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Short Communication

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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR IMPULSIVE EQUATIONS WITH FIXED IMPULSES POINTS

Abstract. The antiperiodic problem for systems of nonlinear impulsive equations with fixed points of impulses actions is considered. The sufficient (among them effective) conditions for the well-posedness of this problem are given.

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Let m_0 be a fixed natural number, ω be a fixed positive real one, and $0 < \tau_1 < \cdots < \tau_{m_0} < \omega$ be fixed points (we assume $\tau_0 = 0$ and $\tau_{m_0+1} = \omega$, if necessary). Let $T = \{\tau_l + m\omega : \ l = 1, \dots, m_0; \ m = 0, \pm 1, \pm 2, \dots\}$.

Consider the system of nonlinear impulsive equations with fixed impulses points

$$\frac{dx}{dt} = f(t, x) \ \text{ almost everywhere on } \ \mathbb{R} \setminus T,$$
$$x(\tau+) - x(\tau-) = I(\tau, x(\tau)) \ \text{ for } \ \tau \in T$$

with the ω-antiperiodic condition

$$x(t + \omega) = -x(t) \ \text{ for } \ t \in \mathbb{R},$$

where $f = (f_i)_{i=1}^n$ is a vector-function belonging to the Carathéodory class $Car(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $I = (I_i)_{i=1}^n : T \times R^n \to \mathbb{R}^n$ is a vector-function such that $I(\tau, \cdot\,)$ is continuous for every $\tau \in T$.

We assume that

$$f(t + \omega, x) = -f(t, -x) \ \text{ and } \ I(\tau + \omega, x) = -I(\tau, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ \tau \in T, \ \ x \in \mathbb{R}^n.$$

Due to the above condition, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given system, then the vector-function $y(t) = -x(t + \omega)$ $(t \in \mathbb{R})$ will likewise be a solution of that system. Moreover, it is evident that if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given ω-antiperiodic problem, then its restriction on the closed interval $[0, \omega]$ will be a solution of the problem

$$\frac{dx}{dt} = f(t, x) \ \text{ almost everywhere on } \ [0, \omega] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1}$$
$$x(\tau_l+) - x(\tau_l-) = I(\tau_l, x(\tau_l)) \ \ (l = 1, \dots, m_0); \tag{2}$$
$$x(0) = -x(\omega). \tag{3}$$

Let now $x:[0,\omega]\to\mathbb{R}^n$ be a solution of system (1), (2) on $[0,\omega]$. By x we designate the continuation of this function on the whole R just as a solution of system (1), (2), as well. As above, the vector-function $y(t)=-x(t+\omega)$ $(t\in\mathbb{R})$ will be a solution of system (1), (2). On the other hand, according to equality (3), we have $y(0)=-x(\omega)=x(0)$. So, if we assume that system (1), (2) under the Cauchy condition $x(0)=c$ is uniquely solvable for every $c\in\mathbb{R}^n$, then $x(t+\omega)=-x(t)$ for $t\in\mathbb{R}$, i.e., x is ω-antiperiodic. This means that the set of restrictions of the ω-antiperiodic solutions of system (1), (2) on $[0,\omega]$ coincides with the set of solutions of problem (1), (2); (3).

In this connection, we consider the boundary value problem (1), (2); (3) on the closed interval $[0,\omega]$. Below, we will give the sufficient conditions guaranteeing the well-posedness of this problem.

Consider a sequence of vector-functions $f_k\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$ $(k=1,2,\dots)$, sequences of points τ_{lk} $(k=1,2,\dots;\ l=1,\dots,m_0)$, $0<\tau_{1k}<\dots<\tau_{m_0k}<\omega$, and sequences of operators $I_k:\{\tau_{1k},\dots,\tau_{m_0k}\}\times R^n\to\mathbb{R}^n$ $(k=1,2,\dots)$ such that $I_k(\tau_{lk},\cdot)$ $(k=1,2,\dots;\ l=1,\dots,m_0)$ are continuous.

In this paper, we establish the sufficient conditions guaranteeing both the solvability of the impulsive systems

$$\frac{dx}{dt}=f_k(t,x)\ \text{ almost everywhere on }\ [0,\omega]\setminus\{\tau_{1k},\dots,\tau_{m_0k}\}, \tag{1_k}$$

$$x(\tau_{lk}+)-x(\tau_{lk}-)=I_k(\tau_{lk},x(\tau_{lk}))\ \ (l=1,\dots,m_0) \tag{2_k}$$

$(k=1,2,\dots)$ under condition (3) for any sufficiently large k and the convergence of their solutions to a solution of problem (1), (2); (3), as $k\to+\infty$.

We assume that the above-described concept is fulfilled for problems $(1_k),(2_k);(3)$ $(k=1,2,\dots)$, as well.

The well-posed problem for the linear boundary value problem for impulsive systems with a finite number of impulses points has been investigated in [5], where the necessary and sufficient conditions were given for the case. Analogous problems are investigated in [1, 11–13] (see also the references therein) for the linear and nonlinear boundary value problems for ordinary differential systems.

A good many issues on the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [2–4, 6–9, 14–16] and the references therein). But the above-mentioned works do not, as we know, contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$; $[a,b]$ $(a,b\in R)$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$.

$|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+=\frac{|X|+X}{2}$.

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$\mathbb{R}^{(n\times n)\times m}=\mathbb{R}^{n\times n}\times\dots\times\mathbb{R}^{n\times n}$ $(m-\text{times})$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=R_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix, inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n\times n$-matrix.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X:[a,b]\to R^{n\times m}$, i.e., the sum of total variations of components of X; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a)=0$, $v(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\leq b$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:[a,b]\to R^{n\times m}$ at the point t (we will assume $X(t)=X(a)$ for $t\leq a$ and $X(t)=X(b)$ for $t\geq b$, if necessary).

$\mathrm{BV}([a,b],R^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$).

$C([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all continuous matrix-functions $X:[a,b]\to D$.

Let $T_{m_0}=\{\tau_1,\dots,\tau_{m_0}\}$.

$C([a,b],D;T_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}\}$ belong to $C([c,d],D)$.

$C_s([a,b],\mathbb{R}^{n\times m};T_{m_0})$ is the Banach space of all $X\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

If $y\in C_s([a,b],\mathbb{R};T_{m_0})$ and $r\in]0,+\infty[$, then $U(y;r)=\{x\in C_s([a,b],\mathbb{R}^n;T_{m_0}):\ \|x-y\|_s<r\}$.

$D(y,r)$ is the set of all $x\in\mathbb{R}^n$ such that $\inf\{\|x-y(t)\|:\ t\in[a,b]\}<r$.

$\widetilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$\widetilde{C}([a,b],D;T_{m_0})$ is the set of all matrix-functions $X:[a,b]\to D$, having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

An operator $\varphi:C([a,b],\mathbb{R}^{n\times m};T_{m_0})\to R^n$ is called nondecreasing if the inequality $\varphi(x)(t)\le\varphi(y)(t)$ for $t\in[a,b]$ holds for every $x,y\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ such that $x(t)\le y(t)$ for $t\in[a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1\subset\mathbb{R}^n$ and $D_2\subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\,\cdot\,,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\,\cdot\,):D_1\to D_2$ is continuous for almost every $t\in[a,b]$, and $\sup\{|f_{kj}(\,\cdot\,,x)|:\ x\in D_0\}\in L([a,b],R;g_{ik})$ for every compact $D_0\subset D_1$.

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(k=1,\dots,n;\ j=1,\dots,m;)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with a bounded total variation.

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$, consisting of a matrix-function $X\in L([a,b],\mathbb{R}^{n\times n})$ and of a sequence of constant $n\times n$ matrices $\{Y_l\}_{l=1}^m\}$, satisfies the Lappo–Danilevskiĭ condition if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0\in[a,b]$ such that

$$\int\limits_{t_0}^{t}X(\tau)\,dX(\tau)=\int\limits_{t_0}^{t}dX(\tau)\cdot X(\tau)\ \text{ for }\ t\in[a,b],$$
$$X(t)Y_l=Y_lX(t)\ \text{ for }\ t\in[a,b]\ \ (l=1,\dots,m).$$

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega\in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing and $\omega(t,0)=0$ for every $t\in[a,b]$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x\in\widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ satisfying both system (1) for a.e. on $[0,\omega]\setminus T_{m_0}$ and relation (2) for every $l\in\{1,\dots,m_0\}$.

Definition 1. Let $\ell:C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0:C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair (P,J), consisting of a matrix-function $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a continuous with respect to the last n-variables operator $J:T_{m_0}\times R^n\to\mathbb{R}^n$, satisfies the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi\in L([0,\omega],\mathbb{R}_+^{n\times n})$ and constant matrices $\Psi_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that

$$|P(t,x)|\le\Phi(t)\ \text{ a.e. on }\ [0,\omega],\ \ x\in\mathbb{R}^n,$$
$$|J(\tau_l,x)|\le\Psi_l\ \text{ for }\ x\in\mathbb{R}^n\ \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n}+G_l)\neq 0 \ \ (l=1,\dots,m_0) \tag{4}$$

and the problem

$$\frac{dx}{dt}=A(t)x \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \tag{5}$$

$$x(\tau_l+)-x(\tau_l-)=G_l\,x(\tau_l) \ \ (l=1,\dots,m_0), \tag{6}$$

$$|\ell(x)|\leq \ell_0(x) \tag{7}$$

has only the trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l,\dots,G_{m_0}$ for which there exists a sequence $y_k\in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int\limits_0^t P(\tau,y_k(\tau))\,d\tau=\int\limits_0^t A(\tau)\,d\tau \ \text{ uniformly on } \ [0,\omega],$$

$$\lim_{k\to+\infty} J(\tau_l,y_k(\tau_l))=G_l \ \ (l=1,\dots,m_0).$$

Remark 1. In particular, condition (4) holds if $\|\Psi_l\|<1$ $(l=1,\dots,m_0)$.

As above, we assume that $f=(f_i)_{i=1}^n\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of problem $(1),(2);(3)$, and r be a positive number. Let us introduce the following definition.

Definition 2. The solution x^0 is said to be strongly isolated in the radius r if there exist matrix- and vector-functions $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$, continuous with respect to the last n-variables operators $J,H: T_{m_0}\times R^n\to\mathbb{R}^n$, linear continuous ℓ and $\widetilde{\ell}$ and a positive homogeneous ℓ_0 operators acting from $C_s([0,\omega],\mathbb{R}^n;T_{m_0})$ into $\mathbb{R}^n$ such that

(a) the equalities

$$f(t,x)=P(t,x)x+q(t,x) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ \|x-x^0(t)\|<r,$$

$$I(\tau_l,x)=J(\tau_l,x)x+H(\tau_l,x) \ \text{ for } \ \|x-x^0(\tau_l)\|<r \ \ (l=1,\dots,m_0),$$

$$x(0)+x(\omega)=\ell(x)+\widetilde{\ell}(x) \ \text{ for } \ x\in U(x^0;r)$$

are valid;

(b) the functions $\alpha(t,\rho)=\max\{\|q(t,x)\|:\|x\|\leq\rho\}$, $\beta(\tau_l,\rho)=\max\{\|H(\tau_l,x)\|:\|x\|\leq\rho\}$ $(l=1,\dots,m_0)$ and $\gamma(\rho)=\sup\{[|\widetilde{l}(x)|-l_0(x)]_+:\|x\|_s\leq\rho\}$ satisfy the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\gamma(\rho)+\int\limits_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0; \tag{8}$$

(c) the problem

$$\frac{dx}{dt}=P(t,x)x+q(t,x) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0},$$

$$x(\tau_l+)-x(\tau_l-)=J(\tau_l,x(\tau_l))\,x(\tau_l)+H(\tau_l,x(\tau_l)) \ \ (l=1,\dots,m_0);$$

$$\ell(x)+\widetilde{\ell}(x)=0$$

has no solution different from x^0;

(d) the pair (P,J) satisfies he Opial condition with respect to the pair (ℓ,ℓ_0).

Remark 2. If $\ell(x) \equiv x(0) + x(\omega)$ and $\ell_0(x) \equiv 0$, then we say that the pair (P, J) satisfies the Opial ω-antiperiodic condition. In this case, condition (7) coincides with condition (3), and $\widetilde{\ell}(x) \equiv 0$ and $\gamma(\rho) \equiv 0$ in Definitions 1 and 2.

Definition 3. We say that a sequence (f_k, I_k) $(k = 1, 2, \dots)$ belongs to the set $W_r(f, I; x^0)$ if:

(a) the equalities

$$\lim_{k\to+\infty} \int_0^t f_k(\tau, x)\,d\tau = \int_0^t f(\tau, x)\,d\tau \ \text{ uniformly on } \ [0, \omega],$$
$$\lim_{k\to+\infty} I_k(\tau_{lk}, x) = I(\tau_l, x) \ \ (l = 1, \dots, m_0)$$

are valid for every $x \in D(x^0; r)$;

(b) there exist a sequence of functions $\omega_k \in M([a, b] \times \mathbb{R}_+, \mathbb{R}_+)$ $(k = 1, 2, \dots)$ such that

$$\sup\bigg\{ \int_0^\omega \omega_k(t, r)\,dt : \ k = 1, 2, \dots \bigg\} < +\infty, \tag{9}$$

$$\sup\bigg\{ \sum_{l=1}^{m_0} \omega_k(\tau_{lk}, r) : \ k = 1, 2, \dots \bigg\} < +\infty; \tag{10}$$

$$\lim_{s\to0+} \sup\bigg\{ \int_0^\omega \omega_k(t, s)\,dt : \ k = 1, 2, \dots \bigg\} = 0, \tag{11}$$

$$\lim_{s\to0+} \sup\bigg\{ \sum_{l=1}^{m_0} \omega_k(\tau_{lk}, s) : \ k = 1, 2, \dots \bigg\} = 0; \tag{12}$$

$$\|f_k(t, x) - f_k(t, y)\| \le \omega_k(t, \|x - y\|) \ \text{ for } \ t \in [0, \omega] \setminus T_{m_0}, \ \ x, y \in D(x^0; r) \ \ (k = 1, 2, \dots),$$
$$\|I_k(\tau_{lk}, x) - I_k(\tau_{lk}, y)\| \le \omega_k(\tau_{lk}, \|x - y\|) \ \text{ for } \ x, y \in D(x^0; r) \ \ (l = 1, \dots, m_0; k = 1, 2, \dots).$$

Remark 3. If for every natural m there exists a positive number ν_m such that $\omega_k(t, m\delta) \le \nu_m \omega_k(t, \delta)$ for $\delta > 0$, $t \in [0, \omega] \setminus T_{m_0}$ $(k = 1, 2, \dots)$, then estimate (9) follows from condition (11); analogously, if $\omega_k(\tau_{lk}, m\delta) \le \nu_m \omega_k(\tau_{lk}, \delta)$ for $\delta > 0$ $(l = 1, \dots, m_0;\ k = 1, 2, \dots)$, then estimate (10) follows from condition (12). In particular, the sequences of functions

$$\begin{aligned} \omega_k(t, \delta) &= \max\big\{ \|f_k(t, x) - f_k(t, y)\| : \ x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \big\} \\ &\qquad\qquad \text{for } \ t \in [0, \omega] \setminus T_{m_0} \ \ (k = 1, 2, \dots), \\ \omega_k(\tau_{lk}, \delta) &= \max\big\{ \|I_k(\tau_{lk}, x) - I_k(\tau_{lk}, y)\| : x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \big\} \\ &\qquad\qquad (l = 1, \dots, m_0;\ k = 1, 2, \dots) \end{aligned}$$

have the latters properties, respectively.

Definition 4. Problem $(1), (2); (3)$ is said to be $(x^0; r)$-correct if for every $\varepsilon \in]0, r[$ and $(f_k, I_k)_{k=1}^{+\infty} \in W_r(f, I; x^0)$ there exists a natural number k_0 such that problem $(1_k), (2_k)$ has at last one ω-antiperiodic solution contained in $U(x^0; r)$, and any such solution belongs to the ball $U(x^0; \varepsilon)$ for every $k \ge k_0$.

Definition 5. Problem $(1), (2); (3)$ is said to be correct if it has a unique solution x^0 and is $(x^0; r)$-correct for every $r > 0$.

Theorem 1. *If problem* $(1), (2); (3)$ *has a solution* x^0 *strongly isolated in the radius* r*, then it is* $(x^0; r)$*-correct.*

Theorem 2. *Let the conditions*

$$\|f(t,x)-P(t,x)\,x\| \le \alpha(t,\|x\|) \ \text{ a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{13}$$

$$\|I(\tau_l,x)-J(\tau_l,x)x\| \le \beta(\tau_l,\|x\|) \ \text{ for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0), \tag{14}$$

$$|x(0)+x(\omega)-\ell(x)| \le \ell_0(x)+\ell_1(\|x\|_s) \ \text{ for } x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{15}$$

hold, where $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators, the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\left(\|\ell_1(\rho)\|+\int\limits_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\right)=0. \tag{16}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3. *Let conditions* (13)–(15),

$$P_1(t)\le P(t,x)\le P_2(t) \ \text{ a.e. on } [0,\omega]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n, \tag{17}$$

$$J_{1l}\le J(\tau_l,x)\le J_{2l} \ \text{ for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{18}$$

hold, where $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that condition (16) *holds. Let, moreover, condition* (4) *hold and problem* (5), (6); (7) *have only the trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that*

$$P_1(t)\le A(t)\le P_2(t) \ \text{ a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{19}$$

$$J_{1l}\le G_l\le J_{2l} \ \text{ for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0). \tag{20}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 4. Theorem 3 is interesting only in the case where $P\notin Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem follows immediately from Theorem 2 in the case where $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 4. *Let conditions* (15),

$$|f(t,x)-P(t)\,x|\le Q(t)\,|x|+q(t,\|x\|) \ \text{ a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{21}$$

$$|I_l(x)-J_l\,x|\le H_l\,|x|+h(\tau_l,\|x\|) \ \text{ for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{22}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are the constant matrices, $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\left(\|\ell_1(\rho)\|+\int\limits_0^\omega \|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\right)=0 \tag{23}$$

holds. Let, moreover, the conditions

$$\det(I_{n\times n}+J_l)\ne 0 \ \ (l=1,\dots,m_0) \tag{24}$$

$$\|H_l\|\cdot\|(I_{n\times n}+J_l)^{-1}\|<1 \ \ (j=1,2;\ l=1,\dots,m_0) \tag{25}$$

hold and the system of impulsive inequalities

$$\Big|\frac{dx}{dt} - P(t)x\Big| \le Q(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{26}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_l x(\tau_l)\big| \le H_l|x(\tau_l)| \ \ (l = 1,\dots,m_0) \tag{27}$$

have only the trivial solution satisfying condition (7). *Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1. *Let the conditions*

$$|f(t,x) - P(t)x| \le q(t,\|x\|) \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{28}$$

$$|I(\tau_l,x) - J_l x| \le h(\tau_l,\|x\|) \ \ \text{for } x\in\mathbb{R}^n \ \ (l = 1,\dots,m_0), \tag{29}$$

$$|x(0) + x(\omega) - \ell(x)| \le \ell_1(\|x\|_s) \ \ \text{for } x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{30}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying condition* (24), $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ *is the linear continuous operator;* $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable, and* $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ *is a vector-function such that condition* (23) *holds. Let, moreover, the problem*

$$\frac{dx}{dt} = P(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{31}$$

$$x(\tau_l+) - x(\tau_l-) = J_l\,x(\tau_l) \ \ (l = 1,\dots,m_0); \tag{32}$$

$$\ell(x) = 0. \tag{33}$$

have only the trivial solution. Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 5. Let $Y = (y_1,\dots,y_n)$ be a fundamental matrix, with columns $y_1,\dots,y_n$, of system (31), (32). Then the homogeneous boundary value problem (31), (32); (33) has only the trivial solution if and only if

$$\det(\ell(Y)) \ne 0, \tag{34}$$

where $\ell(Y) = (\ell(y_1),\dots,\ell(y_n))$.

If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the fundamental matrix Y $(Y(0) = I_{n\times n})$ of the homogeneous system (31), (32) has the form

$$Y(t) \equiv \exp\bigg(\int_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n} + J_l).$$

Theorem 5. *Let the conditions*

$$\big|f(t,x) - f(t,y) - P(t)(x-y)\big| \le Q(t)|x-y| \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x,y\in\mathbb{R}^n, \tag{35}$$

$$\big|I(\tau_l,x) - I(\tau_l,y) - J_l\cdot(x-y)\big| \le H_l|x-y| \ \ \text{for } x,y\in\mathbb{R}^n \ \ (k = l,\dots,m_0), \tag{36}$$

$$\big|x(0) - y(0) + x(\omega) - y(\omega) - \ell(x-y)\big| \le \ell_0(x-y) \ \ \text{for } x,y\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ *and* $H_l\in\mathbb{R}^{n\times n}_+$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25), $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ *and* $\ell_0 : C_s([0,\omega],\,\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, problem* (26), (27); (7) *have only the trivial solution. Then problem* (1), (2); (3) *is correct.*

Corollary 2. *Let there exist a solution* x^0 *of problem* (1), (2); (3) *and a positive number* $r>0$ *such that the conditions*

$$\big|f(t,x) - f(t,x^0(t)) - P(t)(x - x^0(t))\big| \le Q(t)|x - x^0(t)| \ \ \text{a.a. } [0,\omega]\setminus T_{m_0}, \ \ \|x - x^0(t)\| < r,$$

$$\big|I(\tau_l,x) - I(\tau_l,x^0(\tau_l)) - J_l\cdot(x - x^0(\tau_l))\big| \le H_l|x - x^0(\tau_l)| \ \ \text{for } \|x - x^0(\tau_l)\| < r \ \ (l = l,\dots,m_0),$$

$$\big|x(0) - x^0(0) + x(\omega) - x^0(\omega) - \ell(x - x^0)\big| \le \ell^*(|x - x^0|) \ \ \text{for } x\in U(x^0,r)$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $Q \in L([0,\omega],\mathbb{R}_+^{n\times n})$, J_l *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25), $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ *and* $\ell^* : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities*

$$\Big|\frac{dx}{dt} - P(t)x\Big| \le Q(t)x \ \text{ a.e. on } [0,\omega]\setminus T_{m_0},$$
$$\big|x(\tau_l+) - x(\tau_l-) - J_l\cdot x(\tau_l)\big| \le H_l\cdot x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only the trivial solution under the condition $|\ell(x)| \le \ell^*(|x|)$. *Then problem* (1), (2); (3) *is* $(x^0;r)$*-correct.*

Corollary 3. *Let the components of the vector-functions* f *and* I_l $(l = 1,\dots,n)$ *have partial derivatives by the last* n *variables belonging to the Carathéodory class* $Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$. *Let, moreover,* x^0 *be a solution of problem* (1), (2); (3) *such that the condition*

$$\det\big(I_{n\times n} + G_l(x^0(\tau_l))\big) \neq 0 \ \ (l = 1,\dots,m_0)$$

hold and the system

$$\frac{dx}{dt} = F(t,x^0(t))x \ \text{ almost everywhere on } [0,\omega]\setminus T_{m_0},$$
$$x(\tau_l+) - x(\tau_l-) = G_l(x^0(\tau_l))\cdot x(\tau_l) \ \ (l = 1,\dots,m_0);$$
$$\ell(x) = 0,$$

where $F(t,x) \equiv \frac{\partial f(t,x)}{\partial x}$ *and* $G_l(x) \equiv \frac{\partial I_l(x)}{\partial x}$, *have only the trivial solution under condition* (3). *Then problem* (1), (2); (3) *is* $(x^0;r)$*-correct for any sufficiently small* r.

In general, it is rather difficult to verify condition (34) directly even in the case if one is able to write out the fundamental matrix of system (31), (32); (33). Therefore, it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (31), (32); (33). Below, we will give the results concerning the question. Analogous results have been obtained in [2] for the general linear boundary value problems for impulsive systems, and in [12] by T. Kiguradze for the case of ordinary differential equations.

In this connection, we introduce the operators. For every matrix-function $X \in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we put

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } \ 0 \le t \le \omega,$$
$$\big[(X,Y_1,\dots,Y_{m_0})(0)\big]_i = O_{n\times n} \ \ (i = 1,2,\dots),$$
$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int\limits_0^t X(\tau)\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau$$
$$+ \sum_{0\le\tau_l<t} Y_l\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } \ 0 < t \le \omega \ \ (i = 1,2,\dots). \quad (37)$$

Corollary 4. *Let conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \int\limits_0^\omega d\mathcal{L}(t)\cdot x(t),$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are the constant matrices satisfying condition* (24), $\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R},\mathbb{R}_+^n)$ *is a vector-function such that condition* (23) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int\limits_0^\omega d\mathcal{L}(t)\cdot\big[(P,J_l,\dots,J_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{38}$$

where the operators $[(P, J_1, \dots, J_{m_0})(t)]_i$ $(i = 0, 1, \dots)$ *are defined by* (37), *and*

$$M_{k,m} = \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(\omega)\big]_m + \sum_{i=0}^{m-1} \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(\omega)\big]_i \int\limits_0^\omega dV(M_k^{-1}\mathcal{L})(t) \cdot \big[\big(|P|, |J_1|, \dots, |J_{m_0}|\big)(t)\big]_k.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5. *Let conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \sum_{j=1}^{n_0} \mathcal{L}_j x(t_j), \tag{39}$$

$P \in L([0,\omega], \mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices satisfying condition* (24), $t_j \in [0,\omega]$ *and* $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$, $\mathcal{L} \in L([0,\omega], \mathbb{R}^{n\times n})$, $\ell : C_s([0,\omega], \mathbb{R}^n; T_{m_0}) \to \mathbb{R}^n$ *is the linear continuous operator;* $q \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}^n_+)$ *and* $h \in C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}^{n\times n}_+)$ *are the vector-functions nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is a vector-function such that condition* (23) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = \sum_{j=1}^{n_0} \sum_{i=0}^{k-1} \mathcal{L}_j \big[(P, J_l, \dots, J_{m_0})(t_j)\big]_i$$

is nonsingular and inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1} \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(\omega)\big]_i\Big) \sum_{j=1}^{n_0} |M_k^{-1}\mathcal{L}_j| \cdot \big[\big(|P|, |J_l|, \dots, |J_{m_0}|\big)(t_j)\big]_k.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5 for $k = 1$ and $m = 1$ has the following form.

Corollary 6. *Let conditions* (28)–(30) *hold, where the operator* ℓ *is defined by* (39), $P \in L([0,\omega], \mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices satisfying condition* (24), $t_j \in [0,\omega]$ *and* $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$; $q \in Car([0,\omega] \times \mathbb{R}_+, \mathbb{R}^n_+)$ *and* $h \in C(T_{m_0} \times \mathbb{R}_+; \mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable, and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is the vector-function such that condition* (23) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0} \mathcal{L}_j\Big) \neq 0 \ \text{ and } \ r(\mathcal{L}_0\, A_0) < 1,$$

where

$$\mathcal{L}_0 = I_{n\times n} + \Big|\Big(\sum_{j=1}^{n_0} \mathcal{L}_j\Big)^{-1}\Big| \cdot \sum_{j=1}^{n_0} |\mathcal{L}_j| \ \text{ and } \ A_0 = \int\limits_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0} |J_l|.$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 6. If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then condition (34) has the forms

$$\det\bigg(\int\limits_0^\omega d\mathcal{L}(t)\cdot\exp\bigg(\int\limits_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n}+J_l)\bigg)\neq 0,$$

$$\det\bigg(\sum_{j=1}^{n_0}L_j\exp\bigg(\int\limits_0^{t_j}P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t_j}(I_{n\times n}+J_l)\bigg)\neq 0$$

for the operators ℓ defined, respectively, in Corollary 4 and Corollary 5.

By Remark 2, in the case if $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, the results given above have, respectively, the following forms.

Theorem 2′. *Let conditions* (13) *and* (14) *hold, where the pair* (P,J) *satisfies the Opial* ω*-antiperiodic condition;* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *is nondecreasing in the second variable function such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int\limits_0^\omega\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{40}$$

Then problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3′. *Let conditions* (13), (14), (17), (18) *and* (40) *hold, where* $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$*;* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function, nondecreasing in the second variable, and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *is nondecreasing in the second variable function. Let, moreover, condition* (4) *hold and problem* (5), (6); (3) *have only the trivial solution for every matrix-function* $A\in L([0,\omega],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *satisfying conditions* (19) *and* (20)*. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 4′. *Let conditions* (21) *and* (22) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25)*,* $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$*, and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int\limits_0^\omega\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{41}$$

Let, moreover, the system of impulsive inequalities (26), (27) *have only the trivial solution satisfying condition* (3)*. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1′. *Let conditions* (28), (29) *and* (40) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying condition* (24)*,* $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, problem* (31), (32), (3) *have only the trivial solution. Then problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 5′. *Let conditions* (35) *and* (36) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying conditions* (24) *and* (25)*. Let, moreover, problem* (26), (27); (7) *have only the trivial solution. Then problem* (1), (2); (3) *is correct.*

Corollary 5′. *Let conditions* (28), (29) *and* (41) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying condition* (24)*;* $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in$

$C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ are the vector-functions, nondecreasing in the second variable. Let, moreover, there exist natural numbers k and m such that the matrix

$$M_k=\sum_{i=0}^{k-1}\big[(P,J_l,\dots,J_{m_0})(\omega)\big]_i$$

is nonsingular and inequality (38) holds, where

$$M_{k,m}=\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1}\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_i\Big)|M_k^{-1}|\cdot\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_k.$$

Then problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Corollary 5′ for $k=1$ and $m=1$ has the following form.

Corollary 6′. *Let conditions (28), (29) and (41) hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ ($l=1,\dots,m_0$) are the constant matrices satisfying condition (24); $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ are the vector-functions, nondecreasing in the second variable. Let, moreover,*

$$r(A_0)<\frac{1}{2}\,,$$

where

$$A_0=\int_0^\omega |P(t)|\,dt+\sum_{l=1}^{m_0}|J_l|.$$

Then problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Remark 7. In the conditions of Corollary 6′, if the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then condition (34) has the form

$$\det\bigg(I_{n\times n}+\exp\bigg(\int_0^\omega P(\tau)\,d\tau\bigg)\cdot\prod_{l=1}^{m_0}(I_{n\times n}+J_l)\bigg)\neq 0.$$

The analogous questions are investigated in [7] for system (1), (2) under the general nonlinear boundary condition $h(x)=0$, where $h:C([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ is a continuous vector-functional, nonlinear, in general. The results given in the paper are the particular cases of the results obtained in [7] for $h(x)\equiv x(0)+x(\omega)$.

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References

[1] M. Ashordia, On the stability of solutions of linear boundary value problems for a system of ordinary differential equations. *Georgian Math. J.* (1994), no. 2, 115–126.

[2] M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems. *Mem. Differential Equations Math. Phys.* **36** (2005), 1–80.

[3] M. Ashordia, On the two-point boundary value problems for linear impulsive systems with singularities. *Georgian Math. J.* **19** (2012), no. 1, 19–40.

[4] Sh. Akhalaia, M. Ashordia, and N. Kekelia, On the necessary and sufficient conditions for the stability of linear generalized ordinary differential, linear impulsive and linear difference systems. *Georgian Math. J.* **16** (2009), no. 4, 597–616.

[5] M. Ashordia and G. Ekhvaia, Criteria of correctness of linear boundary value problems for systems of impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **37** (2006), 154–157.

[6] M. Ashordia and G. Ekhvaia, On the solvability of a multipoint boundary value problem for systems of nonlinear impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **43** (2008), 153–158.

[7] M. Ashordia, G. Ekhvaia, and N. Kekelia, On the solvability of general boundary value problems for systems of nonlinear impulsive equations with finite and fixed points of impulse actions. *Bound. Value Probl.* **2014**, 2014:157, 17 pp.

[8] D. D. Baĭnov and P. S. Simeonov, *Systems with Impulse Effect. Stability, Theory and Applications.* Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1989.

[9] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions.* Contemporary Mathematics and Its Applications, 2. Hindawi Publishing Corporation, New York, 2006.

[10] I. T. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations.* (Russian) Izdat. Tbilis. Univ., Tbilisi, 1975.

[11] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Itogi Nauki i Tekhniki, *Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987; translation in *J. Soviet Math.* **43** (1988), no. 2, 2259–2339.

[12] I. Kiguradze, *The Initial Value Problem and Boundary Value Problems for Systems of Ordinary Differential Equations. Vol. I. Linear Theory.* (Russian) "Metsniereba", Tbilisi, 1997.

[13] M. A.Krasnosel'skiĭ and S. G. Kreĭn, On the principle of averaging in nonlinear mechanics. (Russian) *Uspehi Mat. Nauk (N.S.)* **10** (1955), no. 3(65), 147–152.

[14] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations.* Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[15] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, and N. V. Skripnik, *Differential Equations with Impulse Effects. Multivalued Right-Hand Sides with Discontinuities.* De Gruyter Studies in Mathematics, 40. Walter de Gruyter & Co., Berlin, 2011.

[16] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations.* World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 14. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

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