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Sergo Kharibegashvili and Otar Jokhadze

THE CAUCHY–DARBOUX PROBLEM FOR WAVE EQUATIONS WITH A NONLINEAR DISSIPATIVE TERM

Abstract. The Cauchy–Darboux problem for wave equations with a nonlinear dissipative term is investigated. The questions on the existence, uniqueness and nonexistence of a global solution of the problem are considered. The question of local solvability of the problem is also discussed.

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რეზიუმე. გამოკვლეულია კოში-დარბუს ამოცანა ტალღის განტოლებისთვის არაწრფივი დისიპატიური წევრით. განხილულია არსებობის, ერთადერთობის და გლობალური ამონახსნის არარსებობის საკითხები. ასევე განიხილება დასმული ამოცანის ლოკალური ამოხსნადობის საკითხი.

1. Statement of the Problem

In a plane of independent variables x and t we consider a wave equation with a nonlinear dissipative term (see [16, p. 57], [17])

$$Lu := u_{tt} - u_{xx} + g(x,t,u)u_t = f(x,t), \tag{1.1}$$

where f, g are the given and u is an unknown real functions.

By $D_T := \{(x,t) :\ 0 < x < \widetilde{k}t,\ 0 < t < T\}$ we denote a triangular domain lying inside of the characteristic angle $t > |x|$ and bounded by the segments $\widetilde{\gamma}_{1,T} :\ x = \widetilde{k}t$, $\widetilde{\gamma}_{2,T} :\ x = 0$, $0 \le t \le T$ and $\widetilde{\gamma}_{3,T} :\ t = T$, $0 \le x \le \widetilde{k}T$, $0 < \widetilde{k} < 1$. For $T = +\infty$, we assume that $D_\infty := \{(x,t) \in \mathbb{R}^2 :\ 0 < x < \widetilde{k}t,\ 0 < t < +\infty\}$.

For the equation (1.1), we consider the Cauchy–Darboux problem on finding a solution $u(x,t)$ in the domain D_T by the conditions [2, p. 284]

$$u\big|_{\widetilde{\gamma}_{1,T}} = 0, \quad u_x\big|_{\widetilde{\gamma}_{2,T}} = 0. \tag{1.2}$$

Note that, the problem

$$u_{tt} - u_{xx} + a(x,t)u_x + b(x,t)u_t + c(x,t)u + d(x,t,u) = f(x,t),$$
$$(\alpha_i u_x + \beta_i u_t + \gamma_i u)\big|_{\gamma_{i,T}} = 0,\ \ i = 1,2;\ \ u(0,0) = 0$$

in a linear case has been investigated in [4,11,12,18,22,23] and in a nonlinear case in [1,6–8,10,13–15]. As is mentioned in [4,23], the problems of such a matter arise under mathematical simulation of small harmonic wedge oscillations in a supersonic flow and of string oscillations in a cylinder filled with a viscous liquid. It should also be noted that when passing from the nonlinearity $d(x,t,u)$ appearing in [1, 6–8, 10, 13–15] to the nonlinearity of type $g(x,t,u)u_t$ in the equation (1.1), as it will be seen below when studying the boundary value problem, there arise difficulties, and not only of technical character.

Below, we will show that under definite requirements imposed on the nonlinear function g the problem (1.1), (1.2) is locally solvable. The conditions of global solvability of the problem will be obtained, violation of these conditions may, generally speaking, give rise to a soluion destruction after a lapse of a finite time interval. The question on the uniqueness of a solution of the problem (1.1), (1.2) will also be considered in the present work.

Definition 1.1. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$. The function u is said to be a general solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T if $u \in C^1(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T)$ such that $u_n \to u$ and $Lu_n \to f$, as $n \to \infty$, respectively, in the spaces $C^1(\overline{D}_T)$ and $C(\overline{D}_T)$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T) := \{v \in C^2(\overline{D}_T) :\ v|_{\widetilde{\gamma}_{1,T}} = 0,\ v_x|_{\widetilde{\gamma}_{2,T}} = 0\}$, $\widetilde{\Gamma}_T := \widetilde{\gamma}_{1,T} \cup \widetilde{\gamma}_{2,T}$.

Remark 1.1. Below, for the sake of simplicity of our exposition, sometimes instead of a generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T we will speak about a generalized solution of that problem.

Remark 1.2. Obviously, a classical solution of the problem (1.1), (1.2) from the space $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \widetilde{\Gamma}_T)$ is a generalized solution of that problem. In its turn, if a generalized solution of the problem (1.1), (1.2) belongs to the space $C^2(\overline{D}_T)$, it will also be a classical solution of the problem.

Definition 1.2. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$. We say that the problem (1.1), (1.2) is locally solvable in the class C^1, if there is a positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$, this problem has a generalized solution of the class C^1 in the domain D_{T_1}.

Definition 1.3. Let $f \in C(\overline{D}_\infty)$, $g \in C(\overline{D}_\infty \times \mathbb{R})$. We say that the problem (1.1), (1.2) is globally solvable in the class C^1, if for any finite $T > 0$ this problem has a generalized solution of the class C^1 in the domain D_{T_1}.

When investigating the problem (1.1), (1.2), below, in Section 4, we will need to study the following mixed problem: in the domain $D_{t_1,t_2} := D_T \cap \{t_1 < t < t_2\}$, where $0 < t_1 < t_2 \le T$, find a solution $u(x,t)$ of the equation (1.1) by the initial

$$u\big|_{t=t_1} = \varphi, \quad u_t\big|_{t=t_1} = \psi \tag{1.3}$$

and boundary

$$u\big|_{\partial D_{t_1,t_2}\cap\widetilde{\gamma}_{1,T}}=0,\quad u_x\big|_{\partial D_{t_1,t_2}\cap\widetilde{\gamma}_{2,T}}=0 \tag{1.4}$$

conditions.

Remark 1.3. Analogously, just as in the case of the problem (1.1), (1.2), we introduce the notions of a generalized solution, local and global solvability of the problem (1.1), (1.3), (1.4).

2. EQUIVALENT REDUCTION OF THE PROBLEM (1.1), (1.2) TO THE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF VOLTERRA TYPE

In new independent variables $\xi=\frac{1}{2}\,(t+x)$, $\eta=\frac{1}{2}\,(t-x)$, the domain D_T will pass into a triangular domain E_T with vertices at the points $O(0,0)$, $Q_1(\frac{T}{1+k},\frac{kT}{1+k})$, $Q_2(\frac{T}{2},\frac{T}{2})$ of the plane of variables ξ, η, and the problem (1.1), (1.2) will pass into the problem (see Figure 2.1)

$$\widetilde{L}\widetilde{u}:=\widetilde{u}_{\xi\eta}+\frac{1}{2}\,g(\xi-\eta,\xi+\eta,\widetilde{u})(\widetilde{u}_{\xi}+\widetilde{u}_{\eta})=\widetilde{f}(\xi,\eta),\quad(\xi,\eta)\in E_T, \tag{2.1}$$

$$\widetilde{u}\big|_{\gamma_{1,T}}=0,\quad(\widetilde{u}_{\xi}-\widetilde{u}_{\eta})\big|_{\gamma_{2,T}}=0, \tag{2.2}$$

with respect to a new unknown function $\widetilde{u}(\xi,\eta):=u(\xi-\eta,\xi+\eta)$ with the right-hand side $\widetilde{f}(\xi,\eta):=f(\xi-\eta,\xi+\eta)$. Here,

$$\gamma_{1,T}:\eta=k\xi,\ \ 0\leq\xi\leq\xi_T:=\frac{T}{1+k},\quad\gamma_{2,T}:\xi=\eta,\ \ 0\leq\eta\leq\eta_T:=\frac{T}{2}, \tag{2.3}$$

$$0<k:=\frac{1-\widetilde{k}}{1+\widetilde{k}}<1. \tag{2.4}$$

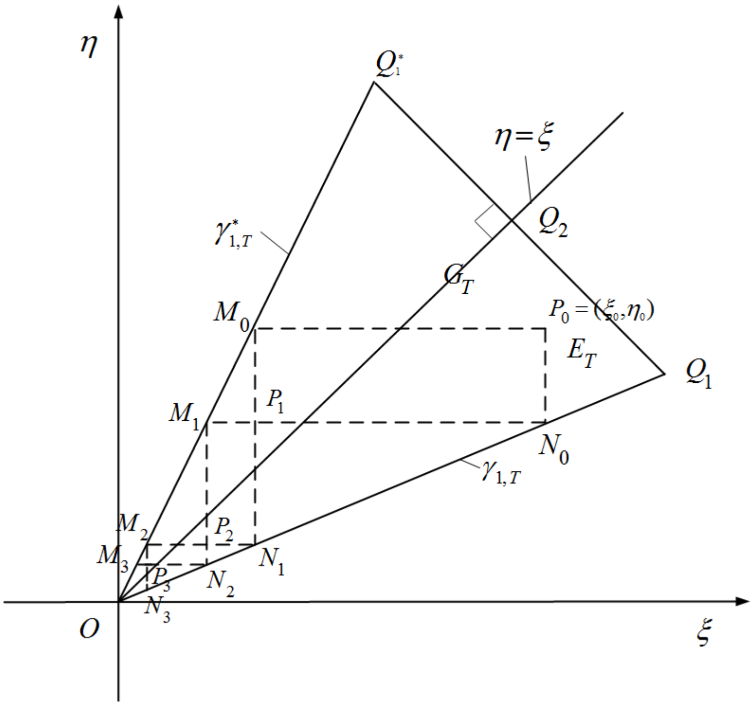

FIGURE 1

Remark 2.1. According to Definition 1.1, we introduce the notion of a generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) of the class C^1 in the domain E_T, i.e., there exists a sequence of function $\widetilde{u}_n\in\overset{\circ}{C}{}^2(\overline{E}_T,\Gamma_T):=\{w\in C^2(\overline{E}_T):\ w|_{\gamma_{1,T}}=0,\ (w_{\xi}-w_{\eta})|_{\gamma_{2,T}}=0\}$ such that

$$\lim_{n\to\infty}\|\widetilde{u}_n-\widetilde{u}\|_{C(\overline{E}_T)}=0,\quad\lim_{n\to\infty}\|\widetilde{L}\widetilde{u}_n-\widetilde{f}\|_{C(\overline{E}_T)}=0, \tag{2.5}$$

where $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Note that, if u is a generalized solution of the problem (1.1), (1.2) in a sense of Definition 1.1, then $\widetilde{u}$ will be a generalized solution of the problem (2.1), (2.2) in a sense of the given definition, and vice versa.

By G_T we denote a triangular domain with vertices at the points O, $Q_1, Q_1^*(\frac{kT}{1+k}, \frac{T}{1+k})$, symmetric with respect to the straight line $\xi = \eta$, and as is easily seen, $G_T \cap \{\eta < \xi\} = E_T$.

We continue the functions $\widetilde{u}_n$ and $\widetilde{f}$ evenly with respect to the straight line $\xi = \eta$ into the domain G_T retaining for them previous notation, i.e.,

$$\widetilde{u}_n(\xi,\eta) = \widetilde{u}_n(\eta,\xi), \quad \widetilde{f}(\xi,\eta) = \widetilde{f}(\eta,\xi), \ \ (\xi,\eta) \in G_T. \tag{2.6}$$

Remark 2.2. Since $\widetilde{f}|_{\overline{E}_T} \in C(\overline{E}_T)$ and $\widetilde{u}_n|_{\overline{E}_T} \in \overset{\circ}{C}{}^2(\overline{E}_T, \Gamma_T)$, taking into account (2.6), we have

$$\widetilde{f} \in C(\overline{G}_T), \quad \widetilde{u}_n \in C^2(\overline{G}_T), \tag{2.7}$$

$$\widetilde{u}_n\big|_{\gamma_{1,T}} = 0, \quad \widetilde{u}_n\big|_{\gamma_{1,T}^*} = 0, \tag{2.8}$$

where $\gamma_{1,T}^* := OQ_1^* \in \partial G_T$, i.e., $\gamma_{1,T}^* : \ \xi = k\eta$, $0 \leq \eta \leq \frac{T}{1+k}$.

Remark 2.3. Note that, for the functions $\widetilde{u}_n$, $\widetilde{f}$, continued to the domain G_T, the limiting equalities of type (2.5) remain valid, i.e.,

$$\lim_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}\|_{C(\overline{G}_T)} = 0, \quad \lim_{n\to\infty} \|\widetilde{L}\widetilde{u}_n - \widetilde{f}\|_{C(\overline{G}_T)} = 0. \tag{2.9}$$

If $P_0 = (\xi_0, \eta_0) \in E_T$, we denote by $P_1M_0P_0N_0$ the characteristic with respect to the equation (2.1) rectangle whose vertices N_0 and M_0 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{1,T}^*$, i.e., by virtue of (2.3): $N_0 = (\xi_0, k\xi_0)$, $M_0 = (k\eta_0, \eta_0)$, $P_1 = (k\eta_0, k\xi_0)$. Since $P_1 \in G_T$, we construct analogously the characteristic rectangle $P_2M_1P_1N_1$ with vertices N_1 and M_1 lying, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{1,T}^*$. Continuing this process, we get the characteristic rectangle $P_{i+1}M_iP_iN_i$ for which $N_i \in \gamma_{1,T}$, $M_i \in \gamma_{1,T}^*$, where $N_i = (\xi_i, k\xi_i)$, $M_i = (k\eta_i, \eta_i)$, $P_{i+1} = (k\eta_i, k\xi_i)$, if $P_i = (\xi_i, \eta_i)$, $i = 0, 1, \dots$.

It is easily seen that

$$\begin{aligned} P_{2m} &= (k^{2m}\xi_0, k^{2m}\eta_0), \quad P_{2m+1} = (k^{2m+1}\eta_0, k^{2m+1}\xi_0), \\ M_{2m} &= (k^{2m+1}\eta_0, k^{2m}\eta_0), \quad M_{2m+1} = (k^{2m+2}\xi_0, k^{2m+1}\xi_0), \quad m = 0, 1, 2, \dots . \\ N_{2m} &= (k^{2m}\xi_0, k^{2m+1}\xi_0), \quad N_{2m+1} = (k^{2m+1}\eta_0, k^{2m+2}\eta_0), \end{aligned} \tag{2.10}$$

As is known, for any function v of the class C^2 in the closed characteristic rectangle $P_{i+1}M_iP_iN_i$ the equality (see, e.g., [3, p. 173])

$$v(P_i) = v(M_i) + v(N_i) - v(P_{i+1}) + \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, v \; d\xi_1 d\eta_1, \ \ i = 0, 1, \dots, \tag{2.11}$$

where $\widetilde{\square} = \frac{\partial^2}{\partial\xi\partial\eta}$, is valid.

From (2.10), by virtue of (2.8), we have $\widetilde{u}_n(M_i) = \widetilde{u}_n(N_i) = 0$, $i = 0, 1, 2, \dots$. Therefore, (2.11) for $v = \widetilde{u}_n$ results in

$$\begin{aligned} \widetilde{u}_n(\xi_0, \eta_0) &= \widetilde{u}_n(P_0) = \widetilde{u}_n(M_0) + \widetilde{u}_n(N_0) - \widetilde{u}_n(P_1) + \int\limits_{P_1M_0P_0N_0} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 \\ &= -\widetilde{u}_n(P_1) + \int\limits_{P_1M_0P_0N_0} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 \\ &= -\widetilde{u}_n(M_1) - \widetilde{u}_n(N_1) + \widetilde{u}_n(P_2) - \int\limits_{P_2M_1P_1N_1} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 + \int\limits_{P_1M_0P_0N_0} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 \\ &= \widetilde{u}_n(P_2) - \int\limits_{P_2M_1P_1N_1} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 + \int\limits_{P_1M_0P_0N_0} \widetilde{\square}\,\widetilde{u}_n \; d\xi_1 \, d\eta_1 = \cdots \end{aligned}$$

$$= (-1)^m \widetilde{u}_n(P_m) + \sum_{i=0}^{m-1} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1, \ \ (\xi_0,\eta_0) \in E_T. \tag{2.12}$$

Since the point P_m from (2.12) tends to the point O, as $m \to \infty$, by virtue of (2.8), we have $\lim\limits_{m\to\infty} \widetilde{u}_n(P_m) = 0$. Hence, passing in the equality (2.12) to the limit, as $m \to \infty$, for the function $\widetilde{u}_n \in C^2(\overline{G}_T)$ in the domain E_T we obtain the following integral representation:

$$\widetilde{u}_n(\xi_0,\eta_0) = \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1, \ \ (\xi_0,\eta_0) \in E_T. \tag{2.13}$$

Remark 2.4. Since $\widetilde{\square}\, \widetilde{u}_n \in C(\overline{E}_T)$ and there are the inequalities (2.4), and owing to (2.10),

$$\operatorname{mes} P_{i+1}M_iP_iN_i = k^{2i}(\xi_0 - k\eta_0)(\eta_0 - k\xi_0), \tag{2.14}$$

therefore the series in the right-hand side of the equality (2.13) is uniformly and absolutely convergent.

It can be easily seen that by virtue of (2.4) and (2.14),

$$\begin{aligned} &\bigg| \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{\square}\, \widetilde{u}_n \, d\xi_1 \, d\eta_1 - \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1 \bigg| \\ &\leq \sum_{i=0}^{\infty} \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)} \operatorname{mes} P_{i+1}M_iP_iN_i = \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)} \sum_{i=0}^{\infty} k^{2i}(\xi_0 - k\eta_0)(\eta_0 - k\xi_0) \\ &\leq \frac{\xi_0\eta_0}{1-k^2} \big\| \widetilde{\square}\, \widetilde{u}_n - \widetilde{f} \big\|_{C(\overline{G}_T)}. \end{aligned} \tag{2.15}$$

Remark 2.5. By (2.5) for $g = 0$ and (2.15), passing in the equality (2.13) to the limit, as $n \to \infty$, for a generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) we obtain the following integral representation:

$$\widetilde{u}(\xi_0,\eta_0) = \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1, \ \ (\xi_0,\eta_0) \in E_T. \tag{2.16}$$

Remark 2.6. From the above reasonings it follows that for any $\widetilde{f} \in C(\overline{E}_T)$, the linear problem (2.1), (2.2) has a unique generalized solution $\widetilde{u}$ which is representable in the form of a uniformly and absolutely convergent series (2.16) and for $\widetilde{f} \in C^1(\overline{E}_T)$ is a classical solution of that problem, i.e., $\widetilde{u} \in \overset{\circ}{C}{}^2(\overline{E}_T, \Gamma_T)$.

According to (2.16), we introduce into consideration the operator $\widetilde{\square}^{-1} : C(\overline{E}_T) \to C(\overline{E}_T)$ acting by the formula

$$(\widetilde{\square}^{-1} \widetilde{f})(\xi,\eta) := \sum_{i=0}^{\infty} (-1)^i \int\limits_{P_{i+1}M_iP_iN_i} \widetilde{f} \, d\xi_1 \, d\eta_1, \ \ (\xi,\eta) \in E_T. \tag{2.17}$$

In the integrand here, according to (2.6), under $\widetilde{f}$ we mean the right-hand side of the equation (2.1) which is continued evenly from the domain E_T to the domain G_T with respect to the straight line $\xi = \eta$, and due to (2.7), we have $\widetilde{f} \in C(\overline{E}_T)$.

Remark 2.7. By virtue of (2.17) and Remark 2.6, a unique generalized solution $\widetilde{u}$ of the problem (2.1), (2.2) is representable in the form $\widetilde{u} = \widetilde{\square}^{-1} \widetilde{f}$, and in view of (2.4), (2.14), the estimate

$$\begin{aligned} |\widetilde{u}(\xi,\eta)| &\leq \sum_{i=0}^{\infty} \int\limits_{P_{i+1}M_iP_iN_i} |\widetilde{f}| \, d\xi_1 \, d\eta_1 \leq \xi\eta \|\widetilde{f}\|_{C(\overline{E}_T)} \sum_{i=0}^{\infty} k^{2i} \\ &\leq \frac{\xi^2+\eta^2}{2(1-k^2)} \|\widetilde{f}\|_{C(\overline{E}_T)} \leq \frac{T^2}{1-k^2} \|\widetilde{f}\|_{C(\overline{E}_T)} \end{aligned}$$

holds which in its turn yields

$$\|\widetilde{\square}^{-1}\|_{C(\overline{E}_T) \longrightarrow C(\overline{E}_T)} \leq \frac{T^2}{1-k^2}\,. \tag{2.18}$$

Remark 2.8. Standard reasonings (see, e.g., [9]) show that the function $\widetilde{u} \in C^1(\overline{E}_T)$ is the generalized solution of the problem (2.1), (2.2), if and only if it is a solution of the following nonlinear Volterra type integro-differential equation

$$\widetilde{u}(\xi,\eta) + \frac{1}{2}\,\widetilde{\Box}^{-1}\big(g(\xi-\eta,\xi+\eta,\widetilde{u})(\widetilde{u}_\xi + \widetilde{u}_\eta)\big)(\xi,\eta) = (\widetilde{\Box}^{-1}\widetilde{f})(\xi,\eta), \quad (\xi,\eta) \in E_T. \tag{2.19}$$

3. LOCAL SOLVABILITY OF THE PROBLEM (1.1), (1.2)

Lemma 3.1. *The operator* $\widetilde{\Box}^{-1}$ *defined by the formula* (2.17) *is the linear continuous operator acting from the space* $C(\overline{E}_T)$ *to the space* $C^1(\overline{E}_T)$*.*

Proof. To this end, we first show that for $\widetilde{f} \in C(\overline{E}_T)$, the series from the right-hand side of (2.17), differentiated formally with respect to ξ and to η converges uniformly on the set $\overline{E}_T$. Indeed, as it can be easily verified, we have

$$\begin{aligned}(L_1\widetilde{f})(\xi,\eta) &:= \frac{\partial}{\partial\xi}\,\big[(\widetilde{\Box}^{-1}\widetilde{f})(\xi,\eta)\big] \\ &= \sum_{n=0}^{\infty}\bigg[k^{2n}\int\limits_{N_{2n}P_{2n}} \widetilde{f}\,d\eta_1 + k^{2n+2}\int\limits_{P_{2n+2}M_{2n+1}} \widetilde{f}\,d\eta_1 - k^{2n+1}\int\limits_{M_{2n+1}N_{2n}} \widetilde{f}\,d\xi_1\bigg],\end{aligned} \tag{3.1}$$

$$\begin{aligned}(L_2\widetilde{f})(\xi,\eta) &:= \frac{\partial}{\partial\eta}\,\big[(\widetilde{\Box}^{-1}\widetilde{f})(\xi,\eta)\big] \\ &= \sum_{n=0}^{\infty}\bigg[k^{2n}\int\limits_{M_{2n}P_{2n}} \widetilde{f}\,d\xi_1 + k^{2n+2}\int\limits_{P_{2n+2}N_{2n+1}} \widetilde{f}\,d\xi_1 - k^{2n+1}\int\limits_{N_{2n+1}M_{2n}} \widetilde{f}\,d\eta_1\bigg].\end{aligned} \tag{3.2}$$

By virtue of (2.10), we have the equalities

$$\begin{aligned}&|N_{2m}P_{2m}| = k^{2m}(\eta - k\xi), \quad |P_{2m+2}M_{2m+1}| = k^{2m+1}(\xi - k\eta), \quad |M_{2m+1}N_{2m}| = k^{2m}(1-k^2)\xi, \\ &|M_{2m}P_{2m}| = k^{2m}(\xi - k\eta), \quad |P_{2m+2}N_{2m+1}| = k^{2m+1}(\eta - k\xi), \quad |N_{2m+1}M_{2m}| = k^{2m}(1-k^2)\eta,\end{aligned}$$

which in view of (2.4) imply that the series (3.1) and (3.2) are uniformly and absolutely convergent, and the estimate

$$\max\big\{\|L_1\widetilde{f}\|_{C(\overline{E}_T)}, \|L_2\widetilde{f}\|_{C(\overline{E}_T)}\big\} \le \frac{3T}{1-k^4}\,\|\widetilde{f}\|_{C(\overline{E}_T)} \tag{3.3}$$

holds.

From (3.3), in view of (2.18) and the fact that $\|v\|_{C^1} := \max\{\|v\|_C, \|v_\xi\|_C, \|v_\eta\|_C\}$, it follows that Lemma 3.1 is valid. $\Box$

Introducing the notation $v_1 := \widetilde{u}$, $v_2 := \widetilde{u}_\xi$, $v_3 := \widetilde{u}_\eta$ and differentiating formally the equality (2.19) with respect to ξ and η for $(\xi,\eta) \in E_T$, we obtain

$$\begin{cases} v_1(\xi,\eta) = -\frac{1}{2}\,\widetilde{\Box}^{-1}\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (\widetilde{\Box}^{-1}\widetilde{f})(\xi,\eta), \\ v_2(\xi,\eta) = -\frac{1}{2}L_1\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_1\widetilde{f})(\xi,\eta), \\ v_3(\xi,\eta) = -\frac{1}{2}L_2\big(g(\xi-\eta,\xi+\eta,v_1)(v_2+v_3)\big) + (L_2\widetilde{f})(\xi,\eta), \end{cases} \tag{3.4}$$

where the linear operators L_1 and L_2 are defined by the equalities (3.1) and (3.2).

Remark 3.1. It is not difficult to check that if $\widetilde{u} \in C^1(\overline{E}_T)$ is a solution of the nonlinear equation (2.19), then the functions $v_1 := \widetilde{u}$, $v_2 := \widetilde{u}_\xi$, $v_3 := \widetilde{u}_\eta$ of the class $C(\overline{E}_T)$ satisfy the system of nonlinear equations (3.4), and vice versa, if the functions v_1, v_2 and v_3 of the class $C(\overline{E}_T)$ satisfy the system of equations (3.4), then $v_1 \in C^1(\overline{E}_T)$ and $v_{1\xi} = v_2$, $v_{2\eta} = v_3$, and $\widetilde{u} = v_1$ will be a solution of the equation (2.19) of the class $C^1(\overline{E}_T)$.

We will now proceed to the proof of the local solvability of the system of nonlinear integral equations (3.4).

Let us consider the following conditions:

$$|g(x,t,s)| \le m(r), \quad \big|g(x,t,s_2) - g(x,t,s_1)\big| \le c(r)|s_2 - s_1|, \quad (x,t) \in \overline{D}_T, \quad |s|, |s_1|, |s_2| \le r, \tag{3.5}$$

where $m(r)$ and $c(r)$ are some nonnegative continuous functions of argument $r \ge 0$. Obviously, the conditions (3.5) will be fulfilled if $g, g_s \in C(\overline{D}_T \times \mathbb{R})$.

Theorem 3.1. *Let $f \in C(\overline{D}_T)$ and the function $g \in C(\overline{D}_T \times \mathbb{R})$ satisfy the conditions* (3.5)*. Then there exists a positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$ the problem* (1.1), (1.2) *has at least one generalized solution in the domain D_{T_1}.*

Proof. By Remarks 2.1 and 2.8, the problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of nonlinear integral equations (3.4) in the class $C(\overline{E}_T)$. Below, we will prove the solvability of the system (3.4) by using the principle of contracted mappings (see, e.g., [21, p. 390]).

Assume $V := (v_1, v_2, v_3)$ and introduce the vector operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$ acting by the formula

$$\begin{cases} (\Phi_1 V)(\xi,\eta) = -\dfrac{1}{2}\,\widetilde{\square}^{-1}\big(g(\xi-\eta, \xi+\eta, v_1)(v_2+v_3)\big) + (\widetilde{\square}^{-1}\widetilde{f})(\xi,\eta), \\ (\Phi_2 V)(\xi,\eta) = -\dfrac{1}{2}\,L_1\big(g(\xi-\eta, \xi+\eta, v_1)(v_2+v_3)\big) + (L_1\widetilde{f})(\xi,\eta), \\ (\Phi_3 V)(\xi,\eta) = -\dfrac{1}{2}\,L_2\big(g(\xi-\eta, \xi+\eta, v_1)(v_2+v_3)\big) + (L_2\widetilde{f})(\xi,\eta). \end{cases} \tag{3.6}$$

Taking into account (3.6), the system (3.4) can be rewritten in the vector form

$$V = \Phi V. \tag{3.7}$$

Let

$$\|V\|_{X_T} := \max_{1\le i\le 3}\big\{\|v_i\|_{C(\overline{E}_T)}\big\}, \quad V \in X_T := C(\overline{E}_T; \mathbb{R}^3),$$

where $C(\overline{E}_T; \mathbb{R}^3)$ is a set of continuous vector functions $V : \overline{E}_T \to \mathbb{R}^3$.

We fix the number $R > 0$ and denote by $B_R(T) := \{V \in X_T : \ \|V\|_{X_T} \le R\}$ a closed ball of radius R in the Banach space X_T with center in a zero element.

Below, we will prove that there exists the positive number $T_0 = T_0(f,g) \le T$ such that for any $T_1 < T_0$:

(i) Φ maps the ball $B_R(T_1)$ into itself;

(ii) Φ is a contractive mapping on the set $B_R(T_1)$.

Indeed, by the estimates (2.18), (3.3) and the first inequality (3.5), from (3.6) for $V \in B_R(T_1)$, when $T_1 < T$, we have

$$\begin{aligned} \big|(\Phi_1 V)(\xi,\eta)\big| &\le \frac{T_1^2}{1-k^2}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big), \quad (\xi,\eta) \in E_{T_1}, \\ \big|(\Phi_i V)(\xi,\eta)\big| &\le \frac{3T_1}{1-k^4}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big), \quad (\xi,\eta) \in E_{T_1}, \quad i = 2,3. \end{aligned} \tag{3.8}$$

From these estimates, owing to the fact that $k^2 < 1$, it follows that

$$\|\Phi V\|_{X_{T_1}} \le \frac{T_1(T_1+3)}{1-k^2}\,\big(RM(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big). \tag{3.9}$$

For the fixed $R > 0$, we require for the value T_1 to be so small that

$$\frac{T_1(T_1+3)}{1-k^2}\,\big(Rm(R) + \|\widetilde{f}\|_{C(\overline{E}_T)}\big) \le R. \tag{3.10}$$

Then from (3.9) and (3.10) it follows that $\Phi U \in B_R(T_1)$, and hence the condition (i) is fulfilled.

Next, by (2.18) and (3.5), from (3.6), for $V_i = (v_i^1, v_i^2, v_i^3) \in B_R(T_1)$, $i = 1,2$, we have

$$\begin{aligned} \big|(\Phi_1 V_2 - \Phi_1 V_1)(\xi,\eta)\big| &= \frac{1}{2}\,\Big|\widetilde{\square}^{-1}\Big[g(\xi-\eta, \xi+\eta, v_2^1)(v_2^2+v_2^3) - g(\xi-\eta, \xi+\eta, v_1^1)(v_1^2+v_1^3)\Big]\Big| \\ &= \frac{1}{2}\,\Big|\widetilde{\square}^{-1}\Big[\big(g(\xi-\eta, \xi+\eta, v_2^1) - g(\xi-\eta, \xi+\eta, v_1^1)\big)(v_2^2+v_2^3) + g(\xi-\eta, \xi+\eta, v_1^1)(v_2^2-v_1^2+v_2^3-v_1^3)\Big]\Big| \\ &\le \frac{T_1^2}{1-k^2}\,(Rc(R) + m(R))\|V_2 - V_1\|_{X_{T_1}}. \end{aligned}$$

Analogously, taking into account (3.3), we have

$$\big|(\Phi_i V_2-\Phi_i V_1)(\xi,\eta)\big| \le \frac{3T_1}{1-k^4}\,(Rc(R)+m(R))\|V_2-V_1\|_{X_{T_1}},\quad i=2,3. \tag{3.11}$$

We now choose the number T_1 so small that

$$\frac{T_1(T_1+3)}{1-k^2}\,(Rc(R)+m(R))\le q=const<1, \tag{3.12}$$

and hence $\|\Phi V_2-\Phi V_1\|_{X_{T_1}}\le q\|V_2-V_1\|_{X_{T_1}}$. Thus the operator Φ is a contractive mapping on the set $B_R(T_1)$, i.e., the condition (ii) is fulfilled.

It follows from (3.11) and (3.12) that there exists the number $T_0=T_0(f,g)\le T$ such that for any $T_1<T_0$, both conditions (i) and (ii) are fulfilled for the mapping $\Phi: B_R(T_1)\to B_R(T_1)$. Therefore, by the principle of contracted mappings, there exists the solution V of the equation (3.7) in the space $C(\overline{E}_{T_1};\mathbb{R}^3)$. □

Remark 3.2. From the above reasonings as in proving Theorem 3.1 dealt with the contraction of the mapping Φ, it immediately follows that if u_1 and u_2 are two possible solutions of the problem (1.1), (1.2) of the class $C^1(\overline{D}_T)$, then there exists the positive number $T_1=T_1(\|u_1\|,\|u_2\|)\le T$ such that $u_1|_{D_{T_1}}=u_2|_{D_{T_1}}$.

4. A PRIORI ESTIMATES OF A SOLUTION OF THE PROBLEM (1.1), (1.3), (1.4) IN THE CLASSES $C(\overline{D}_{t_1,t_2})$ AND $C^1(\overline{D}_{t_1,t_2})$

Assume

$$\begin{aligned}\omega_\tau&:=\overline{D}_{t_1,t_2}\cap\{t=\tau\},\quad t_1\le\tau\le t_2,\\ \gamma_{i;t_1,t_2}&:=\overline{D}_{t_1,t_2}\cap\widetilde{\gamma}_{i,T},\quad i=1,2,\\ \Gamma_{t_1,t_2}&:=\gamma_{1;t_1,t_2}\cup\gamma_{2;t_1,t_2},\end{aligned}$$

and introduce into consideration the space

$$\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2}):=\Big\{v\in C^2(\overline{D}_{t_1,t_2}):\ v\big|_{\gamma_{1;t_1,t_2}}=0,\ v_x\big|_{\gamma_{2;t_1,t_2}}=0\Big\}.$$

Let

$$f\in C(\overline{D}_T),\ \ g\in C(\overline{D}_T\times\mathbb{R}),\ \ \varphi\in C^1(\overline{\omega}_{t_1}),\ \ \psi\in C(\overline{\omega}_{t_1}). \tag{4.1}$$

Definition 4.1. The function $u\in C^1(\overline{D}_{t_1,t_2})$ is said to be a generalized solution of the problem (1.1), (1.3), (1.4) if there exists a sequence of functions $u_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities

$$\lim_{n\to\infty}\|u_n-u\|_{C^1(\overline{D}_{t_1,t_2})}=0,\quad \lim_{n\to\infty}\|Lu_n-f\|_{C(\overline{D}_{t_1,t_2})}=0 \tag{4.2}$$

and

$$\lim_{n\to\infty}\|u_n|_{\overline{\omega}_{t_1}}-\varphi\|_{C^1(\overline{\omega}_{t_1})}=0,\quad \lim_{n\to\infty}\|u_{nt}|_{\overline{\omega}_{t_1}}-\psi\|_{C(\overline{\omega}_{t_1})}=0 \tag{4.3}$$

hold.

Lemma 4.1. *Let the conditions* (4.1) *and*

$$g(x,t,s)\ge -M_T,\ \ (x,t,s)\in\overline{D}_T\times\mathbb{R},\ \ M_T:=const>0, \tag{4.4}$$

be fulfilled. Then for a generalized solution $u\in C^1(\overline{D}_{t_1,t_2})$ of the problem (1.1), (1.3), (1.4) *an a priori estimate*

$$\|u\|_{C(\overline{D}_{t_1,t_2})}\le c_1\big(\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})}\big) \tag{4.5}$$

with the positive constant $c_1=c_1(T)$, independent of u, f, φ, and ψ is valid.

Proof. Let u be a generalized solution of the problem (1.1), (1.3), (1.4). Then by Definition 4.1, there exists the sequence of functions $u_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities (4.2), (4.3) are valid.

Consider the function $u_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ as a solution of the following mixed problem

$$Lu_n=f_n, \tag{4.6}$$

$$u_n\big|_{\overline{\omega}_{t_1}} = \varphi_n, \quad u_{nt}\big|_{\overline{\omega}_{t_1}} = \psi_n, \tag{4.7}$$

$$u_n\big|_{\gamma_{1;t_1,t_2}} = 0, \quad u_{nx}\big|_{\gamma_{2;t_1,t_2}} = 0. \tag{4.8}$$

Here,

$$\varphi_n := u_n\big|_{\overline{\omega}_{t_1}}, \quad \psi_n := u_{nt}\big|_{\overline{\omega}_{t_1}}, \quad f_n := Lu_n. \tag{4.9}$$

Multiplying both parts of the equality (4.6) by u_{nt} and integrating the obtained equality with respect to the domain $D_{t_1,t_2;\tau} := \{(x,t) \in D_{t_1,t_2} :\ t_1 < t < \tau\}$, $t_1 < \tau \le t_2$, we have

$$\frac{1}{2}\int\limits_{D_{t_1,t_2;\tau}} (u_{nt}^2)_t\,dx\,dt - \int\limits_{D_{t_1,t_2;\tau}} u_{nxx}u_{nt}\,dx\,dt + \int\limits_{D_{t_1,t_2;\tau}} g(x,t,u_n)u_{nt}^2\,dx\,dt = \int\limits_{D_{t_1,t_2;\tau}} f_nu_{nt}\,dx\,dt.$$

Taking into account (4.8) and applying Green's formula to the left-hand side of the last equality, we obtain

$$\begin{aligned}\int\limits_{D_{t_1,t_2;\tau}} f_nu_{nt}\,dx\,dt &= \int\limits_{\gamma_{1;t_1,\tau}} \frac{1}{2\nu_t}\left[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)\right] ds \\ &\quad + \frac{1}{2}\int\limits_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)\,dx - \frac{1}{2}\int\limits_{\omega_{t_1}}(u_{nx}^2 + u_{nt}^2)\,dx + \int\limits_{D_{t_1,t_2;\tau}} g(x,t,u_n)u_{nt}^2\,dx\,dt, \end{aligned} \tag{4.10}$$

where $\nu := (\nu_x, \nu_t)$ is a unit vector of the outer normal to $D_{t_1,t_2;\tau}$.

Taking into account the fact that the operator $\nu_t\,\frac{\partial}{\partial x} - \nu_x\,\frac{\partial}{\partial t}$ is the directional derivative, tangent to $\gamma_{1;t_1,\tau}$, owing to the first condition (4.8), we have

$$(u_{nx}\nu_t - u_{nt}\nu_x)\big|_{\gamma_{1;t_1,\tau}} = 0. \tag{4.11}$$

Since $\nu_x = \frac{1}{\sqrt{1+\widetilde{k}^2}}$, $\nu_t = \frac{-\widetilde{k}}{\sqrt{1+\widetilde{k}^2}}$ and $0 < \widetilde{k} < 1$, therefore

$$(\nu_t^2 - \nu_x^2)\big|_{\gamma_{1;t_1,\tau}} < 0. \tag{4.12}$$

Consequently, by (4.4), (4.11), (4.12), from (4.10), we have

$$w_n(\tau) := \int\limits_{\omega_\tau}(u_{nx}^2 + u_{nt}^2)\,dx \le \int\limits_{\omega_{t_1}}(u_{nx}^2 + u_{nt}^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_nu_{nt}\,dx\,dt + 2M_T\int\limits_{D_{t_1,t_2;\tau}} u_{nt}^2\,dx\,dt. \tag{4.13}$$

Bearing in mind the inequality $2f_nu_{nt} \le u_{nt}^2 + f_n^2$, by (4.7) and (4.13), we get

$$w_n(\tau) \le (1 + 2M_T)\int\limits_{D_{t_1,t_2;\tau}} u_{nt}^2\,dx\,dt + \int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + \int\limits_{\omega_{t_1}}(\varphi_n'^2 + \psi_n^2)\,dx,$$

whence, in view of the expression for the function $w_n(\tau)$, it follows that

$$w_n(\tau) \le m_T\int\limits_0^\tau w_n(\sigma)\,d\sigma + \|f_n\|^2_{L_2(D_{t_1,t_2;\tau})} + \|\varphi_n'\|^2_{L_2(\omega_{t_1})} + \|\psi_n\|^2_{L_2(\omega_{t_1})},$$

where $m_T := 1+2M_T$. Hence, since the value $\|f_n\|^2_{L_2(D_{t_1,t_2;\tau})}$, being the function of τ, is nondecreasing, by the Gronwall's lemma (see, e.g., [5, p. 13]), we have

$$w_n(\tau) \le \exp(m_T\tau)\Big[\|f_n\|^2_{L_2(D_{t_1,t_2;\tau})} + \|\varphi_n'\|^2_{L_2(\omega_{t_1})} + \|\psi_n\|^2_{L_2(\omega_{t_1})}\Big]. \tag{4.14}$$

If $(x,t) \in \overline{D}_{t_1,t_2}$, then by virtue of the first condition (4.8), we obtain the equality

$$u_n(x,t) = u_n(x,t) - u_n(\widetilde{k}t,t) = \int\limits_{\widetilde{k}t}^{x} u_{nx}(\sigma,t)\,d\sigma,$$

which owing to the Schwartz inequality and (4.14) results in

$$
\begin{aligned}
|u_n(x,t)|^2 &\le \int\limits_x^{\widetilde{k}t} d\sigma \int\limits_x^{\widetilde{k}t} [u_{nx}(\sigma,t)]^2\,d\sigma \le (\widetilde{k}t-x)\int\limits_{\omega_t}[u_{nx}(\sigma,t)]^2\,d\sigma \le (\widetilde{k}t-x)w_n(t) \le \widetilde{k}tw_n(t)\\
&\le \widetilde{k}t_2\exp(m_Tt_2)\Big[\|f_n\|^2_{C(\overline{D}_T)}\operatorname{mes}D_{t_1,t_2;\tau} + \operatorname{mes}\omega_{t_1}\big(\|\varphi_n\|^2_{C^1(\overline{\omega}_{t_1})} + \|\psi_n\|^2_{C(\overline{\omega}_{t_1})}\big)\Big]\\
&= \frac{1}{2}\,\widetilde{k}^2t_2(t_2^2-t_1^2)\exp(m_Tt_2)\|f_n\|^2_{C(\overline{D}_T)} + \widetilde{k}^2t_1t_2\exp(m_Tt_2)\|\varphi_n\|^2_{C^1(\overline{\omega}_{t_1})}\\
&\qquad + \widetilde{k}^2t_1t_2\exp(m_Tt_2)\|\psi_n\|^2_{C(\overline{\omega}_{t_1})}.
\end{aligned}
\tag{4.15}
$$

Thus, using the obvious inequality

$$
\Big(\sum_{i=1}^n a_i^2\Big)^{\frac{1}{2}} \le \sum_{i=1}^n |a_i|,
$$

we obtain

$$
\begin{aligned}
\|u_n\|_{C(\overline{D}_{t_1,t_2})} &\le T\widetilde{k}\sqrt{\frac{T}{2}}\,\exp\Big(\frac{Tm_T}{2}\Big)\|f_n\|_{C(\overline{D}_T)}\\
&\qquad + T\widetilde{k}\exp\Big(\frac{Tm_T}{2}\Big)\|\varphi_n\|_{C^1(\overline{\omega}_{t_1})} + T\widetilde{k}\exp\Big(\frac{Tm_T}{2}\Big)\|\psi_n\|_{C(\overline{\omega}_{t_1})}.
\end{aligned}
$$

Passing in the last inequality to the limit, as $n\to\infty$, by virtue of (4.2), (4.3), (4.9), we obtain the estimate (4.5) in which

$$
c_1(T) = T\widetilde{k}\exp\Big(\frac{Tm_T}{2}\Big)\max\Big\{\sqrt{\frac{T}{2}}\,,1\Big\}. \qquad \square
$$

Remark 4.1. Repeating the same reasoning as in Lemma 4.1, for a generalized solution of the problem (1.1), (1.2) we obtain an a priori estimate

$$
\|u\|_{C(\overline{D}_T)} \le c_0\|f\|_{C(\overline{D}_T)},
$$

where

$$
c_0 = T\widetilde{k}\sqrt{\frac{T}{2}}\,\exp\Big(\frac{m_TT}{2}\Big).
$$

Below, using the classical method of characteristics and taking into account (4.5), we obtain a priori estimate in the space $C^1(\overline{D}_{t_1,t_2})$ for a generalized solution of the problem (1.1), (1.3), (1.4).

We have the following

Lemma 4.2. *Under the conditions of Lemma* 4.1, *if*

$$
t_2 - t_1 \le \frac{1}{2}\,\widetilde{k}t_1, \tag{4.16}
$$

for a generalized solution of the problem (1.1), (1.3), (1.4) *an a priori estimate*

$$
\|u\|_{C^1(\overline{D}_{t_1,t_2})} \le \big(2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\big)\exp\big[2(K_{\varphi,\psi}+1)T\big] \tag{4.17}
$$

holds. Here,

$$
K_{\varphi,\psi} := K\big(\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\big), \tag{4.18}
$$

where

$$
K(s) := \sup_{(x,t)\in D_T,\,|s_1|\le c_1s} |g(x,t,s_1)| < +\infty,
$$

c is the constant from the a priori estimate (4.5), *and*

$$
\|u\|_{C^1(\overline{D}_{t_1,t_2})} := \max\Big\{\|u\|_{C(\overline{D}_{t_1,t_2})}, \|u_x\|_{C(\overline{D}_{t_1,t_2})}, \|u_t\|_{C(\overline{D}_{t_1,t_2})}\Big\}.
$$

Proof. Let u be a generalized solution of the problem (1.1), (1.3), (1.4). The limiting equalities (4.2), (4.3) are valid, where u_n can be considered as a solution of the problem (4.6)–(4.8) with right-hand sides f_n, φ_n, ψ_n from (4.9). For the fixed natural n we introduce the functions

$$
u_n^1 := u_{nt} - u_{nx}, \quad u_n^2 := u_{nt} + u_{nx}, \quad u_n^3 := u_n, \tag{4.19}
$$

which in view of (4.7), (4.8) for $t_1 \le t \le t_2$ satisfy the initial and boundary conditions

$$u_n^1\big|_{\omega_{t_1}} = \psi_n - \varphi_n', \quad u_n^2\big|_{\omega_{t_1}} = \psi_n + \varphi_n', \quad u_n^3\big|_{\omega_{t_1}} = \varphi_n, \tag{4.20}$$

$$\Big(u_n^2 + \frac{1-\widetilde{k}}{1+\widetilde{k}}\,u_n^1\Big)\Big|_{\gamma_{1;t_1,t_2}} = 0, \quad u_n^3\big|_{\gamma_{1;t_1,t_2}} = 0, \quad (u_n^1 - u_n^2)\big|_{\gamma_{2;t_1,t_2}} = 0. \tag{4.21}$$

By virtue of (1.1), and (4.19), the unknown functions u_n^1, u_n^2, u_n^3 satisfy the following system of partial differential equations of the first order

$$\begin{cases} \dfrac{\partial u_n^1}{\partial t} + \dfrac{\partial u_n^1}{\partial x} = f_n(x,t) - \dfrac{1}{2}\,g(x,t,u_n^3)(u_n^1 + u_n^2), \\ \dfrac{\partial u_n^2}{\partial t} - \dfrac{\partial u_n^2}{\partial x} = f_n(x,t) - \dfrac{1}{2}\,g(x,t,u_n^3)(u_n^1 + u_n^2), \\ \dfrac{\partial u_n^3}{\partial t} - \dfrac{\partial u_n^3}{\partial x} = u_n^1. \end{cases} \tag{4.22}$$

Taking into account (4.16), we divide the domain D_{t_1,t_2} into three subdomains

$$D_{1;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ t - t_1 < x < (1+\widetilde{k})t_1 - t\big\},$$

$$D_{2;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ 0 < x < t - t_1\big\},$$

$$D_{3;t_1,t_2} := \big\{(x,t) \in D_{t_1,t_2} :\ (1+\widetilde{k})t_1 - t < x < \widetilde{k}t\big\}.$$

For $(x,t) \in D_{1;t_1,t_2}$, integration equations of the system (4.22) along the corresponding characteristic curves and bearing in mind the initial conditions (4.20), we obtain

$$\begin{cases} u_n^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau, u_n^3(P_\tau))\big(u_n^1(P_\tau) + u_n^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(P_\tau)\,d\tau + \psi_n(x-t+t_1) - \varphi_n'(x-t+t_1), \\ u_n^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u_n^3(Q_\tau))\big(u_n^1(Q_\tau) + u_n^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f_n(Q_\tau)\,d\tau + \psi_n(x+t-t_1) + \varphi_n'(x+t-t_1), \\ u_n^3(x,t) = \displaystyle\int_{t_1}^{t} u_n^1(Q_\tau)\,d\tau + \varphi_n(x+t-t_1), \end{cases}$$

where $P_\tau := (x - t + \tau, \tau)$, $Q_\tau := (x + t - \tau, \tau)$. Passing in this system to the limit, as $n \to \infty$, in the space $C(\overline{D}_{1;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.10), we have

$$\begin{cases} u^1(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(P_\tau, u^3(P_\tau))\big(u^1(P_\tau) + u^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f(P_\tau)\,d\tau + \psi(x-t+t_1) \\ \qquad\qquad -\varphi'(x-t+t_1), \\ u^2(x,t) = -\dfrac{1}{2}\displaystyle\int_{t_1}^{t} g(Q_\tau, u^3(Q_\tau))\big(u^1(Q_\tau) + u^2(Q_\tau)\big)\,d\tau + \int_{t_1}^{t} f(Q_\tau)\,d\tau + \psi(x+t-t_1) \\ \qquad\qquad +\varphi'(x+t-t_1), \\ u^3(x,t) = \displaystyle\int_{t_1}^{t} u^1(Q_\tau)\,d\tau + \varphi(x+t-t_1). \end{cases} \tag{4.23}$$

Here, by (4.2) and (4.19),

$$u^1 := u_t - u_x, \quad ;u^2 := u_t + u_x, \quad u^3 := u. \tag{4.24}$$

In case $(x,t)\in D_{2;t_1,t_2}$, integrating equations of the system (4.22) along the corresponding characteristic curves and taking into account the initial conditions (4.20), we obtain

$$\begin{cases} u_n^1(x,t)=u_n^1(0,t-x)-\dfrac{1}{2}\displaystyle\int\limits_{t-x}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int\limits_{t-x}^{t} f_n(P_\tau)\,d\tau,\\ u_n^2(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f_n(Q_\tau)\,d\tau+\psi_n(x+t-t_1)\\ \qquad\qquad +\varphi_n'(x+t-t_1),\\ u_n^3(x,t)=\displaystyle\int\limits_{t_1}^{t} u_n^1(Q_\tau)d\tau+\varphi_n(x+t-t_1). \end{cases} \tag{4.25}$$

Since due to (4.21) the equality $u_n^1(0,t-x)=u_n^2(0,t-x)$ holds, bearing in mind the second equality of the obtained system and the notation $P_\tau^2:=(t-x-\tau,\tau)$, we can rewrite the system (4.25) in the form

$$\begin{cases} u_n^1(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t-x} g(P_\tau^2,u_n^3(P_\tau^2))\big(u_n^1(P_\tau^2)+u_n^2(P_\tau^2)\big)\,d\tau+\int\limits_{t_1}^{t-x} f_n(P_\tau^2)\,d\tau+\psi_n(t-x-t_1)\\ \qquad\qquad +\varphi_n'(t-x-t_1)-\dfrac{1}{2}\displaystyle\int\limits_{t-x}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int\limits_{t-x}^{t} f_n(P_\tau)\,d\tau,\\ u_n^2(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f_n(Q_\tau)\,d\tau+\psi_n(x+t-t_1)+\varphi_n'(x+t-t_1),\\ u_n^3(x,t)=\displaystyle\int\limits_{t_1}^{t} u_n^1(Q_\tau)d\tau+\varphi_n(x+t-t_1). \end{cases}$$

Passing here to the limit as $n\to\infty$ in the space $C(\overline{D}_{2;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.19), we have

$$\begin{cases} u^1(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t-x} g(P_\tau^2,u^3(P_\tau^2))\big(u^1(P_\tau^2)+u^2(P_\tau^2)\big)\,d\tau+\int\limits_{t_1}^{t-x} f(P_\tau^2)\,d\tau+\psi(t-x-t_1)\\ \qquad\qquad +\varphi'(t-x-t_1)-\dfrac{1}{2}\displaystyle\int\limits_{t-x}^{t} g(P_\tau,u^3(P_\tau))\big(u^1(P_\tau)+u^2(P_\tau)\big)\,d\tau+\int\limits_{t-x}^{t} f(P_\tau)\,d\tau,\\ u^2(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(Q_\tau,u^3(Q_\tau))\big(u^1(Q_\tau)+u^2(Q_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f(Q_\tau)\,d\tau+\psi(x+t-t_1)\\ \qquad\qquad +\varphi'(x+t-t_1),\\ u^3(x,t)=\displaystyle\int\limits_{t_1}^{t} u^1(Q_\tau)\,d\tau+\varphi(x+t-t_1). \end{cases} \tag{4.26}$$

For $(x,t)\in D_{3;t_1,t_2}$, integrating equations of the system (4.22) along the characteristic curves, in view of the initial and boundary conditions (4.20), (4.21), we obtain

$$
\begin{cases}
u_n^1(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f_n(P_\tau)\,d\tau+\psi_n(x-t+t_1)-\varphi_n'(x-t+t_1),\\[2mm]
u_n^2(x,t)=u_n^2\Big(\dfrac{\widetilde{k}(x+t)}{\widetilde{k}+1}\,,\dfrac{x+t}{\widetilde{k}+1}\Big)-\dfrac{1}{2}\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} f_n(Q_\tau)\,d\tau,\\[2mm]
u_n^3(x,t)=\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} u_n^1(Q_\tau)\,d\tau.
\end{cases}
\tag{4.27}
$$

Since by (4.21) there is on $\gamma_{1;t_1,t_2}$ the equality $u_n^2=\frac{\widetilde{k}-1}{\widetilde{k}+1}\,u_n^1$, due to the first equality of the obtained system and the notation $P_\tau^3:=(\frac{\widetilde{k}-1}{\widetilde{k}+1}\,(x+t)+\tau,\tau)$, the system (4.27) can be rewritten in the form

$$
\begin{cases}
u_n^1(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(P_\tau,u_n^3(P_\tau))\big(u_n^1(P_\tau)+u_n^2(P_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f_n(P_\tau)\,d\tau+\psi_n(x-t+t_1)-\varphi_n'(x-t+t_1),\\[2mm]
u_n^2(x,t)=\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\bigg[-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{\frac{x+t}{\widetilde{k}+1}} g(P_\tau^3,u_n^3(P_\tau^3))\big(u_n^1(P_\tau^3)+u_n^2(P_\tau^3)\big)\,d\tau+\int\limits_{t_1}^{\frac{x+t}{\widetilde{k}+1}} f_n(P_\tau^3)\,d\tau\\
\qquad+\psi_n\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\,(x+t)+t_1\Big)-\varphi_n'\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)\bigg]\\
\qquad-\dfrac{1}{2}\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u_n^3(Q_\tau))\big(u_n^1(Q_\tau)+u_n^2(Q_\tau)\big)\,d\tau+\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} f_n(Q_\tau)\,d\tau,\\[2mm]
u_n^3(x,t)=\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} u_n^1(Q_\tau)\,d\tau.
\end{cases}
$$

Passing in this system to the limit, as $n\to\infty$, in the space $C(\overline{D}_{3;t_1,t_2})$ and taking into account (4.2), (4.3), (4.6), (4.7), (4.9) and (4.10), we have

$$
\begin{cases}
u^1(x,t)=-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{t} g(P_\tau,u^3(P_\tau))\big(u^1(P_\tau)+u^2(P_\tau)\big)\,d\tau+\int\limits_{t_1}^{t} f(P_\tau)\,d\tau+\psi(x-t+t_1)-\varphi'(x-t+t_1),\\[2mm]
u^2(x,t)=\dfrac{\widetilde{k}-1}{\widetilde{k}+1}\bigg[-\dfrac{1}{2}\displaystyle\int\limits_{t_1}^{\frac{x+t}{\widetilde{k}+1}} g(P_\tau^3,u^3(P_\tau^3))\big(u^1(P_\tau^3)+u^2(P_\tau^3)\big)\,d\tau+\int\limits_{t_1}^{\frac{x+t}{\widetilde{k}+1}} f(P_\tau^3)\,d\tau\\
\qquad+\psi\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)-\varphi'\Big(\dfrac{\widetilde{k}-1}{\widetilde{k}+1}(x+t)+t_1\Big)\bigg]\\
\qquad-\dfrac{1}{2}\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} g(Q_\tau,u^3(Q_\tau))\big(u^1(Q_\tau)+u^2(Q_\tau)\big)\,d\tau+\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} f(Q_\tau)\,d\tau,\\[2mm]
u^3(x,t)=\displaystyle\int\limits_{\frac{x+t}{\widetilde{k}+1}}^{t} u^1(Q_\tau)\,d\tau.
\end{cases}
\tag{4.28}
$$

By the a priori estimate (4.5), for a generalized solution $u^3=u$ of the problem (1.1), (1.3), (1.4) we get

$$\big|g(x,t,u^3(x,t))\big|\le K_{\varphi,\psi},\ \ (x,t)\in\overline{D}_{t_1,t_2}, \tag{4.29}$$

where $K_{\varphi,\psi}$ is defined in (4.18).

Let

$$v^i(t):=\sup_{(\xi,\tau)\in\overline{D}_{t_1,t}}|u^i(\xi,\tau)|,\ \ i=1,2,3,\qquad F(t):=\sup_{(\xi,\tau)\in\overline{D}_{t_1,t}}|f(\xi,\tau)|. \tag{4.30}$$

It follows from (4.23), (4.26) and (4.28) by virtue of (4.29) and (4.30) that

$$|u^i(x,t)|\le(K_{\varphi,\psi}+1)\int\limits_{t_1}^{t}\big[v^1(\tau)+v^2(\tau)\big]\,d\tau+2t\|f\|_{C(\overline{D}_{t_1,t})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})},\ \ i=1,2,3.$$

whence taking into account the fact that the right-hand sides of these inequalities are nondecreasing, by virtue of (4.30), we obtain

$$|v^i(t)|\le(K_{\varphi,\psi}+1)\int\limits_{t_1}^{t}\big[v^1(\tau)+v^2(\tau)\big]\,d\tau+2t_2\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})},$$
$$t_1\le t\le t_2,\ \ i=1,2,3.$$

Putting $v(t):=\max\limits_{1\le i\le 3}v^i(t)$, the obtained inequalities result in

$$v(t)\le 2(K_{\varphi,\psi}+1)\int\limits_{t_1}^{t}v(\tau)\,d\tau+2t_2\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})},\ \ t_1\le t\le t_2. \tag{4.31}$$

From (4.31), applying Gronwall's lemma, we obtain

$$v(t)\le\Big[2t_2\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})}\Big]\exp\big[2(K_{\varphi,\psi}+1)t\big],\ \ t_1\le t\le t_2.$$

From (4.24) and (4.30), it now easily follows that

$$\|u\|_{C^1(\overline{D}_{t_1,t_2})}\le\Big[2t_2\|f\|_{C(\overline{D}_{t_1,t_2})}+\|\varphi\|_{C^1(\overline{\omega}_{t_1})}+\|\psi\|_{C(\overline{\omega}_{t_1})}\big]\exp[2(K_{\varphi,\psi}+1)t_2\Big],$$

which proves Lemma 4.2. □

5. The Uniqueness of a Solution of the Problems (1.1), (1.2) and (1.1), (1.3), (1.4)

Lemma 5.1. *Let the conditions* (3.5), (4.1), (4.4), (4.16) *be fulfilled. Then the problem* (1.1), (1.3), (1.4) *may have no more than one generalized solution of the class* $C^1(\overline{D}_{t_1,t_2})$.

Proof. Indeed, assume that the problem (1.1), (1.3), (1.4) has two possible different generalized solutions u^1 and u^2 of the class C^1 in the domain D_{t_1,t_2}. According to Definition 1.1, there exists a sequence of functions $u_n^i\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ such that the limiting equalities

$$\lim_{n\to\infty}\|u_n^i-u^i\|_{C^1(\overline{D}_{t_1,t_2})}=0,\quad \lim_{n\to\infty}\|Lu_n^i-f\|_{C(\overline{D}_{t_1,t_2})}=0 \tag{5.1}$$

and

$$\lim_{n\to\infty}\|u_n^i|_{\overline{\omega}_{t_1}}-\varphi\|_{C^1(\overline{\omega}_{t_1})}=0,\quad \lim_{n\to\infty}\|u_{nt}^i|_{\overline{\omega}_{t_1}}-\psi\|_{C(\overline{\omega}_{t_1})}=0,\ \ i=1,2, \tag{5.2}$$

hold.

We take advantage here the well-known notation $\Box:=\partial^2/\partial t^2-\partial^2/\partial x^2$ and put $\omega_n:=u_n^2-u_n^1$. It can be easily seen that the function $\omega_n\in\overset{\circ}{C}{}^2(\overline{D}_{t_1,t_2},\Gamma_{t_1,t_2})$ satisfies the following equalities:

$$\Box\omega_n+g_n=f_n,, \tag{5.3}$$

$$\omega_n\big|_{\overline{\omega}_{t_1}}=\widetilde{\varphi}_n,\quad \omega_{nt}\big|_{\overline{\omega}_{t_1}}=\widetilde{\psi}_n, \tag{5.4}$$

$$\omega_n\big|_{\gamma_{1;t_1,t_2}}=0,\quad \omega_{nx}\big|_{\gamma_{2;t_1,t_2}}=0, \tag{5.5}$$

where

$$g_n:=g(x,t,u_n^2)u_{nt}^2-g(x,t,u_n^1)u_{nt}^1,\quad f_n:=Lu_n^2-Lu_n^1, \tag{5.6}$$

$$\widetilde{\varphi}_n := \omega_n\big|_{\overline{\omega}_{t_1}}, \quad \widetilde{\psi}_n := \omega_{nt}\big|_{\overline{\omega}_{t_1}}, \tag{5.7}$$

and by virtue of (5.2) and (5.7), the equalities

$$\lim_{n\to\infty} \|\widetilde{\varphi}_n\|_{C^1(\overline{\omega}_{t_1})} = 0, \quad \lim_{n\to\infty} \|\widetilde{\psi}_n\|_{C(\overline{\omega}_{t_1})} = 0, \ \ i = 1, 2, \tag{5.8}$$

hold.

By the first equality of (5.1), there is the number $A = const > 0$, independent of the indices i and n, such that

$$\|u_n^i\|_{C^1(\overline{D}_{t_1,t_2})} \le A. \tag{5.9}$$

According to the second equalities of (5.1) and (5.6), we have

$$\lim_{n\to\infty} \|f_n\|_{C(\overline{D}_{t_1,t_2})} = 0. \tag{5.10}$$

By (3.5), (5.9) and the first equality of (5.6), it is not difficult to see that

$$g_n^2 = \Big(g(x,t,u_n^2)\omega_{nt} + \big(g(x,t,u_n^2) - g(x,t,u_n^1)\big)u_{nt}^1\Big)^2 \le 2m^2(A)\omega_{nt}^2 + 2A^2c^2(A)\omega_n^2. \tag{5.11}$$

Multiplying both parts of the equality (5.3) by ω_{nt} and integrating the obtained equality with respect to the domain D_{t_1,t_2}, by virtue of (5.4), (5.5), just in the same manner as when obtaining inequality (4.13), from (4.10)–(4.12), we have

$$w_n(\tau) := \int\limits_{\omega_\tau} (\omega_{nx}^2 + \omega_{nt}^2)\,dx \le \int\limits_{\omega_{t_1}} \big(\widetilde{\varphi}_n{}'^2 + \widetilde{\psi}_n^2\big)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt. \tag{5.12}$$

By virtue of the estimate (5.11) and the Cauchy inequality, we obtain

$$\begin{aligned}
2\int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt &\le \int\limits_{D_{t_1,t_2;\tau}} (f_n - g_n)^2 dxdt + \int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + 2\int\limits_{D_{t_1,t_2;\tau}} g_n^2\,dx\,dt + \int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + 4A^2c^2(A)\int\limits_{D_{t_1,t_2;\tau}} \omega_n^2\,dx\,dt + (1 + 4m^2(A))\int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt.
\end{aligned} \tag{5.13}$$

Next, in view of the equality

$$\omega_n(x,t) = \int\limits_{\widetilde{k}t}^{x} \omega_{nx}(\xi,t)\,d\xi, \quad (x,t) \in \overline{D}_{t_1,t_2;\tau}$$

which follows from the first equality of (5.5), reasoning in a standard manner, we obtain the following inequality:

$$\int\limits_{D_{t_1,t_2;\tau}} \omega_n^2\,dx\,dt \le (\widetilde{k}T)^2 \int\limits_{D_{t_1,t_2;\tau}} \omega_{nx}^2\,dx\,dt. \tag{5.14}$$

It follows from (5.12)–(5.14) that

$$\begin{aligned}
w_n(\tau) &\le \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2 + \widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt \\
&\qquad + 4k^2T^2A^2c^2(A)\int\limits_{D_{t_1,t_2;\tau}} \omega_{nx}^2\,dx\,dt + (1 + 4m^2(A))\int\limits_{D_{t_1,t_2;\tau}} \omega_{nt}^2\,dx\,dt \\
&\le \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2 + \widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt + \big(4k^2T^2A^2c^2(A) + 1 + 4m^2(A)\big)\int\limits_{D_{t_1,t_2;\tau}} (\omega_{nx}^2 + \omega_{nt}^2)\,dx\,dt
\end{aligned}$$

$$= \big(4k^2T^2A^2c^2(A)+1+4m^2(A)\big)\int\limits_{t_1}^{\tau} w_n(\sigma)\,d\sigma + \int\limits_{\omega_{t_1}} (\widetilde{\varphi}_n'^2+\widetilde{\psi}_n^2)\,dx + 2\int\limits_{D_{t_1,t_2;\tau}} f_n^2\,dx\,dt,$$

whence due to the Gronwall's lemma, we find that

$$w_n(\tau) \le c_2\big(\|\widetilde{\varphi}'_n\|^2_{L_2(\omega_{t_1})} + \|\widetilde{\psi}_n\|^2_{L_2(\omega_{t_1})} + 2\|f_n\|^2_{L_2(D_{t_1,t_2})}\big), \;\; t_1<\tau\le t_2, \tag{5.15}$$

where

$$c_2 := \exp\big(4k^2T^2A^2c^2(A)+1+4m^2(A)\big)(t_2-t_1).$$

Reasoning analogously as in the obtaining estimate (4.15) and taking into account obvious inequalities

$$\|f_n\|^2_{L_2(D_{t_1,t_2})} \le \|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\operatorname{mes} D_{t_1,t_2}, \quad \|\widetilde{\varphi}'_n\|^2_{L_2(\omega_{t_1})} \le \|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})}\operatorname{mes}\omega_{t_1},$$
$$\|\widetilde{\psi}_n\|^2_{L_2(\omega_{t_1})} \le \|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})}\operatorname{mes}\omega_{t_1},$$

by virtue of (5.15), for $(x,t)\in\overline{D}_{t_1,t_2}$ we have

$$\begin{aligned}|\omega_n(x,t)|^2 \le \widetilde{k}t w_n(t) \le \widetilde{k}Tc_2\Big(\operatorname{mes}\omega_{t_1}\|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})} + \operatorname{mes}\omega_{t_1}\|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})} + 2\operatorname{mes} D_{t_1,t_2}\|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\Big)\\ \le c_2(\widetilde{k}T)^2(1+T)\big(\|\widetilde{\varphi}'_n\|^2_{C(\overline{\omega}_{t_1})} + \|\widetilde{\psi}_n\|^2_{C(\overline{\omega}_{t_1})} + \|f_n\|^2_{C(\overline{D}_{t_1,t_2})}\big).\end{aligned}$$

Hence it immediately follows that

$$\|\omega_n\|_{C(\overline{D}_{t_1,t_2})} \le \widetilde{k}T\sqrt{c_2(1+T)}\,\big(\|\widetilde{\varphi}'_n\|_{C(\overline{\omega}_{t_1})} + \|\widetilde{\psi}_n\|_{C(\overline{\omega}_{t_1})} + \|f_n\|_{C(\overline{D}_{t_1,t_2})}\big). \tag{5.16}$$

According to the definition of the function ω_n and the first equality of (5.1), we can easily see that

$$\lim_{n\to\infty}\|\omega_n\|_{C^1(\overline{D}_{t_1,t_2})} = \|u^2-u^1\|_{C^1(\overline{D}_{t_1,t_2})}$$

and all the more,

$$\lim_{n\to\infty}\|\omega_n\|_{C(\overline{D}_{t_1,t_2})} = \|u^2-u^1\|_{C(\overline{D}_{t_1,t_2})}.$$

Therefore, passing in the inequality (5.16) to the limit, as $n\to\infty$, and taking into account (5.8) and (5.10), we obtain $\|u^2-u^1\|_{C(\overline{D}_{t_1,t_2})}=0$, i.e. $u^1=u^2$. □

Theorem 5.1. *Let the conditions* (3.5), (4.1), (4.4) *be fulfilled. Then the problem* (1.1), (1.2) *may have no more than one generalized solution of the class* $C^1(\overline{D}_T)$.

Proof. We take a natural number n so large that $\Delta = \frac{T-T_1}{n} < \frac{1}{2}\widetilde{k}T_1$, where T_1 is the number appearing in Remark 3.2, and put $T_i := T_1+(i-1)\Delta$, $i=2,\dots,n+1$. Then if u_1 and u_2 are the two possible solutions of the problem (1.1), (1.2) of the class $C^1(\overline{D}_T)$, then owing to Remark 3.2, we have $u_1|_{D_{T_1}} = u_2|_{D_{T_1}}$, whence by virtue of Lemma 5.1, we find that $u_1|_{D_{T_1,T_2}} = u_2|_{D_{T_1,T_2}}$. Further, continuing analogous reasoning step by step, in the domains $D_{T_2,T_3}, D_{T_3,T_4},\dots,D_{T_n,T_{n+1}}$ we find that $u_1|_{D_{T_i,T_{i+1}}} = u_2|_{D_{T_i,T_{i+1}}}$, $i=2,\dots,n$, and hence $u_1|_{D_T}=u_2|_{D_T}$. Thus this proves the uniqueness of a solution of the problem (1.1), (1.2) in the class $C^1(\overline{D}_T)$. □

6. Solvability of the Problem (1.1), (1.2)

As is known, if a global a priori estimate of a solution is obtained and the existence of a local solution of the evolution problem is established, then reasoning in a standard manner, we obtain the existence of the global solution of that problem (see, e.g., [20]). In our case, the a priori estimate of a solution of the problem (1.1), (1.3), (1.4) is obtained under the assumption that the height $\Delta t := t_2-t_1$ of the trapezoid D_{t_1,t_2} is less than the defined value (see (4.16)). Therefore, in this case, to prove the existence of the global solution, we have to modify the above-mentioned general approach, making it convenient for our case.

Remark 6.1. In the assumption that the condition (4.16) is fulfilled, we consider first the question on the solvability of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{t_1,t_2} taking into account that if u is a generalized solution of that problem of the class C^1 in the domain D_{t_1,t_2}, then $u^1 := u_t - u_x$, $u^2 := u_t + u_x$, $u^3 := u$ is a continuous solution of the system of nonlinear Volterra type integral equations (4.23), (4.26), (4.28), respectively, in the domains $D_{1;t_1,t_2}$, $D_{2;t_1,t_2}$, $D_{3;t_1,t_2}$, and vice versa, if u^1, u^2, u^3 is a continuous solution of the above-mentioned system, then $u := u^3$ is a generalized solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{t_1,t_2}, and the equalities $u^1 := u_t - u_x$, $u^2 := u_t + u_x$ are valid.

We rewrite the systems (4.23), (4.26) and (4.28) in the vector form

$$U(P) = (\Phi U)(P), \quad P \in D_{t_1,t_2}, \tag{6.1}$$

where $U := (u^1, u^2, u^3)$ and $\Phi := (\Phi^1, \Phi^2, \Phi^3)$, and the operators

$$\Phi^1(U) := \Phi(U)\big|_{D_{1;t_1,t_2}}, \quad \Phi^2(U) := \Phi(U)\big|_{D_{2;t_1,t_2}}, \quad \Phi^3(U) := \Phi(U)\big|_{D_{3;t_1,t_2}} \tag{6.2}$$

are defined by the right-hand sides of the systems (4.23), (4.26) and (4.28), respectively.

Let

$$\|U\|_{X_{t_1,t_2}} := \max_{1\le i\le 3} \big\{\|u^i\|_{C(\overline{D}_{t_1,t_2})}\big\}, \quad U \in X_{t_1,t_2} := C(\overline{D}_{t_1,t_2}; \mathbb{R}^3).$$

We fix the number $R > 0$ and denote by $B_R(t_1,t_2) := \{U \in X_{t_1,t_2} : \ \|U\|_{X_{t_1,t_2}} \le R\}$ a closed ball of radius R in the Banach space X_{t_1,t_2} with the center in a zero element.

Below, it will be shown that there exists the positive number $t_2^0 \in (t_1, T]$ such that for any $t_2 < t_2^0$:

(i) Φ maps the ball $B_R(t_1,t_2)$ into itself;

(ii) Φ is a contracting mapping on the set $B_R(t_1,t_2)$.

Assume

$$R = 2\big(2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}\big).$$

For $\|U\|_{X_{t_1,t_2}} \le R$, by virtue of (6.1), from (4.31), we have

$$\begin{aligned}\big|(\Phi U)(x,t)\big| &\le 2(K_{\varphi,\psi}+1)\int_{t_1}^{t} v(\tau)\,d\tau + 2t_2\|f\|_{C(\overline{D}_{t_1,t_2})} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})} \\ &\le 2(K_{\varphi,\psi}+1)R(t-t_1) + 2T\|f\|_{C(\overline{D}_T)} + \|\varphi\|_{C^1(\overline{\omega}_{t_1})} + \|\psi\|_{C(\overline{\omega}_{t_1})}, \quad t_1 \le t \le t_2,\end{aligned}$$

whence for

$$\Delta t_1 := t_2 - t_1 \le \frac{1}{4(K_{\varphi,\psi}+1)}\,, \tag{6.3}$$

we obtain

$$|(\Phi U)(x,t)| \le R, \quad (x,t) \in D_{t_1,t_2}. \tag{6.4}$$

The value K here is defined in Lemma 4.2.

Thus, by (6.4), in the case (6.3), the operator Φ maps the ball $B_R(t_1,t_2)$ into itself, i.e., item (i) is fulfilled.

Let us now show that item (ii) is likewise fulfilled, that is, the operator Φ is a contracted mapping in that ball. Indeed, for $U_i := (u_i^1, u_i^2, u_i^3)$, $i = 1,2$, and $P \in D_{1;t_1,t_2}$, from (4.23), by virtue of (3.5) for

$$(\Phi_1^1 U)(P) := -\frac{1}{2}\int_{t_1}^{t} g(P_\tau, u^3(P_\tau))\big(u^1(P_\tau)+u^2(P_\tau)\big)\,d\tau + \int_{t_1}^{t} f(P_\tau)\,d\tau + \psi(x-t+t_1) - \varphi'(x-t+t_1),$$

we have

$$\begin{aligned}\big|(\Phi_1^1 U_2 - \Phi_1^1 U_1)(x,t)\big| \le \frac{1}{2}\int_{t_1}^{t} \Big(&\big|g(P_\tau, u_2^3(P_\tau)) - g(P_\tau, u_1^3(P_\tau))\big|\,\big|u_2^1(P_\tau) + u_2^2(P_\tau)\big| \\ &+ \big|g(P_\tau, u_1^3(P_\tau))\big|\,\big|u_2^1(P_\tau) - u_1^1(P_\tau) + u_2^2(P_\tau) - u_1^2(P_\tau)\big|\Big)\,d\tau\end{aligned}$$

$$\leq c(R)R\Delta t_1\|u_2^3-u_1^3\|_{C(\overline{D}_{1;t_1,t_2})}+\frac{1}{2}\,m(R)\Delta t_1\big(\|u_2^1-u_1^1\|_{C(\overline{D}_{1;t_1,t_2})}+\|u_2^2-u_1^2\|_{C(\overline{D}_{1;t_1,t_2})}\big)$$
$$\leq (c(R)R+m(R))\Delta t_1\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},$$

whence in view of (4.23) and (6.3), for

$$\Delta t_1:=t_2-t_1=\min\Big\{\frac{1}{2}\,\widetilde{k}t_1,\,\frac{1}{4(K_{\varphi,\psi}+1)}\,,\,\frac{1}{2(c(R)R+m(R))}\Big\} \tag{6.5}$$

we obtain

$$\big|(\Phi_1^1U_2-\Phi_1^1U_1)(x,t)\big|\leq\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},\ \ (x,t)\in D_{1;t_1,t_2}. \tag{6.6}$$

The estimates, analogous to (6.6) are likewise valid for the operators

$$(\Phi_2^1U)(P):=-\frac{1}{2}\int\limits_{t_1}^{t}g(Q_\tau,u^3(Q_\tau))\big(u^1(Q_\tau)+u^2(Q_\tau)\big)\,d\tau+\int\limits_{t_1}^{t}f(Q_\tau)\,d\tau+\psi(x+t-t_1)+\varphi'(x+t-t_1)$$

and

$$(\Phi_3^1U)(P):=\int\limits_{t_1}^{t}u^1(Q_\tau)\,d\tau+\varphi(x+t-t_1)$$

from (6.2), namely,

$$\big|(\Phi_i^1U_2-\Phi_i^1U_1)(x,t)\big|\leq\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{1;t_1,t_2})},\ \ (x,t)\in D_{1;t_1,t_2},\ \ i=2,3. \tag{6.7}$$

The same reasonings in the case (6.5) result in the following estimates:

$$\big|(\Phi_j^iU_2-\Phi_j^iU_1)(x,t)\big|\leq\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{i;t_1,t_2})},\ \ (x,t)\in D_{i;t_1,t_2},\ \ i=2,3;\ \ j=1,2,3. \tag{6.8}$$

Bearing in mind (6.1), (6.2), (6.5)–(6.8), the estimate

$$\|\Phi U_2-\Phi U_1\|_{C(\overline{D}_{t_1,t_2})}\leq\frac{1}{2}\,\|U_2-U_1\|_{C(\overline{D}_{t_1,t_2})},\ \ (x,t)\in D_{t_1,t_2} \tag{6.9}$$

holds.

Thus, in the case (6.5), by virtue of (6.4), (6.9) and theorem on the contracted mapping it follows that the system (6.1) in the class $C(\overline{D}_{t_1,t_2})$ is solvable, and hence the following lemma is valid.

Lemma 6.1. *The problem* (1.1), (1.3), (1.4) *has a unique solution of the class* C^1 *in the domain* D_{t_1,t_2} *if the condition* (6.5) *is fulfilled.*

Let $t_1=T_1<T$, where T_1 is taken from Theorem 3.1 when the problem (1.1), (1.2) has a unique generalized solution of the class C^1 in the triangular domain D_{T_1}.

We take a natural number n so large that the inequality

$$\frac{T-T_1}{n}<\frac{1}{2}\,\widetilde{k}T_1 \tag{6.10}$$

holds.

Accordingly, we divide the interval $[T_1,T]$ into n equal segments $[T_1,T_2],[T_2,T_3],\dots,[T_n,T_{n+1}]$ of the same length $\Delta:=\frac{T-T_1}{n}$.

In the domain D_{T_1,T_2}, consider the problem (1.1), (1.3), (1.4) in which as the initial functions φ and ψ we take traces of the solution u and its derivative u_t of the problem (1.1), (1.2) in the domain D_{T_1} on the interval ω_{T_1}. In view of (6.10), the condition (4.16) of Lemma 4.2 is fulfilled, and hence we have the following a priori estimate

$$\|u\|_{C^1(\overline{D}_{T_1,T_2})}\leq L_1:=\big(2T\|f\|_{C(\overline{D}_T)}+\|\varphi\|_{C^1(\overline{\omega}_{T_1})}+\|\psi\|_{C(\overline{\omega}_{T_1})}\big)\exp\big[2(K_{\varphi,\psi}+1)T\big]. \tag{6.11}$$

Remark 6.2. From the definition of the value $K=K(s)$, $s\geq 0$ it is easy to see that it is the nondecreasing function with respect to the variable s.

Remark 6.3. It is not difficult to see that by virtue of (6.11) and (4.17), if u is a solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{T_1,T_2}, then the estimate

$$\|u|_{t=\tau}\|_{C^1(\overline{\omega}_\tau)}+\|u_t|_{t=\tau}\|_{C(\overline{\omega}_\tau)}\leq 2L_1\ \ \forall\,\tau\in[T_1,T_2] \tag{6.12}$$

is valid, and hence

$$K_{\varphi_\tau,\psi_\tau}=K\big(\|f\|_{C(\overline{D}_T)}+\|u|_{t=\tau}\|_{C^1(\overline{\omega}_\tau)}+\|u_t|_{t=\tau}\|_{C(\overline{\omega}_\tau)}\big)\leq K\big(\|f\|_{C(\overline{D}_T)}+2L_1\big)\ \ \forall\,\tau\in[T_1,T_2]. \tag{6.13}$$

By Lemma 6.1, in view of (6.5) and (6.13), for the value Δt_1 for which there exists the unique solution of the problem (1.1), (1.3), (1.4) of the class C^1 in the domain D_{T_1,t_2}, where $t_2=T_1+\Delta t_1$, the following lower bound

$$\Delta t_1\geq\min\Big\{\frac{1}{2}\,\widetilde{k}t_1,\ \frac{1}{4(K(\|f\|_{C(\overline{D}_T)}+2L_1)+1)}\,,\ \frac{1}{2(c(R)R+m(R))}\Big\} \tag{6.14}$$

is valid.

Continuing this process of constructing a local solution of the problem (1.1), (1.3), (1.4) in the domains D_{t_{i-1},t_i}, by (6.14), for the length Δt_i of the interval $[t_{i-1},t_i]$, independently on the step number i, there exists the natural number i_0 such that $t_{i_0}\geq t_2$. This latter means that the problem (1.1), (1.3), (1.4) has the unique solution in the domain D_{T_1,T_2}. The same process, owing to the estimate (6.14), allows one to construct step by step a unique solution of the problem (1.1), (1.3), (1.4) in the domains $D_{T_2,T_3},\dots,D_{T_n,T_{n+1}}$, and since $T_{n+1}=T$, this proves the existence of a generalized solution of the problem (1.1), (1.2) in the domain D_T.

Thus the following theorem is valid.

Theorem 6.1. *Let $f\in C(\overline{D}_T)$, $g\in C(\overline{D}_T\times\mathbb{R})$ and the conditions* (3.5) *and* (4.4) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique generalized solution of the class C^1 in the domain D_T.*

Remark 6.4. From Theorem 6.1 we arrive at the global solvability of the problem (1.1), (1.2) in the sense of Definition 1.3.

7. The Case of Nonexistence of a Global Solution of the Problem (1.1), (1.2)

Below, we will show that violation of the condition (4.4) may result in the nonexistence of global solvability of the problem (1.1), (1.2) in the sense of Definition 1.3. To simplify our exposition, we consider the case $\widetilde{k}=1$, i.e., when $\widetilde{\gamma}_{1,T}$ is the characteristic of the equation (1.1). Indeed, let $g(x,t,s)=-|s|^\alpha s$, $s\in\mathbb{R}$ and the nonlinearity exponent $\alpha>-1$.

Lemma 7.1. *Let u be a strong generalized solution of the problem* (1.1), (1.2) *of the class C^1 in the domain D_T in the sense of Definition* 1.1. *Then the following integral equality*

$$\int\limits_{D_T}u\,\square\,\varphi\,dx\,dt=\int\limits_{D_T}|u|^\alpha uu_t\varphi\,dx\,dt+\int\limits_{D_T}f\varphi\,dx\,dt \tag{7.1}$$

is valid for any function φ such that

$$\varphi\in C^2(\overline{D}_T),\ \ \varphi\big|_{\widetilde{\gamma}_{3,T}}=0,\ \ \varphi_t\big|_{\widetilde{\gamma}_{3,T}}=0,\ \ \varphi_x\big|_{\widetilde{\gamma}_{2,T}}=0. \tag{7.2}$$

Proof. According to the definition of a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T, the function $u\in C^1(\overline{D}_T)$ and there exists the sequence of functions $u_n\in\overset{\circ}{C}{}^2(\overline{D}_T,\widetilde{\Gamma}_T)$ such that the equalities

$$\lim_{n\to\infty}\|u_n-u\|_{C^1(\overline{D}_T)}=0,\quad \lim_{n\to\infty}\|Lu_n-f\|_{C(\overline{D}_T)}=0 \tag{7.3}$$

are valid.

Assume $f_n:=Lu_n$. We multiply both parts of the equality $Lu_n=f_n$ by the function φ and integrate the obtained equality with respect to the domain D_T. As a result of integration by parts of the left part of that equality, in view of (7.2) and the conditions (1.2), we obtain

$$\int\limits_{D_T}u_n\,\square\,\varphi\,dx\,dt=\int\limits_{D_T}|u_n|^\alpha u_nu_{nt}\varphi\,dx\,dt+\int\limits_{D_T}f_n\varphi\,dx\,dt.$$

Passing in this equality to the limit, as $n\to\infty$, owing to (7.3), we obtain (7.1). $\square$

Below, the use will be made of the test functions method (see, e.g., [19, pp. 10–12]). We introduce into consideration the function $\varphi^0 := \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty),\ \ \varphi^0 + \varphi^0_t \le 0,\ \ \varphi^0\big|_{D_{T=1}} > 0,\ \ \varphi^0_x\big|_{\widetilde{\gamma}_{2,\infty}} = 0,\ \ \varphi^0\big|_{t\ge 1} = 0 \tag{7.4}$$

and

$$\kappa_0 := \int\limits_{D_{T=1}} \frac{|\Box\, \varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\, dx\, dt < +\infty,\ \ p' = \frac{\alpha+2}{\alpha+1}. \tag{7.5}$$

It can be easily verified that in the capacity of the function φ^0 satisfying the conditions (7.4) and (7.5), we can take the function

$$\varphi^0(x,t) = \begin{cases} [x(1-t)]^n, & (x,t) \in D_{T=1}, \\ 0, & t \ge 1 \end{cases}$$

for a sufficiently large positive n.

Put $\varphi_T(x,t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, $T > 0$. By virtue of (7.4), it can be easily seen that

$$\varphi_T \in C^2(\overline{D}_T),\ \ \varphi_T + T\,\frac{\partial \varphi_T}{\partial t} \le 0,\ \ \varphi_T\big|_{D_T} > 0,\ \ \frac{\partial \varphi_T}{\partial x}\bigg|_{\widetilde{\gamma}_{2,T}} = 0,\ \ \varphi_T\big|_{\widetilde{\gamma}_{3,T}} = 0,\ \ \frac{\partial \varphi_T}{\partial t}\bigg|_{\widetilde{\gamma}_{3,T}} = 0. \tag{7.6}$$

Given f, we consider the function

$$\zeta(T) := \int\limits_{D_T} f \varphi_T\, dx\, dt,\ \ T > 0. \tag{7.7}$$

The following theorem on the nonexistence of global solvability of the problem (1.1), (1.2) holds.

Theorem 7.1. *Let $g(x,t,s) = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$, $f \in C(\overline{D}_\infty)$, and $f \ge 0$ in the domain D_∞. Then if*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{7.8}$$

there exists the positive number $T^* := T^*(f)$ such that for $T > T^*$ the problem (1.1), (1.2) *fails to have a strong generalized solution u of the class C^1 in the domain D_T.*

Proof. Suppose that in the conditions of this theorem there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T. Then by Lemma 7.1, there is the equality (7.1) in which, due to (7.6), in the capacity of the function φ is taken the function $\varphi = \varphi_T$, i.e.,

$$\int\limits_{D_T} u\, \Box\, \varphi_T\, dx\, dt = \int\limits_{D_T} |u|^\alpha u u_t \varphi_T\, dx\, dt + \int\limits_{D_T} f\varphi_T\, dx\, dt. \tag{7.9}$$

Taking into account (1.2) and (7.6), we have

$$\begin{aligned}\int\limits_{D_T} |u|^\alpha u u_t \varphi_T\, dx\, dt &= \frac{1}{\alpha+2} \int\limits_{D_T} \varphi_T\, \frac{\partial}{\partial t}\, |u|^{\alpha+2}\, dx\, dt \\ &= -\frac{1}{\alpha+2} \int\limits_{D_T} |u|^{\alpha+2}\, \frac{\partial \varphi_T}{\partial t}\, dx\, dt \ge \frac{1}{(\alpha+2)T} \int\limits_{D_T} |u|^{\alpha+2} \varphi_T\, dx\, dt.\end{aligned}$$

Hence by (7.7), it follows from (7.9) that

$$\frac{1}{pT} \int\limits_{D_T} |u|^p \varphi_T\, dx\, dt \le \int\limits_{D_T} u\, \Box\, \varphi_T\, dx\, dt - \zeta(T),\ \ p := \alpha + 2 > 1. \tag{7.10}$$

If in the Young's inequality with parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{p}\, a^p + \frac{1}{p'\varepsilon^{p'-1}}\, b^{p'};\ \ a, b \ge 0,\ \ \frac{1}{p} + \frac{1}{p'} = 1,\ \ p > 1,$$

we take $a = |u|\varphi_T^{\frac{1}{p}}$, $b = \frac{|\Box\,\varphi_T|}{\varphi_T^{\frac{1}{p}}}$, $\varepsilon = \frac{1}{T}$, then in view of the fact that $\frac{p'}{p} = p' - 1$, we obtain

$$|u\,\Box\,\varphi_T| = |u|\varphi_T^{\frac{1}{p}}\,\frac{|\Box\,\varphi_T|}{\varphi_T^{\frac{1}{p}}} \le \frac{1}{pT}\,|u|^p\varphi_T + \frac{T^{p'-1}}{p'}\,\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,.$$

By virtue of (7.10) and the last inequality, we have

$$0 \le \frac{T^{p'-1}}{p'}\int\limits_{D_T}\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \zeta(T). \tag{7.11}$$

Since $\varphi_T(x,t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, in view of (7.4), (7.5), after the change of variables $x = Tx_1$, $t = Tt_1$, it can be easily verified that

$$\int\limits_{D_T}\frac{|\Box\,\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt = \frac{1}{T^{2(p'-1)}}\int\limits_{D_{T=1}}\frac{|\Box\,\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx_1d\,t_1 = \frac{\kappa_0}{T^{2(p'-1)}}\,.$$

Hence, bearing in mind (7.11), we obtain

$$0 \le \frac{\kappa_0}{p'T^{p'-1}} - \zeta(T). \tag{7.12}$$

Since $p' = \frac{p}{p-1} > 1$, by virtue of (7.5), we have

$$\lim_{T\to+\infty}\frac{\kappa_0}{p'T^{p'-1}} = 0.$$

Therefore, owing to (7.8), there exists the positive number $T^* := T^*(f)$ such that for $T > T^*$, the right-hand side of the inequality (7.12) is negative, whereas the left-hand side equals zero. The obtained contradiction shows that if u is a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T, then necessarily $T \le T^*$, which proves Theorem 7.1. $\Box$

Remark 7.1. It is easy to check that if $f \in C(\overline{D}_\infty)$, $f \ge 0$, and $f(x,t) \ge ct^{-m}$ for $t \ge 1$, where $c = const > 0$, $0 \le m = const \le 2$, then the condition (7.8) is fulfilled and hence for $g = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$ the problem (1.1), (1.2) for sufficiently large T fails to have a strong generalized solution u of the class C^1 in the domain D_T.

Indeed, introducing in (7.7)the transformation of independent variables x and t by formula $x = Tx_1$, $t = Tt_1$, after simple transformations we will have

$$\begin{aligned}\zeta(T) &= T^2\int\limits_{D_{T=1}} f(Tx_1,Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1\\ &\ge cT^{2-m}\int\limits_{D_{T=1}\cap\{t_1\ge T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1 + T^2\int\limits_{D_{T=1}\cap\{t_1<T^{-1}\}} f(Tx_1,Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1\end{aligned}$$

in the assumption that $T > 1$. Further, let $T_1 > 1$ be an arbitrary fixed number. Then from the last inequality, when $T \ge T_1 > 1$, for the function ζ we have

$$\zeta(T) \ge cT^{2-m}\int\limits_{D_{T=1}\cap\{t_1\ge T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1 \ge c\int\limits_{D_{T=1}\cap\{t_1\ge T_1^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1,$$

which immediately results in the validity of (7.8).

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References

1. G. K. Berikelashvili, O. M. Jokhadze, B. G. Midodashvili and S. S. Kharibegashvili, On the existence and nonexistence of global solutions of the first Darboux problem for nonlinear wave equations. (Russian) *Differ. Uravn.* **44** (2008), no. 3, 359–372, 430; translation in *Differ. Equ.* **44** (2008), no. 3, 374–389.
2. A. V. Bitsadze, Some classes of partial differential equations (Russian) *Nauka, Moscow*, 1981.
3. A. V. Bitsadze, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1982.
4. O. G. Goman, A reflected-wave equation (Boundary reflected wave intensity equation, proving solution uniqueness and existence in gas dynamics problem). *Moskov. Univ., Vestnik, Ser. I-Matemat., Mekhan.* **23** (1968), 84–87.
5. D. Henri, Geometric theory of semilinear parabolic equations. (Russian) *Mir, Moscow*, 1985.
6. O. Jokhadze, On existence and nonexistence of global solutions of Cauchy–Goursat problem for nonlinear wave equations. *J. Math. Anal. Appl.* **340** (2008), no. 2, 1033–1045.
7. O. Jokhadze, Cauchy–Goursat problem for one-dimensional semilinear wave equations. *Comm. Partial Differential Equations 34* (2009), no. 4-6, 367–382.
8. O. M. Jokhadze and S. S. Kharibegashvili, First Darboux problem for nonlinear hyperbolic equations of second order. (Russian) *Mat. Zametki* **84** (2008), no. 5, 693–712; translation in *Math. Notes* **84** (2008), no. 5, 646–663.
9. O. Jokhadze and S. Kharibegashvili, On the Cauchy and Cauchy-Darboux problems for semilinear wave equations. *Georgian Math. J.* **22** (2015), no. 1, 81–104.
10. O. Jokhadze and B. Midodashvili, The first Darboux problem for wave equations with a nonlinear positive source term. *Nonlinear Anal.* **69** (2008), no. 9, 3005–3015.
11. S. S. Kharibegashvili, A boundary value problem for a second-order hyperbolic equation. (Russian) *Dokl. Akad. Nauk SSSR* **280** (1985), no. 6, 1313–1316.
12. S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Math. Phys.* **4** (1995), 127 pp.
13. S. S. Kharibegashvili and O. M. Jokhadze, The Cauchy–Darboux problem for the one-dimensional wave equation with power nonlinearity. (Russian) *Sib. Mat. Zh.* **54** (2013), no. 6, 1407–1426; translation in *Sib. Math. J.* **54** (2013), no. 6, 1120–1136.
14. S. S. Kharibegashvili and O. M. Jokhadze, Second Darboux problem for the wave equation with a power-law nonlinearity. (Russian) *Differ. Uravn.* **49** (2013), no. 12, 1623–1640; translation in *Differ. Equ.* **49** (2013), no. 12, 1577–1595.
15. S. Kharibegashvili and O. Jokhadze, Boundary value problem for a wave equation with power nonlinearity in the angular domains. *Proc. A. Razmadze Math. Inst.* **164** (2014), 116–120.
16. J.-L. Lions, Certain methods for the solution of nonlinear boundary value problems. (Russian) *Mir, Moscow*, 1972.
17. J.-L. Lions and W. A. Strauss, Some non-linear evolution equations. *Bull. Soc. Math. France* **93** (1965), 43–96.
18. Z. O. Mel'nik, Example of a nonclassical boundary value problem for the vibrating string equation. (Russian) *Ukrain. Mat. Zh.* **32** (1980), no. 5, 671–674.
19. È. Mitidieri and S. I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. (Russian) *Tr. Mat. Inst. Steklova* **234** (2001), 1–384; translation in *Proc. Steklov Inst. Math.* **2001**, no. 3 (234), 1–362.
20. I. Segal, Non-linear semi-groups. *Ann. of Math. (2)* **78** (1963), 339–364.
21. V. A. Trenogin, Functional analysis. (Russian) *Nauka, Moscow*, 1980.
22. S. D. Troitskaya, On a well-posed boundary-value problem for hyperbolic equations with two independent variables. *Uspekhi Mat. Nauk* **50** (1995), no. 4, 124–125; translation in *Russian Math. Surveys* **50** (1995), no. 4.
23. S. D. Troitskaya, On a boundary value problem for hyperbolic equations. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **62** (1998), no. 2, 193–224; translation in *Izv. Math.* **62** (1998), no. 2, 399–428.

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Authors' addresses:

Sergo Kharibegashvili

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia;

2. Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia.

E-mail: `kharibegashvili@yahoo.com`

Otar Jokhadze

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia;

2. Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 13 University Str., Tbilisi 0186, Georgia.

E-mail: `ojokhadze@yahoo.com`