{\eta_k\}^+\big\rangle_{S_2}\bigg|$$
$$\leq\Big(\big\|\mathcal{G}(t+\Delta t)-\mathcal{G}(t)\big\|_{[L_2(\Omega)]^6}+\big\|f(t+\Delta t)-f(t)\big\|_{L_2(S_2)}$$
$$+\big\|\varphi(t+\Delta t)-\varphi(t)\big\|_{[H^{-1/2}(S_2)]^3}\Big)\|W_k\|_{[H^1(\Omega)]^6},$$

where $W_k = (\xi_k, \eta_k)^\top \in \mathcal{K}_0$.

In what follows, we assume that

$$\mathcal{G}, \mathcal{G}', \mathcal{G}'' \in L_2(0,T;[L_2(\Omega)]^6), \quad f \in L_\infty(S_2), \quad \varphi, \varphi', \varphi'' \in L_2(0,T;[H^{-1/2}(S_2)]^3). \tag{6.14}$$

Note that the further analysis of the problem shows that g cannot be dependent on t, and hence f also cannot be dependent on t. Assumptions $\mathcal{G}$, f, and φ are continuously differentiable with respect to t almost everywhere in the interval $(0;T)$, and hence $|\mathcal{P}_k(t+\Delta t) - \mathcal{P}_k(t)| \to 0$ as $\Delta t \to 0$, implying that the function $\mathcal{L}$ is continuous with respect to the first argument.

To prove the continuity of the function $\mathcal{L}$ with respect to $Y_{\varepsilon m}$, it suffices to consider only the term $\Phi(S_{\varepsilon m})$. By formula (6.2), we have

$$\Phi_k(S_{\varepsilon m}) = \Big\langle j'_\varepsilon\Big(\sum_{j=1}^m S_{j\varepsilon m} W_j\Big), W_k\Big\rangle_{S_2} = \int_{S_2} g(x) \frac{\Big(\sum\limits_{j=1}^m S_{j\varepsilon m}\{(\xi_j)_s\}^+\Big)\cdot\{(\xi_k)_s\}^+}{\sqrt{\Big|\sum\limits_{j=1}^m S_{j\varepsilon m}\{(\xi_j)_s\}^+\Big|^2 + \varepsilon^2}}\, dS.$$

It is easily seen that Φ_k is continuous and continuously differentiable with respect to the variables $S_{j\varepsilon m}$. Moreover, Φ_k and its derivatives with respect to $S_{j\varepsilon m}$ are bounded by an absolute constant depending on ε. Therefore, the function $\mathcal{L}$ satisfies the Lipschitz condition in the second argument. Consequently, system (6.13) possesses at most one solution.

Any vector function $Y_{\varepsilon m}$ that is a solution to problem (6.13) possesses second order continuous derivatives with respect to t. The same is valid for $U_{\varepsilon m}(t)$ defined by formula (6.8) with $C_{j\varepsilon m}(t)$, being a solution of problem (6.13). It can be shown that $U_{\varepsilon m}(t)$ possesses actually continuous third order derivatives with respect to t and solves problem (6.5)–(6.7).

In the next subsections we derive some a priori estimates which we need to perform the limiting procedure with respect to the dimension m.

6.2 A priori estimates I

Insert the solution of system (6.13) in (6.8) and then substitute $U'_{\varepsilon m}(t)$ instead of V into (6.5) to obtain

$$\begin{aligned}(PU''_{\varepsilon m}(t), U'_{\varepsilon m}(t)) &+ \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U'_{\varepsilon m}(t)) \\ &+ \mathcal{B}^{(1)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t)) + \big\langle j'_\varepsilon(U'_{\varepsilon m}(t)), U'_{\varepsilon m}(t)\big\rangle_{S_2} = \big\langle \Psi(t), U'_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.\end{aligned}$$

Since

$$\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)), U'_{\varepsilon m}(t)\big\rangle_{S_2} = \int_{S_2} g(x)\, \frac{|\{(u'_{\varepsilon m}(t))_s\}^+|^2}{\sqrt{|\{(u'_{\varepsilon m}(t))_s\}^+|^2 + \varepsilon^2}}\, dS \geq 0$$

and $\mathcal{B}^{(1)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t)) \geq 0$, from the preceding equality we have

$$\frac{d}{dt}\Big\{\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U_{\varepsilon m}(t))\Big\} \leq 2\big\langle \Psi(t), U'_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.$$

Consequently, due to the homogeneous initial conditions, we arrive at the inequality

$$\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + \mathcal{B}^{(0)}(U_{\varepsilon m}(t), U_{\varepsilon m}(t)) \leq 2\int_0^t \big\langle \Psi(\sigma), U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma.$$

By virtue of (2.4), we get

$$\big\|\sqrt{P}\, U'_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} + C_1\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \leq C_2\|U_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} + 2\int_0^t \big\langle \Psi(\sigma), U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\, d\sigma \tag{6.15}$$

with C_1 and C_2 from (2.4). Since $U_{\varepsilon m}(0)=0$, we can write

$$U_{\varepsilon m}(t)=\int\limits_0^t U'_{\varepsilon m}(\sigma)\,d\sigma,$$

whence

$$\|U_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}\leq\int\limits_0^t \|U'_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}\,d\sigma. \tag{6.16}$$

For the last term in (6.15) we have

$$\begin{aligned}2\int\limits_0^t \big\langle\Psi(\sigma),U'_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\,d\sigma &= 2\big\langle\Psi(t),U_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}-2\int\limits_0^t \big\langle\Psi'(\sigma),U_{\varepsilon m}(\sigma)\big\rangle_{\mathcal{K}_0}\,d\sigma\\ &\leq \frac{1}{\delta}\,\|\Psi(t)\|^2_{\mathcal{K}'_0}+\delta\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}+\int\limits_0^t \big(\|\Psi'(\sigma)\|^2_{\mathcal{K}'_0}+\|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\big)\,d\sigma\\ &\leq C_3+\delta\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}+\int\limits_0^t \|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\,d\sigma. \end{aligned}\tag{6.17}$$

Taking into account estimates (6.16) and (6.17) and choosing δ in inequality (6.17) smaller than C_1 from (6.15), we finally get

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\leq C_4\int\limits_0^t \big(\|U'_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\big)\,d\sigma+C_5$$

with some constants C_4 and C_5 independent of m and ε. Now, by using Gronwall's lemma, we obtain

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6}+\|U_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\leq C \tag{6.18}$$

with the constant C independent of m and ε.

6.3 A priori estimates II

Differentiating (6.5) with respect to t and replacing V with the vector-function $U''_{\varepsilon m}(t)$, we obtain

$$\begin{aligned}&(PU'''_{\varepsilon m}(t),U''_{\varepsilon m}(t))+\mathcal{B}^{(0)}(U'_{\varepsilon m}(t),U''_{\varepsilon m}(t))\\ &\qquad+\mathcal{B}^{(1)}(U''_{\varepsilon m}(t),U''_{\varepsilon m}(t))+\Big\langle\frac{d}{dt}\,j'_\varepsilon(U'_{\varepsilon m}(t)),U''_{\varepsilon m}(t)\Big\rangle_{S_2}=\big\langle\Psi'(t),U''_{\varepsilon m}(t)\big\rangle_{\mathcal{K}_0}.\end{aligned}\tag{6.19}$$

Due to formula (6.2), for every $W=(\xi,\eta)^\top\in\mathcal{K}_0$ and $V=(v,w)^\top\in\mathcal{K}_0$, we have

$$\langle j'_\varepsilon(W(t)),V\rangle_{S_2}=\int\limits_{S_2} g(x)Q_\varepsilon(\xi_s(t))\cdot\{v_s\}^+\,dS, \tag{6.20}$$

where

$$Q_\varepsilon\big(\xi_s(t)\big):=\frac{r_{S_2}\{\xi_s(t)\}^+}{\sqrt{|r_{S_2}\{\xi_s(t)\}^+|^2+\varepsilon^2}}\,.$$

Equality (6.20) yields

$$\Big\langle\frac{d}{dt}\,j'_\varepsilon(W(t)),V\Big\rangle_{S_2}=\int\limits_{S_2} g(x)\lim_{h\to 0}\frac{1}{h}\,\big[Q_\varepsilon(\xi_s(t+h))-Q_\varepsilon(\xi_s(t))\big]\cdot\{v_s\}^+\,dS.$$

Replace here V by the vector-function $W'(t)$, then

$$\Big\langle \frac{d}{dt}\, j'_\varepsilon(W(t)), W'(t) \Big\rangle_{S_2} = \int\limits_{S_2} g(x) \lim_{h\to 0} \frac{1}{h} \left[Q_\varepsilon(\xi_s(t+h)) - Q_\varepsilon(\xi_s(t)) \right] \cdot \frac{1}{h} \left\{ \xi_s(t+h) - \xi_s(t) \right\}^+ dS.$$

Since j_ε is a convex differentiable functional on $\mathcal{K}_0$, the operator $j'_\varepsilon : \mathcal{K}_0 \to \mathcal{K}'_0$ is monotone and we have

$$\begin{aligned} 0 &\le \Big\langle j'_\varepsilon(W(t+h)) - j'_\varepsilon(W(t)), W(t+h) - W(t) \Big\rangle_{S_2} \\ &= \int\limits_{S_2} g(x) Q_\varepsilon(\xi_s(t+h)) \cdot \{\xi_s(t+h) - \xi_s(t)\}^+ dS + \int\limits_{S_2} g(x) Q_\varepsilon(\xi_s(t)) \cdot \big\{ \xi_s(t) - \xi_s(t+h) \big\}^+ dS \\ &= \int\limits_{S_2} g(x) \big[Q_\varepsilon(\xi_s(t+h)) - Q_\varepsilon(\xi_s(t)) \big] \cdot \big\{ \xi_s(t+h) - \xi_s(t) \big\}^+ dS. \end{aligned}$$

Thus we obtain

$$\Big\langle \frac{d}{dt}\, j'_\varepsilon(W(t)), W'(t) \Big\rangle_{S_2} \ge 0. \tag{6.21}$$

Taking into account (6.21), it follows from (6.19) that

$$(PU'''_{\varepsilon m}(t), U''_{\varepsilon m}(t)) + \mathcal{B}^{(0)}(U'_{\varepsilon m}(t), U''_{\varepsilon m}(t)) + \mathcal{B}^{(1)}(U''_{\varepsilon m}(t), U''_{\varepsilon m}(t)) \le \big\langle \Psi'(t), U''_{\varepsilon m}(t) \big\rangle_{\mathcal{K}_0},$$

whence, since $\mathcal{B}^{(1)}(U''_{\varepsilon m}(t), U''_{\varepsilon m}(t))$ is nonnegative, we have

$$\frac{1}{2} \frac{d}{dt} \Big\{ \big\| \sqrt{P}\, U''_{\varepsilon m}(t) \big\|^2_{[L_2(\Omega)]^6} + B^{(0)}(U'_{\varepsilon m}(t), U'_{\varepsilon m}(t)) \Big\} \le \big\langle \Psi'(t), U''_{\varepsilon m}(t) \big\rangle_{\mathcal{K}_0}.$$

Using (2.4) and the homogeneous initial condition (6.7), by the integration of the foregoing formula we get

$$\begin{aligned} &\big\| \sqrt{P}\, U''_{\varepsilon m}(t) \big\|^2_{[L_2(\Omega)]^6} + C_1 \| U'_{\varepsilon m}(t) \|^2_{[H^1(\Omega)]^6} \\ &\qquad \le C_2 \| U'_{\varepsilon m}(t) \|^2_{[L_2(\Omega)]^6} + \big\| \sqrt{P}\, U''_{\varepsilon m}(0) \big\|^2_{[L_2(\Omega)]^6} + 2 \int\limits_0^t \big\langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma) \big\rangle_{\mathcal{K}_0} d\sigma \end{aligned} \tag{6.22}$$

with C_1 and C_2 from (2.4). Since

$$\int\limits_0^t \big\langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma) \big\rangle_{\mathcal{K}_0} d\sigma = \big\langle \Psi'(t), U'_{\varepsilon m}(t) \big\rangle_{\mathcal{K}_0} - \int\limits_0^t \big\langle \Psi''(\sigma), U'_{\varepsilon m}(\sigma) \big\rangle_{\mathcal{K}_0} d\sigma, \tag{6.23}$$

using the inclusions (6.14), we infer that $\Psi'' \in L_2(0,T;\mathcal{K}'_0)$, and hence for an arbitrary positive δ it follows from (6.23) that

$$\begin{aligned} \int\limits_0^t \langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma) \rangle_{\mathcal{K}_0} d\sigma &\le \frac{1}{2\delta} \|\Psi'(t)\|^2_{\mathcal{K}'_0} + \frac{\delta}{2} \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \\ &\quad + C_3 \int\limits_0^t \|\Psi''(\sigma)\|^2_{\mathcal{K}'_0} d\sigma + C_4 \int\limits_0^t \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6} d\sigma. \end{aligned} \tag{6.24}$$

Taking now into account the inequality

$$\|\Psi'(t)\|^2_{\mathcal{K}'_0} \le 2 \int\limits_0^t \|\Psi''(\sigma)\|^2_{\mathcal{K}'_0} d\sigma + 2 \|\Psi'(0)\|^2_{\mathcal{K}'_0} \le C_5,$$

from (6.24) we get

$$\int_0^t \langle \Psi'(\sigma), U''_{\varepsilon m}(\sigma)\rangle_{\mathcal{K}_0}\, d\sigma \le C_6 + \frac{\delta}{2}\|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} + C_4 \int_0^t \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\, d\sigma. \tag{6.25}$$

Choosing δ sufficiently small and taking into account estimates (6.25) and

$$\|U'_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} \le \int_0^t \|U''_{\varepsilon m}(\sigma)\|^2_{[L_2(\Omega)]^6}\, d\sigma,$$

from (6.22) we derive

$$\begin{aligned}\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} &+ \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\\ &\le C_7\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} + C_8\int_0^t \Big[\big\|\sqrt{P}\,U''_{\varepsilon m}(\sigma)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\Big]\, d\sigma + C_9.\end{aligned} \tag{6.26}$$

Let us now estimate $\|\sqrt{P}U''_{\varepsilon m}(0)\|_{[L_2(\Omega)]^6}$. Substituting $t = 0$ in (6.5), we obtain

$$(PU''_{\varepsilon m}(0), V) = \langle \Psi(0), V\rangle_{\mathcal{K}_0} \quad \forall\, V \in \mathbf{W}_m, \tag{6.27}$$

where, in view of (6.4),

$$\langle \Psi(0), V\rangle_{\mathcal{K}_0} = (\mathcal{G}(0), V) + \int_{S_2} f(0)\{v_n\}^+\, dS + \langle \varphi(0), r_{S_2}\{w\}^+\rangle_{S_2}.$$

Here we formulate one more restriction on the data of the problem: we assume that there exists a vector-function $U_0 \in [L_2(\Omega)]^6$ such that

$$\langle \Psi(0), V\rangle_{\mathcal{K}_0} = (U_0, V) \quad \forall\, V \in \mathcal{K}_0. \tag{6.28}$$

Note that if $\varphi \in L_2(0,T;[L_2(S_2)]^3)$, then (6.28) holds.

Since $U''_{\varepsilon m}(0) \in \mathbf{W}_m$, we can take $U''_{\varepsilon m}(0)$ instead of V in (6.27) and, using (6.28), we arrive at the inequality

$$\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} = (U_0, U''_{\varepsilon m}(0)) \le \|U_0\|_{[L_2(\Omega)]^6}\|U''_{\varepsilon m}(0)\|_{[L_2(\Omega)]^6},$$

whence

$$\big\|\sqrt{P}\,U''_{\varepsilon m}(0)\big\|^2_{[L_2(\Omega)]^6} \le C_{10}$$

with C_{10} independent of ε and m. Therefore (6.26) takes the form

$$\begin{aligned}\big\|\sqrt{P}\,U''_{\varepsilon m}(t)\big\|^2_{[L_2(\Omega)]^6} &+ \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6}\\ &\le C_{11} + C_{12}\int_0^t \Big[\big\|\sqrt{P}\,U''_{\varepsilon m}(\sigma)\big\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(\sigma)\|^2_{[H^1(\Omega)]^6}\Big]\, d\sigma.\end{aligned}$$

Using again Gronwall's lemma, we find

$$\|U''_{\varepsilon m}(t)\|^2_{[L_2(\Omega)]^6} + \|U'_{\varepsilon m}(t)\|^2_{[H^1(\Omega)]^6} \le C, \tag{6.29}$$

where C does not depend on ε and m.

6.4 The basic existence theorem

First, we pass to the limit with respect to the dimension m. The estimates (6.18) and (6.29) show that $U_{\varepsilon m}$ and $U'_{\varepsilon m}$ (respectively, $U''_{\varepsilon m}$) are bounded by the constants independent of ε and m in the space $L_\infty(0,T;\mathcal{K}_0)$ (respectively, in the space $L_\infty(0,T;[L_2(\Omega)]^6)$. Thus we can choose from the sequence $U_{\varepsilon m}$ a subsequence, which we again denote by $U_{\varepsilon m}$, such that

$$\begin{aligned} &U_{\varepsilon m} \to U_\varepsilon \;\; *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0) \text{ as } m\to\infty,\\ &U'_{\varepsilon m} \to U'_\varepsilon \;\; *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0) \text{ as } m\to\infty,\\ &U''_{\varepsilon m} \to U''_\varepsilon \;\; *\text{-weakly in } L_\infty(0,T;[L_2(\Omega)]^6) \text{ as } m\to\infty. \end{aligned} \tag{6.30}$$

Let us show that the limiting function U_ε satisfies the regularized variational equation (6.3) with the homogeneous initial conditions for $t=0$. We proceed as follows. Let $\vartheta_j\in C^1([0;T])$, $\vartheta_j(T)=0$, $j=\overline{1,\infty}$, be smooth scalar functions and consider the vector-function $\Phi(t)=\sum\limits_{j=1}^{m_0}\vartheta_j(t)W_j$ with a natural number m_0. It is easy to see that $\Phi\in\mathbf{W}_m$ for every $m\geq m_0$ and $\forall\, t\in[0;T]$ and, consequently, from (6.5) we have

$$\begin{aligned} &(PU''_{\varepsilon m}(t),\Phi(t))+\mathcal{B}^{(0)}(U_{\varepsilon m}(t),\Phi(t))\\ &\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon m}(t),\Phi(t))+\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)),\Phi(t)\big\rangle_{S_2}=\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}. \end{aligned} \tag{6.31}$$

Integrate (6.31) with respect to t from 0 to T,

$$\begin{aligned} &\int_0^T\Big[(PU''_{\varepsilon m}(t),\Phi(t))+\mathcal{B}^{(0)}(U_{\varepsilon m}(t),\Phi(t))\\ &\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon m}(t),\Phi(t))+\big\langle j'_\varepsilon(U'_{\varepsilon m}(t)),\Phi(t)\big\rangle_{S_2}\Big]\,dt=\int_0^T\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\,dt. \end{aligned}$$

Taking now into account (6.30) and passing to the limit in the last equality as $m\to\infty$, we get

$$\begin{aligned} &\int_0^T\Big[(PU''_{\varepsilon}(t),\Phi(t))+\mathcal{B}^{(0)}(U_{\varepsilon}(t),\Phi(t))\\ &\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon}(t),\Phi(t))+\big\langle j'_\varepsilon(U'_{\varepsilon}(t)),\Phi(t)\big\rangle_{S_2}\Big]\,dt=\int_0^T\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\,dt. \end{aligned} \tag{6.32}$$

Since the finite linear combinations $\sum\limits_j\vartheta_j(t)W_j$ are dense in $\mathcal{K}_0$ for every $t\in[0;T]$, equality (6.32) allows us to conclude that

$$\begin{aligned} &\int_0^T\Big[(PU''_{\varepsilon}(t),V)+\mathcal{B}^{(0)}(U_{\varepsilon}(t),V)\\ &\qquad +\mathcal{B}^{(1)}(U'_{\varepsilon}(t),V)+\big\langle j'_\varepsilon(U'_{\varepsilon}(t)),V\big\rangle_{S_2}-\big\langle\Psi(t),\Phi(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt=0\quad\forall\,V\in\mathcal{K}_0. \end{aligned} \tag{6.33}$$

To obtain equality (6.3), it remains to derive a pointwise equation from the integral equality (6.33). To this end, we take an arbitrary fixed number $\tau\in(0;T)$ and an arbitrary vector-function $W\in\mathcal{K}_0$. Consider the family of neighborhoods of the point τ,

$$\Gamma_k=\Big(\tau-\frac{1}{k}\,,\;\tau+\frac{1}{k}\Big),$$

and define the function $V(t)$ as follows:

$$V(t)=\begin{cases}0, & \text{if } t\notin\Gamma_k,\\ W, & \text{if } t\in\Gamma_k.\end{cases}$$

Denoting the measure of Γ_k by $|\Gamma_k|$, from (6.33) we find that

$$\bigg(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} PU_\varepsilon''(t)\,dt,W\bigg)+\mathcal{B}^{(0)}\bigg(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U_\varepsilon(t)\,dt,W\bigg)+\mathcal{B}^{(1)}\bigg(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U_\varepsilon'(t)\,dt,W\bigg)\\ +\Big\langle j_\varepsilon'\bigg(\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k} U_\varepsilon'(t)\,dt\bigg),W\Big\rangle_{S_2}-\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k}\big\langle\Psi(t),W\big\rangle_{\mathcal{K}_0}\,dt=0. \quad (6.34)$$

According to the Lebesgue theorem, since

$$\frac{1}{|\Gamma_k|}\int\limits_{\Gamma_k}\psi(t)\,dt\longrightarrow\psi(\tau)\ \text{ as }\ k\to\infty$$

for almost all τ, it follows from (6.34) that

$$(PU_\varepsilon''(\tau),W)+\mathcal{B}^{(0)}(U_\varepsilon(\tau),W)+\mathcal{B}^{(1)}(U_\varepsilon'(\tau),W)+\big\langle j_\varepsilon'(U_\varepsilon'(\tau)),W\big\rangle_{S_2}=\big\langle\Psi(\tau),W\big\rangle_{\mathcal{K}_0}\quad\forall\,W\in\mathcal{K}_0,$$

that is, the limiting function U_ε satisfies the regularized variational equation (6.3). As for the initial conditions for $t=0$, we notice that the conditions (6.30) allow us to conclude that $U_\varepsilon(t)$ and $U_\varepsilon'(t)$ are the continuous mappings of the interval $[0;T]$ onto $\mathcal{K}_0$. Thus $U_\varepsilon(0)$ and $U_\varepsilon'(0)$ are well defined and, in view of (6.30), we see that $U_{\varepsilon m}(0)$ and $U_{\varepsilon m}'(0)$ converge weakly in $\mathcal{K}_0$ to $U_\varepsilon(0)$ and $U_\varepsilon'(0)$, respectively. Since $U_{\varepsilon m}(0)=0$ and $U_{\varepsilon m}'(0)=0$, we can show that $U_\varepsilon(0)=0$ and $U_\varepsilon'(0)=0$, i.e., the initial conditions are fulfilled.

It remains to pass to the limit in equality (6.3) with respect to the parameter ε. Repeating the arguments applied above, we can derive the estimate

$$\|U_\varepsilon(t)\|_{[H^1(\Omega)]^6}+\|U_\varepsilon'(t)\|_{[H^1(\Omega)]^6}+\|U_\varepsilon''(t)\|_{[L_2(\Omega)]^6}\le C$$

with the constant C independent of ε. Thus from the sequence $\{U_\varepsilon(t)\}$ we can choose a subsequence, which we denote again by $\{U_\varepsilon\}$, such that

$$\begin{aligned}&U_\varepsilon\to U\ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0)\ \text{ as }\ \varepsilon\to0,\\ &U_\varepsilon'\to U'\ \ *\text{-weakly in } L_\infty(0,T;\mathcal{K}_0)\ \text{ as }\ \varepsilon\to0,\\ &U_\varepsilon''\to U''\ \ *\text{-weakly in } L_\infty(0,T;[L_2(\Omega)]^6)\ \text{ as }\ \varepsilon\to0.\end{aligned}$$

Let us show that the limiting function U satisfies the variational inequality (3.11). Replacing in (6.3) V by the vector-function $W-U_\varepsilon'(t)$, where $W\in\mathcal{K}_0$ is arbitrary, we have

$$(PU_\varepsilon''(t),W-U_\varepsilon'(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),W-U_\varepsilon'(t))\\ +\mathcal{B}^{(1)}(U_\varepsilon'(t),W-U_\varepsilon'(t))+j_\varepsilon(W)-j_\varepsilon(U_\varepsilon'(t))-\big\langle\Psi(t),W-U_\varepsilon'(t)\big\rangle_{\mathcal{K}_0}\\ =j_\varepsilon(W)-j_\varepsilon(U_\varepsilon'(t))-\big\langle j_\varepsilon'(U_\varepsilon'(t)),W-U_\varepsilon'(t)\big\rangle_{S_2}\quad\forall\,W\in\mathcal{K}_0. \quad (6.35)$$

The right-hand side of the above inequality is non-negative. Indeed, since the functional j_ε is convex, we find that

$$j_\varepsilon(W)-j_\varepsilon(U_\varepsilon'(t))-\big\langle j_\varepsilon'(U_\varepsilon'(t)),W-U_\varepsilon'(t)\big\rangle_{S_2}\\ =j_\varepsilon(W)-j_\varepsilon(U_\varepsilon'(t))-\lim_{h\to0}\frac{1}{h}\big[j_\varepsilon(hW+(1-h)U_\varepsilon'(t))-j_\varepsilon(U_\varepsilon'(t))\big]\ge0.$$

Taking into account the last inequality, from (6.35) we have

$$\int_0^T \Big[(PU''_\varepsilon(t),W)+\mathcal{B}^{(0)}(U_\varepsilon(t),W)+\mathcal{B}^{(1)}(U'_\varepsilon(t),W)+j_\varepsilon(W)-\big\langle\Psi(t),W-U'_\varepsilon(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt$$
$$\geq\int_0^T\Big[(PU''_\varepsilon(t),U'_\varepsilon(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),U'_\varepsilon(t))+\mathcal{B}^{(1)}(U'_\varepsilon(t),U'_\varepsilon(t))+j_\varepsilon(U'_\varepsilon(t))\Big]\,dt.$$

On the other hand, the equality

$$\int_0^T\Big[(PU''_\varepsilon(t),U'_\varepsilon(t))+\mathcal{B}^{(0)}(U_\varepsilon(t),U'_\varepsilon(t))+\mathcal{B}^{(1)}(U'_\varepsilon(t),U'_\varepsilon(t))+j_\varepsilon(U'_\varepsilon(t))\Big]\,dt$$
$$=\frac{1}{2}\Big[\big\|\sqrt{P}\,U'_\varepsilon(T)\big\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(U_\varepsilon(T),U_\varepsilon(T))\Big]+\int_0^T\big[\mathcal{B}^{(1)}(U'_\varepsilon(t),U'_\varepsilon(t))+j_\varepsilon(U'_\varepsilon(t))\big]\,dt$$

with the help of the inequality

$$\liminf_{\varepsilon\to0}\mathcal{B}^{(0)}(U_\varepsilon(T),U_\varepsilon(T))\geq\mathcal{B}^{(0)}(U(T),U(T))$$

leads to the inequality

$$\int_0^T\Big[(PU''(t),W-U'(t))+\mathcal{B}^{(0)}(U(t),W-U'(t))+\mathcal{B}^{(1)}(U'(t),W-U'(t))$$
$$+j(W)-j(U'(t))-\big\langle\Psi(t),W-U'(t)\big\rangle_{\mathcal{K}_0}\Big]\,dt\geq0\quad\forall\,W\in\mathcal{K}_0.\qquad(6.36)$$

From the integral relation (6.36) we can derive as above the pointwise inequality

$$(PU''(t),W-U'(t))+\mathcal{B}^{(0)}(U(t),W-U'(t))$$
$$+\mathcal{B}^{(1)}(U'(t),W-U'(t))+j(W)-j(U'(t))-\big\langle\Psi(t),W-U'(t)\big\rangle_{\mathcal{K}_0}\geq0\quad\forall\,W\in\mathcal{K}_0,$$

and by an analogous reasoning we conclude that the homogeneous initial conditions are fulfilled. Thus we have proved the following existence theorem.

Theorem 6.1. *Let conditions* (6.14) *be fulfilled, g be independent of t, and let there exist a vector-function $U_0\in[L_2(\Omega)]^6$ such that*

$$(U_0,V)=\big(\mathcal{G}(0),V\big)+\int_{S_2}f(0)\,\{v_n\}^+\,dS+\big\langle\varphi(0),r_{S_2}\{w\}^+\big\rangle_{S_2}\quad\forall\,V=(v,w)^\top\in\mathcal{K}_0.$$

Then there exists one and only one function $U\in\mathcal{K}$ which is a solution of the variational inequality (3.11) *and, according to Theorem* 4.1*, it is a solution of problem* (A_0)*, as well.*

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Author's addresses:

1. Andrea Razmadze Mathematical Institute of Ivane Javakishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

2. Department of Mathematics, Georgian Technical University, 77 M. Kostava Str., Tbilisi 0171, Georgia.

E-mail: r.gachechiladze@yahoo.com