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Francesco Dondi and Massimo Lanza de Cristoforis

REGULARIZING PROPERTIES OF THE DOUBLE LAYER POTENTIAL OF SECOND ORDER ELLIPTIC DIFFERENTIAL OPERATORS

Abstract. We prove the validity of regularizing properties of a double layer potential associated to the fundamental solution of a nonhomogeneous second order elliptic differential operator with constant coefficients in Schauder spaces by exploiting an explicit formula for the tangential derivatives of the double layer potential itself. We also introduce ad hoc norms for kernels of integral operators in order to prove continuity results of integral operators upon variation of the kernel, which we apply to the layer potentials.

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Key words and phrases. Double layer potential, second order differential operators with constant coefficients.

რეზიუმე. ორმაგი ფენის პოტენციალის მხები წარმოებულის ცხადი ფორმულის გამოყენებით დამტკიცებულია იმ ორმაგი ფენის პოტენციალის რეგულარული თვისებები, რომელიც დაკავშირებულია არაერთგვაროვანი მეორე რიგის მუდმივკოეფიციენტებიანი ელიფსური დიფერენციალური ოპერატორის ფუნდამენტურ ამონახსნთან შაუდერის სივრცეებში. აგრეთვე შემოღებულია სპეციალური ნორმები ინტეგრალური ოპერატორების გულებისთვის, რათა დამტკიცებულ იქნას ინტეგრალური ოპერატორების უწყვეტობა გულის ცვლილებისას, რომელიც გამოყენებულია ორმაგი ფენის პოტენციალებისთვის.

1 Introduction

In this paper, we consider the double layer potential associated to the fundamental solution of a second order differential operator with constant coefficients. Throughout the paper, we assume that

$$n \in \mathbb{N} \setminus \{0,1\}\,,$$

where $\mathbb{N}$ denotes a set of natural numbers including 0. Let $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Let $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the external unit normal to $\partial\Omega$. Let N_2 denote the number of multi-indices $\gamma \in \mathbb{N}^n$ with $|\gamma| \leq 2$. For each

$$\mathbf{a} \equiv (a_\gamma)_{|\gamma|\leq 2} \in \mathbb{C}^{N_2}\,, \tag{1.1}$$

we set

$$a^{(2)} \equiv (a_{lj})_{l,j=1,\dots,n}, \quad a^{(1)} \equiv (a_j)_{j=1,\dots,n}, \quad a \equiv a_0\,,$$

with $a_{lj} \equiv 2^{-1} a_{e_l+e_j}$ for $j \neq l$, $a_{jj} \equiv a_{e_j+e_j}$, and $a_j \equiv a_{e_j}$, where $\{e_j : j = 1,\dots,n\}$ is the canonical basis of $\mathbb{R}^n$. We note that the matrix $a^{(2)}$ is symmetric. Then we assume that $\mathbf{a} \in \mathbb{C}^{N_2}$ satisfies the following ellipticity assumption

$$\inf_{\xi\in\mathbb{R}^n, |\xi|=1} \mathrm{Re}\Big\{\sum_{|\gamma|=2} a_\gamma \xi^\gamma\Big\} > 0\,, \tag{1.2}$$

and we consider the case in which

$$a_{lj} \in \mathbb{R} \quad \forall\, l,j = 1,\dots,n\,. \tag{1.3}$$

Introduce the operators

$$P[\mathbf{a},D]u \equiv \sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j}u) + \sum_{l=1}^{n} a_l \partial_{x_l} u + au\,,$$
$$B^*_\Omega v \equiv \sum_{l,j=1}^{n} \overline{a}_{jl}\nu_l \partial_{x_j} v - \sum_{l=1}^{n} \nu_l \overline{a}_l v\,,$$

for all $u, v \in C^2(\overline{\Omega})$, a fundamental solution $S_\mathbf{a}$ of $P[\mathbf{a},D]$, and the double layer potential

$$\begin{aligned} w\big[\partial\Omega,\mathbf{a},S_\mathbf{a},\mu\big](x) &\equiv \int_{\partial\Omega} \mu(y)\overline{B^*_{\Omega,y}}\big(S_\mathbf{a}(x-y)\big)\,d\sigma_y \\ &= -\int_{\partial\Omega} \mu(y) \sum_{l,j=1}^{n} a_{jl}\nu_l(y)\,\frac{\partial S_\mathbf{a}}{\partial x_j}(x-y)\,d\sigma_y - \int_{\partial\Omega} \mu(y)\sum_{l=1}^{n} \nu_l(y) a_l S_\mathbf{a}(x-y)\,d\sigma_y \quad \forall\, x \in \mathbb{R}^n\,, \end{aligned} \tag{1.4}$$

where the density (or the moment) μ is a function from $\partial\Omega$ to $\mathbb{C}$. Here the subscript y of $\overline{B^*_{\Omega,y}}$ means that we take y as a variable of the differential operator $\overline{B^*_{\Omega,y}}$. The role of the double layer potential in the solution of boundary value problems for the operator $P[\mathbf{a},D]$ is well known (cf. e.g., Günter [14], Kupradze, Gegelia, Basheleishvili and Burchuladze [20], Mikhlin [23]).

The analysis of the continuity and compactness properties of the integral operator associated to the double layer potential is a classical topic. In particular, it has long been known that if μ is of the class $C^{m,\alpha}$, then the restriction of the double layer potential to the sets

$$\Omega^+ \equiv \Omega\,, \quad \Omega^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\Omega$$

can be extended to a function of $C^{m,\alpha}(\mathrm{cl}\,\Omega^+)$ and to a function of $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$, respectively (cf., e.g., Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5]).

In case $n = 3$ and Ω is of the class $C^{1,\alpha}$ and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, it has long been known that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is a linear and compact operator in $C^{1,\alpha}(\partial\Omega)$ and is linear and continuous from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ (cf. Schauder [30], [31], Miranda [24].)

In case $n = 3$, $m \geq 1$ and Ω is of the class C^{m+1} and if $P[\mathbf{a}, D]$ is the Laplace operator, Günter [14, Ch. II, § 21, Thm. 3] has proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{m-1,\alpha'}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ for $\alpha' \in]\alpha, 1[$ and, accordingly, is compact in $C^{m,\alpha}(\partial\Omega)$.

Fabes, Jodeit and Rivière [12] have proved that if Ω is of the class C^1 and if $P[\mathbf{a}, D]$ is the Laplace operator, then $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is compact in $L^p(\partial\Omega)$ for $p \in]1, +\infty[$. Later, Hofmann, M. Mitrea and Taylor [16] have proved the same compactness result under more general conditions on $\partial\Omega$.

In case $n = 2$ and Ω is of the class $C^{2,\alpha}$, and if $P[\mathbf{a}, D]$ is the Laplace operator, Schippers [32] has proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^0(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$.

In case $n = 3$ and Ω is of the class C^2, and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Colton and Kress [2] have developed works of Günter [14] and Mikhlin [23] and proved that the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ and, accordingly, is compact in $C^{1,\alpha}(\partial\Omega)$.

Wiegner [36] has proved that if $\gamma \in \mathbb{N}^n$ has odd length and Ω is of the class $C^{m,\alpha}$, then the operator with kernel $(x-y)^\gamma |x-y|^{-(n-1)-|\gamma|}$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\mathrm{cl}\,\Omega)$ (and a corresponding result holds for the exterior of Ω).

In case $n = 3$, $m \geq 2$ and Ω is of the class $C^{m,\alpha}$, and if $P[\mathbf{a}, D]$ is the Helmholtz operator, Kirsch [18] has proved that the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is bounded from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ and, accordingly, is compact in $C^{m,\alpha}(\partial\Omega)$.

von Wahl [35] has considered the case of Sobolev spaces and proved that if Ω is of the class C^∞ and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary.

Later on, Heinemann [15] has developed the ideas of von Wahl in the frame of Schauder spaces and proved that if Ω is of the class C^{m+5} and $S_{\mathbf{a}}$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary, *i.e.*, $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m,\alpha}(\partial\Omega)$ to $C^{m+1,\alpha}(\partial\Omega)$.

Maz'ya and Shaposhnikova [22] have proved that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous in fractional Sobolev spaces under sharp regularity assumptions on the boundary and if $P[\mathbf{a}, D]$ is the Laplace operator.

Mitrea [26] has proved that the double layer of second order equations and systems is compact in $C^{0,\beta}(\partial\Omega)$ for $\beta \in]0, \alpha[$ and bounded in $C^{0,\alpha}(\partial\Omega)$ under the assumption that Ω is of the class $C^{1,\alpha}$. Then by exploiting a formula for the tangential derivatives such results have been extended to the compactness and boundedness results in $C^{1,\beta}(\partial\Omega)$ and $C^{1,\alpha}(\partial\Omega)$, respectively.

Mitrea, Mitrea and Verdera [28] have proved that if q is a homogeneous polynomial of odd order, then the operator with kernel $q(x-y)|x-y|^{-(n-1)-\deg(q)}$ maps $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$.

In this paper, of special interest are the regularizing properties of the operator $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ in Schauder spaces under the assumption that Ω is of the class $C^{m,\alpha}$. We prove our statements by exploiting tangential derivatives and an inductive argument to reduce the problem to the case of the action of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ on $C^{0,\alpha}(\partial\Omega)$ instead of flattening the boundary with parametrization functions as done by the other authors. We mention that the idea of exploiting an inductive argument together with the formula for the tangential gradient in order to prove the continuity and compactness properties of the double layer potential has been used by Kirsch [18, Thm. 3.2] in case $n = 3$, $P[\mathbf{a}, D]$ equals the Helmholtz operator and $S_{\mathbf{a}}$ is the fundamental solution satisfying the radiation condition. The tangential derivatives of $f \in C^1(\partial\Omega)$ are defined by the equality

$$M_{lr}[f] \equiv \nu_l \frac{\partial \widetilde{f}}{\partial x_r} - \nu_r \frac{\partial \widetilde{f}}{\partial x_l} \quad \text{on } \partial\Omega$$

for all $l, r \in \{1, \ldots, n\}$. Here $\widetilde{f}$ denotes an extension of f to an open neighbourhood of $\partial\Omega$, and one can easily verify that $M_{lr}[f]$ is independent of the specific choice of the extension $\widetilde{f}$ of f. Then we prove an explicit formula for

$$M_{lr}\big[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]\big](x) - w\big[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, M_{lr}[\mu]\big](x) \quad \forall\, x \in \partial\Omega \tag{1.5}$$

for all $\mu \in C^1(\partial\Omega)$ and $l, r \in \{1, \dots, n\}$ (see formula (9.1)).

We note that Günter [14, Ch. II, § 10, (42)] presents the formula for the partial derivatives of the double layer with respect to the variables in $\mathbb{R}^n$ in case $n = 3$ and $P[\mathbf{a}, D]$ equals the Laplace operator (see (7.1) for the case of the Laplace operator). A similar formula can be found in Kupradze, Gegelia, Basheleishvili and Burchuladze [20, Ch. V, § 6, (6.11)] for the elastic double layer potential in case $n = 3$. Schwab and Wendland [33] have proved that the difference in (1.5) can be written in terms of pseudodifferential operators of order -1. Dindoš and Mitrea have proved a number of properties of the double layer potential. In particular, [7, Prop. 3.2] proves the existence of integral operators such that the gradient of the double layer potential corresponding to the Stokes system can be written as a sum of such integral operators applied to the gradient of the moment of the double layer. Duduchava, Mitrea, and Mitrea [11] analyze various properties of the tangential deriatives. Duduchava [10] investigates partial differential equations on hypersurfaces and the Bessel potential operators. In particular, [10, point B of the proof of Lem. 2.1] analyzes the commutator properties both of the Bessel potential operator and of a tangential derivative. Hofmann, Mitrea and Taylor [16, (6.2.6)] prove a general formula for the tangential derivatives of the double layer potential corresponding to the second order elliptic *homogeneous* equations and systems in explicit terms.

Formula (9.1), we have computed here, extends the formula of [21] for the Laplace operator, which has been computed with arguments akin to those of Günter [14, Ch. II, § 10, (42)], and a formula of [8] for the Helmholtz operator, and can be considered as a variant of the formula of Hofmann, Mitrea and Taylor [16, (6.2.6)] for the second order *nonhomogeneous* elliptic differential operator $P[\mathbf{a}, D]$.

Formula (9.1) involves auxiliary operators, which we analyze in Section 8. We have based our analysis of the auxiliary operators involved in formula (9.1) on the introduction of boundary norms for weakly singular kernels and on the result of the joint continuity of weakly singular integrals both on the kernel of the integral and on the functional variable of the corresponding integral operator (see Section 6). For fixed choices of the kernel and for some choices of the parameters, such lemmas are known (cf. e.g., Kirsch and Hettlich [19, Thm. 3.17, p. 121]). The authors believe that the methods of Section 6 may be applied to simplify also the exposition of other classical proofs of properties of layer potentials.

By exploiting formula (9.1), we can prove that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a linear and continuous operator from $C^m(\partial\Omega)$ to the generalized Schauder space $C^{m,\omega_\alpha}(\partial\Omega)$ of functions with m-th order derivatives which satisfy a generalized ω_α-Hölder condition with

$$\omega_\alpha(r) \sim r^\alpha |\ln r| \quad \text{as} \quad r \to 0,$$

and that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a linear and continuous operator from $C^{m,\beta}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ for all $\beta \in]0, \alpha]$. In particular, the double layer potential has a regularizing effect on the boundary if Ω is of the class $C^{m,\alpha}$. As a consequence of our result, $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ induces a compact operator from $C^m(\partial\Omega)$ to itself, and from $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ to itself, and from $C^{m,\alpha}(\partial\Omega)$ to itself when Ω is of the class $C^{m,\alpha}$.

2 Notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [6]. If A is a matrix with real or complex entries, then A^t denotes the transpose matrix of A. The set $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in]0, +\infty[$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$ or, more simply, by $C^m(\Omega)$. Let $s \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^s$. Then Df denotes the Jacobian matrix of f. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted by $C^m(\mathrm{cl}\,\Omega)$.

The subspace of $C^m(\mathrm{cl}\ \Omega)$ whose derivatives up to order m are bounded is denoted by $C^m_b(\mathrm{cl}\ \Omega)$. Then $C^m_b(\mathrm{cl}\ \Omega)$ endowed with the norm $\|f\|_{C^m_b(\mathrm{cl}\ \Omega)} \equiv \sum_{|\eta|\leq m} \sup_{\mathrm{cl}\ \Omega} |D^\eta f|$ is a Banach space.

Now, let ω be a function of $]0,+\infty[$ to itself such that

$$\omega \ \text{ is increasing and } \ \lim_{r\to 0^+} \omega(r) = 0\,, \tag{2.1}$$

$$\sup_{(a,t)\in[1,+\infty[\times]0,+\infty[} \frac{\omega(at)}{a\omega(t)} < +\infty,$$

and

$$\sup_{r\in\,]0,1[} \omega^{-1}(r) r < \infty\,. \tag{2.2}$$

If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^n$ to $\mathbb{C}$, we set

$$|f:\mathbb{D}|_{\omega(\cdot)} \equiv \sup\left\{\frac{|f(x)-f(y)|}{\omega(|x-y|)} :\ x,y\in\mathbb{D},\ x\neq y\right\}.$$

If $|f:\mathbb{D}|_{\omega(\cdot)} < \infty$, we say that the function f is $\omega(\cdot)$-Hölder continuous. Sometimes we simply write $|f|_{\omega(\cdot)}$ instead of $|f:\mathbb{D}|_{\omega(\cdot)}$. If $\omega(r)=r$ and $|f:\mathbb{D}|_{\omega(\cdot)}<\infty$, then we say that f is Lipschitz continuous and we set $\mathrm{Lip}(f) \equiv |f:\mathbb{D}|_{\omega(\cdot)}$. The subspace of $C^0(\mathbb{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted by $C^{0,\omega(\cdot)}(\mathbb{D})$, and the subspace of $C^0(\mathbb{D})$ whose functions are Lipschitz continuous is denoted by $\mathrm{Lip}(\mathbb{D})$.

Let Ω be an open subset of $\mathbb{R}^n$. The subspace of $C^m(\mathrm{cl}\ \Omega)$ whose functions have m-th order derivatives that are $\omega(\cdot)$-Hölder continuous is denoted by $C^{m,\omega(\cdot)}(\mathrm{cl}\ \Omega)$. We set

$$C^{m,\omega(\cdot)}_b(\mathrm{cl}\ \Omega) \equiv C^{m,\omega(\cdot)}(\mathrm{cl}\ \Omega) \cap C^m_b(\mathrm{cl}\ \Omega)\,.$$

The space $C^{m,\omega(\cdot)}_b(\mathrm{cl}\,\Omega)$, equipped with its usual norm

$$\|f\|_{C^{m,\omega(\cdot)}_b(\mathrm{cl}\ \Omega)} = \|f\|_{C^m_b(\mathrm{cl}\ \Omega)} + \sum_{|\eta|=m} |D^\eta f:\Omega|_{\omega(\cdot)}\,,$$

is well-known to be a Banach space.

Obviously, $C^{m,\omega(\cdot)}_b(\mathrm{cl}\ \Omega) = C^{m,\omega(\cdot)}(\mathrm{cl}\ \Omega)$ if Ω is bounded (in this case, we shall always drop the subscript b). The subspace of $C^m(\mathrm{cl}\ \Omega)$ of those functions f such that $f_{|\mathrm{cl}\,(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\omega(\cdot)}(\mathrm{cl}\,(\Omega\cap\mathbb{B}_n(0,R)))$ for all $R\in]0,+\infty[$ is denoted by $C^{m,\omega(\cdot)}_{\mathrm{loc}}(\mathrm{cl}\ \Omega)$. Clearly, $C^{m,\omega(\cdot)}_{\mathrm{loc}}(\mathrm{cl}\ \Omega) = C^{m,\omega(\cdot)}(\mathrm{cl}\ \Omega)$ if Ω is bounded.

Of particular importance is the case in which $\omega(\cdot)$ is the function r^α for some fixed $\alpha\in]0,1]$. In this case, we simply write $|\cdot:\mathrm{cl}\,\Omega|_\alpha$ instead of $|\cdot:\mathrm{cl}\,\Omega|_{r^\alpha}$, $C^{m,\alpha}(\mathrm{cl}\ \Omega)$ instead of $C^{m,r^\alpha}(\mathrm{cl}\ \Omega)$, and $C^{m,\alpha}_b(\mathrm{cl}\ \Omega)$ instead of $C^{m,r^\alpha}_b(\mathrm{cl}\ \Omega)$. We observe that property (2.2) implies that

$$C^{m,1}_b(\mathrm{cl}\ \Omega) \subseteq C^{m,\omega(\cdot)}_b(\mathrm{cl}\ \Omega)\,.$$

For the definition of a bounded open Lipschitz subset of $\mathbb{R}^n$, we refer, e.g., to Nečas [29, §1.3]. Let $m\in\mathbb{N}\setminus\{0\}$. We say that a bounded open subset Ω of $\mathbb{R}^n$ is of the class $C^{m,\alpha}$ if for every $P\in\partial\Omega$ there exist an open neighborhood W of P in $\mathbb{R}^n$, and a diffeomorphism $\psi\in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n,\mathbb{R}^n)$ of $\mathbb{B}_n \equiv \{x\in\mathbb{R}^n :\ |x|<1\}$ onto W such that $\psi(0)=P$, $\psi(\{x\in\mathbb{B}_n :\ x_n=0\}) = W\cap\partial\Omega$, $\psi(\{x\in\mathbb{B}_n :\ x_n<0\}) = W\cap\Omega$ (ψ is said to be a parametrization of $\partial\Omega$ around P). Now, let Ω be bounded and of class $C^{m,\alpha}$. By the compactness of $\partial\Omega$ and by definition of a set of the class $C^{m,\alpha}$, there exist $P_1,\dots,P_r\in\partial\Omega$, and parametrizations $\{\psi_i\}_{i=1,\dots,r}$, with $\psi_i\in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n,\mathbb{R}^n)$ such that $\bigcup_{i=1}^r \psi_i(\{x\in\mathbb{B}_n :\ x_n=0\}) = \partial\Omega$. Let $h\in\{1,\dots,m\}$. Let ω be as in (2.1), (2.2). Let

$$\sup_{r\in\,]0,1[} \omega^{-1}(r) r^\alpha < \infty\,. \tag{2.3}$$

We denote by $C^{h,\omega(\cdot)}(\partial\Omega)$ the linear space of functions f of $\partial\Omega$ to $\mathbb{C}$ such that $f\circ\psi_i(\cdot,0)\in C^{h,\omega(\cdot)}(\mathrm{cl}\,\mathbb{B}_{n-1})$ for all $i=1,\dots,r$, and we set

$$\|f\|_{C^{h,\omega(\cdot)}(\partial\Omega)}\equiv\sup_{i=1,\dots,r}\|f\circ\psi_i(\cdot,0)\|_{C^{h,\omega(\cdot)}(\mathrm{cl}\,\mathbb{B}_{n-1})}\quad\forall f\in C^{h,\omega(\cdot)}(\partial\Omega).$$

It is well known that by choosing a different finite family of parametrizations as $\{\psi_i\}_{i=1,\dots,r}$, we would obtain an equivalent norm. In case $\omega(\cdot)$ is the function r^α, we have the spaces $C^{h,\alpha}(\partial\Omega)$.

It is known that $(C^{h,\omega(\cdot)}(\partial\Omega),\|\cdot\|_{C^{h,\omega(\cdot)}(\partial\Omega)})$ is complete. Moreover, condition (2.3) implies that the restriction operator is linear and continuous from $C^{h,\omega(\cdot)}(\mathrm{cl}\,\Omega)$ to $C^{h,\omega(\cdot)}(\partial\Omega)$.

We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$ and retain the standard notation for the Lebesgue spaces.

Remark 2.1. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$.

Let ω be as in (2.1), (2.2). If $h\in\{1,\dots,m\}$, $h<m$, then $m-1\geq 1$ and Ω is of the class $C^{m-1,1}$, and condition (2.2) implies the validity of condition (2.3) with α replaced by 1. Thus we can consider the space $C^{h,\omega(\cdot)}(\partial\Omega)$ even if we do not assume condition (2.3). If instead of h we take m, the definition we gave requires (2.3).

Remark 2.2. Let ω be as in (2.1), $\mathbb{D}$ be a subset of $\mathbb{R}^n$ and let f be a bounded function from $\mathbb{D}$ to $\mathbb{C}$, $a\in]0,+\infty[$. Then

$$\sup_{x,y\in\mathbb{D},\,|x-y|\geq a}\frac{|f(x)-f(y)|}{\omega(|x-y|)}\leq\frac{2}{\omega(a)}\sup_{\mathbb{D}}|f|\,.$$

Thus the difficulty of estimating the Hölder quotient $\frac{|f(x)-f(y)|}{\omega(|x-y|)}$ of a bounded function f lies entirely in case $0<|x-y|<a$. Then we have the following well known extension result. For a proof, we refer to Troianiello [34, Thm. 1.3, Lem. 1.5].

Lemma 2.1. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$, $j\in\{0,\dots,m\}$, Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$, and let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Then there exists a linear and continuous extension operator '$\sim$' of $C^{j,\alpha}(\partial\Omega)$ to $C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$, which takes $\mu\in C^{j,\alpha}(\partial\Omega)$ to a map $\widetilde{\mu}\in C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ such that $\widetilde{\mu}_{|\partial\Omega}=\mu$ and the support of μ is compact and contained in $\mathbb{B}_n(0,R)$. The same statement holds by replacing $C^{m,\alpha}$ by C^m and $C^{j,\alpha}$ by C^j.*

Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class C^1. The tangential gradient $D_{\partial\Omega}f$ of $f\in C^1(\partial\Omega)$ is defined as

$$D_{\partial\Omega}f\equiv D\widetilde{f}-(\nu\cdot D\widetilde{f})\nu\quad\text{on}\quad\partial\Omega\,,$$

where $\widetilde{f}$ is an extension of f of the class C^1 in an open neighborhood of $\partial\Omega$, and we have

$$\frac{\partial\widetilde{f}}{\partial x_r}-(\nu\cdot D\widetilde{f})\nu_r=\sum_{l=1}^{n}M_{lr}[f]\nu_l\quad\text{on}\quad\partial\Omega$$

for all $r\in\{1,\dots,n\}$. If $\mathbf{a}$ is as in (1.1), (1.2), then we also set

$$D_{\mathbf{a}}f\equiv(D_{\mathbf{a},r}f)_{r=1,\dots,n}\equiv D\widetilde{f}-\frac{D\widetilde{f}a^{(2)}\nu}{\nu^ta^{(2)}\nu}\,\nu\quad\text{on}\quad\partial\Omega\,.$$

Since

$$D_{\mathbf{a},r}f=\frac{\partial\widetilde{f}}{\partial x_r}-\frac{D\widetilde{f}a^{(2)}\nu}{\nu^ta^{(2)}\nu}\,\nu_r=\sum_{l=1}^{r}M_{lr}[f]\left(\frac{\sum_{h=1}^{n}a_{lh}\nu_h}{\nu^ta^{(2)}\nu}\right)\quad\text{on}\quad\partial\Omega\tag{2.4}$$

for all $r\in\{1,\dots,n\}$, $D_{\mathbf{a}}f$ is independent of the specific choice of the extension $\widetilde{f}$ of f. We also need the following well known consequence of the Divergence Theorem.

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of the class C^1. If φ, $\psi \in C^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} M_{lj}[\varphi]\psi \, d\sigma = -\int_{\partial\Omega} \varphi M_{lj}[\psi] \, d\sigma$$

for all $l, j \in \{1, \dots, n\}$.

Next, we introduce the following auxiliary Lemmas, whose proof is based on the definition of the norm in a Schauder space.

Lemma 2.3. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. Let ω be as in (2.1), (2.2), (2.3), and let Ω be a bounded open connected subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *A function $f \in C^1(\partial\Omega)$ belongs to $C^{m,\omega(\cdot)}(\partial\Omega)$ if and only if $M_{lr}[f] \in C^{m-1,\omega(\cdot)}(\partial\Omega)$ for all $l, r \in \{1, \dots, n\}$.*

(ii) *The norm $\|\cdot\|_{C^{m,\omega(\cdot)}(\partial\Omega)}$ is equivalent to the norm on $C^{m,\omega(\cdot)}(\partial\Omega)$ defined by*

$$\|f\|_{C^0(\partial\Omega)} + \sum_{l,r=1}^{n} \left\|M_{lr}[f]\right\|_{C^{m-1,\omega(\cdot)}(\partial\Omega)} \quad \forall\, f \in C^{m,\omega(\cdot)}(\partial\Omega)\,.$$

We have the following (see also Remark 2.1)

Lemma 2.4. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, and let $h \in \{1, \dots, m\}$. Then the following statements hold:*

(i) *Let $h < m$ and ω be as in (2.1), (2.2). Then M_{lj} is linear and continuous from $C^{h,\omega(\cdot)}(\partial\Omega)$ to $C^{h-1,\omega(\cdot)}(\partial\Omega)$ for all $l, j \in \{1, \dots, n\}$. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(ii) *Let $h < m$, ω be as in (2.1), (2.2), and let $\mathbf{a}$ be as in (1.1), (1.2). Then the function from $C^{h,\omega(\cdot)}(\partial\Omega)$ to $C^{h-1,\omega(\cdot)}(\partial\Omega, \mathbb{R}^n)$, which takes f to $D_{\mathbf{a}}f$ is linear and continuous. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(iii) *Let $h < m$ and ω be as in (2.1), (2.2). Then the space $C^{h,\omega(\cdot)}(\partial\Omega)$ is continuously imbedded into $C^{h-1,1}(\partial\Omega)$. If we further assume that ω satisfies condition (2.3), then the same statement holds also for $h = m$.*

(iv) *Let $h < m$. Let ψ_1, ψ_2 be as in (2.1), (2.2), and let the condition $\sup_{r \in]0,1[} \psi_2^{-1}(r)\psi_1(r) < \infty$ hold. Then $C^{h,\psi_1(\cdot)}(\partial\Omega)$ is continuously imbedded into $C^{h,\psi_2(\cdot)}(\partial\Omega)$. If we further assume that ψ_j satisfies condition (2.3) for $j \in \{1, 2\}$, then the same statement holds also for $h = m$.*

(v) *Let $h < m$. Let ψ_1, ψ_2, ψ_3 be as in (2.1), (2.2), and let the conditions $\sup_{j=1,2} \sup_{r \in]0,1[} \psi_j(r)\psi_3^{-1}(r) < \infty$ hold. Then the pointwise product is bilinear and continuous from $C^{h,\psi_1(\cdot)}(\partial\Omega) \times C^{h,\psi_2(\cdot)}(\partial\Omega)$ to $C^{h,\psi_3(\cdot)}(\partial\Omega)$. If we further assume that ψ_j satisfies condition (2.3) for $j \in \{1, 2, 3\}$, then the same statement holds also for $h = m$.*

Lemma 2.5. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let ψ_1, ψ_2, ψ_3 be as in (2.1), (2.2), and let the conditions $\sup_{j=1,2} \sup_{r \in]0,1[} \psi_j(r)\psi_3^{-1}(r) < \infty$ hold. Then the pointwise product is bilinear and continuous from $C^{0,\psi_1(\cdot)}(\partial\Omega) \times C^{0,\psi_2(\cdot)}(\partial\Omega)$ to $C^{0,\psi_3(\cdot)}(\partial\Omega)$.*

3 Preliminary inequalities

We first introduce the following elementary lemma on matrices.

Lemma 3.1. *Let $\Lambda \in M_n(\mathbb{R})$ be invertible. Let $|\Lambda| \equiv \sup_{|x|=1} |\Lambda x|$. Then the following statements hold:*

(i) *If $\tau_\Lambda \equiv \max\{|\Lambda|, |\Lambda^{-1}|\}$, then*

$$\tau_\Lambda^{-1}|x| \leq |\Lambda x| \leq \tau_\Lambda |x| \quad \forall\, x \in \mathbb{R}^n \,.$$

(ii) *If $r \in]0, +\infty[$, then*

$$|\Lambda^{-1}x|^{-r} \leq |\Lambda|^r |x|^{-r} \quad \forall\, x \in \mathbb{R}^n \setminus \{0\} \,.$$

Proof. Statement (i) is well known. We now consider statement (ii). Let $x \in \mathbb{R}^n \setminus \{0\}$. Then we have

$$|x| = |\Lambda(\Lambda^{-1}x)| \leq |\Lambda|\, |\Lambda^{-1}x| \,.$$

Hence, $|\Lambda^{-1}x| \geq |\Lambda|^{-1}|x|$ and the statement follows. □

Then we introduce the following elementary lemma, which collects either the known inequalities or the variants of the known inequalities, which we will need in the sequel.

Lemma 3.2. *Let $\gamma \in \mathbb{R}$ and $\Lambda \in M_n(\mathbb{R})$ be invertible. The following statements hold:*

(i)

$$\frac{1}{2}\,|x' - y| \leq |x'' - y| \leq 2|x' - y| \,,$$

$$\frac{1}{2\tau_\Lambda^2}\,|\Lambda x' - \Lambda y| \leq |\Lambda x'' - \Lambda y| \leq 2\tau_\Lambda^2 |\Lambda x' - \Lambda y| \,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(ii)

$$|x' - y|^\gamma \leq 2^{|\gamma|}|x'' - y|^\gamma \,, \quad |x'' - y|^\gamma \leq 2^{|\gamma|}|x' - y|^\gamma \,,$$

$$|\Lambda x' - \Lambda y|^\gamma \leq (2\tau_\Lambda^2)^{|\gamma|}|\Lambda x'' - \Lambda y|^\gamma \,, \quad |\Lambda x'' - \Lambda y|^\gamma \leq (2\tau_\Lambda^2)^{|\gamma|}|\Lambda x' - \Lambda y|^\gamma \,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(iii)

$$\big||x' - y|^\gamma - |x'' - y|^\gamma\big| \leq (2^{|\gamma|} - 1)|x' - y|^\gamma \quad \forall\, y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|) \,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

(iv) *There exist m_γ, $m_\gamma(\Lambda) \in]0, +\infty[$ such that*

$$\big||x' - y|^\gamma - |x'' - y|^\gamma\big| \leq m_\gamma |x' - x''|\, |x' - y|^{\gamma - 1},$$

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq m_\gamma(\Lambda)|\Lambda x' - \Lambda x''|\, |\Lambda x' - \Lambda y|^{\gamma - 1}$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$, $y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)$.

(v)

$$\big|\ln|x' - y| - \ln|x'' - y|\big| \leq 2|x' - x''|\, |x' - y|^{-1} \quad \forall\, y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|) \,,$$

for all $x', x'' \in \mathbb{R}^n$, $x' \neq x''$.

Proof. The first two inequalities of statement (i) follow by the triangular inequality. Further, we have

$$|\Lambda x' - \Lambda y| \leq \tau_\Lambda |x' - y| \leq \tau_\Lambda 2|x'' - y| \leq 2\tau_\Lambda^2 |\Lambda x'' - \Lambda y| \,,$$

and thus the first of the second two inequalities of statement (i) holds true. The second of the second two inequalities of statement (i) can be proved by interchanging the roles of x' and x''.

It now suffices to prove only the second inequalities in statements (ii), (iv). Indeed, the first inequalities follow by the second ones and by the equality $\tau_\Lambda = 1$ when Λ is the identity matrix.

The first of the second inequalities in (ii) for $\gamma \geq 0$ follows by raising the inequality $|\Lambda x' - \Lambda y| \leq (2\tau_\Lambda^2)|\Lambda x'' - \Lambda y|$ of statement (i) to the power γ. For $\gamma < 0$ the same inequality follows by raising the inequality $|\Lambda x'' - \Lambda y| \leq (2\tau_\Lambda^2)|\Lambda x' - \Lambda y|$ of statement (i) to the power γ. The second of the second inequalities of (ii) can be proved by interchanging the roles of x' and x''.

Statement (iii) follows by a direct application of (ii). To prove (iv) and (v), we follow Cialdea [1, § 8]. First consider (iv) and assume that $|\Lambda x' - \Lambda y| \leq |\Lambda x'' - \Lambda y|$. By the Lagrange Theorem, there exists $\zeta \in [|\Lambda x' - \Lambda y|, |\Lambda x'' - \Lambda y|]$ such that

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq \big|\gamma|\zeta^{\gamma-1}|\,|\Lambda x' - \Lambda y| - |\Lambda x'' - \Lambda y|\big|\,.$$

If $\gamma \geq 1$, then the inequality $\zeta \leq |\Lambda x'' - \Lambda y|$ and (i) imply

$$\zeta^{\gamma-1} \leq |\Lambda x'' - \Lambda y|^{\gamma-1} \leq (2\tau_\Lambda^2)^{|\gamma-1|}|\Lambda x' - \Lambda y|^{\gamma-1}\,.$$

If $\gamma < 1$, then the inequalities $\zeta \geq |\Lambda x' - \Lambda y|$ and $\tau_\Lambda \geq 1$ imply

$$\zeta^{\gamma-1} \leq |\Lambda x' - \Lambda y|^{\gamma-1} \leq (2\tau_\Lambda^2)^{|\gamma-1|}|\Lambda x' - \Lambda y|^{\gamma-1}\,.$$

Then we have

$$\big||\Lambda x' - \Lambda y|^\gamma - |\Lambda x'' - \Lambda y|^\gamma\big| \leq |\gamma|(2\tau_\Lambda^2)^{|\gamma-1|}\big||\Lambda x' - \Lambda y| - |\Lambda x'' - \Lambda y|\big|\,|\Lambda x' - \Lambda y|^{\gamma-1}\,, \tag{3.1}$$

which implies the validity of (iv). Similarly, in case $|\Lambda x' - \Lambda y| > |\Lambda x'' - \Lambda y|$, we can prove that (3.1) holds with x' and x'' interchanged. Thus (ii) implies the validity of (iv).

We now consider statement (v) and assume that $|x' - y| \leq |x'' - y|$. By the Lagrange Theorem, there exists $\zeta \in [|x' - y|, |x'' - y|]$ such that

$$\big|\ln|x' - y| - \ln|x'' - y|\big| \leq \zeta^{-1}|\,|x' - y| - |x'' - y|\,| \leq \zeta^{-1}|x' - x''|\,. \tag{3.2}$$

By the above assumption, $\zeta^{-1} \leq |x' - y|^{-1}$, and thus statement (v) follows. Similarly, if $|x' - y| > |x'' - y|$, we can prove that (3.2) holds with x' and x'' interchanged and (i) implies that $\zeta^{-1} \leq |x'' - y|^{-1} \leq 2|x' - y|^{-1}$, which yields the validity of (v). □

Lemma 3.3. *Let G be a nonempty bounded subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $F \in \mathrm{Lip}(\partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)])$ with*

$$\mathrm{Lip}(F) \equiv \left\{\frac{|F(\theta', r') - F(\theta'', r'')|}{|\theta' - \theta''| + |r' - r''|} :\ (\theta', r'), (\theta'', r'') \in \partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)],\ (\theta', r') \neq (\theta'', r'')\right\}.$$

Then

$$\left|F\Big(\frac{x' - y}{|x' - y|}, |x' - y|\Big) - F\Big(\frac{x'' - y}{|x'' - y|}, |x'' - y|\Big)\right| \leq \mathrm{Lip}(F)(2 + \mathrm{diam}\,(G))\,\frac{|x' - x''|}{|x' - y|} \qquad \forall\, y \in G \setminus \mathbb{B}_n\big(x', 2|x' - x''|\big) \tag{3.3}$$

for all $x', x'' \in G$, $x' \neq x''$. In particular, if $f \in C^1(\partial\mathbb{B}_n \times \mathbb{R}, \mathbb{C})$, then

$$M_{f,G} \equiv \sup\left\{\left|f\Big(\frac{x' - y}{|x' - y|}, |x' - y|\Big) - f\Big(\frac{x'' - y}{|x'' - y|}, |x'' - y|\Big)\right|\frac{|x' - y|}{|x' - x''|} :\ x', x'' \in G,\ \ x' \neq x'',\ \ y \in G \setminus \mathbb{B}_n\big(x', 2|x' - x''|\big)\right\} < \infty\,.$$

(ii) *Let W be an open neighbourhood of $\mathrm{cl}\,(G - G)$. Let $f \in C^1(W, \mathbb{C})$. Then*

$$\widetilde{M}_{f,G} \equiv \sup\Big\{\big|f(x' - y) - f(x'' - y)\big|\,|x' - x''|^{-1} :\ x', x'' \in G,\ x' \neq x'',\ y \in G\Big\} < \infty\,.$$

Here $G - G \equiv \{y_1 - y_2 :\ y_1, y_2 \in G\}$.

Proof. First we consider statement (i). The Lipschitz continuity of F implies that the left-hand side of (3.3) is less or equal to

$$
\begin{aligned}
&\mathrm{Lip}(F)\left\{\left|\frac{x'-y}{|x'-y|}-\frac{x''-y}{|x''-y|}\right|+\big||x'-y|-|x''-y|\big|\right\}\\
&\quad\le \mathrm{Lip}(F)\left\{|x''-y|\left|\frac{1}{|x''-y|}-\frac{1}{|x'-y|}\right|+\frac{1}{|x'-y|}\big||x''-y|-|x'-y|\big|+|x'-x''|\right\}\\
&\le \mathrm{Lip}(F)\left\{|x''-y|\,\frac{|x'-x''|}{|x''-y|\,|x'-y|}+\frac{|x'-x''|}{|x'-y|}+|x'-x''|\right\}\le \mathrm{Lip}(F)|x'-x''|\left\{\frac{2+|x'-y|}{|x'-y|}\right\},
\end{aligned}
$$

and thus inequality (3.3) holds true.

Since $\partial\mathbb{B}_n\times\mathbb{R}$ is a manifold of the class C^∞ imbedded into $\mathbb{R}^{n+1}$, there exists $F\in C^1(\mathbb{R}^{n+1})$ which extends f. Since $\partial\mathbb{B}_n\times[0,\mathrm{diam}\,(G)]$ is a compact subset of $\mathbb{R}^{n+1}$, F is Lipschitz continuous on $\partial\mathbb{B}_n\times[0,\mathrm{diam}\,(G)]$, and the second part of statement (i) follows by inequality (3.3).

We now consider statement (ii). Since $f\in C^1(W,\mathbb{C})$, f is Lipschitz continuous on the compact set $\mathrm{cl}\,(G-G)$, and statement (ii) follows. □

We have the following well-known statement.

Lemma 3.4. *Let $\alpha\in]0,1]$ and Ω be a bounded open connected subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then there exists $c_{\Omega,\alpha}>0$ such that*

$$\big|\nu(y)\cdot(x-y)\big|\le c_{\Omega,\alpha}|x-y|^{1+\alpha}\quad\forall\,x,y\in\partial\Omega\,.$$

Next, we introduce a list of classical inequalities which can be verified by exploiting the local parametrizations of $\partial\Omega$.

Lemma 3.5. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $\gamma\in]-\infty,n-1[$. Then*

$$c'_{\Omega,\gamma}\equiv\sup_{x\in\partial\Omega}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^\gamma}<+\infty\,.$$

(ii) *Let $\gamma\in]-\infty,n-1[$. Then*

$$c''_{\Omega,\gamma}\equiv\sup_{x',x''\in\partial\Omega,\ x'\neq x''}|x'-x''|^{-(n-1)+\gamma}\int_{\mathbb{B}_n(x',3|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x'-y|^\gamma}<+\infty\,.$$

(iii) *Let $\gamma\in]n-1,+\infty[$. Then*

$$c'''_{\Omega,\gamma}\equiv\sup_{x',x''\in\partial\Omega,\ x'\neq x''}|x'-x''|^{-(n-1)+\gamma}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^\gamma}$$

is finite.

(iv)

$$c^{iv}_{\Omega}\equiv\sup_{x',x''\in\partial\Omega,\ 0<|x'-x''|<1/e}\big|\ln|x'-x''|\big|^{-1}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{n-1}}<+\infty\,.$$

4 Preliminaries on the fundamental solution

First we introduce a formula for the fundamental solution of $P[\mathbf{a}, D]$. For this, we follow a formulation of Dalla Riva [3, Thm. 5.2, 5.3] and Dalla Riva, Morais and Musolino [5, Thm. 3.1, 3.2] (see also John [17], Miranda [24] for homogeneous operators, and Mitrea and Mitrea [27, p. 203]).

Theorem 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then there exist a real analytic function* A_0 *from* $\partial\mathbb{B}_n$ *to* $\mathbb{C}$, *a real analytic function* A_1 *from* $\partial\mathbb{B}_n \times \mathbb{R}$ *to* $\mathbb{C}$, $b_0 \in \mathbb{C}$, *a real analytic function* B_1 *from* $\mathbb{R}^n$ *to* $\mathbb{C}$, $B_1(0) = 0$, *and a real analytic function* C *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that*

$$S_{\mathbf{a}}(x) = |x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) + |x|^{3-n} A_1\Big(\frac{x}{|x|}, |x|\Big) + b_0 \ln|x| + B_1(x) \ln|x| + C(x) \tag{4.1}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$, *and both* b_0 *and* B_1 *equal zero if* n *is odd. Moreover,*

$$|x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) + \delta_{2,n} b_0 \ln|x|$$

is a fundamental solution for the principal part $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$. *Here* $\delta_{2,n}$ *denotes the Kronecker symbol. Namely,*

$$\delta_{2,n} = 1 \ \ \textit{if} \ \ n = 2, \quad \delta_{2,n} = 0 \ \ \textit{if} \ \ n > 2\,.$$

Corollary 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then the following statements hold:*

(i) *If* $n \geq 3$, *then there exists one and only one fundamental solution of the principal part* $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$ *which is positively homogeneous of degree* $2 - n$ *in* $\mathbb{R}^n \setminus \{0\}$.

(ii) *If* $n = 2$, *then there exists one and only one fundamental solution* $S(x)$ *of the principal part* $\sum_{l,j=1}^{n} \partial_{x_l}(a_{lj}\partial_{x_j})$ *of* $P[\mathbf{a}, D]$ *such that*

$$\beta_0 \equiv \lim_{x\to 0} \frac{S(x)}{\ln|x|} \in \mathbb{C}\,, \qquad \int_{\partial\mathbb{B}_n} S\, d\sigma = 0\,,$$

and $S(x) - \beta_0 \ln|x|$ *is positively homogeneous of degree* 0 *in* $\mathbb{R}^n \setminus \{0\}$.

Proof. We retain the notation of Theorem 4.1. We first consider statement (i). By Theorem 4.1, the function $|x|^{2-n}A_0(\frac{x}{|x|})$ is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and is, clearly, positively homogeneous of degree $2-n$. Now assume that u is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and u is positively homogeneous of degree $2 - n$ in $\mathbb{R}^n \setminus \{0\}$. Then the difference

$$w(x) \equiv |x|^{2-n} A_0\Big(\frac{x}{|x|}\Big) - u(x)$$

defines an entire real analytic function in $\mathbb{R}^n$ and is positively homogeneous of degree $2-n$ in $\mathbb{R}^n \setminus \{0\}$. In particular,

$$\lambda^{n-2} w(\lambda x) = w(x) \quad \forall\, (\lambda, x) \in]0, +\infty[\, \times (\mathbb{R}^n \setminus \{0\})\,,$$

and, accordingly,

$$\lambda^{(n-2)+|\beta|} D^\beta w(\lambda x) = D^\beta w(x) \quad \forall\, (\lambda, x) \in]0, +\infty[\, \times (\mathbb{R}^n \setminus \{0\})$$

for all $\beta \in \mathbb{N}^n$. Then by letting λ tend to 0^+, we obtain $D^\beta w(0) = 0$ for all $\beta \in \mathbb{N}^n$. Since w is real analytic, we deduce that w is equal to 0 in $\mathbb{R}^n$ and thus statement (i) holds.

Now assume that $n = 2$. By Theorem 4.1, the function

$$S(x) \equiv A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma + b_0 \ln |x|$$

is a fundamental solution of the principal part of $P[\mathbf{a}, D]$ and satisfies the conditions of statement (ii). Suppose that u is another fundamental solution. Then the difference

$$w(x) \equiv A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma + b_0 \ln |x| - u(x)$$

defines an entire real analytic function in $\mathbb{R}^n$ and we have

$$0 = \lim_{x \to 0} \frac{w(x)}{\ln |x|} = \lim_{x \to 0} \frac{A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma}{\ln |x|} + b_0 - \lim_{x \to 0} \frac{u(x)}{\ln |x|} \,,$$

and, accordingly,

$$b_0 = \lim_{x \to 0} \frac{u(x)}{\ln |x|} \equiv \beta_0 \in \mathbb{C} \,.$$

Then our assumption implies that the real analytic function

$$u(x) - \beta_0 \ln |x| = u(x) - b_0 \ln |x|$$

is positively homogeneous of degree 0 in $\mathbb{R}^n \setminus \{0\}$. Hence, there exists a function g_0 from $\partial\mathbb{B}_n$ to $\mathbb{C}$ such that

$$u(x) - b_0 \ln |x| = g_0\Big(\frac{x}{|x|}\Big) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\} \,.$$

In particular, g_0 is real analytic and

$$\begin{aligned} w(x) &= A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma + b_0 \ln |x| - \Big(g_0\Big(\frac{x}{|x|}\Big) + b_0 \ln |x|\Big) \\ &= A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma - g_0\Big(\frac{x}{|x|}\Big) \,. \end{aligned}$$

Moreover, w must be positively homogeneous of degree 0 in $\mathbb{R}^n \setminus \{0\}$. Since w is continuous at 0, w must be constant in the whole $\mathbb{R}^n$. Since

$$\int_{\partial\mathbb{B}_n} w \, d\sigma = \int_{\partial\mathbb{B}_n} S \, d\sigma - \int_{\partial\mathbb{B}_n} u \, d\sigma = 0,$$

such a constant must equal 0 and thus

$$A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma = g_0\Big(\frac{x}{|x|}\Big) \ \text{ for all } \ x \in \mathbb{R}^n \setminus \{0\}.$$

Hence,

$$u(x) = A_0\Big(\frac{x}{|x|}\Big) - \frac{1}{2\pi} \int_{\partial\mathbb{B}_n} A_0 \, d\sigma + b_0 \ln |x|$$

and statement (ii) follows. □

We can introduce the following

Definition 4.1. Let $\mathbf{a}$ be as in (1.1), (1.2). We define the normalized fundamental solution of the principal part of $P[\mathbf{a}, D]$, to be the only fundamental solution of Corollary 4.1.

By Theorem 4.1 and Corollary 4.1, the normalized fundamental solution of the principal part of $P[\mathbf{a},D]$ equals

$$|x|^{2-n}A_0\Big(\frac{x}{|x|}\Big)$$

if $n\geq 3$, and

$$A_0\Big(\frac{x}{|x|}\Big)-\frac{1}{2\pi}\int_{\partial\mathbb{B}_n}A_0\,d\sigma+b_0\ln|x|$$

if $n=2$, where A_0 and b_0 are as in Theorem 4.1. We now see that if the principal coefficients of $P[\mathbf{a},D]$ are real, then the normalized fundamental solution of the principal part of $P[\mathbf{a},D]$ has a very specific form. To do so, we introduce the fundamental solution S_n of the Laplace operator. Namely, we set

$$S_n(x)\equiv\begin{cases}\frac{1}{s_n}\ln|x| & \forall\,x\in\mathbb{R}^n\setminus\{0\},\ \text{ if }\ n=2\,,\\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall\,x\in\mathbb{R}^n\setminus\{0\},\ \text{ if }\ n>2\,,\end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n$. Then we have the following elementary statement, which can be verified by the chain rule and by Corollary 4.1 (cf. e.g., Dalla Riva [4]).

Lemma 4.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3). *Then there exists an invertible matrix* $T\in M_n(\mathbb{R})$ *such that*

$$a^{(2)}=TT^t \tag{4.2}$$

and the function

$$S_{a^{(2)}}(x)\equiv\frac{1}{\sqrt{\det a^{(2)}}}\,S_n(T^{-1}x)\quad\forall\,x\in\mathbb{R}^n\setminus\{0\}\,,$$

coincides with the normalized fundamental solution of the principal part of $P[\mathbf{a},D]$ *if* $n\geq 3$, *and coincides with the normalized fundamental solution of the principal part of* $P[\mathbf{a},D]$ *up to an additive constant if* $n=2$.

Theorem 4.1, Corollary 4.1 and Lemma 4.1 imply the validity of the following

Corollary 4.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T\in M_n(\mathbb{R})$ *be as in* (4.2) *and let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$.

Then there exist a real analytic function A_1 *from* $\partial\mathbb{B}_n\times\mathbb{R}$ *to* $\mathbb{C}$, *a real analytic function* B_1 *from* $\mathbb{R}^n$ *to* $\mathbb{C}$, $B_1(0)=0$, *and a real analytic function* C *from* $\mathbb{R}^n$ *to* $\mathbb{C}$ *such that*

$$S_{\mathbf{a}}(x)=\frac{1}{\sqrt{\det a^{(2)}}}\,S_n(T^{-1}x)+|x|^{3-n}A_1\Big(\frac{x}{|x|},|x|\Big)+\big(B_1(x)+b_0(1-\delta_{2,n})\big)\ln|x|+C(x)\,, \tag{4.3}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$, *and both* b_0 *and* B_1 *equal zero if* n *is odd. Moreover,*

$$\frac{1}{\sqrt{\det a^{(2)}}}\,S_n(T^{-1}x)$$

is a fundamental solution for the principal part of $P[\mathbf{a},D]$.

Next we prove the following technical statement.

Lemma 4.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$ *and let* G *be a nonempty bounded subset of* $\mathbb{R}^n$.

(i) *Let* $\gamma\in[0,1[$. *Then*

$$C_{0,S_{\mathbf{a}},G,n-1-\gamma}\equiv\sup_{0<|x|\leq\operatorname{diam}(G)}|x|^{n-1-\gamma}|S_{\mathbf{a}}(x)|<+\infty\,. \tag{4.4}$$

If $n>2$, *then* (4.4) *holds also for* $\gamma=1$.

(ii)

$$\widetilde{C}_{0,S_{\mathbf{a}},G} \equiv \sup\Big\{ \frac{|x'-y|^{n-1}}{|x'-x''|} \big| S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y) \big| : \\ x', x'' \in G, x' \neq x'',\ y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|) \Big\} < \infty .$$

Proof. Statement (i) is an immediate consequence of formula (4.1). Now prove statement (ii). For this, we resort to formula (4.1) and set

$$A(\theta, r) \equiv A_0(\theta) + r A_1(\theta, r) \quad \forall (\theta, r) \in \partial\mathbb{B}_n \times \mathbb{R}, \\ B(x) \equiv b_0 + B_1(x) \quad \forall x \in \mathbb{R}^n .$$

Then Lemmas 3.2 and 3.3 imply

$$\begin{aligned} \big| S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y) \big| &\leq |x'-y|^{2-n} \Big| A\Big(\frac{x'-y}{|x'-y|}, |x'-y| \Big) - A\Big(\frac{x''-y}{|x''-y|}, |x''-y| \Big) \Big| \\ &+ \Big| A\Big(\frac{x''-y}{|x''-y|}, |x''-y| \Big) \Big| \big| |x'-y|^{2-n} - |x''-y|^{2-n} \big| + \big| \ln|x'-y| \big| \big| B(x'-y) - B(x''-y) \big| \\ &+ |B(x''-y)| \big| \ln|x'-y| - \ln|x''-y| \big| + \big| C(x'-y) - C(x''-y) \big| \\ &\leq |x'-y|^{2-n} M_{A,G} \frac{|x'-x''|}{|x'-y|} + \Big(\sup_{\partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)]} |A| \Big) m_{2-n} \frac{|x'-x''|}{|x'-y|^{n-1}} \\ &+ \big| \ln|x'-y| \big| \widetilde{M}_{B,G} |x'-x''| + \sup_{G-G} |B|\, 2 \frac{|x'-x''|}{|x'-y|} + \widetilde{M}_{C,G} |x'-x''| . \end{aligned}$$

Since A is continuous on the compact set $\partial\mathbb{B}_n \times [0, \mathrm{diam}\,(G)]$, and B and C are continuous on the compact set $\mathrm{cl}\,(G-G)$, there exists $c > 0$ such that

$$\begin{aligned} \big| S_{\mathbf{a}}(x'-y) - S_{\mathbf{a}}(x''-y) \big| &\leq c|x'-x''| \Big\{ |x'-y|^{1-n} + \frac{1}{|x'-y|} + \ln|x'-y| + 1 \Big\} \\ &\leq c|x'-x''|\, |x'-y|^{1-n} \Big\{ 1 + |x'-y|^{n-2} + |x'-y|^{n-1} \ln|x'-y| + |x'-y|^{n-1} \Big\} , \end{aligned}$$

and thus statement (ii) holds. □

Lemma 4.3. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T \in M_n(\mathbb{R})$ *be as in* (4.2). *Let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, B_1, C *be as in Corollary* 4.2, *and let* G *be a nonempty bounded subset of* $\mathbb{R}^n$. *Then the following statements hold:*

(i) *There exists a real analytic function* A_2 *from* $\partial\mathbb{B}_n \times \mathbb{R}$ *to* $\mathbb{C}^n$ *such that*

$$DS_{\mathbf{a}}(x) = \frac{1}{s_n \sqrt{\det a^{(2)}}} |T^{-1}x|^{-n} x^t (a^{(2)})^{-1} \\ + |x|^{2-n} A_2\Big(\frac{x}{|x|}, |x| \Big) + DB_1(x) \ln|x| + DC(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\} . \quad (4.5)$$

(ii)

$$C_{1,S_{\mathbf{a}},G} \equiv \sup_{0<|x|\leq \mathrm{diam}\,(G)} |x|^{n-1} |DS_{\mathbf{a}}(x)| < +\infty .$$

(iii)

$$\widetilde{C}_{1,S_{\mathbf{a}},G} \equiv \sup\Big\{ \frac{|x'-y|^{n}}{|x'-x''|} \big| DS_{\mathbf{a}}(x'-y) - DS_{\mathbf{a}}(x''-y) \big| : \\ x', x'' \in G, x' \neq x'',\ y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|) \Big\} < \infty .$$

Proof. By formula (4.3) and by the chain rule, we have

$$
\begin{aligned}
DS_{\mathbf{a}}(x) = {} & \frac{1}{s_n\sqrt{\det a^{(2)}}}\,|T^{-1}x|^{-n}x^t(a^{(2)})^{-1} + (3-n)|x|^{2-n}\,\frac{x^t}{|x|}A_1\left(\frac{x}{|x|},|x|\right) \\
& + |x|^{3-n}\left\{DA_1\Big(\frac{x}{|x|},|x|\Big)\big[|x|I - x\otimes x|x|^{-1}\big]|x|^{-2} + \frac{\partial A_1}{\partial r}\Big(\frac{x}{|x|},|x|\Big)\,\frac{x^t}{|x|}\right\} \\
& + DB_1(x)\ln|x| + \big(B_1(x) + b_0(1-\delta_{2,n})\big)\,\frac{x^t}{|x|^2} + DC(x) \qquad (4.6)
\end{aligned}
$$

for all $x \in \mathbb{R}^n \setminus \{0\}$, where by A_1 we have still denote any real analytic extension of the function A_1 of Corollary 4.2 to an open neighbourhood of $\partial\mathbb{B}_n \times \mathbb{R}$ in $\mathbb{R}^{n+1}$ and where $x \otimes x$ denotes the matrix $(x_l x_j)_{l,j=1,\dots,n}$. Next, we consider the term $B_1(x)/|x|$. By the Fundamental Theorem of Calculus, we have

$$
\frac{B_1(x)}{|x|} = \int_0^1 DB_1\Big(t\,\frac{x}{|x|}|x|\Big)\,\frac{x}{|x|}\,dt \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,. \qquad (4.7)
$$

Thus, if we set

$$
\beta(\theta, r) = \int_0^1 DB_1(t\theta r)\theta\, dt \quad \forall\,(\theta, r) \in \mathbb{R}^n \times \mathbb{R}\,,
$$

the function β will be real analytic and will satisfy the equality

$$
\frac{B_1(x)}{|x|} = \beta\Big(\frac{x}{|x|},|x|\Big) \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}\,. \qquad (4.8)
$$

Define

$$
\begin{aligned}
A_2(\theta, r) \equiv {} & (3-n)\theta^t A_1(\theta, r) + DA_1(\theta, r)[I - \theta\otimes\theta] + \frac{\partial A_1}{\partial r}(\theta, r)\theta^t r \\
& + \beta(\theta, r)r^{n-2}\theta^t + r^{n-3}\theta^t b_0(1-\delta_{2,n}) \quad \forall\,(\theta, r) \in \partial\mathbb{B}_n \times \mathbb{R}\,.
\end{aligned}
$$

By the real analyticity of A_1 and β, and by the equality $r^{n-3}\theta^t b_0(1-\delta_{2,n}) = 0$ if $n = 2$, the function A_2 is real analytic. Hence, equalities (4.6) and (4.8) imply the validity of statement (i).

Next, we turn to the proof of statement (ii). By Lemma 3.1(ii) and by the Schwartz inequality, we have

$$
|T^{-1}x|^{-n}\,|x^t(a^{(2)})^{-1}| \le |x|^{1-n}|T|^n\big|(a^{(2)})^{-1}\big|\,.
$$

Hence, formula (4.5) implies that

$$
\begin{aligned}
|x|^{n-1}|DS_{\mathbf{a}}(x)| \le {} & \frac{1}{s_n\sqrt{\det a^{(2)}}}\,|T|^n\big|(a^{(2)})^{-1}\big| \\
& + \left\{|x|A_2\Big(\frac{x}{|x|},|x|\Big) + \big(|x|^{n-1}\ln|x|\big)DB_1(x) + |x|^{n-1}DC(x)\right\}
\end{aligned}
$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Then the continuity of A_2 on the compact set $\partial\mathbb{B}_n \times [0, \operatorname{diam}(G)]$ and the continuity of DB_1 and DC on the compact set $\operatorname{cl}\mathbb{B}_n(0, \operatorname{diam}(G))$ imply the validity of statement (ii).

We now turn to statement (iii). Let $x', x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x' - x''|)$. By statement (i), we have

$$
\begin{aligned}
& \big|DS_{\mathbf{a}}(x'-y) - DS_{\mathbf{a}}(x''-y)\big| \\
& \quad \le \frac{1}{s_n\sqrt{\det a^{(2)}}}\left||T^{-1}(x'-y)|^{-n}(x'-y)^t(a^{(2)})^{-1} - |T^{-1}(x''-y)|^{-n}(x''-y)^t(a^{(2)})^{-1}\right| \\
& \qquad + \left||x'-y|^{2-n}A_2\Big(\frac{x'-y}{|x'-y|},|x'-y|\Big) - |x''-y|^{2-n}A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\right| \\
& \qquad + \Big|\ln|x'-y|DB_1(x'-y) - \ln|x''-y|DB_1(x''-y)\Big| + \big|DC(x'-y) - DC(x''-y)\big|\,. \qquad (4.9)
\end{aligned}
$$

We first estimate the first summand in the right-hand side of inequality (4.9). By the triangular inequality, we have

$$
\begin{aligned}
\Big||T^{-1}(x'-y)|^{-n}(x'-y)^t(a^{(2)})^{-1} - |T^{-1}(x''-y)|^{-n}(x''-y)^t(a^{(2)})^{-1}\Big| \\
\leq |x'-y|\,\big|(a^{(2)})^{-1}\big|\,\Big||T^{-1}(x'-y)|^{-n} - |T^{-1}(x''-y)|^{-n}\Big| \\
+ |x'-x''|\,\big|(a^{(2)})^{-1}\big|\,|T^{-1}(x''-y)|^{-n}\,. \qquad (4.10)
\end{aligned}
$$

Thus Lemmas 3.1(ii), 3.2(ii),(iv) with $\gamma = -n$, $\Lambda = T^{-1}$ imply that

$$
\begin{aligned}
\big||T^{-1}(x'-y)|^{-n} - |T^{-1}(x''-y)|^{-n}\big| \leq m_{-n}(T^{-1})|T^{-1}x' - T^{-1}x''|\,|T^{-1}x' - T^{-1}y|^{-n-1} \\
\leq m_{-n}(T^{-1})|T^{-1}|\,|T|^{n+1}|x'-x''|\,|x'-y|^{-n-1}\,, \qquad (4.11) \\
|T^{-1}(x''-y)|^{-n} \leq |T|^n|x''-y|^{-n}\,, \quad |x''-y|^{-n} \leq 2^n|x'-y|^{-n}\,.
\end{aligned}
$$

Next, we estimate the second summand in the right-hand side of inequality (4.9). By Lemmas 3.2(iv) and 3.3(i), the second summand is less or equal to

$$
\begin{aligned}
\big||x'-y|^{2-n} - |x''-y|^{2-n}\big|\,\left|A_2\Big(\frac{x''-y}{|x''-y|}, |x''-y|\Big)\right| \\
+ |x'-y|^{2-n}\left|A_2\Big(\frac{x'-y}{|x'-y|}, |x'-y|\Big) - A_2\Big(\frac{x''-y}{|x''-y|}, |x''-y|\Big)\right| \\
\leq m_{2-n}|x'-x''|\,|x'-y|^{2-n-1}\sup_{\partial\mathbb{B}_n\times[0,\mathrm{diam}(G)]}|A_2| + |x'-y|^{2-n}\Big(\sum_{j=1}^n M_{A_{2,j},G}\Big)|x'-x''||x'-y|^{-1}\,. \qquad (4.12)
\end{aligned}
$$

Further, we estimate the third summand in the right-hand side of inequality (4.9). By Lemmas 3.2(v) and 3.3(ii), the third summand is less or equal to

$$
\begin{aligned}
\big|\ln|x'-y| - \ln|x''-y|\big|\,|DB_1(x''-y)| + \big|\ln|x'-y|\big|\,\big|DB_1(x'-y) - DB_1(x''-y)\big| \\
\leq 2|x'-x''|\,|x'-y|^{-1}\sup_{G-G}|DB_1| + \Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial B_1}{\partial x_j},G}\Big)|x'-x''|\,\big|\ln|x'-y|\big| \\
\leq |x'-x''|\,|x'-y|^{-n}\Big\{2|x'-y|^{n-1}\sup_{G-G}|DB_1| + \Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial B_1}{\partial x_j},G}\Big)|x'-y|^n\big|\ln|x'-y|\big|\Big\}\,. \qquad (4.13)
\end{aligned}
$$

Finally, Lemma 3.3(ii) implies that

$$
\begin{aligned}
|DC(x'-y) - DC(x''-y)| \leq \Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial C}{\partial x_j},G}\Big)|x'-x''| \\
\leq |x'-x''|\,|x'-y|^{-n}\Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial C}{\partial x_j},G}\Big)\sup_{(x',y)\in G\times G}|x'-y|^n\,. \qquad (4.14)
\end{aligned}
$$

Thus inequalities (4.9)–(4.14) imply the validity of statement (iii). □

5 Preliminary inequalities on the boundary operator

Let us turn to estimate the kernel $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ of the double layer potential of (1.4). We will do it under assumption (1.3). For this, we introduce some basic inequalities for $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ by means of the following

Lemma 5.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $T \in M_n(\mathbb{R})$ *be as in* (4.2) *and let* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$.

Let $\alpha \in]0,1]$ *and* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$. *Then the following statements hold:*

(i) *If* $\alpha \in]0,1[$ *, then*

$$b_{\Omega,\alpha} \equiv \sup\left\{|x-y|^{n-1-\alpha}\left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))\right| :\ x,y \in \partial\Omega,\ x \neq y\right\} < +\infty\,. \tag{5.1}$$

If $n > 2$*, then* (5.1) *holds also for* $\alpha = 1$*.*

(ii)

$$\begin{aligned}\widetilde{b}_{\Omega,\alpha} \equiv \sup\Bigg\{ & \frac{|x'-y|^{n-\alpha}}{|x'-x''|}\left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x'-y)) - \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x''-y))\right| : \\ & x',x'' \in \partial\Omega,\ x' \neq x'',\ y \in \partial\Omega \setminus \mathbb{B}_n(x',2|x'-x''|)\Bigg\} < +\infty\,.\end{aligned}$$

Proof. By Lemma 4.3(i), we have

$$\begin{aligned}\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) &= -\,DS_{\mathbf{a}}(x-y)a^{(2)}\nu(y) - \nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y)\\ &= -\,\frac{1}{s_n\sqrt{\det a^{(2)}}}|T^{-1}(x-y)|^{-n}(x-y)^t\nu(y)\\ &\quad -|x-y|^{2-n}A_2\Big(\frac{x-y}{|x-y|},|x-y|\Big)a^{(2)}\nu(y) - DB_1(x-y)a^{(2)}\nu(y)\ln|x-y|\\ &\quad - DC(x-y)a^{(2)}\nu(y) - \nu^t(y)a^{(1)}S_{\mathbf{a}}(x-y) \quad \forall\, x,y\in\partial\Omega,\ \ x\neq y\,.\end{aligned} \tag{5.2}$$

By Lemmas 3.1(ii), 3.4, 4.2(i), and by the equality in (5.2), we have

$$\begin{aligned}|x-y|^{n-1-\alpha}\left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))\right| &\leq \frac{1}{s_n\sqrt{\det a^{(2)}}}\,c_{\Omega,\alpha}|T|^n|x-y|^{-n+1+\alpha+n-1-\alpha}\\ &+|x-y|^{2-1-\alpha}|a^{(2)}|\left|A_2\Big(\frac{x-y}{|x-y|},|x-y|\Big)\right| + |x-y|^{n-1-\alpha}\big|\ln|x-y|\big|\,|a^{(2)}|\,|DB_1(x-y)|\\ &+|x-y|^{n-1-\alpha}|a^{(2)}|\,|DC(x-y)| + |a^{(1)}|C_{0,S_{\mathbf{a}},\partial\Omega,n-1-\alpha}\end{aligned}$$

for all $x,y\in\partial\Omega$, $x\neq y$. If either $\alpha\in]0,1[$ or $\alpha\in]0,1]$ and $n>2$, then the right-hand side is bounded for $x,y\in\partial\Omega$, $x\neq y$. Hence, we conclude that statement (i) holds true.

Next, we consider statement (ii).

$$\begin{aligned}&\left|\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x'-y)) - \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x''-y))\right|\\ &\quad\leq \frac{\left||T^{-1}(x'-y)|^{-n}(x'-y)^t\nu(y) - |T^{-1}(x''-y)|^{-n}(x''-y)^t\nu(y)\right|}{s_n\sqrt{\det a^{(2)}}}\\ &\quad+|a^{(2)}|\left|A_2\Big(\frac{x'-y}{|x'-y|},|x'-y|\Big) - A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\right|\,|x'-y|^{2-n}\\ &\quad+|a^{(2)}|\left|A_2\Big(\frac{x''-y}{|x''-y|},|x''-y|\Big)\right|\,\left||x'-y|^{2-n}-|x''-y|^{2-n}\right|\\ &\quad+|a^{(2)}|\,\left|DB_1(x'-y)-DB_1(x''-y)\right|\,\big|\ln|x'-y|\big| + |a^{(2)}|\,|DB_1(x''-y)|\,\big|\ln|x'-y|-\ln|x''-y|\big|\\ &\quad+|a^{(2)}|\,\left|DC(x'-y)-DC(x''-y)\right| + |a^{(1)}|\,\left|S_{\mathbf{a}}(x'-y)-S_{\mathbf{a}}(x''-y)\right|\end{aligned} \tag{5.3}$$

for all $x',x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Denote by J_1 the first term in the right-hand side of (5.3). By Lemmas 3.1(ii), 3.2(ii),(iv) with $\gamma = -n$, $\Lambda = T^{-1}$, and by Lemma 3.4, we have

$$\begin{aligned} J_1 &\leq \frac{1}{s_n\sqrt{\det a^{(2)}}} \\ &\quad \times \Big\{ \big||T^{-1}(x'-y)|^{-n} - |T^{-1}(x''-y)|^{-n}\big| \, \big|(x'-y)^t\nu(y)\big| + |T^{-1}(x''-y)|^{-n}\big|(x'-x'')^t\nu(y)\big| \Big\} \\ &\leq \frac{1}{s_n\sqrt{\det a^{(2)}}} \\ &\quad \times \Big\{ m_{-n}(T^{-1})\big||T^{-1}x' - T^{-1}x''| \, |T^{-1}x' - T^{-1}y|^{-n-1}\big| \, |x'-y|^{1+\alpha} c_{\Omega,\alpha} \\ &\qquad + 2^n|T|^n|x'-y|^{-n}\big|(x'-x'')^t\nu(y)\big| \Big\} \end{aligned} \tag{5.4}$$

for all $x', x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Note that

$$\begin{aligned} \big|(x'-x'')^t\nu(y)\big| &\leq \big|(x'-x'')^t(\nu(y)-\nu(x'))\big| + \big|(x'-x'')^t\nu(x')\big| \\ &\leq |x'-x''| \, |\nu|_\alpha |x'-y|^\alpha + c_{\Omega,\alpha}|x'-x''|^{1+\alpha} \leq |x'-x''| \, |x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha}) \end{aligned}$$

and, accordingly,

$$\begin{aligned} J_1 \leq \frac{|x'-x''|}{s_n\sqrt{\det a^{(2)}}} \Big\{ & m_{-n}(T^{-1})|T^{-1}| \, |T|^{n+1}|x'-y|^{-n-1}|x'-y|^{1+\alpha}c_{\Omega,\alpha} \\ & + 2^n|T|^n|x'-y|^{-n}|x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha}) \Big\} \end{aligned} \tag{5.5}$$

for all $x', x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Next, we denote by J_2 the sum of the terms different from J_1 in the right-hand side of (5.3). Then Lemma 3.2(iv),(v) and Lemmas 3.3, 4.2(ii) imply that

$$\begin{aligned} J_2 &\leq |a^{(2)}|\Big(\sum_{j=1}^n M_{A_{2,j},\partial\Omega}\Big)\frac{|x'-x''|}{|x'-y|}\,|x'-y|^{2-n} + |a^{(2)}|\sup_{\partial\mathbb{B}_n\times[0,\mathrm{diam}(\partial\Omega)]}|A_2|m_{2-n}|x'-x''|\,|x'-y|^{1-n} \\ &\quad + |a^{(2)}|\Big(\sum_{j=1}^n \widetilde{M}_{\frac{\partial B_1}{\partial x_j},\partial\Omega}\Big)|x'-x''|\,\big|\ln|x'-y|\big| + |a^{(2)}|\sup_{\partial\Omega-\partial\Omega}|DB_1|\,2\,\frac{|x'-x''|}{|x'-y|} \\ &\quad + \widetilde{M}_C|x'-x''| + \widetilde{C}_{0,S_{\mathbf{a}},\partial\Omega}|a^{(1)}|\,\frac{|x'-x''|}{|x'-y|^{n-1}} \end{aligned} \tag{5.6}$$

for all $x', x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. By inequalities (5.3), (5.5), (5.6), we conclude that statement (ii) holds. □

6 Boundary norms for kernels

For each subset A of $\mathbb{R}^n$, we find it convenient to set

$$\Delta_A \equiv \big\{(x,y) \in A \times A : \; x = y\big\}.$$

We now introduce a class of functions on $(\partial\Omega)^2 \setminus \Delta_{\partial\Omega}$ which may carry a singularity as the variable tends to a point of the diagonal, just as in the case of the kernels of integral operators corresponding to layer potentials defined on the boundary of an open subset Ω of $\mathbb{R}^n$.

Definition 6.1. Let G be a nonempty bounded subset of $\mathbb{R}^n$. Let $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$. We denote by $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)$ the set of continuous functions K from $(G \times G) \setminus \Delta_G$ to $\mathbb{C}$ such that

$$\begin{aligned} &\|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)} \equiv \sup\Big\{|x-y|^{\gamma_1}|K(x,y)| : \; x, y \in G, \; x \neq y\Big\} \\ &+ \sup\Big\{\frac{|x'-y|^{\gamma_2}}{|x'-x''|^{\gamma_3}}\,|K(x',y) - K(x'',y)| : \; x', x'' \in G, \; x' \neq x'', \; y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)\Big\} < +\infty. \end{aligned}$$

One can easily verify that $(\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G), \|\cdot\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(G)})$ is a Banach space.

Remark 6.1. Let $\mathbf{a}$ be as in (1.1), (1.2) and $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$.

(i) Let G be a nonempty bounded subset of $\mathbb{R}^n$. Then Lemma 4.2 implies that $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-1-\gamma,n-1,1}(G)$ for all $\gamma \in [0,1[$ and the same membership holds also for $\gamma = 1$ if $n > 2$. If we further assume that $\mathbf{a}$ satisfies (1.3), then Lemma 4.3 implies that $\frac{\partial}{\partial x_j} S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-1,n,1}(G)$ for all $j \in \{1,\dots,n\}$.

(ii) Let $\mathbf{a}$ satisfy (1.3), $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then Lemma 5.1 implies that $\overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y)) \in \mathcal{K}_{n-1-\alpha,n-\alpha,1}(\partial\Omega)$.

For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $]0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} r^\theta |\ln r|, & r \in]0, r_\theta], \\ r_\theta^\theta |\ln r_\theta|, & r \in]r_\theta, +\infty[, \end{cases}$$

where

$$r_\theta \equiv \begin{cases} \min\left\{e^{-1/\theta}, e^{\frac{2\theta-1}{\theta(1-\theta)}}\right\} & \text{if } \theta \in]0,1[, \\ e^{-1} & \text{if } \theta = 1. \end{cases}$$

Obviously, $\omega_\theta(\cdot)$ is concave and satisfies (2.1), (2.2), and (2.3) with $\alpha = \theta$. We also note that if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then the continuous imbedding

$$C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

holds for all $\theta' \in]0,\theta[$. We now consider the properties of an integral operator with a kernel in the class $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)$.

Proposition 6.1. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma_1 \in]-\infty, n-1[$, γ_2, $\gamma_3 \in \mathbb{R}$. Then the following statements hold:*

(i) *If $(K,\mu) \in \mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$, then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in \partial\Omega$, and the function $u[\partial\Omega,K,\mu]$ from $\partial\Omega$ to $\mathbb{C}$ defined by*

$$u[\partial\Omega,K,\mu](x) \equiv \int_{\partial\Omega} K(x,y)\mu(y)\,d\sigma_y \quad \forall\, x \in \partial\Omega \tag{6.1}$$

is continuous. Moreover, the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^0(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega,K,\mu]$, is continuous.

(ii) *If $\gamma_1 \in [n-2,n-1[$, $\gamma_2 \in]n-1,+\infty[$, $\gamma_3 \in]0,1]$, $(n-1)-\gamma_2+\gamma_3 \in]0,1]$, then the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to the space $C^{0,\min\{(n-1)-\gamma_1,(n-1)-\gamma_2+\gamma_3\}}(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega,K,\mu]$, is continuous.*

(iii) *If $\gamma_1 \in [n-2,n-1[$, $\gamma_2 = n-1$, $\gamma_3 \in]0,1]$, then the bilinear map from $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega) \times L^\infty(\partial\Omega)$ to the space $C^{0,\max\{r^{(n-1)-\gamma_1},\omega_{\gamma_3}(r)\}}(\partial\Omega)$, which takes (K,μ) to $u[\partial\Omega,K,\mu]$ is continuous.*

Proof. By definition of the norm in $\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)$, we have

$$\left|K(x,y)\mu(y)\right| \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{L^\infty(\partial\Omega)} \frac{1}{|x-y|^{\gamma_1}} \quad \forall\, (x,y) \in (\partial\Omega)^2 \setminus D_{\partial\Omega}\,.$$

Then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in \partial\Omega$, and the Vitali Convergence Theorem implies that $u[\partial\Omega,K,\mu]$ is continuous on $\partial\Omega$ (cf., e.g., Folland [13, (2.33), pp. 60, 180].) By Lemma 3.5(i), we also have

$$\left|\int_{\partial\Omega} K(x,y)\mu(y)\,d\sigma_y\right| \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{L^\infty(\partial\Omega)} c'_{\Omega,\gamma_1}\,. \tag{6.2}$$

Hence, statement (i) follows. Next, we turn to estimate the Hölder coefficient of $u[\partial\Omega, K, \mu]$ under the assumptions of statements (ii) and (iii). Let $x', x'' \in \partial\Omega$, $x' \neq x''$. By Remark 2.2, there is no loss of generality in assuming that $0 < |x' - x''| \leq r_{\gamma_3}$. Then the inclusion $\mathbb{B}_n(x', 2|x' - x''|) \subseteq \mathbb{B}_n(x'', 3|x' - x''|)$ and the triangular inequality imply that

$$\left|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\right| \leq \|\mu\|_{L^\infty(\partial\Omega)} \Bigg\{ \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, d\sigma_y$$
$$+ \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, d\sigma_y + \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \left|K(x', y) - K(x'', y)\right| d\sigma_y \Bigg\}. \quad (6.3)$$

From Lemma 3.5(ii) it follows that

$$\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, d\sigma_y + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, d\sigma_y$$
$$\leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \Bigg\{ \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_1}} + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|x'' - y|^{\gamma_1}} \Bigg\}$$
$$\leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} 2c''_{\Omega,\gamma_1} |x' - x''|^{(n-1)-\gamma_1}. \quad (6.4)$$

Moreover, we have

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \left|K(x', y) - K(x'', y)\right| d\sigma_y \leq \|K\|_{\mathcal{K}_{\gamma_1,\gamma_2,\gamma_3}(\partial\Omega)} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \quad (6.5)$$

both in case $\gamma_2 \in]n - 1, +\infty[$ and $\gamma_2 = n - 1$ and for all $\gamma_3 \in]0, 1]$.

Under the assumptions of statement (ii), Lemma 3.5(iii) yields

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \leq c'''_{\Omega,\gamma_2} |x' - x''|^{(n-1)-\gamma_2+\gamma_3}. \quad (6.6)$$

Instead, under the assumptions of statement (iii), Lemma 3.5(iv) implies that

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}}\, d\sigma_y \leq c^{iv}_{\Omega} |x' - x''|^{\gamma_3} \left|\ln|x' - x''|\right|. \quad (6.7)$$

Inequalities (6.2)–(6.7) imply the validity of statements (ii), (iii). □

Note that Proposition 6.1(ii) for $n = 3$, $\gamma_1 = 2 - \alpha$, $\gamma_2 = 3 - \alpha$, $\gamma_3 = 1$ and for fixed K is known (see Kirsch and Hettlich [19, § 3.1.3, Thm. 3.17 (a)]). Next, we introduce two technical lemmas, which we need to define an auxiliary integral operator.

Lemma 6.1. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$, $\alpha, \beta \in]0, 1[$ and $\gamma_2 \in \mathbb{R}$, $\gamma_3 \in]0, 1]$.*

If $\gamma_2 - \beta > n - 1$, we further require that $\gamma_3 + (n - 1) - (\gamma_2 - \beta) > 0$.

Then there exists $c > 0$ such that the function $u[\partial\Omega, K, \mu]$ defined by (6.1) satisfies the inequality

$$\left|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\right| \leq c\|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \omega(|x' - x''|)$$
$$+ \|\mu\|_{C^0(\partial\Omega)} \left|u[\partial\Omega, K, 1](x') - u[\partial\Omega, K, 1](x'')\right| \quad \forall\, x', x'' \in \partial\Omega \quad (6.8)$$

for all $(K, \mu) \in \mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$, where

$$\omega(r) \equiv \begin{cases} r^{\min\{\alpha+\beta,\gamma_3\}} & \text{if } \gamma_2 - \beta < n - 1, \\ \max\left\{r^{\alpha+\beta}, \omega_{\gamma_3}(r)\right\} & \text{if } \gamma_2 - \beta = n - 1, \\ r^{\min\{\alpha+\beta,\gamma_3+(n-1)-(\gamma_2-\beta)\}} & \text{if } \gamma_2 - \beta > n - 1, \end{cases} \qquad \forall\, r \in]0, +\infty[.$$

Proof. By Remark 2.2 and Proposition 6.1(i), it suffices to consider the case $0 < |x' - x''| < r_{\gamma_3}$. By the triangular inequality, we have

$$\begin{aligned}&\big|u[\partial\Omega, K, \mu](x') - u[\partial\Omega, K, \mu](x'')\big| \\ &\quad \leq \bigg|\int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big](\mu(y) - \mu(x'))\, d\sigma_y\bigg| + |\mu(x')| \bigg|\int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big]\, d\sigma_y\bigg|. \end{aligned} \tag{6.9}$$

By exploiting the inclusion $\mathbb{B}_n(x', 2|x' - x''|) \subseteq \mathbb{B}_n(x'', 3|x' - x''|)$, the triangular inequality, Lemmas 3.2(i), 3.5(ii), and the inequality

$$|y - x'|^\beta \leq |y - x''|^\beta + |x' - x''|^\beta ,$$

we have

$$\begin{aligned}&\bigg|\int_{\partial\Omega} \big[K(x', y) - K(x'', y)\big](\mu(y) - \mu(x'))\, d\sigma_y\bigg| \\
&\leq \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(x', y)|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\quad + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(x'', y)|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\quad + \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \big|K(x', y) - K(x'', y)\big|\, |y - x'|^\beta\, d\sigma_y \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\leq \|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \bigg\{ \int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|y - x'|^{(n-1)-(\alpha+\beta)}} \\
&\quad + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x' - x''|^\beta}{|y - x''|^{(n-1)-\alpha}}\, d\sigma_y \\
&\quad + \int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{d\sigma_y}{|y - x''|^{(n-1)-(\alpha+\beta)}} \\
&\quad + \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x' - x''|^{\gamma_3} |x' - y|^\beta}{|x' - y|^{\gamma_2}}\, d\sigma_y \bigg\} \\
&\leq \|K\|_{\mathcal{K}_{(n-1)-\alpha,\gamma_2,\gamma_3}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} \\
&\quad \times \bigg\{ 2c''_{\Omega,(n-1)-(\alpha+\beta)} |x' - x''|^{\alpha+\beta} + |x' - x''|^\beta c''_{\Omega,(n-1)-\alpha} |x' - x''|^\alpha \\
&\qquad + |x' - x''|^{\gamma_3} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \bigg\}. \end{aligned} \tag{6.10}$$

At this point we distinguish three cases. If $\gamma_2 - \beta < n - 1$, then by Lemma 3.5(i)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \leq \int_{\partial\Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \leq c'_{\Omega,\gamma_2-\beta} ,$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = r^{\min\{\alpha+\beta,\gamma_3\}}$. If $\gamma_2 - \beta = n - 1$, then by Lemma 3.5(iv)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2-\beta}} \leq c^{iv}_\Omega \big|\ln|x' - x''|\big| ,$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = \max\{r^{\alpha+\beta}, \omega_{\gamma_3}(r)\}$. If $\gamma_2 - \beta > n-1$, then by Lemma 3.5(iii)

$$\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{d\sigma_y}{|x'-y|^{\gamma_2-\beta}} \leq c'''_{\Omega,\gamma_2-\beta}|x'-x''|^{(n-1)-(\gamma_2-\beta)},$$

and thus inequalities (6.9) and (6.10) imply that there exists $c > 0$ such that inequality (6.8) holds with $\omega(r) = r^{\min\{\alpha+\beta,\gamma_3+(n-1)-(\gamma_2-\beta)\}}$. □

We also point out the validity of the following 'folklore' Lemma.

Lemma 6.2. *Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$, $\gamma_1 \in]-\infty, n-1[$, G be a subset of $\mathbb{R}^n$. Let $K \in C^0((G \times \partial\Omega)\setminus\Delta_{\partial\Omega})$ be such that*

$$\kappa_{\gamma_1} \equiv \sup_{(x,y)\in(G\times\partial\Omega)\setminus\Delta_{\partial\Omega}} |x-y|^{\gamma_1}|K(x,y)| < +\infty.$$

Let $\mu \in L^\infty(\partial\Omega)$. Then the function $K(x,\cdot)\mu(\cdot)$ is integrable in $\partial\Omega$ for all $x \in G$ and the function $u^\sharp[\partial\Omega, K, \mu]$ from G to $\mathbb{C}$ defined by

$$u^\sharp[\partial\Omega, K, \mu](x) \equiv \int_{\partial\Omega} K(x,y)\mu(y)\,d\sigma_y \quad \forall\, x \in G$$

is continuous. If $\sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}} < \infty$, then $u^\sharp[\partial\Omega, K, \mu]$ satisfies the inequality

$$\left|u^\sharp[\partial\Omega, K, \mu](x)\right| \leq \sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}}\,\kappa_{\gamma_1}\|\mu\|_{L^\infty(\partial\Omega)} \quad \forall\, x \in G. \tag{6.11}$$

Proof. The integrability of $K(x,\cdot)\mu(\cdot)$ follows from the inequality

$$\left|K(x,y)\mu(y)\right| \leq \frac{\kappa_{\gamma_1}\|\mu\|_{L^\infty(\partial\Omega)}}{|x-y|^{\gamma_1}} \quad \text{a.a. } y \in \partial\Omega.$$

Since $\sup_{x\in G}\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{\gamma_1}} < \infty$, inequality (6.11) follows and the Vitali Convergence Theorem implies that $u^\sharp[\partial\Omega, K, \mu]$ is continuous on G (cf., e.g., Folland [13, (2.33) pp. 60, 180]). □

We now introduce an auxiliary integral operator and deduce some properties which we will need in the sequel by applying Proposition 6.1 and Lemma 6.1.

Lemma 6.3. *Let $\theta \in]0,1]$ and Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold:*

(i) *Let $Z \in C^0((\mathrm{cl}\,\Omega \times \partial\Omega)\setminus\Delta_{\partial\Omega})$ satisfy the inequality*

$$\kappa_{n-1}[Z] \equiv \sup_{(x,y)\in(\mathrm{cl}\,\Omega\times\partial\Omega)\setminus\Delta_{\partial\Omega}} |x-y|^{n-1}|Z(x,y)| < +\infty. \tag{6.12}$$

Let $(f,\mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial\Omega)$ and $H^\sharp[Z,f]$ be the function from $(\mathrm{cl}\,\Omega \times \partial\Omega)\setminus\Delta_{\partial\Omega}$ to $\mathbb{C}$ defined by

$$H^\sharp[Z,f](x,y) \equiv (f(x)-f(y))Z(x,y) \quad \forall\,(x,y) \in (\mathrm{cl}\,\Omega \times \partial\Omega)\setminus\Delta_{\partial\Omega}.$$

If $x \in \mathrm{cl}\,\Omega$, then the function $H^\sharp[Z,f](x,\cdot)$ is Lebesgue integrable in $\partial\Omega$ and the function $Q^\sharp[Z,f,\mu]$ from $\mathrm{cl}\,\Omega$ to $\mathbb{C}$ defined by

$$Q^\sharp[Z,f,\mu](x) \equiv \int_{\partial\Omega} H^\sharp[Z,f](x,y)\mu(y)\,d\sigma_y \quad \forall\, x \in \mathrm{cl}\,\Omega$$

is continuous.

(ii) *The map H from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega)$ to $\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega)$, which takes (Z,g) to the function from $(\partial\Omega)^2 \setminus \Delta_{\partial\Omega}$ to $\mathbb{C}$ defined by*

$$H[Z,g](x,y) \equiv (g(x)-g(y))Z(x,y) \quad \forall\,(x,y) \in (\partial\Omega)^2 \setminus \Delta_{\partial\Omega}\,,$$

is bilinear and continuous.

(iii) *The map Q from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\Omega)$ to $C^{0,\omega_\theta(\cdot)}(\partial\Omega)$, which takes (Z,g,μ) to the function from $\partial\Omega$ to $\mathbb{C}$ defined by*

$$Q[Z,g,\mu](x) \equiv \int_{\partial\Omega} H[Z,g](x,y)\mu(y)\,d\sigma_y \quad \forall\,x \in \partial\Omega\,,$$

is trilinear and continuous.

(iv) *Let $\alpha \in]0,1[$, $\beta \in]0,1]$. Then there exists $q \in]0,+\infty[$ such that*

$$\begin{aligned}\big|Q[Z,g,\mu](x') - Q[Z,g,\mu](x'')\big| \leq q\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C^{0,\beta}(\partial\Omega)}|x'-x''|^\alpha \\ + \|\mu\|_{C^0(\partial\Omega)}\big|Q[Z,g,1](x') - Q[Z,g,1](x'')\big| \quad \forall\,x',x'' \in \partial\Omega\end{aligned}$$

for all $(Z,g,\mu) \in \mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$.

Proof. By assumption (6.12) and by the Hölder continuity of f, we have

$$\big|H^\sharp[Z,f](x,y)\big| \leq \frac{|f|_\theta}{|x-y|^{(n-1)-\theta}}\kappa_{n-1}[Z]$$

for all $(x,y) \in (\mathrm{cl}\,\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}$. Thus Lemma 6.2 implies the validity of statement (i).

By the Hölder continuity of g, we have

$$\big|H[Z,g](x,y)\big| \leq \frac{|g|_\theta}{|x-y|^{(n-1)-\theta}}\,\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)} \quad \forall\,(x,y) \in (\partial\Omega)^2 \setminus \Delta_{\partial\Omega}\,. \tag{6.13}$$

Now, let x', $x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Then we have

$$\begin{aligned}\big|H[Z,g](x',y) - H[Z,g](x'',y)\big| &\leq |g(x')-g(y)|\,\big|Z(x',y)-Z(x'',y)\big| + |g(x')-g(x'')|\,|Z(x'',y)| \\ &\leq \|g\|_{C^{0,\theta}(\partial\Omega)}\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\left\{\frac{|x'-y|^\theta|x'-x''|}{|x'-y|^n} + \frac{|x'-x''|^\theta}{|x''-y|^{n-1}}\right\}.\end{aligned} \tag{6.14}$$

Since $|x'-x''| \leq |x'-y|$, we have $|x'-x''|^{1-\theta} \leq |x'-y|^{1-\theta}$. Moreover, Lemma 3.2(i) implies that $|x''-y| \geq \frac{1}{2}|x'-y|$ and thus the term in braces in the right-hand side of (6.14) is less or equal to

$$\frac{|x'-y|\,|x'-x''|^\theta}{|x'-y|^n} + \frac{2^{n-1}|x'-x''|^\theta}{|x'-y|^{n-1}} \leq (1+2^{n-1})\,\frac{|x'-x''|^\theta}{|x'-y|^{n-1}}\,. \tag{6.15}$$

Thus inequalities (6.13)–(6.15) imply that

$$\big\|H[Z,g]\big\|_{\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega)} \leq 2^n\|Z\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)}\|g\|_{C^{0,\theta}(\partial\Omega)}\,. \tag{6.16}$$

Hence statement (ii) holds true. We now turn to prove (iii). By Proposition 6.1(iii) with $\gamma_1 = n-1-\theta$, $\gamma_2 = n-1$, $\gamma_3 = \theta$, the map $u[\partial\Omega,\cdot,\cdot]$ is continuous from $\mathcal{K}_{n-1-\theta,n-1,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^{0,\max\{r^{(n-1)-[(n-1)-\theta]},\omega_\theta(r)\}}(\partial\Omega) = C^{0,\omega_\theta(\cdot)}(\partial\Omega)$. Then statement (ii) implies that $u[\partial\Omega, H[\cdot,\cdot],\cdot]$ is continuous from $\mathcal{K}_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ to $C^{0,\omega_\theta(\cdot)}(\partial\Omega)$. Since

$$u\big[\partial\Omega, H[Z,g],\mu\big] = \int_{\partial\Omega} H[Z,g](x,y)\mu(y)\,d\sigma_y \quad \forall\,x \in \partial\Omega\,, \tag{6.17}$$

statement (iii) holds true. Since $C^{0,\beta_1}(\partial\Omega)$ is continuously imbedded into $C^{0,\beta_2}(\partial\Omega)$ whenever $0 < \beta_2 \leq \beta_1 \leq 1$, we can assume that $\alpha+\beta < 1$. Then by equality (6.17), by Lemma 6.1 with $\gamma_2 = n-1$, $\gamma_3 = \alpha$ and by statement (ii) with $\theta = \alpha$, statement (iv) holds true. □

7 Preliminaries on layer potentials

Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$ and let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $\mu \in L^\infty(\partial\Omega)$, Lemma 4.2(i) ensures the convergence of the integral

$$v[\partial\Omega, S_{\mathbf{a}}, \mu](x) \equiv \int_{\partial\Omega} S_{\mathbf{a}}(x-y)\mu(y)\, d\sigma_y \quad \forall\, x \in \mathbb{R}^n\,,$$

which defines the single layer potential relative to μ, $S_{\mathbf{a}}$. We collect in the following statement some known properties of the single layer potential which we will exploit in the sequel (cf. Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5] and the references therein).

Theorem 7.1. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function $v^+[\partial\Omega, S_{\mathbf{a}}, \mu] \equiv v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\Omega}$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and the function $v^-[\partial\Omega, S_{\mathbf{a}}, \mu] \equiv v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\Omega^-}$ belongs to $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$. Moreover, the map which takes μ to the function $v^+[\partial\Omega, S_{\mathbf{a}}, \mu]$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and the map from the space $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R) \setminus \Omega)$ which takes μ to $v^-[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ is continuous for all $R \in]0,+\infty[$ such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.*

(ii) *Let $l \in \{1,\dots,n\}$. If $\mu \in C^{0,\alpha}(\partial\Omega)$, then we have the following jump relation*

$$\frac{\partial}{\partial x_l} v^\pm[\partial\Omega, S_{\mathbf{a}}, \mu](x) = \mp \frac{\nu_l(x)}{2\nu(x)^t a^{(2)} \nu(x)}\, \mu(x) + \int_{\partial\Omega} \partial_{x_l} S_{\mathbf{a}}(x-y)\mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega\,,$$

where the integral in the right-hand side exists in the sense of the principal value.

We now introduce the following refinement of a classical result for the homogeneous second order elliptic operators (cf. Miranda [25]).

Theorem 7.2. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$, Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$ and let $\gamma \in]0,1[$. Then the operator $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ from $L^\infty(\partial\Omega)$ to $C^{0,\gamma}(\partial\Omega)$ which takes μ to $v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ is continuous.*

If, in addition, we assume that $n > 2$, then $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]_{|\partial\Omega}$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\omega_1(\,\cdot\,)}(\partial\Omega)$.

Proof. By Lemma 4.2, we have $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{(n-1)-\gamma, n-1, 1}(\partial\Omega)$, and also $S_{\mathbf{a}}(x-y) \in \mathcal{K}_{n-2,n-1,1}(\partial\Omega)$ if we assume that $n > 2$. Since

$$v[\partial\Omega, S_{\mathbf{a}}, \mu]_{|\partial\Omega} = u\big[\partial\Omega, S_{\mathbf{a}}(x-y), \mu\big]\,,$$

Proposition 6.1(iii) implies that $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\max\{r^\gamma, \omega_1(r)\}}(\partial\Omega) = C^{0,\gamma}(\partial\Omega)$, and also that $v[\partial\Omega, S_{\mathbf{a}}, \cdot\,]$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\max\{r,\omega_1(r)\}}(\partial\Omega) = C^{0,\omega_1(r)}(\partial\Omega)$ if we assume that $n > 2$. □

Next, we turn to the double layer potential and introduce the following technical result (cf. Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5] and the references therein).

Theorem 7.3. *Let $\mathbf{a}$ be as in (1.1), (1.2), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{m,\alpha}$. Then the following statements hold:*

(i) *If $\mu \in C^{0,\alpha}(\partial\Omega)$, then the restriction $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\Omega}$ can be extended uniquely to a continuous function $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ from $\mathrm{cl}\,\Omega$ to $\mathbb{C}$, and $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\Omega^-}$ can be extended uniquely to a continuous function $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ from $\mathrm{cl}\,\Omega^-$ to $\mathbb{C}$, and we have the following jump relation*

$$w^\pm[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) = \pm\frac{1}{2}\,\mu(x) + w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) \quad \forall\, x \in \partial\Omega\,.$$

(ii) *If* $\mu \in C^{m,\alpha}(\partial\Omega)$, *then* $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *and* $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *belongs to* $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$. *Moreover, the map from the space* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *which takes* μ *to* $w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]$ *is continuous and the map from the space* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R) \setminus \Omega)$ *which takes* μ *to* $w^-[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous for all* $R \in]0, +\infty[$ *such that* $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.

(iii) *Let* $r \in \{1, \dots, n\}$. *If* $\mu \in C^{m,\alpha}(\partial\Omega)$ *and* U *is an open neighborhood of* $\partial\Omega$ *in* $\mathbb{R}^n$ *and* $\tilde{\mu} \in C^m(U)$, $\tilde{\mu}_{|\partial\Omega} = \mu$, *then the equality*

$$\begin{aligned}\frac{\partial}{\partial x_r} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu](x) = &\sum_{j,l=1}^{n} a_{lj} \frac{\partial}{\partial x_l} \Big\{ \int_{\partial\Omega} S_{\mathbf{a}}(x-y) \Big[\nu_r(y) \frac{\partial \tilde{\mu}}{\partial y_j}(y) - \nu_j(y) \frac{\partial \tilde{\mu}}{\partial y_r}(y) \Big] d\sigma_y \Big\} \\ &+ \int_{\partial\Omega} \big[DS_{\mathbf{a}}(x-y) a^{(1)} + a S_{\mathbf{a}}(x-y) \big] \nu_r(y) \mu(y)\, d\sigma_y \\ &- \int_{\partial\Omega} \partial_{x_r} S_{\mathbf{a}}(x-y) \nu^t(y) a^{(1)} \mu(y)\, d\sigma_y \quad \forall\, x \in \mathbb{R}^n \setminus \partial\Omega \end{aligned} \tag{7.1}$$

holds.

Note that formula (7.1) for the Laplace operator with $n = 3$ can be found in Günter [14, Ch. 2, § 10, (42)]. By combining Theorems 7.1 and 7.3, we deduce that under the assumptions of Theorem 7.3(iii), the equality

$$\begin{aligned}\frac{\partial}{\partial x_r} w^+[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu] = &\sum_{j,l=1}^{n} a_{lj} \frac{\partial}{\partial x_l} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{rj}[\mu]\big] + Dv^+[\partial\Omega, S_{\mathbf{a}}, \nu_r \mu] a^{(1)} \\ &+ a v^+[\partial\Omega, S_{\mathbf{a}}, \nu_r \mu] - \frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t a^{(1)})\mu] \ \text{ on } \ \mathrm{cl}\,\Omega \end{aligned} \tag{7.2}$$

holds.

Next, we introduce a result proved by Schauder [30, Hilfsatz VII, p. 112] for the Laplace operator, which we extend here to the second order elliptic operators by exploiting Proposition 6.1.

Theorem 7.4. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, $\alpha \in]0,1[$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$. *If* $\mu \in L^\infty(\partial\Omega)$, *then* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} \in C^{0,\alpha}(\partial\Omega)$. *Moreover, the operator from* $L^\infty(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$ *which takes* μ *to* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ *is continuous.*

Proof. By Lemma 5.1, the function $K_{\mathbf{a}}(x,y) \equiv \overline{B^*_{\Omega,y}}(S_{\mathbf{a}}(x-y))$ belongs to $\mathcal{K}_{(n-1)-\alpha, n-\alpha, 1}(\partial\Omega)$. Since

$$w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} = u[\partial\Omega, K_{\mathbf{a}}, \mu]\,,$$

Proposition 6.1(ii) implies that the function $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $L^\infty(\partial\Omega)$ to $C^{0,\min\{\alpha,(n-1)-(n-\alpha)+1\}}(\partial\Omega) = C^{0,\alpha}(\partial\Omega)$. □

8 Auxiliary integral operators

In order to compute the tangential derivatives of the double layer potential, we introduce the following two statements which concern two auxiliary integral operators. To shorten our notation, we define the function Θ from $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n}$ to $\mathbb{R}^n \setminus \{0\}$ as follows:

$$\Theta(x,y) \equiv x - y \quad \forall\, (x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n}\,. \tag{8.1}$$

Theorem 8.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$ *and* $r \in \{1, \dots, n\}$. *Then the following statements hold:*

(i) *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$ *and* $\theta \in]0,1]$. *If* $(f,\mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial\Omega)$, *then the function*

$$Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu\Big](x) = \int_{\partial\Omega} (f(x) - f(y))\, \frac{\partial S_{\mathbf{a}}}{\partial x_r}(x-y)\mu(y)\, d\sigma_y \quad \forall\, x \in \mathrm{cl}\,\Omega$$

is continuous.

(ii) *Let* $\alpha \in]0,1[$, $\beta, \theta \in]0,1]$, $m \in \mathbb{N} \setminus \{0\}$ *and* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$. *Then the map* $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ *from* $C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\min\{\alpha,\beta,\theta\}}(\mathrm{cl}\,\Omega)$ *which takes* (f,μ) *to* $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu]$ *is bilinear and continuous.*

Proof. By Lemma 4.3(ii), statement (i) is an immediate consequence of Lemma 6.3(i). Consider statement (ii). By treating separately the cases $x \in \partial\Omega$ and $x \in \Omega$, and exploiting Theorem 7.1(ii), we have

$$Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, f, \mu\Big](x) = f(x)\, \frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, \mu](x) - \frac{\partial}{\partial x_r} v^+[\partial\Omega, S_{\mathbf{a}}, f\mu](x),$$

for all $x \in \mathrm{cl}\,\Omega$. Then the statement follows by Theorem 7.1(i) and by the continuity of the pointwise product in Schauder spaces. □

Theorem 8.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3) *and* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$. *Then the following statement holds:*

(i) *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$ *and* $\theta \in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ *from* $C^{0,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)$ *to* $C^{0,\omega_\theta(\cdot)}(\partial\Omega)$, *which takes* (g,μ) *to the function*

$$Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x) = \int_{\partial\Omega} (g(x) - g(y))\, \frac{\partial S_{\mathbf{a}}}{\partial x_r}\, (x-y)\mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega\,, \tag{8.2}$$

is continuous.

(ii) *Let* $\alpha \in]0,1[$, $\beta \in]0,1]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ *from* $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$, *which takes* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu]$, *is continuous.*

Proof. By Lemma 4.3, we have $\frac{\partial S_{\mathbf{a}}}{\partial x_r} \in \mathcal{K}_{n-1,n,1}(\partial\Omega)$. Then Lemma 6.3(iii) implies the validity of statement (i).

We now consider statement (ii). By statement (i) and by the continuity of the inclusion of $C^{0,\beta}(\partial\Omega)$ into $L^\infty(\partial\Omega)$, we already know that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ to $C^0(\partial\Omega)$. Then it suffices to show that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)$ to the semi-normed space $(C^{0,\alpha}(\partial\Omega), |\cdot : \partial\Omega|_\alpha)$. By Lemma 6.3(iv), there exists $q \in]0,+\infty[$ such that

$$\begin{aligned}
&\Big| Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x') - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big](x'')\Big| \\
&\qquad \le q \Big\| \frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta \Big\|_{\mathcal{K}_{n-1,n,1}(\partial\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C^{0,\beta}(\partial\Omega)} |x' - x''|^\alpha \\
&\qquad\qquad + \|\mu\|_{C^0(\partial\Omega)} \Big| Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x') - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x'')\Big|
\end{aligned} \tag{8.3}$$

for all $x', x'' \in \partial\Omega$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator as in Lemma 2.1, defined on $C^{0,\alpha}(\partial\Omega)$. Since

$$Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, 1\Big](x) = Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \widetilde{g}, 1\Big](x) \quad \forall\, x \in \partial\Omega\,,$$

Theorem 8.1(ii) implies that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,1]$ is continuous from $C^{0,\alpha}(\partial\Omega)$ to itself and, accordingly, there exists $q'\in]0,+\infty[$ such that

$$\left\|Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,1\Big]\right\|_{C^{0,\alpha}(\partial\Omega)}\leq q'\|g\|_{C^{0,\alpha}(\partial\Omega)}\quad\forall\, g\in C^{0,\alpha}(\partial\Omega)\,. \tag{8.4}$$

Combining inequalities (8.3) and (8.4), we deduce that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\cdot,\cdot]$ is continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,\beta}(\partial\Omega)$ to $(C^{0,\alpha}(\partial\Omega),|\cdot:\partial\Omega|_\alpha)$ and thus the proof is complete. □

In the next lemma, we introduce a formula for the tangential derivatives of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$.

Lemma 8.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha\in]0,1[$, $\theta\in]0,1]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{2,\alpha}$, $r\in\{1,\ldots,n\}$ *and let* $g\in C^{1,\theta}(\partial\Omega)$, $\mu\in C^1(\partial\Omega)$. *Then* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]\in C^1(\partial\Omega)$ *and the formula*

$$\begin{aligned}
M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu\Big]\Big]&=\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,D_{\mathbf{a},j}g,\mu\Big](x)-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,D_{\mathbf{a},l}g,\mu\Big](x)\\
&+\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\sum_{s=1}^n M_{sj}\Big[\sum_{h=1}^n\frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\mu\Big]\Big](x)\\
&-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\sum_{s=1}^n M_{sl}\Big[\sum_{h=1}^n\frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\mu\Big]\Big](x)\\
&+\sum_{s,h=1}^n a_{sh}\nu_l(x)\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\nu_j,\frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x)+Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}\\
&-\sum_{s,h=1}^n a_{sh}\nu_j(x)\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,\nu_l,\frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x)+Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}\\
&-\sum_{s=1}^n a_s\Big\{\nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta,g,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big\}\\
&-a\Big\{g(x)\Big[\nu_l(x)v[\partial\Omega,S_{\mathbf{a}},\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big]\\
&-\Big[\nu_l(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)-\nu_j(x)v\Big[\partial\Omega,S_{\mathbf{a}},g\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big]\Big\}
\end{aligned}\tag{8.5}$$

holds for all $x\in\partial\Omega$ *and* $l,j\in\{1,\ldots,n\}$. *(For* Q *see* (8.2).*)*

Proof. Let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator as in Lemma 2.1, defined either on $C^{1,\theta}(\partial\Omega)$ or on $C^{1,\alpha}(\partial\Omega)$ depending on whether it has been applied to $g\in C^{1,\theta}(\partial\Omega)$ or to $\nu_l\in C^{1,\alpha}(\partial\Omega)$ for $l=1,\ldots,n$.

Now, fix $\beta\in]0,\min\{\theta,\alpha\}[$ and first prove the formula under the assumption $\mu\in C^{1,\beta}(\partial\Omega)$. By Theorem 8.1(ii), we already know that $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,g,\mu]$ belongs to $C^1(\mathrm{cl}\,\Omega)$. Then we find it convenient to introduce the notation

$$M^\sharp_{lj}[f](x)\equiv\widetilde{\nu}_l(x)\,\frac{\partial f}{\partial x_j}\,(x)-\widetilde{\nu}_j(x)\,\frac{\partial f}{\partial x_l}\,(x)\quad\forall\, x\in\mathrm{cl}\,\Omega$$

for all $f\in C^1(\mathrm{cl}\,\Omega)$. If necessary, we write $M^\sharp_{lj,x}$ to emphasize that we are taking x as variable of the differential operator $M^\sharp_{lj}$. Next, we fix $x\in\Omega$ and compute

$$\widetilde{\nu}_l(x)\,\frac{\partial}{\partial x_j}\,Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x)-\widetilde{\nu}_j(x)\,\frac{\partial}{\partial x_l}\,Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x)\,.$$

Clearly,

$$\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x) = \int_{\partial\Omega} \frac{\partial \widetilde{g}}{\partial x_l}(x)\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y + \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial^2}{\partial x_l \partial x_r} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,.$$

To shorten our notation, we set

$$J_1(x) \equiv \int_{\partial\Omega} \frac{\partial \widetilde{g}}{\partial x_l}(x)\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,.$$

Then we have

$$\begin{aligned}
\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x)
&= J_1(x) - \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) \sum_{s,h=1}^{n} \frac{\nu_s(y)a_{sh}\nu_h(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\frac{\partial}{\partial y_l}\Big[\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\Big]\mu(y)\,d\sigma_y \\
&= J_1(x) - \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) \sum_{s=1}^{n} \Big(\nu_s(y)\,\frac{\partial}{\partial y_l} - \nu_l(y)\,\frac{\partial}{\partial y_s}\Big)\Big[\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\Big] \\
&\qquad\times \sum_{h=1}^{n} \frac{a_{sh}\nu_h(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad - \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) \sum_{s=1}^{n} \frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\Big] \sum_{h=1}^{n} a_{sh}\nu_h(y)\,\frac{\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,.
\end{aligned}$$

By Lemma 2.2, the second term in the right-hand side takes the form

$$\begin{aligned}
&\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) \sum_{s=1}^{n} M_{sl,y}\Big[\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\Big]\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad = -\int_{\partial\Omega} \sum_{s=1}^{n} M_{sl,y}\big[\widetilde{g}(x)-\widetilde{g}(y)\big]\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\qquad - \int_{\partial\Omega} \sum_{s=1}^{n} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y) M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\,d\sigma_y\,.
\end{aligned}$$

Since $M_{sl,y}[\widetilde{g}(x)-\widetilde{g}(y)] = -M_{sl}[\widetilde{g}](y)$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_l} Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big](x) &= \frac{\partial \widetilde{g}}{\partial x_l}(x) \int_{\partial\Omega} \frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\quad - \int_{\partial\Omega} \sum_{s=1}^{n} M_{sl}[\widetilde{g}](y)\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&+ \int_{\partial\Omega} \sum_{s=1}^{n} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y) M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\,d\sigma_y \\
&\quad - \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) \sum_{s=1}^{n} \frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r} S_{\mathbf{a}}(x-y)\Big](a^{(2)}\nu(y))_s\,\frac{\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,.
\end{aligned}$$

Accordingly, we have

$$
\begin{aligned}
M^\sharp_{lj}\Big[Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\mu\Big]\Big](x) &= M^\sharp_{lj}[\widetilde{g}](x)\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&- \int_{\partial\Omega}\sum_{s=1}^{n}\Big\{\widetilde{\nu}_l(x)M_{sj}[\widetilde{g}](y)-\widetilde{\nu}_j(x)M_{sl}[\widetilde{g}](y)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&+ \int_{\partial\Omega}\sum_{s=1}^{n}(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big\{\widetilde{\nu}_l(x)M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)-\widetilde{\nu}_j(x)M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^t a^{(2)}\nu}\,\mu\Big](y)\Big\}\,d\sigma_y \\
&- \int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s=1}^{n}\frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r}S_{\mathbf{a}}(x-y)\Big](a^{(2)}\nu)_s(y)\,\frac{\widetilde{\nu}_l(x)\nu_j(y)-\widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\,. \qquad (8.6)
\end{aligned}
$$

We now consider the first two terms in the right-hand side of formula (8.6). By the obvious identity

$$
M^\sharp_{lj}[\widetilde{g}] = \widetilde{\nu}_l\Big[\frac{\partial}{\partial x_j}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\widetilde{\nu}^t a^{(2)}\widetilde{\nu}}\,\widetilde{\nu}_j\Big]-\widetilde{\nu}_j\Big[\frac{\partial}{\partial x_l}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\widetilde{\nu}^t a^{(2)}\widetilde{\nu}}\,\widetilde{\nu}_l\Big] \quad \text{in } \operatorname{cl}\Omega\,,
$$

by the corresponding formula for $M_{lj}[\widetilde{g}]$ on $\partial\Omega$, by formula (2.4) and by straightforward computations, we obtain

$$
\begin{aligned}
& M^\sharp_{lj}[\widetilde{g}](x)\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad - \int_{\partial\Omega}\sum_{s=1}^{n}\Big\{\widetilde{\nu}_l(x)M_{sj}[\widetilde{g}](y)-\widetilde{\nu}_j(x)M_{sl}[\widetilde{g}](y)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\,\frac{(a^{(2)}\nu(y))_s}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\qquad = \widetilde{\nu}_l(x)\Big[\frac{\partial}{\partial x_j}\,\widetilde{g}(x)-\frac{D\widetilde{g}(x)a^{(2)}\widetilde{\nu}(x)}{\widetilde{\nu}^t(x)a^{(2)}\widetilde{\nu}(x)}\,\widetilde{\nu}_j(x)\Big]\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad - \widetilde{\nu}_j(x)\Big[\frac{\partial}{\partial x_l}\,\widetilde{g}(x)-\frac{D\widetilde{g}(x)a^{(2)}\widetilde{\nu}(x)}{\widetilde{\nu}^t(x)a^{(2)}\widetilde{\nu}(x)}\,\widetilde{\nu}_l(x)\Big]\int_{\partial\Omega}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
& - \widetilde{\nu}_l(x)\int_{\partial\Omega}\Big[\frac{\partial}{\partial y_j}\,\widetilde{g}(y)-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\,\widetilde{\nu}_j(y)\Big]\Big(\sum_{s,h=1}^{n}\widetilde{\nu}_s(y)\,\frac{a^{(2)}_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad + \widetilde{\nu}_l(x)\int_{\partial\Omega}\widetilde{\nu}_j(y)\Big\{\sum_{s,h=1}^{n}\frac{\partial}{\partial y_s}\,\widetilde{g}(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} \\
&\qquad\qquad \times\Big(\widetilde{\nu}_s(y)\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
& + \widetilde{\nu}_j(x)\int_{\partial\Omega}\Big[\frac{\partial}{\partial y_l}\,\widetilde{g}(y)-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\,\widetilde{\nu}_l(y)\Big]\Big(\sum_{s,h=1}^{n}\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&\qquad - \widetilde{\nu}_j(x)\int_{\partial\Omega}\widetilde{\nu}_l(y)\Big\{\sum_{s,h=1}^{n}\frac{\partial}{\partial y_s}\,\widetilde{g}(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)} \\
&\qquad\qquad \times\Big(\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\Big\}\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\,. \qquad (8.7)
\end{aligned}
$$

Since

$$
\widetilde{\nu}(y)=\nu(y)\,,\qquad \Big(\sum_{s,h=1}^{n}\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)=1\quad \forall\, y\in\partial\Omega\,,
$$

we have

$$\Big\{\sum_{s,h=1}^{n}\frac{\partial}{\partial y_s}\,\widetilde{g}(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}-\frac{D\widetilde{g}(y)a^{(2)}\widetilde{\nu}(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big(\widetilde{\nu}_s(y)\,\frac{a_{sh}\nu_h(y)}{\widetilde{\nu}^t(y)a^{(2)}\widetilde{\nu}(y)}\Big)\Big\}=0$$

for all $y\in\partial\Omega$ and, accordingly, the right-hand side of (8.7) equals

$$\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\frac{\partial}{\partial x_j}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^ta^{(2)}\nu}\,\widetilde{\nu}_j,\mu\Big](x)-\widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\frac{\partial}{\partial x_l}\,\widetilde{g}-\frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^ta^{(2)}\nu}\,\widetilde{\nu}_l,\mu\Big](x)\,.$$

Consider the third term in the right-hand side of formula (8.6) and note that

$$\begin{aligned}\int_{\partial\Omega}\sum_{s=1}^{n}(\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big\{\widetilde{\nu}_l(x)M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^ta^{(2)}\nu}\,\mu\Big](y)-\widetilde{\nu}_j(x)M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^ta^{(2)}\nu}\,\mu\Big](y)\Big\}\,d\sigma_y\\ =\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\sum_{s=1}^{n}M_{sj}\Big[\frac{(a^{(2)}\nu)_s}{\nu^ta^{(2)}\nu}\,\mu\Big]\Big](x)\\ -\widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\widetilde{g},\sum_{s=1}^{n}M_{sl}\Big[\frac{(a^{(2)}\nu)_s}{\nu^ta^{(2)}\nu}\,\mu\Big]\Big](x)\,.\end{aligned}\tag{8.8}$$

Next, we consider the last integral in the right-hand side of formula (8.6) and note that if $x\in\Omega$ and $y\in\partial\Omega$, we have

$$\sum_{s,h=1}^{n}\frac{\partial}{\partial x_h}\Big[a_{sh}\,\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)\Big]+\sum_{s=1}^{n}a_s\,\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)+aS_{\mathbf{a}}(x-y)=0\,.$$

Thus we obtain

$$\begin{aligned}\sum_{s,h=1}^{n}a_{sh}\nu_h(y)\,\frac{\partial}{\partial x_r}\Big[\frac{\partial}{\partial y_s}\,S_{\mathbf{a}}(x-y)\Big]\\ =\sum_{s,h=1}^{n}a_{sh}\Big(\nu_h(y)\,\frac{\partial}{\partial y_r}-\nu_r(y)\,\frac{\partial}{\partial y_h}\Big)\Big[\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)\Big]+\nu_r(y)\sum_{s=1}^{n}a_s\,\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)+\nu_r(y)aS_{\mathbf{a}}(x-y)\,,\end{aligned}$$

and we note that the first parenthesis in the right-hand side equals $M_{hr,y}$. The last integral in the right-hand side of formula (8.6) equals

$$\begin{aligned}\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\sum_{s,h=1}^{n}a_{sh}\nu_h(y)\,\frac{\partial}{\partial y_s}\Big[\frac{\partial}{\partial x_r}\,S_{\mathbf{a}}(x-y)\Big]\frac{\widetilde{\nu}_l(x)\nu_j(y)-\widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\ =\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\Big\{\sum_{s,h=1}^{n}a_{sh}M_{hr,y}\Big[\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)\Big]\\ +\nu_r(y)\sum_{s=1}^{n}a_s\,\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)+\nu_r(y)aS_{\mathbf{a}}(x-y)\Big\}\frac{\widetilde{\nu}_l(x)\nu_j(y)-\widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\ =\sum_{s,h=1}^{n}a_{sh}\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))M_{hr,y}\Big[\frac{\partial}{\partial x_s}\,S_{\mathbf{a}}(x-y)\Big]\\ \times\frac{\widetilde{\nu}_l(x)(\widetilde{\nu}_j(y)-\widetilde{\nu}_j(x))+\widetilde{\nu}_j(x)(\widetilde{\nu}_l(x)-\widetilde{\nu}_l(y))}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y\\ +\int_{\partial\Omega}(\widetilde{g}(x)-\widetilde{g}(y))\Big[\sum_{s=1}^{n}a_s\,\frac{\partial}{\partial x_s}S_{\mathbf{a}}(x-y)+aS_{\mathbf{a}}(x-y)\Big]\frac{\widetilde{\nu}_l(x)\nu_j(y)-\widetilde{\nu}_j(x)\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\,\nu_r(y)\mu(y)\,d\sigma_y\,.\end{aligned}\tag{8.9}$$

We now consider separately each of the terms in the right-hand side of (8.9). By Lemma 2.2 and the equality $-M_{hr,y}[\widetilde{g}(x)-\widetilde{g}(y)] = M_{hr,y}[\widetilde{g}(y)]$, the first integral in the right-hand side of (8.9) equals

$$\begin{aligned}
&\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y)) M_{hr,y}\Big[\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\Big] \\
&\qquad\qquad \times \frac{\widetilde{\nu}_l(x)(\widetilde{\nu}_j(y)-\widetilde{\nu}_j(x)) + \widetilde{\nu}_j(x)(\widetilde{\nu}_l(x)-\widetilde{\nu}_l(y))}{\nu^t(y) a^{(2)} \nu(y)}\,\mu(y)\,d\sigma_y \\
&= \int_{\partial\Omega} M_{hr}[\widetilde{g}]\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\Big(-\widetilde{\nu}_l(x)\,\frac{\widetilde{\nu}_j(x)-\nu_j(y)}{\nu^t(y)a^{(2)}\nu(y)} + \widetilde{\nu}_j(x)\,\frac{\widetilde{\nu}_l(x)-\nu_l(y)}{\nu^t(y)a^{(2)}\nu(y)}\Big)\mu(y)\,d\sigma_y \\
&\quad + \int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) \\
&\qquad \times \bigg(-\widetilde{\nu}_l(x) M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big](y) + \widetilde{\nu}_j(x) M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big](y)\bigg)\,d\sigma_y \\
&= -\widetilde{\nu}_l(x)\int_{\partial\Omega} (\widetilde{\nu}_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\,\frac{M_{hr}[\widetilde{g}]}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad + \widetilde{\nu}_j(x)\int_{\partial\Omega} (\widetilde{\nu}_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y)\,\frac{M_{hr}[\widetilde{g}]}{\nu^t(y)a^{(2)}\nu(y)}\,\mu(y)\,d\sigma_y \\
&\quad - \widetilde{\nu}_l(x)\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big](y)\,d\sigma_y \\
&\quad + \widetilde{\nu}_j(x)\int_{\partial\Omega} (\widetilde{g}(x)-\widetilde{g}(y))\,\frac{\partial}{\partial x_s} S_{\mathbf{a}}(x-y) M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big](y)\,d\sigma_y \\
&= -\widetilde{\nu}_l(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{\nu}_j, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\nu}\Big]\bigg](x)\bigg\} \\
&\quad + \widetilde{\nu}_j(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{\nu}_l, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\bigg](x)\bigg\}. \qquad (8.10)
\end{aligned}$$

Next, we note that the second integral in the right-hand side of (8.9) equals

$$\begin{aligned}
&\sum_{s=1}^n a_s\bigg\{\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\bigg\} \\
&\quad + a\bigg\{\widetilde{g}(x)\Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big] \\
&\qquad - \Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\mu\Big](x)\Big]\bigg\}.
\end{aligned}$$

By combining formulas (8.6)–(8.10), we obtain

$$\begin{aligned}
&M^\sharp_{lj}\Big[Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \mu\Big]\Big](x) = \widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \frac{\partial}{\partial x_j}\widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\,\widetilde{\nu}_j, \mu\Big](x) \\
&- \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \frac{\partial}{\partial x_l}\widetilde{g} - \frac{D\widetilde{g}a^{(2)}\widetilde{\nu}}{\nu^t a^{(2)}\nu}\,\widetilde{\nu}_l, \mu\Big](x) + \widetilde{\nu}_l(x)Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \sum_{s=1}^n M_{sj}\Big[\sum_{h=1}^n \frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\,\mu\Big]\bigg](x) \\
&\qquad - \widetilde{\nu}_j(x)Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \widetilde{g}, \sum_{s=1}^n M_{sl}\Big[\sum_{h=1}^n \frac{a_{sh}\nu_h}{\nu^t a^{(2)}\nu}\,\mu\Big]\bigg](x) \\
&+ \sum_{s,h=1}^n a_{sh}\widetilde{\nu}_l(x)\bigg\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \nu_j, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\bigg[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_j\mu}{\nu^t a^{(2)}\,\nu}\Big]\bigg](x)\bigg\}
\end{aligned}$$

$$- \sum_{s,h=1}^{n} a_{sh}\widetilde{\nu}_j(x)\Big\{Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \nu_l, \frac{M_{hr}[g]\mu}{\nu^t a^{(2)}\nu}\Big](x) + Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, M_{hr}\Big[\frac{\nu_l\mu}{\nu^t a^{(2)}\nu}\Big]\Big](x)\Big\}$$
$$- \sum_{s=1}^{n} a_s\Big\{\widetilde{\nu}_l(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x) - \widetilde{\nu}_j(x)Q^\sharp\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \widetilde{g}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big\}$$
$$- a\Big\{g(x)\Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, \frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big]$$
$$- \Big[\widetilde{\nu}_l(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_j\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x) - \widetilde{\nu}_j(x)v\Big[\partial\Omega, S_{\mathbf{a}}, g\,\frac{\nu_l\nu_r}{\nu^t a^{(2)}\nu}\,\mu\Big](x)\Big]\Big\}. \quad (8.11)$$

Under our assumptions, the first argument of the maps $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot]$ and $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \cdot, \cdot]$, which appear in the right-hand side of (8.11) belongs to the space $C^{0,\min\{\alpha,\theta\}}(\mathrm{cl}\,\Omega)$ and the second argument of the maps $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot]$, $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_s}\circ\Theta, \cdot, \cdot]$, which appear in the right-hand side of (8.11) belongs to $C^0(\partial\Omega)$. By Theorem 7.1(i) with $m=1$, the single layer potentials in the right-hand side of (8.11) are continuous in $x\in\mathrm{cl}\,\Omega$. Then Theorem 8.1(i) implies that the right-hand side of (8.11) defines a continuous function of the variable $x\in\mathrm{cl}\,\Omega$. Since Ω is of the class $C^{2,\alpha}$ and $\widetilde{g}\in C^{1,\theta}(\mathrm{cl}\,\Omega)$ and since we are assuming that $\mu\in C^{1,\beta}(\partial\Omega)$, Theorem 8.1(ii) implies that $M^\sharp_{lj}[Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]]$ belongs to $C^0(\mathrm{cl}\,\Omega)$. Hence, the equation of (8.11) must hold for all $x\in\mathrm{cl}\,\Omega$ and, in particular, for all $x\in\partial\Omega$. Since $Q^\sharp[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot] = Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot]$ and $M^\sharp_{lj}=M_{lj}$ on $\partial\Omega$, we conclude that (8.5) holds.

Next, we assume that $\mu\in C^1(\partial\Omega)$. We denote by $P_{ljr}[g,\mu]$ the right-hand side of (8.5). By Theorem 8.2(i), the operators $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \cdot]$, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, D_{\mathbf{a},j}g, \cdot]$, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \nu_l, \cdot]$ are linear and continuous from the space $C^0(\partial\Omega)$ to $C^0(\partial\Omega)$. By Theorem 7.2 and by the continuity of the pointwise product in $C^0(\partial\Omega)$, the operator $P_{ljr}[g,\cdot]$ is continuous from $C^0(\partial\Omega)$ to $C^0(\partial\Omega)$. In particular, $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]$, $P_{ljr}[g,\mu]\in C^0(\partial\Omega)$.

We now show that the weak M_{lj}-derivative of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \cdot]$ in $\partial\Omega$ coincides with $P_{ljr}[g,\mu]$.

Considering both an extension of μ of the class C^1 with a compact support in $\mathbb{R}^n$ and a sequence of mollifiers of such an extension, and then taking the restriction to $\partial\Omega$, we can conclude that there exists a sequence of functions $\{\mu_b\}_{b\in\mathbb{N}}$ in $C^2(\partial\Omega)$ converging to μ in $C^1(\partial\Omega)$. We note that if $\varphi\in C^1(\partial\Omega)$, then the validity of (8.5) for $\mu_b\in C^2(\partial\Omega)\subseteq C^{1,\beta}(\partial\Omega)$, the membership of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu_b]$ in $C^1(\partial\Omega)$ (see Theorem 8.1(ii)) and Lemma 2.2 imply that

$$\int_{\partial\Omega} Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu\Big]M_{lj}[\varphi]\,d\sigma = \lim_{b\to\infty}\int_{\partial\Omega} Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu_b\Big]M_{lj}[\varphi]\,d\sigma$$
$$= -\lim_{b\to\infty}\int_{\partial\Omega} M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu_b\Big]\Big]\varphi\,d\sigma = -\lim_{b\to\infty}\int_{\partial\Omega} P_{ljr}[g,\mu_b]\varphi\,d\sigma = -\int_{\partial\Omega} P_{ljr}[g,\mu]\varphi\,d\sigma\,.$$

Hence, $P_{ljr}[g,\mu]$ coincides with the weak M_{lj}-derivative of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]$ for all $l,j\in\{1,\dots,n\}$. Since both $P_{ljr}[g,\mu]$ and $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]$ are the continuous functions, it follows that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]\in C^1(\partial\Omega)$ and $M_{lj}[Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]] = P_{ljr}[g,\mu]$, classically. Hence (8.5) holds also for $\mu\in C^1(\partial\Omega)$. □

By exploiting formula (8.5), we can prove the following theorem.

Theorem 8.3. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$ *and let* $r\in\{1,\dots,n\}$. *Then the following statements hold:*

(i) *Let* $\theta\in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot]$ *from the space* $C^{m-1,\theta}(\partial\Omega)\times C^{m-1}(\partial\Omega)$ *to* $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, *which takes a pair* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]$, *is continuous.*

(ii) *Let* $\beta\in]0,1]$. *Then the bilinear map* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, \cdot, \cdot]$ *from the space* $C^{m-1,\alpha}(\partial\Omega)\times C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$, *which takes a pair* (g,μ) *to* $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta, g, \mu]$, *is continuous.*

Proof. We first prove statement (i). We proceed by induction on m. Case $m = 1$ holds by Theorem 8.2(i). We now prove that if the statement holds for m, then it holds for $m+1$. Thus we now assume that Ω is of the class $C^{m+1,\alpha}$, and we turn to prove that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is bilinear and continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^{m,\omega_\theta(\cdot)}(\partial\Omega)$. By Lemma 2.3(ii), it suffices to prove that the following two statements hold:

(j) $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ is continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^0(\partial\Omega)$;

(jj) $M_{lj}[Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]]$ is continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to the space $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$ for all l, $j \in \{1, \dots, n\}$.

Statement (j) holds by the case $m = 1$, and by the imbedding of $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ into $C^{0,\theta}(\partial\Omega) \times C^0(\partial\Omega)$. We now prove statement (jj). Since $m + 1 \geq 2$, Lemma 8.1 and the inductive assumption imply that we can actually apply M_{lj} to $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$. We find it convenient to denote by $P_{ljr}[g, \mu]$ the right-hand side of formula (8.5). Then we have

$$M_{lj}\Big[Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, g, \mu\Big]\Big] = P_{ljr}[g, \mu] \quad \forall\, (g, \mu) \in C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)\,.$$

By Lemma 2.4 and the membership of ν in $C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$, which is contained in $C^{m-1,1}(\partial\Omega, \mathbb{R}^n)$, by the continuity of the pointwise product in Schauder spaces, by the continuity of the imbedding of $C^m(\partial\Omega)$ into $C^{m-1}(\partial\Omega)$ and of $C^{m,\alpha}(\partial\Omega)$ into $C^{m-1,\theta}(\partial\Omega)$, by the inductive assumption on the continuity of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$, by the continuity of $v[\partial\Omega, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega) \subseteq C^{m-1,\theta}(\partial\Omega)$, and by the continuity of the imbedding of $C^m(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$ and of $C^m(\partial\Omega)$ into $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, and by the continuity of $D_{\mathbf{a}}$ from $C^{m,\theta}(\partial\Omega)$ to $C^{m-1,\theta}(\partial\Omega)$, we conclude that $P_{ljr}[\cdot, \cdot]$ is bilinear and continuous from $C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)$ to $C^{m-1,\omega_\theta(\cdot)}(\partial\Omega)$, and the proof of statement (jj) and, accordingly, of statement (i) is complete. The proof of statement (ii) follows the lines of the proof of statement (i), by replacing the use of Theorem 8.2(i) with that of Theorem 8.2(ii). □

Definition 8.1. Let $\mathbf{a}$ be as in (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$, $\alpha \in]0, 1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of the class $C^{1,\alpha}$. Then we set

$$\begin{aligned} R[g, h, \mu] \equiv \sum_{r=1} a_r \Big\{ Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, gh, \mu\Big] - gQ\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, h, \mu\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, h, g\mu\Big]\Big\} \\ + a\Big\{ gv[\partial\Omega, S_{\mathbf{a}}, h\mu] - hv[\partial\Omega, S_{\mathbf{a}}, g\mu]\Big\} \end{aligned}$$

for all $(g, h, \mu) \in (C^{0,\alpha}(\partial\Omega))^2 \times C^0(\partial\Omega)$.

Since

$$g(x)h(y) - g(y)h(x) = \big[g(x)h(x) - g(y)h(y)\big] - g(x)[h(x) - h(y)] - g(y)[h(x) - h(y)] \quad \forall\, x, y \in \partial\Omega\,,$$

we have

$$R[g, h, \mu] = \int_{\partial\Omega} \Big\{ \sum_{r=1}^{n} a_r \frac{\partial}{\partial x_r} S_{\mathbf{a}}(x - y) + aS_{\mathbf{a}}(x - y)\Big\} \big[g(x)h(y) - g(y)h(x)\big] \mu(y)\, d\sigma_y \quad \forall\, x \in \partial\Omega\,.$$

Since R is a composition of the operator $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ and of a single layer potential, Theorems 7.1, 7.2 and 8.3, the continuity of the product in Schauder spaces and also of the imbeddings of $C^{m-1}(\partial\Omega)$ into $C^{m-2,\alpha}(\partial\Omega)$ for $m \geq 2$, of $C^{m-1,\alpha}(\partial\Omega)$ into $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ and also of $C^{m,\beta}(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$, imply that the following theorem is valid.

Theorem 8.4. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, $\alpha \in]0, 1[$, $m \in \mathbb{N}\setminus\{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$. *Then the following statements hold:*

(i) *The trilinear map* R *from the space* $(C^{m-1,\alpha}(\partial\Omega))^2 \times C^{m-1}(\partial\Omega)$ *to* $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$*, which takes a triple* (g,h,μ) *to* $R[g,h,\mu]$*, is continuous.*

(ii) *Let* $\beta \in]0,1]$*. Then the trilinear map* R *from the space* $(C^{m-1,\alpha}(\partial\Omega))^2 \times C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$*, which takes a triple* (g,h,μ) *to* $R[g,h,\mu]$*, is continuous.*

9 Tangential derivatives and regularizing properties of the double layer potential

We now exploit Theorems 7.3, 7.4, Lemma 8.1 and Theorems 8.3, 8.4 in order to prove a formula for the tangential derivatives of the double layer potential, which generalizes the corresponding formula of Hofmann, Mitrea and Taylor [16, (6.2.6)] for homogeneous operators. We do so by means of the following

Theorem 9.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$, $\alpha \in]0,1[$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{1,\alpha}$*. If* $\mu \in C^1(\partial\Omega)$*, then* $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega} \in C^1(\partial\Omega)$ *and*

$$\begin{aligned} M_{lj}\big[w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu\big]_{|\partial\Omega}\big] &= w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},M_{lj}[\mu]\big]_{|\partial\Omega} \\ &\quad + \sum_{b,r=1}^{n} a_{br}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_l,M_{jr}[\mu]\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_j,M_{lr}[\mu]\Big]\Big\} \\ &\qquad + \nu_l Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu\cdot a^{(1)},\mu\Big] - \nu_j Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu\cdot a^{(1)},\mu\Big] \\ &\quad + \nu\cdot a^{(1)}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu_j,\mu\Big] - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu_l,\mu\Big]\Big\} \\ &\quad - \nu\cdot a^{(1)} v\big[\partial\Omega,S_{\mathbf{a}},M_{lj}[\mu]\big] + v\big[\partial\Omega,S_{\mathbf{a}},\nu\cdot a^{(1)}M_{lj}[\mu]\big] + R[\nu_l,\nu_j,\mu] \quad \text{on } \partial\Omega \end{aligned} \tag{9.1}$$

for all $l,j \in \{1,\dots,n\}$*. (For* Q *see* (8.2).)

Proof. Fix $\beta \in]0,\alpha[$. First consider the specific case in which $\mu \in C^{1,\beta}(\partial\Omega)$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\sim$' be an extension operator of $C^{1,\beta}(\partial\Omega)$ to $C^{1,\beta}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ as in Lemma 2.1. By Theorem 7.3(i),(ii), we have $w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] \in C^{1,\beta}(\mathrm{cl}\,\Omega)$ and

$$M_{lj}\left[w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\right] = \frac{1}{2}\,M_{lj}[\mu] + M_{lj}\left[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\right]. \tag{9.2}$$

By the definition of M_{lj} and by equality (7.2), we obtain

$$\begin{aligned} M_{lj}\left[w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}\right] &= \nu_l\,\frac{\partial}{\partial x_j}\,w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] - \nu_j\,\frac{\partial}{\partial x_l}\,w^+[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu] \\ &= \nu_l\Big[\sum_{b,r=1}^{n} a_{br}\,\frac{\partial}{\partial x_b}\,v^+\big[\partial\Omega,S_{\mathbf{a}},M_{jr}[\mu]\big] + \sum_{b=1}^{n} a_b\,\frac{\partial}{\partial x_b}\,v^+[\partial\Omega,S_{\mathbf{a}},\nu_j\mu] \\ &\qquad - \frac{\partial}{\partial x_j}\,v^+[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})\mu] + a v^+[\partial\Omega,S_{\mathbf{a}},\nu_j\mu]\Big] \\ &\quad - \nu_j\Big[\sum_{b,r=1}^{n} a_{br}\,\frac{\partial}{\partial x_b}\,v^+\big[\partial\Omega,S_{\mathbf{a}},M_{lr}[\mu]\big] + \sum_{b=1}^{n} a_b\,\frac{\partial}{\partial x_b}\,v^+[\partial\Omega,S_{\mathbf{a}},\nu_l\mu] \\ &\qquad - \frac{\partial}{\partial x_l}\,v^+[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})\mu] + a v^+[\partial\Omega,S_{\mathbf{a}},\nu_l\mu]\Big] \\ &= \sum_{b,r=1}^{n} a_{br}\Big\{\nu_l\,\frac{\partial}{\partial x_b}\,v^+\big[\partial\Omega,S_{\mathbf{a}},M_{jr}[\mu]\big] - \nu_j\,\frac{\partial}{\partial x_b}\,v^+\big[\partial\Omega,S_{\mathbf{a}},M_{lr}[\mu]\big]\Big\} \end{aligned}$$

$$+\sum_{b=1}^{n} a_b\Big\{\nu_l \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_j\mu] - \nu_j \frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_l\mu]\Big\}$$
$$-\Big\{\nu_l \frac{\partial}{\partial x_j} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu] - \nu_j \frac{\partial}{\partial x_l} v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu]\Big\}$$
$$+a\Big\{\nu_l v[\partial\Omega, S_{\mathbf{a}}, \nu_j\mu] - \nu_j v[\partial\Omega, S_{\mathbf{a}}, \nu_l\mu]\Big\} \quad \text{on } \partial\Omega\,. \tag{9.3}$$

We now consider the first term in braces in the right–hand side of (9.3) and note that

$$\begin{aligned}
&\Big\{\nu_l(x)\frac{\partial}{\partial x_b} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{jr}[\mu]\big](x) - \nu_j \frac{\partial}{\partial x_b} v^+\big[\partial\Omega, S_{\mathbf{a}}, M_{lr}[\mu]\big](x)\Big\}\\
&\qquad = -\frac{\nu_l(x)\nu_b(x)}{2\nu^t(x)a^{(2)}\nu(x)} M_{jr}[\mu](x) + \nu_l(x)\int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)M_{jr}[\mu](y)\,d\sigma_y\\
&\quad + \frac{\nu_j(x)\nu_b(x)}{2\nu^t(x)a^{(2)}\nu(x)} M_{lr}[\mu](x) - \nu_j(x)\int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)M_{lr}[\mu](y)\,d\sigma_y\\
&\qquad = \nu_b(x)\,\frac{-\nu_l(x)M_{jr}[\mu](x)+\nu_j(x)M_{lr}[\mu](x)}{2\nu^t(x)a^{(2)}\nu(x)}\\
&\qquad\qquad + \int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)\big\{\nu_l(x)M_{jr}[\mu](y) - \nu_j(x)M_{lr}[\mu](y)\big\}\,d\sigma_y\,.
\end{aligned}\tag{9.4}$$

Further, we note that

$$\big[\nu_l M_{jr}[\mu] - \nu_j M_{lr}[\mu]\big] = \nu_l\nu_j\frac{\partial\mu}{\partial x_r} - \nu_l\nu_r\frac{\partial\mu}{\partial x_j} - \nu_j\nu_l\frac{\partial\mu}{\partial x_r} + \nu_j\nu_r\frac{\partial\mu}{\partial x_l} = -\nu_r M_{lj}[\mu] \quad \text{on } \partial\Omega\,. \tag{9.5}$$

Then we obtain

$$\begin{aligned}
&\sum_{b,r=1}^{n} a_{br}\nu_b\,\frac{-\nu_l M_{jr}[\mu]+\nu_j M_{lr}[\mu]}{2\nu^t a^{(2)}\nu}\\
&\qquad = \sum_{b,r=1}^{n} a_{br}\nu_b\,\frac{\nu_r M_{lj}[\mu]}{2\nu^t a^{(2)}\nu} = \frac{\sum_{b,r=1}^{n}\nu_b a_{br}\nu_r}{2\nu^t a^{(2)}\nu} M_{lj}[\mu] = \frac{1}{2} M_{lj}[\mu] \quad \text{on } \partial\Omega\,.
\end{aligned}\tag{9.6}$$

Consider the term in braces in the argument of the integral in the right-hand side of (9.4) and note that equality (9.5) yields

$$\begin{aligned}
&\nu_l(x)M_{jr}[\mu](y) - \nu_j(x)M_{lr}[\mu](y)\\
&\quad = [\nu_l(x)-\nu_l(y)]M_{jr}[\mu](y) + \big[\nu_l(y)M_{jr}[\mu](y) - \nu_j(y)M_{lr}[\mu](y)\big] - [\nu_j(x)-\nu_j(y)]M_{lr}[\mu](y)\\
&\quad = [\nu_l(x)-\nu_l(y)]M_{jr}[\mu](y) - \nu_r(y)M_{lj}[\mu](y) - [\nu_j(x)-\nu_j(y)]M_{lr}[\mu](y) \quad \forall\, x,y\in\partial\Omega\,.
\end{aligned}\tag{9.7}$$

We now consider the term in the second braces in the right-hand side of equality (9.3) and we note that

$$\begin{aligned}
&\nu_l(x)\frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_j\mu](x) - \nu_j(x)\frac{\partial}{\partial x_b} v^+[\partial\Omega, S_{\mathbf{a}}, \nu_l\mu](x)\\
&\qquad = -\nu_l(x)\frac{\nu_b(x)}{2\nu^t(x)a^{(2)}\nu(x)}\nu_j(x)\mu(x) + \nu_l(x)\int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)\nu_j(y)\mu(y)\,d\sigma_y\\
&\qquad\quad + \nu_j(x)\frac{\nu_b(x)}{2\nu^t(x)a^{(2)}\nu(x)}\nu_l(x)\mu(x) - \nu_j(x)\int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)\nu_l(y)\mu(y)\,d\sigma_y\\
&\qquad = \int_{\partial\Omega}\frac{\partial}{\partial x_b} S_{\mathbf{a}}(x-y)[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)]\mu(y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,.
\end{aligned}\tag{9.8}$$

Next, we consider the term in the third braces in the right-hand side of equality (9.3) and we note that

$$
\begin{aligned}
&\nu_l(x)\,\frac{\partial}{\partial x_j}\,v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu](x) - \nu_j(x)\,\frac{\partial}{\partial x_l}v^+[\partial\Omega, S_{\mathbf{a}}, (\nu^t\cdot a^{(1)})\mu](x)\\
&\quad = -\nu_l(x)\,\frac{\nu_j(x)}{2\nu^t(x)a^{(2)}\nu(x)}\,(\nu^t(x)\cdot a^{(1)})\mu(x) + \nu_l(x)\int_{\partial\Omega}\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\nu^t(y)\cdot a^{(1)}\mu(y)\,d\sigma_y\\
&\quad + \nu_j(x)\,\frac{\nu_l(x)}{2\nu^t(x)a^{(2)}\nu(x)}\,(\nu^t(x)\cdot a^{(1)})\mu(x) - \nu_j(x)\int_{\partial\Omega}\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\nu^t(y)\cdot a^{(1)}\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_l(x)\int_{\partial\Omega}(\nu^t(x)\cdot a^{(1)})\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad - \nu_j(x)\int_{\partial\Omega}(\nu^t(x)\cdot a^{(1)})\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + (\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega}\Big(\nu_l(x)\,\frac{\partial}{\partial x_j} - \nu_j(x)\,\frac{\partial}{\partial x_l}\Big)S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\quad = -\nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&\qquad + \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&+ (\nu^t(x)\cdot a^{(1)})\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y - \int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\Big\}\\
&\qquad + (\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega}\Big(\nu_l(y)\,\frac{\partial}{\partial x_j} - \nu_j(y)\,\frac{\partial}{\partial x_l}\Big)S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y
\end{aligned}
\tag{9.9}
$$

for all $x\in\partial\Omega$. By Lemma 2.2, the last integral in the right-hand side of (9.9) equals

$$
-\int_{\partial\Omega} M_{lj,y}[S_{\mathbf{a}}(x-y)]\mu(y)\,d\sigma_y = \int_{\partial\Omega} S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,. \tag{9.10}
$$

Thus the last term in the right-hand side of (9.9) equals

$$
\begin{aligned}
(\nu^t(x)\cdot a^{(1)})\int_{\partial\Omega} S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y &= \int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)}) - (\nu^t(y)\cdot a^{(1)})\big]S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y\\
&\quad + \int_{\partial\Omega}(\nu^t(y)\cdot a^{(1)})S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,.
\end{aligned}
\tag{9.11}
$$

The last term in braces of equation (9.3) equals

$$\int_{\partial\Omega} S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,. \tag{9.12}$$

Combining (9.2)–(9.4), (9.6)–(9.12), we obtain

$$\begin{aligned}
M_{lj}\big[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]\big](x) &= \sum_{b,r=1}^{n} a_{br}\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{jr}[\mu](y)\,d\sigma_y \\
&- \int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{lr}[\mu](y)\,d\sigma_y - \int_{\partial\Omega}\nu_r(y)\,\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y\Big\} \\
&+ \sum_{b=1}^{n} a_b \int_{\partial\Omega}\frac{\partial}{\partial x_b}\,S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \\
&+ \nu_l(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&- \nu_j(x)\int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y \\
&- (\nu^t(x)\cdot a^{(1)})\Big\{\int_{\partial\Omega}(\nu_l(x)-\nu_l(y))\,\frac{\partial}{\partial x_j}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y - \int_{\partial\Omega}(\nu_j(x)-\nu_j(y))\,\frac{\partial}{\partial x_l}\,S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\Big\} \\
&- \int_{\partial\Omega}\big[(\nu^t(x)\cdot a^{(1)})-(\nu^t(y)\cdot a^{(1)})\big]S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y - \int_{\partial\Omega}(\nu^t(y)\cdot a^{(1)})S_{\mathbf{a}}(x-y)M_{lj}[\mu](y)\,d\sigma_y \\
&+ a\int_{\partial\Omega} S_{\mathbf{a}}(x-y)\big[\nu_l(x)\nu_j(y)-\nu_j(x)\nu_l(y)\big]\mu(y)\,d\sigma_y \quad \forall\, x\in\partial\Omega\,,
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
M_{lj}\big[w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]\big](x) &= \sum_{b,r=1}^{n} a_{br}\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_l,M_{jr}[\mu]\Big](x) - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_j,M_{lr}[\mu]\Big](x)\Big\} \\
&+ \nu_l(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu^t\cdot a^{(1)},\mu\Big](x) - \nu_j(x)Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu^t\cdot a^{(1)},\mu\Big](x) \\
&+ w\big[\partial\Omega,\mathbf{a},S_{\mathbf{a}},M_{lj}[\mu]\big](x) + (\nu^t(x)\cdot a^{(1)})\Big\{Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_l}\circ\Theta,\nu_j,\mu\Big](x) - Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_j}\circ\Theta,\nu_l,\mu\Big](x)\Big\} \\
&- (\nu^t(x)\cdot a^{(1)})v\big[\partial\Omega,S_{\mathbf{a}},M_{lj}[\mu]\big](x) + v\big[\partial\Omega,S_{\mathbf{a}},(\nu^t\cdot a^{(1)})M_{lj}[\mu]\big](x) + R[\nu_l,\nu_j,\mu](x) \quad \forall\, x\in\partial\Omega\,.
\end{aligned}$$

Thus we have proved formula (9.1) for $\mu\in C^{1,\beta}(\partial\Omega)$.

Next, we assume that $\mu\in C^1(\partial\Omega)$. We denote by $T_{lj}[\mu]$ the right-hand side of (9.1). By the continuity of M_{lj} from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$, of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ and $v[\partial\Omega,S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$, of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r}\circ\Theta,\,\cdot\,,\,\cdot\,]$ from $C^{0,\alpha}(\partial\Omega)\times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha}(\partial\Omega)$, of R from $(C^{0,\alpha}(\partial\Omega))^2\times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha}(\partial\Omega)$, and by the continuity of the pointwise product in Schauder spaces, we can conclude that the operators $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot\,]_{|\partial\Omega}$ and $T_{lj}[\,\cdot\,]$ are continuous from $C^1(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ and from $C^1(\partial\Omega)$ to $C^{0,\omega_\alpha(\,\cdot\,)}(\partial\Omega)$, respectively. In particular, $T_{lj}[\mu]$ and $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}$ belong to $C^0(\partial\Omega)$. We now show that the weak M_{lj}-derivative of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu]_{|\partial\Omega}$ coincides with $T_{lj}[\mu]$.

By arguing just as at the end of the proof of Lemma 8.1, there exists a sequence of functions $\{\mu_b\}_{b\in\mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$, which converges to μ in $C^1(\partial\Omega)$. Note that if $\varphi\in C^1(\partial\Omega)$, then the validity of (9.1) for $\mu_b\in C^{1,\alpha}(\partial\Omega)$, the membership of $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu_b]_{|\partial\Omega}$ in $C^{1,\alpha}(\partial\Omega)$, the above-mentioned continuity of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$, and also Lemma 2.2 imply that

$$\int_{\partial\Omega} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} M_{lj}[\varphi]\, d\sigma = \lim_{b\to\infty} \int_{\partial\Omega} w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu_b]_{|\partial\Omega} M_{lj}[\varphi]\, d\sigma$$

$$= -\lim_{b\to\infty} \int_{\partial\Omega} M_{lj}\left[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu_b]_{|\partial\Omega}\right] \varphi\, d\sigma = -\lim_{b\to\infty} \int_{\partial\Omega} T_{lj}[\mu_b]\varphi\, dx = -\int_{\partial\Omega} T_{lj}[\mu]\varphi\, dx\,.$$

Hence, $T_{lj}[\mu]$ coincides with the weak M_{lj}-derivative of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ for all l, j in $\{1, \dots, n\}$. Since both $T_{lj}[\mu]$ and $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}$ are the continuous functions, it follows that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega} \in C^1(\partial\Omega)$ and $M_{lj}[w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \mu]_{|\partial\Omega}] = T_{lj}[\mu]$, classically. Hence (9.1) holds also for $\mu \in C^1(\partial\Omega)$. □

Using formula (9.1), we now prove the following result, which says that the double layer potential on $\partial\Omega$ has a regularizing effect.

Theorem 9.2. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3), $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a}, D]$, $\alpha \in]0, 1[$, $m \in \mathbb{N}\setminus\{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$. *Then the following statements hold:*

(i) *The operator* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^m(\partial\Omega)$ *to* $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$.

(ii) *Let* $\beta \in]0, \alpha]$. *Then the operator* $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^{m,\beta}(\partial\Omega)$ *to* $C^{m,\alpha}(\partial\Omega)$.

Proof. We prove statement (i) by induction on m. As in the previous proof, we denote by $T_{lj}[\mu]$ the right-hand side of formula (9.1). We first consider the case $m = 1$. By Lemma 2.3(ii) and formula (9.1), it suffices to prove that the following two statements hold:

(j) $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$;

(jj) $T_{lj}[\cdot]$ is continuous from $C^1(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ for all $l, j \in \{1, \dots, n\}$.

Theorem 7.4 implies the validity of (j). Statement (jj) follows by the continuity of the pointwise product in Schauder spaces, by the continuity of M_{lj} from $C^1(\partial\Omega)$ to $C^0(\partial\Omega)$, by the continuity of $v[\partial\Omega, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ and of $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ (cf. Theorems 7.2, 7.4), and also by the continuity of $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \cdot, \cdot]$ from $C^{0,\alpha}(\partial\Omega) \times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.2(i)) and by the continuity of R from $\left(C^{0,\alpha}(\partial\Omega)\right)^2 \times C^0(\partial\Omega)$ to $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.4(i).)

Next, we assume that Ω is of the class $C^{m+1,\alpha}$ and we turn to prove that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m+1,\omega_\alpha(\cdot)}(\partial\Omega)$. By Lemma 2.3(ii) and formula (9.1), it suffices to prove that the following two statements hold:

(a) $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^0(\partial\Omega)$;

(b) $T_{lj}[\cdot]$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$. for all $l, j \in \{1, \dots, n\}$.

Statement (a) holds by the inductive assumption. We now prove statement (b). Since Ω is of the class $C^{m+1,\alpha}$, then ν is of the class $C^{m,\alpha}(\partial\Omega)$. Theorem 8.3(i) ensures that $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \nu \cdot a^{(1)}, \cdot]$ and $Q[\frac{\partial S_{\mathbf{a}}}{\partial x_r} \circ \Theta, \nu_j, \cdot]$ are continuous from $C^m(\partial\Omega)$ to $C^{m,\omega_\alpha}(\partial\Omega)$ for all l, j, r in $\{1, \dots, n\}$. Since M_{lj} is continuous from $C^{m+1}(\partial\Omega)$ to $C^m(\partial\Omega)$, the inductive assumption implies that $w[\partial\Omega, \mathbf{a}, S_{\mathbf{a}}, M_{lj}[\cdot]]_{|\partial\Omega}$ is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ for all l, j in $\{1, \dots, n\}$.

Since M_{lj} is continuous from $C^{m+1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ and $v[\partial\Omega, S_{\mathbf{a}}, \cdot]_{|\partial\Omega}$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, $\nu \in (C^{m,\alpha}(\partial\Omega))^n$ and $C^{m,\alpha}(\partial\Omega)$ is continuously imbedded into $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$, we conclude that $v[\partial\Omega, S_{\mathbf{a}}, M_{lj}[\cdot]]_{|\partial\Omega}$ and $v[\partial\Omega, S_{\mathbf{a}}, \nu \cdot a^{(1)} M_{lj}[\cdot]]_{|\partial\Omega}$ are continuous from the space $C^{m+1}(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ for all l, j in $\{1, \dots, n\}$. Moreover, R is continuous from $(C^{m,\alpha}(\partial\Omega))^2 \times C^m(\partial\Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ (cf. Theorem 8.4(i)). Then statement (b) holds true.

Statement (iii) can be proved by the same argument of the proof of statement (i) by exploiting Theorem 8.3(ii) instead of Theorem 8.3(i) and Theorem 8.4(ii) instead of Theorem 8.4(i). □

Since $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ is compactly imbedded into $C^m(\partial\Omega)$ and $C^{m,\alpha}(\partial\Omega)$ is compactly imbedded into $C^{m,\beta}(\partial\Omega)$ for all $\beta\in]0,\alpha[$, we have the following immediate consequence of Theorem 9.2.

Corollary 9.1. *Under the assumptions of Theorem* 9.2*, the linear operator* $w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is compact from* $C^m(\partial\Omega)$ *to itself, from* $C^{m,\omega_\alpha(\cdot)}(\partial\Omega)$ *to itself and from* $C^{m,\alpha}(\partial\Omega)$ *to itself.*

10 Other layer potentials associated to $P[\mathbf{a},D]$

Another relevant layer potential operator associated to the analysis of boundary value problems for the operator $P[\mathbf{a},D]$ is the following

$$w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu](x)\equiv\int_{\partial\Omega}\mu(y)DS_{\mathbf{a}}(x-y)a^{(2)}\nu(x)\,d\sigma_y\quad\forall\,x\in\partial\Omega\,,$$

which we now turn to consider.

Theorem 10.1. *Let* $\mathbf{a}$ *be as in* (1.1), (1.2), (1.3)*,* $S_{\mathbf{a}}$ *be a fundamental solution of* $P[\mathbf{a},D]$*,* $\alpha\in]0,1[$*,* $m\in\mathbb{N}\setminus\{0\}$ *and let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of the class* $C^{m,\alpha}$*. Then the following statements hold:*

(i) *The operator* $w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^{m-1}(\partial\Omega)$ *to* $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$*.*

(ii) *Let* $\beta\in]0,\alpha]$*. Then the operator* $w_*[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is linear and continuous from* $C^{m-1,\beta}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$*.*

Proof. First note that

$$\begin{aligned}
w_*[\partial\Omega,\mathbf{a},&S_{\mathbf{a}},\mu](x)=\sum_{b,r=1}^n a_{br}\int_{\partial\Omega}\nu_r(x)\frac{\partial}{\partial x_b}S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)+\sum_{b,r=1}^n a_{br}\int_{\partial\Omega}\nu_r(y)\,\frac{\partial}{\partial x_b}S_{\mathbf{a}}(x-y)\mu(y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)-\int_{\partial\Omega}\mu(y)\sum_{b,r=1}^n a_{br}\nu_r(y)\,\frac{\partial}{\partial y_b}\,S_{\mathbf{a}}(x-y)\,d\sigma_y\\
&=\sum_{b,r=1}^n a_{br}Q\Big[\frac{\partial S_{\mathbf{a}}}{\partial x_b}\circ\Theta,\nu_r,\mu\Big](x)-w[\partial\Omega,\mathbf{a},S_{\mathbf{a}},\mu](x)-v[\partial\Omega,S_{\mathbf{a}},(a^{(1)}\nu)\mu](x)
\end{aligned}\tag{10.1}$$

for all $x\in\partial\Omega$ and $\mu\in C^0(\partial\Omega)$.

If $m=1$, then Theorem 7.2 implies that $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$.

If $m>1$, then $C^{m-1}(\partial\Omega)$ is continuously imbedded into $C^{m-2,\alpha}(\partial\Omega)$ and Theorem 7.1 implies that $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-2,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. Hence, $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from the space $C^{m-1}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ for all $m\geq 1$. Then formula (10.1), the continuity of the imbedding of $C^{m-1,\alpha}(\partial\Omega)$ into $C^{m-1,\omega_\alpha}(\partial\Omega)$ and Theorems 8.3(i), 9.2(i) imply the validity of statement (i).

We now consider statement (ii). Since $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from $C^{m-1,\beta}(\partial\Omega)$ to $C^{m,\beta}(\partial\Omega)$ and $C^{m,\beta}(\partial\Omega)$ is continuously imbedded into $C^{m-1,\alpha}(\partial\Omega)$, the operator $v[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ is continuous from $C^{m-1,\beta}(\partial\Omega)$ into $C^{m-1,\alpha}(\partial\Omega)$. Then formula (10.1) and Theorems 8.3(ii), 9.2(ii) imply the validity of statement (ii). □

Since the space $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ is compactly imbedded into $C^{m-1}(\partial\Omega)$, and $C^{m-1,\alpha}(\partial\Omega)$ is compactly imbedded into $C^{m-1,\beta}(\partial\Omega)$ for all $\beta\in]0,\alpha[$, we have the following immediate consequence of Theorem 10.1(ii).

Corollary 10.1. *Under the assumptions of Theorem* 10.1*,* $w_*[\partial\Omega,S_{\mathbf{a}},\cdot]_{|\partial\Omega}$ *is compact from* $C^{m-1}(\partial\Omega)$ *to itself, from* $C^{m-1,\omega_\alpha(\cdot)}(\partial\Omega)$ *to itself and from* $C^{m-1,\alpha}(\partial\Omega)$ *to itself.*

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Authors' addresses:

Francesco Dondi

Ascent software, Malta Office, 92/3, Alpha Center, Tarxien Road, Luqa, LQA 1815, Malta.
E-mail: `francesco314@gmail.com`

Massimo Lanza de Cristoforis

Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, Padova 35121, Italy.
E-mail: `mldc@math.unipd.it`