Memoirs on Differential Equations and Mathematical Physics

Volume 77, 2019, 43–57

Vyacheslav M. Evtukhov, Natalia V. Sharay

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THIRD-ORDER DIFFERENTIAL EQUATIONS WITH RAPIDLY VARYING NONLINEARITIES

Abstract. We obtain the existence conditions and asymptotic, as $t \uparrow \omega$ ($\omega \leq +\infty$), representations of one class of solutions of a binomial nonautonomous third-order differential equation with rapidly varying nonlinearity and their derivatives of the first and second order.

2010 Mathematics Subject Classification. 34D05, 34C11.

Key words and phrases. Nonlinear third-order differential equation, rapidly changing functions, asymptotic behavior of $P_\omega(Y_0, \lambda_0)$-solutions.

რეზიუმე. მესამე რიგის დიფერენციალური განტოლებისთვის სწრაფად ცვალებადი არაწრფივობით მიღებულია არსებობის პირობები და ასიმპტოტიკური წარმოდგენები ბინომიალური არაავტონომიური ამონახსნების ერთი კლასისთვის და მათი პირველი და მეორე რიგის წარმოებულებისთვის.

1 Introduction

Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{1.2}$$

Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of the point Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ as } \ y \in \Delta_{Y_0}$$

and conditions (1.2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty. \tag{1.3}$$

Hence, in the equation under consideration, the function φ and its first-order derivative are (see [10, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92]) rapidly varying as $y \to Y_0$.

The asymptotic properties of solutions of binomial second-order differential equations with nonlinearities satisfying condition (1.2) were studied in the works of M. Marić [10], V. M. Evtukhov and his students: N. G. Drik, V. M. Kharkov, A. G. Chernikova [4–6]. Moreover, in the monograph by M. Marić [10, Chapter 3, Section 3.4, pp. 90–99] in the particular case, where $\alpha_0 = 1$, $\omega = +\infty$, $Y_0 = 0$ and p is a properly varying function as $t \to +\infty$, the asymptotic representations of solutions that tend to zero as $t \to +\infty$ were obtained.

In the paper by V. M. Evtukhov and N. G. Drik [5], a special case, where $\varphi(y) = e^{\sigma y}$, $\sigma \neq 0$, was considered.

In [6], V. M. Evtukhov and V. M. Kharkov investigated a class of solutions, which is determined by using the function $\varphi(y)$.

In the paper by V. M. Evtukhov and A. G. Chernikova [4], for the second-order differential equation (1.1) in case φ is a rapidly varying function as $t \to +\infty$, the asymptotic properties of the so-called $P_\omega(Y_0, \lambda_0)$-solutions were completely investigated. It seems natural to try to extend these results to the third-order differential equations.

It should be noted that the results obtained by V. M. Evtukhov and V. N. Shinkarenko [9] on the asymptotic behavior of such solutions of differential equations of higher than the second order in the case, where $\varphi(y) = e^{\sigma y}$, $\sigma \neq 0$, are known.

Definition 1.1. A solution y of the differential equation (1.1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions

$$y(t) \in \Delta_{Y_0} \ \text{ as } \ t \in [t_0, \omega[, \quad \lim_{t\uparrow\omega} y(t) = Y_0,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad k = 1, 2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The aim of the present paper is to obtain the necessary and sufficient existence conditions of $P_\omega(Y_0, \lambda_0)$-solutions of equation (1.1) in a non-particular case, where $\lambda_0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}\}$, as well as asymptotic, as $t \uparrow \omega$, representations of such solutions and their derivatives of order up to two.

2 Functions from the Γ, $\Gamma_{Y_0}(Z_0)$ classes and their asymptotic properties

Without loss of generality, we will further assume that

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[\,, & \text{if } \ \Delta_{Y_0} \ \text{ is a left neighborhood of the point } Y_0, \\ ]Y_0, y_0], & \text{if } \ \Delta_{Y_0} \ \text{ is a right neighborhood of the point } Y_0, \end{cases} \tag{2.1}$$

where $y_0 \in \mathbb{R}$ such that $|y_0| < 1$ as $Y_0 = 0$ and $y_0 > 1$ ($y_0 < -1$) as $Y_0 = +\infty$ (as $Y_0 = -\infty$).

The function $f : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$ satisfying condition (1.2), as $Y_0 = \pm\infty$, and $\lim\limits_{y \to +\infty} f(y) = +\infty$, belongs to the class Γ introduced by L. Khan (see [1, Chapter 3, p. 3.10, p. 175]).

Definition 2.1. The class Γ consists of measurable nondecreasing and right continuous functions $f : [y_0, +\infty[\to]0, +\infty[\,$, for each of which there is a measurable function $g : [y_0, +\infty[\to]0, +\infty[\,$, which complements the function f, such that

$$\lim_{y \to +\infty} \frac{f(y + ug(y))}{f(y)} = e^u \ \text{ for any } \ u \in \mathbb{R}.$$

In [9], the asymptotic properties of functions from this class were investigated in sufficient detail.

Using the change of variables, the class Γ in the paper by of V. M. Evtukhov and A. G. Chernikova [4] was extended to the class $\Gamma_{Y_0}(Z_0)$ of functions $f : \Delta_{Y_0} \to]0, +\infty[\,$, where Y_0 is equal either to zero or to $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of the point Y_0, for which

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = Z_0 = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty \end{cases}$$

Definition 2.2. We say that the function $f : \Delta_{Y_0} \to]0, +\infty[$ belongs to the class of functions $\Gamma_{Y_0}(Z_0)$, if:

(1) the function $f_0(y) = \frac{1}{f(y)}$, as $Y_0 = +\infty$ and $Z_0 = 0$;

(2) the function $f_0(y) = f(-y)$, as $Y_0 = -\infty$ and $Z_0 = +\infty$;

(3) the function $f_0(y) = f(\frac{1}{y})$, as $Y_0 = 0$, where Δ_{Y_0} is a right neighborhood of zero, and $Z_0 = +\infty$;

(4) the function $f_0(y) = \frac{1}{f(\frac{1}{y})}$, as $Y_0 = 0$, where Δ_{Y_0} is a right neighborhood of zero, and $Z_0 = 0$;

(5) the function $f_0(y) = f(-\frac{1}{y})$, as $Y_0 = 0$, where Δ_{Y_0} is a left neighborhood of zero, and $Z_0 = +\infty$;

(6) the function $f_0(y) = \frac{1}{f(-\frac{1}{y})}$, as $Y_0 = 0$, where Δ_{Y_0} is a left neighborhood of zero, and $Z_0 = 0$;

(7) the function $f_0(y) \equiv f(y)$, as $Y_0 = +\infty$ and $Z_0 = +\infty$ belongs to the class Γ.

Using these two definitions, we conclude that for the function $f \in \Gamma_{Y_0}(Z_0)$ the limit relation

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{f(y + ug(y))}{f(y)} = e^u \ \text{ for any } \ u \in \mathbb{R} \tag{2.2}$$

holds, in which the function g, that is complementary for f, in each of the cases 1) - 7) can be expressed through the function g_0, that is complementary for f_0, in the following way (respectively):

(1) $g(y) = -g_0(y)$;

(2) $g(y) = -g_0(-y)$;

(3) $g(y) = -y^2 g_0(\frac{1}{y})$;

(4) $g(y) = y^2 g_0(\frac{1}{y})$;

(5) $g(y) = y^2 g_0(-\frac{1}{y})$;

(6) $g(y) = -y^2 g_0(-\frac{1}{y})$;

(7) $g(y) = g_0(y)$.

Using the properties of the class Γ (see the monograph by Bingham [1]) the following statements were obtained in [4].

Lemma 2.1.

1. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g, then* $\lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{g(y)}{y} = 0$.

2. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g, then for any function $u : \Delta_{Y_0} \to \mathbb{R}$, satisfying the conditions*
$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} u(y) = u_0 \in \mathbb{R}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y + u(y)g(y)) = Z_0,$$
the limit relation
$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y + u(y)g(y))}{f(y)} = e^{u_0}$$
holds.

If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g and, moreover, is continuous and strictly monotone, then there exists a continuous strictly monotone inverse function $f^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$, where
$$\Delta_{Z_0} = \begin{cases} \text{or} & [z_0, Z_0[\,, \\ \text{or} &]Z_0, z_0], \end{cases} \qquad z_0 = f(y_0), \quad Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y).$$

By virtue of Theorems 3.10.4, 3.1.16 from the monograph [1, Chapter 3, p. 3.10, p. 176 and p. 3.1, p. 139] and Definition 2.2, this inverse function has the following properties.

Lemma 2.2. *If $f \in \Gamma_{Y_0}(Z_0)$ with the complementary function g and is a continuous strictly monotone function on the interval Δ_{Y_0}, then the inverse function $f^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$ is slowly varying as $z \to Z_0$ and satisfies the limit relation*
$$\lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}} \frac{f^{-1}(\lambda z) - f(z)}{g(f^{-1}(z))} = \ln\lambda \ \ \textit{for any} \ \ \lambda > 0.$$
Moreover, for any $\Lambda > 1$ this limit relation holds uniformly with respect to $\lambda \in [\frac{1}{\Lambda}, \Lambda]$.

We present some of the important properties of the class of twice continuously differentiable functions $f : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$, where Y_0 is equal either to zero or to $\pm\infty$, and Δ_{Y_0} is some one-sided neighborhood of the point Y_0, each of which satisfies the conditions
$$f'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{f(y)f''(y)}{f'^2(y)} = 1,$$
the proof of which is given in the work of V. M. Evtukhov and A. G. Chernikova [4].

Lemma 2.3. *If a twice continuously differentiable function $f : \Delta_{Y_0} \to]0,+\infty[$ satisfies conditions* (2.1)*, then it belongs to the class $\Gamma_{Y_0}(Z_0)$ with the complementary function $g : \Delta_{Y_0} \to \mathbb{R}$, which is uniquely determined up to the equivalent, as $y \to Y_0$, functions, which can, for example, be one of the following functions:*

$$\frac{\int\limits_Y^y \big(\int\limits_Y^t f(u)\,du\big)\,dt}{\int\limits_Y^y f(x)\,dx} \sim \frac{\int\limits_Y^y f(x)\,dx}{f(y)} \sim \frac{f(y)}{f'(y)} \sim \frac{f'(y)}{f''(y)} \quad \text{as } y \to Y_0,$$

where

$$Y = \begin{cases} y_0, & or \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y) = +\infty,\\ Y_0, & or \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} f(y) = 0. \end{cases}$$

Remark 2.1. The given Lemmas 2.1 and 2.2 refer to the case, where $f : \Delta_{Y_0} \to]0,+\infty[$ (i.e., it takes positive values). In the case of the function $f : \Delta_{Y_0} \to]-\infty,0[$ we will say that it belongs to the class $\Gamma_{Y_0}(Z_0)$, if $(-f) \in \Gamma_{Y_0}(-Z_0)$. Then it is not difficult to verify that Lemmas 2.1 and 2.2 also remain valid.

3 The main results

Let us introduce the necessary auxiliary notation. We assume that the domain of the function φ in equation (1.1) is determined by formula (2.2). Next, we set

$$\mu_0 = \operatorname{sign} \varphi'(y), \quad \nu_0 = \operatorname{sign} y_0, \quad \nu_1 = \begin{cases} 1, & \text{if } \Delta_{Y_0} = [y_0, Y_0[\,,\\ -1, & \text{if } \Delta_{Y_0} =]Y_0, y_0], \end{cases}$$

and introduce the following functions:

$$J(t) = \int\limits_A^t \pi_\omega^2(\tau)p(\tau)\,d\tau, \quad \Phi(y) = \int\limits_B^y \frac{ds}{\varphi(s)}\,,$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty,\\ t-\omega, & \text{if } \omega < +\infty, \end{cases} \tag{3.1}$$

$$A = \begin{cases} \omega, & \text{if } \int\limits_a^\omega \pi_\omega^2(\tau)p(\tau)\,d\tau = const,\\ a, & \text{if } \int\limits_a^\omega \pi_\omega^2(\tau)p(\tau)\,d\tau = \pm\infty, \end{cases} \qquad B = \begin{cases} Y_0, & \text{if } \int\limits_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = const,\\ y_0, & \text{if } \int\limits_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \pm\infty. \end{cases}$$

Taking into account the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution, its first derivative (respectively) in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1 < 0 \text{ if } Y_0 = 0, \quad \nu_0\nu_1 > 0 \text{ if } Y_0 = \pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$, as $y \to Y_0$, and is increasing on Δ_{Y_0}, since on this interval

$\Phi'(y)=\frac{1}{\varphi(y)}>0$. Therefore, there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (1.2) and the monotone increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{or}\quad 0,\\ \text{or}\quad +\infty,\end{cases}\qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\varphi(y_0). \tag{3.2}$$

By virtue of the L'Hospital rule in the form of Stolz and the last of conditions (1.2), we get

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi(y)}{\frac{1}{\varphi'(y)}}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\frac{1}{\varphi(y)}}{-\frac{\varphi''(y)}{\varphi'^2(y)}}=-\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi'^2(y)}{\varphi''(y)\varphi(y)}=-1.$$

Hence,

$$\Phi(y)\sim-\frac{1}{\varphi'(y)}\ \text{ as }\ y\to Y_0\ \text{ and }\ \operatorname{sign}\Phi(y)=-\mu_0\ \text{ as }\ y\in\Delta_{Y_0}. \tag{3.3}$$

From the first of these relations it also follows that

$$\frac{\Phi'(y)}{\Phi(y)}=\frac{\frac{1}{\varphi(y)}}{\Phi(y)}\sim-\frac{\varphi'(y)}{\varphi(y)},\qquad \frac{\Phi''(y)\Phi(y)}{\Phi'^2(y)}=\frac{-\frac{\varphi'(y)}{\varphi^2(y)}\Phi(y)}{\frac{1}{\varphi^2(y)}}\sim 1\ \text{ as }\ y\to Y_0.$$

Therefore, according to Lemma 2.3, $\Phi\in\Gamma_{Y_0}(Z_0)$ with a complementary function, which can be selected as one of the equivalent functions

$$\frac{\Phi'(y)}{\Phi''(y)}\sim\frac{\Phi(y)}{\Phi'(y)}\sim-\frac{\varphi(y)}{\varphi'(y)}\ \text{ as }\ y\to Y_0. \tag{3.4}$$

In addition to the above notation, as $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$, we introduce the auxiliary functions

$$q(t)=\frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\varphi'\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}\,,$$

In addition to the above properties of the twice continuously differentiable functions $f:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$ satisfying conditions (2.1), we will need one more auxiliary statement about a priori asymptotic properties of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) which follows from Corollary 10.1 of [8].

Lemma 3.1. *If $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$, then for each $P_\omega(Y_0,\lambda_0)$-solution of differential equation* (1.1) *the asymptotic relations*

$$\frac{\pi_\omega(t)y'(t)}{y(t)}=\frac{2\lambda_0-1}{\lambda_0-1}\,[1+o(1)],\qquad \frac{\pi_\omega(t)y''(t)}{y'(t)}=\frac{\lambda_o}{\lambda_0-1}\,[1+o(1)],\qquad \frac{\pi_\omega(t)y'''(t)}{y''(t)}=\frac{1+o(1)}{\lambda_0-1} \tag{3.5}$$

as $t\uparrow\omega$ hold, where $\pi_\omega(t)$ is defined by (3.1).

For equation (1.1), the following assertions hold.

Theorem 3.1. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$. Then for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation* (1.1)*, it is necessary that the conditions*

$$\alpha_0\nu_1\lambda_0>0, \tag{3.6}$$

$$\nu_0\nu_1(2\lambda_0-1)(\lambda_0-1)\pi_\omega(t)>0\ \text{ as }\ t\in(a,\omega), \tag{3.7}$$

$$\alpha_0\mu_0\lambda_0J(t)<0\ \text{ as }\ t\in(a,\omega), \tag{3.8}$$

$$\frac{\alpha_0}{\lambda_0}\lim_{t\uparrow\omega}J(t)=Z_0,\qquad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)}=\pm\infty,\qquad \lim_{t\uparrow\omega}q(t)=\frac{2\lambda_0-1}{\lambda_0-1} \tag{3.9}$$

hold. Moreover, each solution of that kind admits the asymptotic representations

$$y(t) = \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big] \quad as \ \ t \uparrow \omega, \tag{3.10}$$

$$y'(t) = \frac{(2\lambda_0-1)}{(\lambda_0-1)} \, \frac{\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\pi_\omega(t)} \, [1+o(1)] \quad as \ \ t \uparrow \omega, \tag{3.11}$$

$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2} \, \frac{\Phi^{-1}\big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\big)}{\pi^2_\omega(t)} [1+o(1)] \quad as \ \ t \uparrow \omega. \tag{3.12}$$

Theorem 3.2. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$, conditions* (3.6)–(3.9) *hold, there exist a limit*

$$\lim_{t\uparrow\omega} \Big[\frac{2\lambda_0-1}{\lambda_0-1} - q(t)\Big] |H(t)|^{\frac{2}{3}} = 0 \tag{3.13}$$

and a finite or equal to $\pm\infty$ limit

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} \sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2}. \tag{3.14}$$

Then the differential equation (1.1) *has at least one $P_\omega(Y_0, \lambda_0)$-solution admitting the asymptotic, as $t \uparrow \omega$, representations*

$$y(t) = \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big], \tag{3.15}$$

$$y'(t) = \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)} \, \Phi^{-1}\Big(\alpha_0 \, \frac{(\lambda_0-1)^2}{\lambda_0} \, J(t)\Big)[1+o(1)H^{-\frac{2}{3}}], \tag{3.16}$$

$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2\pi^2_\omega(t)} \, \Phi^{-1}\Big(\alpha_0 \, \frac{(\lambda_0-1)^2}{\lambda_0} \, J(t)\Big)[1+o(1)H^{-\frac{1}{3}}]. \tag{3.17}$$

Moreover, there exist one-parameter family of such solutions in case $\mu_0\lambda_0\nu_1 < 0$, and two-parameter family, when $\mu_0\lambda_0\nu_1 > 0$.

Proof of Theorem 3.1. Let $y : [t_0, \omega[\to \mathbb{R}$ be an arbitrary $P_\omega(Y_0, \lambda_0)$-solution of the differential equation (1.1). Then, according to Lemma 3.1, the asymptotic relations (3.5) hold. By virtue of these relations and (1.1), this solution and its derivatives of the first, second and third order retain the signs on a certain interval $[t_1, \omega[\subset [t_0, \omega[$, and for these signs the asymptotic relations (3.5) hold, from which follow condition (3.6) and inequality (3.7). In addition, from (1.1), taking into account the second of the asymptotic relations (3.4), it follows that

$$\frac{y'(t)}{\varphi(y(t))} = \alpha_0 \, \frac{(\lambda_0-1)^2}{\lambda_0} \, \pi^2_\omega(t) p(t) [1+o(1)] \quad \text{as} \ \ t \uparrow \omega. \tag{3.18}$$

Integrating this relation from t_0 to t, we get

$$\int\limits_{y(t_0)}^{y(t)} \frac{ds}{\varphi(s)} = \alpha_0 \frac{(\lambda_0-1)^2}{\lambda_0} \int\limits_{t_0}^{t} \pi^2_\omega(\tau) p(\tau) [1+o(1)] \, d\tau \quad \text{as} \ \ t \uparrow \omega.$$

Since, according to the definition of $P_\omega(Y_0, \lambda_0)$-solution, $y(t) \to Y_0$ as $t \uparrow \omega$, it follows that the improper integrals

$$\int\limits_{y(t_0)}^{Y_0} \frac{ds}{\varphi(s)} \quad \text{and} \quad \int\limits_{t_0}^{\omega} \pi^2_\omega(\tau) p(\tau) \, d\tau$$

converge or diverge simultaneously. In view of this fact and the rule for choosing the integration limits A and B in the functions J and Φ, introduced at the beginning of this section, the aforementioned relation can be written as

$$\Phi(y(t)) = \alpha_0 \,\frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{3.19}$$

From here, taking into account (3.2) and (3.3), it follows that inequality (3.8) and the first of conditions (3.9) are true. By virtue of the first of conditions (3.3), it follows from (3.18) and (3.19) that

$$\frac{y''(t)\varphi'(y'(t))}{\varphi(y(t))} = -\frac{\lambda_0\pi_\omega(t)p(t)}{(\lambda_0-1)J(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$

and, therefore, taking into account the first and second of the asymptotic relations (3.4) and the asymptotic relations (3.5),

$$\frac{y(t)\varphi'(y(t))}{\varphi(y(t))} = -\frac{(\lambda_0-1)\pi_\omega^3(t)p(t)}{(2\lambda_0-1)J(t)} \ \text{ as } \ t\uparrow\omega.$$

From this relation, by virtue of (1.3) and the definition of the $P_\omega(Y_0,\lambda_0)$-solution, it directly follows that the second of the limit conditions (3.9) holds.

Now, from (3.19) we find that

$$y(t) = \Phi^{-1}\Big(\alpha_0 \,\frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)[1+o(1)]\Big) \ \text{ as } \ t\uparrow\omega. \tag{3.20}$$

The function Φ, as is stated earlier, belongs to the class $\Gamma_{Y_0}(Z_0)$, where $Z_0 = \lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \Phi(y)$, and the function $g(y) = -\frac{\varphi(y)}{\varphi'(y)}$ can be chosen as its complementary function. Then, according to the conditions $\frac{\alpha_0}{\lambda_0}\lim\limits_{t\uparrow\omega} J(t) = Z_0$ and $\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\in\Delta_{Z_0}$ as $t\in[t_0,\omega[$, which follow from (3.8) and the first condition of (3.1), according to Lemma 2.2, we have

$$\lim_{t\uparrow\omega}\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)[1+o(1)]\big)-\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{-\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}} = \lim_{\substack{z\to Z_0\\ z\in\Delta_{Z_0}}}\frac{\Phi^{-1}(z(1+o(1)))-\Phi^{-1}(z)}{-\frac{\varphi(z)}{\varphi'(z)}} = 0,$$

whence it follows that

$$\begin{aligned}\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)[1+o(1)]\Big)&\\ = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big) &+ \frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,o(1) \ \text{ as } \ t\uparrow\omega.\end{aligned}$$

By virtue of this relation, from (3.20) we obtain the asymptotic representation (3.10). If we consider that

$$\lim_{t\uparrow\omega}\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{y\varphi'(y)}{\varphi(y)} = \pm\infty,$$

then (3.9) can be written as

$$y(t) = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)] \ \text{ as } \ t\uparrow\omega$$

and, therefore, according to the first of the asymptotic relations (3.4), the asymptotic representations (3.11) and (3.12) hold.

It remains to establish the validity of the third of conditions (3.1). According to (3.10), from (3.1) we have

$$y'''(t) = \alpha_0 p(t)\varphi\Big(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big) + \frac{\varphi\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\varphi'\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}\,o(1)\Big) \ \text{ as } \ t\uparrow\omega. \tag{3.21}$$

Since $\varphi\in\Gamma_{Y_0}(Z_0)$, where $Z_0=\lim\limits_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\varphi(y)$, which according to the second conditions of (1.2) is equal either to zero or to $+\infty$, and the function $g(y)=\frac{\varphi(y)}{\varphi'(y)}$ can be chosen as its complementary function, on the basis of Lemma 2.1, taking into account the conditions $\lim\limits_{t\uparrow\omega}\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)=Y_0$ and $\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)\in\Delta_{Y_0}$ as $t\in[t_0,\omega[$, we obtain

$$\lim_{t\uparrow\omega}\frac{\varphi\Big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)) + \frac{\varphi\left(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\right)}{\varphi'\left(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\right)}\,o(1)\Big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)} = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi\big(y+\frac{\varphi(y)}{\varphi'(y)}\,o(1)\big)}{\varphi(y)}=1.$$

Hence,

$$\varphi\Big(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big) + \frac{\varphi\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\varphi'\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}\,o(1)\Big) = \varphi\Big(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big) \ \text{ as } \ t\uparrow\omega$$

and the asymptotic relation (3.21) can be written as

$$y'''(t)=\alpha_0 p(t)\varphi\Big(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big)[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

By virtue of this representation and (3.12),

$$\frac{\pi_\omega(t)y'''(t)}{y''(t)} = \frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\lambda_0(2\lambda_0-1)\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

According to the third of the asymptotic relations (3.5), we obtain the validity of the third of conditions (3.9). ◻

Proof of Theorem 3.2. Suppose that there exists a limit (3.13) that is finite or equal to $\pm\infty$ and for some $\lambda_0\in\mathbb{R}\setminus\{0,1,\frac{1}{2}\}$ conditions (3.7), (3.8) and one of the conditions either (3.14) or (3.16) and (3.17) hold. Under these conditions, we establish the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) that admit asymptotic representations (3.9), (3.10), (3.11) and find the number of such solutions.

First, taking into account the existence of limit (3.13) that is finite or equal to $\pm\infty$, we show that this limit can only be zero. Assume the opposite. Then the relation

$$\frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^{\frac{4}{3}}} = \frac{z(y)}{y^{\frac{2}{3}}}$$

holds, where the function $z:\Delta_{Y_0}\to\mathbb{R}$ is continuous and such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} z(y)=\begin{cases}\text{or} & c=const\neq 0,\\ \text{or} & \pm\infty.\end{cases} \tag{3.22}$$

Integrating this relation on the interval from y_0 to y, we obtain

$$-3\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = c_0+\int_{y_0}^{y}\frac{z(s)}{s^{\frac{2}{3}}}\,ds, \tag{3.23}$$

where c_0 is some constant.

If $\int\limits_{y_0}^{Y_0} \frac{z(s)}{s^{\frac{2}{3}}}\,ds = \pm\infty$, then after dividing by $y^{\frac{1}{3}}$, we have

$$-3\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = \frac{\int\limits_{y_0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\,ds}{y^{\frac{1}{3}}}\,[1+o(1)] \ \text{ as } \ y \to Y_0.$$

Here, the expression on the left, by virtue of (1.3), tends to zero as $y \to Y_0$, and that of on the right, by virtue of condition (3.22), tends either to a nonzero constant or to $\pm\infty$, as according to the L'Hospital rule in the form of Stolz

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\int\limits_{y_0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\,ds}{y^{\frac{1}{3}}} = 3 \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} z(y),$$

which is impossible.

If $\int\limits_{y_0}^{Y_0} \frac{z(s)}{s^{\frac{2}{3}}}\,ds$ converges, which is possible only in the case $Y_0 = 0$, then we rewrite (3.23) in the form

$$-3\mu_0\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = c_1 + \int\limits_{0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\,ds,$$

where $c_1 = c_0 + \int\limits_{y_0}^{0} \frac{z(s)}{s^{\frac{2}{3}}}\,ds$. Let us prove that $c_1 = 0$. Indeed, if $c_1 \neq 0$, then from this relation it follows that

$$\frac{\varphi'(y)}{\varphi(y)} = -\frac{27}{c_1^3} + o(1) \ \text{ as } \ y \to 0.$$

Hence, as a result of integration on the interval from y_0 to y, we get

$$\ln|\varphi(y)| = const + o(1) \ \text{ as } \ y \to 0,$$

which contradicts the second of conditions (1.2). Hence, $c_1 = 0$ and, therefore, we have

$$-3\Big(\frac{\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{3}} = \int\limits_{0}^{y} \frac{z(s)}{s^{\frac{2}{3}}}\,ds.$$

Dividing both sides of this equality by $y^{\frac{1}{3}}$, we note that, by virtue of conditions (1.3), the left-hand side of the resulting relation tends to zero as $y \to 0$, and the right-hand side, by virtue of the L'Hospital rule and (3.22), tends either to a nonzero constant or to $\pm\infty$.

The contradictions obtained in each of the two possible cases lead to the conclusion that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} \sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2} = 0. \tag{3.24}$$

Now, applying the transformation to equation (1.1),

$$\begin{aligned}
y(t) &= \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{y_1}{H(t)}\Big],\\
y'(t) &= \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+y_2(t)],\\
y''(t) &= \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2\pi_\omega^2(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+y_3(t)],
\end{aligned} \tag{3.25}$$

we obtain a system of differential equations

$$\begin{cases} y_1' = \dfrac{H(t)}{\pi_\omega(t)} \left[\dfrac{2\lambda_0-1}{\lambda_0-1} - q(t) + h(t)y_1 + \dfrac{2\lambda_0-1}{\lambda_0-1}\, y_2\right], \\ y_2' = \dfrac{1}{\pi_\omega(t)} \left[\Big(\dfrac{2\lambda_0-1}{\lambda_0-1} - q(t)\Big) + (1-q(t))y_2 + \dfrac{\lambda_0}{\lambda_0-1}\, y_3\right], \\ y_3' = \dfrac{1}{\pi_\omega(t)} \left[2 - \dfrac{2q(t)(\lambda_0-1)}{2\lambda_0-1} + \dfrac{q(t)}{2\lambda_0-1}\, y_1 + (2-q(t))y_3 + \dfrac{q(t)}{2\lambda_0-1}\, R(t,y_1)\right], \end{cases} \tag{3.26}$$

where

$$h(t) = q(t)\, \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\Bigg|_{y=\Phi^{-1}\big(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\big)},$$

$$R(t,y_1) = \frac{\varphi\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, y_1\Big)}{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} - 1 - y_1.$$

We consider this system of equations on the set

$$\Omega = [t_0,\omega[\, \times D_1 \times D_2 \times D_3, \ \text{ where } \ D_i = \{y_i :\ |y_i| \le 1\} \ \ (i=1,2,3),$$

and the number $t_0 \in [a,\omega[$ is chosen, by taking into account conditions (3.2), (3.3), (3.8), the first two conditions (3.9) and (1.3), so that

$$\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t) \in \Delta_{Z_0} \ \text{ as } \ t \in [t_0,\omega[\,,$$

$$\Phi^{-1}\Big(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\Big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, v_1 \in \Delta_{Y_0} \ \text{ as } \ t \in [t_0,\omega[\,, \ \text{ and } \ |v_1| \le 1.$$

On this set, the right-hand sides of the system of differential equations (3.26) are continuous and the function R has on the set $[t_0,\omega[\,\times D_1$ continuous partial derivatives up to the second order inclusive with respect to the variable v_1. At the same time, we have

$$R'_{y_1}(t,y_1) = \frac{\varphi'\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, y_1\Big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} - 1.$$

Here $\varphi' \in \Gamma_{Y_0}(Z_0)$ with the complementary function $g(y) = \frac{\varphi(y)}{\varphi'(y)}$. Therefore,

$$\lim_{t\uparrow\omega} \frac{\varphi'\Big(\Phi^{-1}\big(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, y_1\Big)}{\varphi'\Big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\Big)} = \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\varphi'(y + y_1\frac{\varphi(y)}{\varphi'(y)})}{\varphi'(y)} = e^{y_1}.$$

If, for any fixed $t \in [t_0,\omega[\,$, the function R is expanded according to the Maclaurin formula with the residual Lagrange term to the second-order terms, then we obtain

$$R(t,v_1) = \frac{1}{2}\, \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'^2\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)} \times \varphi''\Big(\Phi^{-1}\Big(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t)\Big) + \frac{\varphi\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\, \frac{(\lambda_0-1)^2}{\lambda_0}\, J(t))\big)}\, \xi\Big) y_1^2,$$

where $|\xi| < |y_1|$. Here, by virtue of the last of conditions (1.2),

$$\varphi''\bigg(\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\bigg)$$
$$=\frac{\varphi'^2\Big(\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\Big)}{\varphi\Big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\Big)}\,\big[1+r_1(t,y_1)\big],$$

where $\lim\limits_{t\uparrow\omega} r_1(t,y_1)=0$ uniformly with respect to $y_1\in D_1$. Therefore, considering that the functions $\varphi,\varphi'\in\Gamma_{Y_0}(Z_0)$ with the complementary function $g(y)=\frac{\varphi(y)}{\varphi'(y)}$, we have

$$\varphi''\bigg(\Phi^{-1}\big(\alpha_0(\lambda_0-1)J(t)\big)+\frac{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,\xi\bigg)$$
$$=\frac{\varphi'^2\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,e^{\xi}\big[1+r_2(t,y_1)\big],$$

where $\lim\limits_{t\uparrow\omega} r_2(t,y_1)=0$ uniformly with respect to $y_1\in D_1$. Therefore, (3.23) can be written as

$$R(t,y_1)=\frac{1}{2}\,e^{\xi}\big[1+r_1(t,y_1)\big]\big[1+r_2(t,y_1)\big]y_1^2.$$

It is clear from the above that for any $\varepsilon>0$ there are $\delta>0$ and $t_1\in[t_0,\omega[$ such that

$$|R(t,y_1)|\le(0.5+\varepsilon)|y_1|^2 \ \text{ as } \ t\in[t_1,\omega[\ \text{ and } \ y_1\in D_{1\delta}=\big\{y_1: \ |y_1|\le\delta\big\}. \tag{3.27}$$

Choosing arbitrarily the number $\varepsilon>0$, we select for it, taking into account the aforementioned about the properties of the function R, the numbers $\delta>0$ and $t_1\in[t_0,\omega[$ such that inequality (3.27) holds, and consider system (3.30) on the set

$$\Omega_1=\Big\{(t,z_1,z_2,z_3)\in\mathbb{R}^4: \ \ t\in[t_1,\omega[,\ \ z_1\in[-\delta,\delta],\ \ z_2\in[-1,1],\ \ z_3\in[-1,1]\Big\}.$$

In addition, in the system of equations (3.26), due to conditions (3.6) – (3.8), (3.13), (1.2) and (1.3),

$$\lim_{t\uparrow\omega}q(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}h(t)=0,\quad \lim_{t\uparrow\omega}H(t)=\pm\infty. \tag{3.28}$$

To establish the existence of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1.1) admitting asymptotic representations (3.10)–(3.12), it is necessary, according to transformation (3.25), to prove the existence of solutions that tend to zero, as $t\uparrow\omega$, of the system of differential equations (3.26). In order to use the well-known results on the existence of solutions of quasilinear systems of differential equations that disappear at a singular point, we reduce system (3.26) to the form that allows us to use such results.

Applying to system (3.26) an additional transformation

$$v_1=z_1,\quad v_2=H^{-\frac{2}{3}}(t)z_2,\quad v_3=H^{-\frac{1}{3}}(t)z_3, \tag{3.29}$$

we get a system of differential equations of the form

$$\begin{cases} z_1'=\dfrac{H^{\frac{1}{3}}(t)}{\pi_\omega(t)}\,\big[f_1(t)+c_{11}(t)z_1+c_{12}(t)z_2+c_{13}(t)z_3\big],\\[2ex] z_2'=\dfrac{H^{\frac{1}{3}}(t)}{\pi_\omega(t)}\,\big[f_2(t)+c_{21}(t)z_1+c_{22}(t)z_2+c_{23}(t)z_3\big],\\[2ex] z_3'=\dfrac{H^{\frac{1}{3}}(t)}{\pi_\omega(t)}\,\Big[f_3(t)+c_{31}(t)z_1+c_{32}(t)z_2+c_{33}(t)z_3+\dfrac{q(t)}{2\lambda_0-1}\,V(t,z_1)\Big],\end{cases} \tag{3.30}$$

where

$$f_1(t)=\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]H^{\frac23}(t),\quad f_2(t)=\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]H^{\frac13}(t),\quad f_3(t)=2-\frac{2q(t)(\lambda_0-1)}{2\lambda_0-1}\,,$$
$$c_{11}(t)=h(t)H^{\frac23}(t),\quad c_{12}(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,,\quad c_{13}(t)=0,\quad c_{21}(t)=0,\quad c_{23}(t)=\frac{\lambda_0}{\lambda_0-1}\,,$$
$$c_{22}(t)=H^{-\frac23}(t)\Big(1-\frac13\,q(t)+\frac23\,q(t)h(t)H(t)\Big),\quad c_{31}(t)=\frac{q(t)}{2\lambda_0-1}\,,\quad c_{32}(t)=0,$$
$$c_{33}(t)=H^{-\frac23}(t)\Big(2-\frac23\,q(t)+\frac13\,q(t)h(t)H(t)\Big),\quad V(t,z_1)=\frac{q(t)}{2\lambda_0-1}\,R(t,z_1).$$

Choosing arbitrarily the number $\varepsilon>0$, we select for it, taking into account the aforementioned about the properties of the function R, the numbers $\delta>0$ and $t_1\in[t_0,\omega[$ such that inequality (3.27) holds, and consider system (3.30) on the set

$$\Omega_1=\Big\{(t,z_1,z_2,z_3)\in\mathbb{R}^4:\ \ t\in[t_1,\omega[\,,\ \ z_1\in[-\delta,\delta],\ \ z_2\in[-1,1],\ \ z_3\in[-1,1]\Big\}.$$

By virtue of (3.28), the replacement of y_1 by z_1 and the first of conditions (3.28),

$$\lim_{z_1\to0}\frac{V(t,z_1)}{z_1^2}=0\ \text{ uniformly with respect to } t\in[t_1,\omega[\,.$$

In addition, according to conditions (3.28), (3.24) and the notation introduced at the beginning of this section, we have $\operatorname{sign}H(t)\pi_\omega(t)=\mu_0\nu_0\pi_\omega(t)$ as $t\in(a,\omega)$ and

$$\lim_{t\uparrow\omega}f_1(t)=0,\quad \lim_{t\uparrow\omega}f_2(t)=0,$$
$$\lim_{t\uparrow\omega}f_3(t)=0,\quad \lim_{t\uparrow\omega}c_{11}(t)=0,\quad \lim_{t\uparrow\omega}c_{12}(t)=\frac{(2\lambda_0-1)}{\lambda_0-1}\,,$$
$$\lim_{t\uparrow\omega}c_{22}(t)=\frac{1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}c_{23}(t)=\frac{\lambda_0}{\lambda_0-1}\,,$$
$$\lim_{t\uparrow\omega}c_{31}(t)=\frac{1}{\lambda_0-1}\,,\quad \lim_{t\uparrow\omega}c_{33}(t)=0,$$
$$\int\limits_{t_1}^{\omega}\frac{|H(\tau)|^{\frac13}}{\pi_\omega(\tau)}\,d\tau=\pm\infty.$$

This, in particular, implies that the limit matrix of coefficients, standing at v_1, v_2 and v_3 in square brackets of system (3.30), has the form

$$C=\begin{pmatrix}0&\dfrac{(2\lambda_0-1)}{\lambda_0-1}&0\\ 0&0&\dfrac{\lambda_0}{\lambda_0-1}\\ \dfrac{1}{\lambda_0-1}&0&0\end{pmatrix}$$

and its characteristic equation is that of the form

$$\rho^3-\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^3}=0. \tag{3.31}$$

If $\lambda_0(2\lambda_0-1)(\lambda_0-1)>0$, then in this case the algebraic equation (3.31) has two complex-conjugate roots with negative real part and one positive real root.

If $\lambda_0(2\lambda_0-1)(\lambda_0-1)<0$, then equation (3.31) has two complex-conjugate roots with a positive real part and one negative real root.

Suppose further that conditions (3.13) are satisfied. It follows that for the system of differential equations (3.30) all the conditions of Theorem 2.2 from [7] are satisfied. According to this theorem, we find that when $\mu_0\nu_1\lambda_0 > 0$, the system of differential equations (3.29) has a two-parameter family of solutions $(z_1, z_2, z_3) : [t_*, \omega[\to \mathbb{R}^3$ ($t_* \in [t_1, \omega[$) that disappear at $t \uparrow \omega$. To each of them, due to substitutions (3.25) and (3.29), there corresponds a solution $y : [t_*, \omega[\to \mathbb{R}$ admitting asymptotic representations (3.10)–(3.12) and (3.15)–(3.17).

If $\mu_0\nu_1\lambda_{<}0$, the system of differential equations (3.30) has a one-parameter family of solutions $(z_1, z_2, z_3) : [t_*, \omega[\to \mathbb{R}^3$ ($t_* \in [t_1, \omega[$) that disappear at $t \uparrow \omega$. To each of them, due to substitutions (3.25) and (3.29), there corresponds a solution $y : [t_*, \omega[\to \mathbb{R}$ admitting asymptotic representations (3.10)–(3.12) and (3.15)–(3.17). □

References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation.* Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1987.

[2] V. . Evtukhov, Asymptotic representations of solutions of non-autonomous ordinary differential equations. *Dissertation of Doctor of Physics and Mathematics, Kiev*, 1998.

[3] V. M. Evtukhov, On solutions vanishing at infinity of real nonautonomous systems of quasilinear differential equations. (Russian) *Differ. Uravn.* **39** (2003), no. 4, 441–452; translation in *Differ. Equ.* **39** (2003), no. 4, 473–484.

[4] V. M. Evtukhov and A. G. Chernikova, Asymptotic behavior of the solutions of second-order ordinary differential equations with rapidly changing nonlinearities. (Russian) *Ukraïn. Mat. Zh.* **69** (2017), no. 10, 1345–1363; translation in *Ukrainian Math. J.* **69** (2018), no. 10, 1561–1582.

[5] V. M. Evtukhov and N. G. Drik, Asymptotic behavior of solutions of a second-order nonlinear differential equation. *Georgian Math. J.* **3** (1996), no. 2, 101–120.

[6] V. M. Evtukhov and V. M. Khar'kov, Asymptotic representations of solutions of second-order essentially nonlinear differential equations. (Russian) *Differ. Uravn.* **43** (2007), no. 10, 1311–1323.

[7] V. M. Evtukhov and A. M. Samoĭlenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. (Russian) *Ukr. Mat. Zh.* **62** (2010), no. 1, 52–80; translation in *Ukr. Math. J.* **62** (2010), no. 1, 56–86.

[8] V. M. Evtukhov and A. M. Samoĭlenko, Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities. (Russian) *Differ. Uravn.* **47** (2011), no. 5, 628–650; translation in *Differ. Equ.* **47** (2011), no. 5, 627–649.

[9] V. M. Evtukhov and V. N. Shinkarenko, Asymptotic representations of solutions of two-term n-order non-autonomous ordinary differential equations with exponential nonlinearity. (Russian) *Differ. Uravn.* **44** (2008), no. 3, 308–322; translation in *Differ. Equ.* **44** (2008), no. 3, 319–333.

[10] V. Marić, *Regular Variation and Differential Equations.* Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.

(Received 08.05.2019)

Authors' address:

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odesa 65026, Ukraine.
e-mail: `emden@farlep.net`; `rusnat36@gmail.com`