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ON A CLASS OF NONLINEAR NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS OF n-TH ORDER

Abstract. Asymptotic representations of solutions of nonautonomous nonlinear ordinary differential n-th order equations that are close, in a certain sense, to linear equations are established.

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1. Introduction and Preliminaries

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) y |\ln|y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1,1\}$, $\sigma \in \mathbb{R}$, $p : [a,\omega[\, \to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$[1].

A solution y of the equation (1.1) is called a $P_\omega(\lambda_0)$-solution, if it is defined on the interval $[t_y,\omega[\, \subset [a,\omega[$, and satisfies the conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty \end{cases} \quad (k=0,1,\dots,n-1), \quad \lim_{t\uparrow\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0. \tag{1.2}$$

For each such solution, the representation $y(t)|\ln|y(t)|| = |y(t)|^{1+o(1)} \operatorname{sign} y(t)$ as $t \uparrow \omega$, holds. Therefore, when we study these solutions, the equation (1.1) is asymptotically close to linear differential equations

$$y^{(n)} = \alpha_0 p(t) y, \tag{1.3}$$

such asymptotic behavior of solutions has been studied extensively (see, e.g., [9, Chapter 1]).

For $n = 2$ and any $\sigma \in \mathbb{R}$, asymptotic behavior as $t \uparrow \omega$ of all possible types of $P_\omega(\lambda_0)$-solutions of the differential equation (1.1) was studied in [1, 2, 3, 5, 7].

We introduce the following auxiliary notation:

$$a_{0k} = (n-k)\lambda_0 - (n-k-1) \;\; (k=1,\dots,n) \;\text{ for }\; \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = t - \omega, \;\text{ if }\; \omega < +\infty,$$

$$I_A(t) = \int_A^t [\pi_\omega(\tau)]^{n-1} p(\tau)\, d\tau,$$

$$A = \omega, \;\text{ if }\; \int_a^\omega |\pi_\omega(\tau)|^{n-1} p(\tau)\, d\tau < +\infty.$$

The following theorem concerning the differential equation (1.1) has been established in [4].

Theorem 1.1. *Let $\sigma \neq n$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\}$. Then for the existence of a $P_\omega(\lambda_0)$-solution of the equation* (1.1) *it is necessary, and if the inequality*

$$\sigma \neq a_{01}\Big(1 + \sum_{k=1}^{n-1} \frac{1}{a_{0k}}\Big) \tag{1.4}$$

holds and the algebraic equation

$$\prod_{j=1}^{n-1}(a_{0j}+\rho) + \sum_{k=1}^{n-1}\prod_{j=1}^{k-1}(a_{0j}+\rho) \prod_{j=k+1}^{n-1} a_{0j} = 0 \tag{1.5}$$

with respect to ρ has no roots with zero real part, then it is sufficient for the inequality

$$\alpha_0\Big(\prod_{k=1}^{n-1} a_{0k}\Big)\big[(\lambda_0-1)\pi_\omega(t)\big]^n > 0 \;\; \textit{for} \;\; t \in [a,\omega[\,, \tag{1.6}$$

and the conditions

$$\lim_{t\uparrow\omega} p^{\frac{1}{n}}(t)|\pi_\omega(t)| \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{\sigma}{n-\sigma}} = \frac{|a_{01}|}{|\lambda_0-1|}\left(\frac{\prod_{k=1}^{n-1}|a_{0k}|^{\frac{1}{n}}}{|a_{01}|}\right)^{\frac{n}{n-\sigma}},$$

[1] We assume that $a > 1$ for $\omega = +\infty$, and $\omega - a < 1$ for $\omega < +\infty$.

to take place. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$*:*

$$\ln|y(t)| = \nu\biggl(\frac{|a_{01}|}{\prod_{k=1}^{n-1}|a_{0k}|^{\frac{1}{n}}}\biggr)^{\frac{n}{n-\sigma}}\Bigl|\frac{n-\sigma}{n}\,J_B(t)\Bigr|^{\frac{n}{n-\sigma}}[1+o(1)], \tag{1.7}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \ (k=1,\dots,n-1), \tag{1.8}$$

where

$$\nu = \operatorname{sign}\bigl[a_{01}(\lambda_0-1)(n-\sigma)\pi_\omega(t)J_B(t)\bigr].$$

In addition to these conditions, if the algebraic equation (1.5) *has the* m*-roots* (*including multiples*)*, the real parts of which have a sign opposite to the sign of the function* $(\lambda_0-1)\pi_\omega(t)$ *on the interval* $[a,\omega[$*, and the inequality*

$$\Bigl(\frac{\sigma}{a_{01}} - 1 - \sum_{k=1}^{n-1}\frac{1}{a_{0k}}\Bigr)\Bigl(1+\sum_{k=1}^{n-1}\frac{1}{a_{0k}}\Bigr) > 0$$

is satisfied, then the equation (1.1) *has* m*-parametric family of solutions with the representations* (1.7) *and* (1.8)*, and when the opposite inequality holds, it has* $m+1$*-parametric family of such solutions.*

From this theorem the following corollary for the linear differential equation (1.3) is obtained.

Corollary 1.1. *For the existence of* $P_\omega(\lambda_0)$*-solution of the equation* (1.3)*, where* $\lambda_0 \in \mathbb{R}\setminus\{0,\frac{1}{2},\dots,\frac{n-2}{n-1},1\}$*, it is necessary, and if the algebraic equation* (1.5) *with respect to* ρ *has no roots with zero real part, then it is sufficient that the inequality* (1.6) *and the condition*

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t) = \frac{\alpha_0\prod_{k=1}^{n-1}a_{0k}}{(\lambda_0-1)^n} \tag{1.9}$$

are satisfied. For each of these solutions the asymptotic representations

$$\ln|y(t)| = \frac{\alpha_0(\lambda_0-1)^{n-1}I_A(t)}{\prod_{k=2}^{n-1}a_{0k}}\,[1+o(1)], \tag{1.10}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)] \ \ (k=1,\dots,n-1), \tag{1.11}$$

take place as $t\uparrow\omega$*. Moreover, if in addition to these conditions, the algebraic equation* (1.5) *has the* m*-roots* (*including multiples*)*, the real parts of which have a sign, opposite to that of the function* $(\lambda_0-1)\pi_\omega(t)$ *on the interval* $[a,\omega[$*, then for the equation* (1.1) *there exists* $m+1$*-parametric family of solutions with the representations* (1.10) *and* (1.11)*.*

We note that this corollary refers to the case where the differential equation (1.3) is asymptotically close to the Euler equations.

If

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^n(t) = c_0 \neq 0$$

and the next algebraic equation with respect to λ_0

$$c_0(\lambda_0-1)^n = \alpha_0\prod_{k=1}^{n-1}\bigl[(n-k)\lambda_0-(n-k-1)\bigr],$$

which we obtain from (1.9) by taking into account the inequality (1.6), has n distinct real roots λ_{0j} $(j = 1, \dots, n)$, then the fundamental system of solutions y_j $(j = 1, \dots, n)$ of the differential equation (1.3) admits as $t \uparrow \omega$ the following asymptotic representations:

$$\ln |y_j(t)| = \frac{\alpha_0(\lambda_{0j} - 1)^{n-1} I_A(t)}{\prod_{k=2}^{n-1}[(n-j)\lambda_{0j} - (n-j-1)]}\,[1 + o(1)],$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{(n-j)\lambda_{0j} - (n-j-1)}{(\lambda_{0j} - 1)\pi_\omega(t)}\,[1 + o(1)] \ \ (k = 1, \dots, n-1; \ \ j = 1, \dots, n).$$

From the previous statements it is clear that the case for $\lambda_0 = 1$ is a special one in the study of $P_\omega(\lambda_0)$-solutions. This case is the subject of this work.

2. The Main Result and the Necessary Auxiliary Statements for its Establishment

We introduce the function $J_B(t)$, setting

$$J_B(t) = \int\limits_B^t p^{\frac{1}{n}}(\tau)\,d\tau, \quad B = \begin{cases} a, & \text{if} \displaystyle\int\limits_a^\omega p^{\frac{1}{n}}(\tau)\,d\tau = +\infty, \\ \omega, & \text{if} \displaystyle\int\limits_a^\omega p^{\frac{1}{n}}(\tau)\,d\tau < +\infty. \end{cases}$$

The main result of this paper is the following

Theorem 2.1. *Let $\sigma \neq n$. Then for the existence of $P_\omega(1)$-solution of the equation* (1.1) *it is necessary that for some $\mu \in \{-1, 1\}$ the inequality*

$$\alpha_0 \mu^n > 0 \tag{2.1}$$

and the condition

$$\lim_{t \uparrow \omega} |\pi_\omega(t)| p^{\frac{1}{n}}(t) |J_B(t)|^{\frac{\sigma}{n-\sigma}} = +\infty \tag{2.2}$$

hold. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$

$$\ln |y(t)| = \nu \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{n}{n-\sigma}} [1 + o(1)], \tag{2.3}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \mu p^{\frac{1}{n}}(t) \Big|\frac{n-\sigma}{n} J_B(t)\Big|^{\frac{\sigma}{n-\sigma}} [1 + o(1)] \ \ (k = 1, \dots, -1), \tag{2.4}$$

where

$$\nu = \mu \operatorname{sign}\Big(\frac{n-\sigma}{n}\, J_B(t)\Big).$$

If the function $p : [a, \omega[\to]0, +\infty[$ is continuously differentiable, and there exists the limit (finite or equal to $\pm\infty$)

$$\lim_{t \uparrow \omega} \frac{(p^{\frac{1}{n}}(t) |J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t) |J_B(t)|^{\frac{2\sigma}{n-\sigma}}}, \tag{2.5}$$

and if (2.1) *and* (2.2) *hold, then the equation* (1.1) *has at least one $P_\omega(1)$-solution which admits the asymptotic representations* (2.3), (2.4) *as $t \uparrow \omega$. If $\mu = 1$ and $\sigma > n$, then there exists $(n-1)$-parametric family of solutions, if $\mu = 1$ and $\sigma < n$, then we get n-parametric family of solutions, if $\mu = -1$ and $\sigma < n$, then we obtain one parametric family of solutions.*

To prove Theorem 2.1, we will use the following lemma which can be deduced from Lemmas 10.1–10.6 in [6].

Lemma 2.1. *Let $y : [t_0, \omega[\to \mathbb{R} \setminus \{0\}$ be an arbitrary $P_\omega(1)$-solution of the equation* (1.1)*. Then we have the following asymptotic relations:*

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} \sim \frac{y'(t)}{y(t)} \quad (k = 1, \dots, n) \quad \text{as} \quad t \uparrow \omega \tag{2.6}$$

and

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) y^{(k)}(t)}{y^{(k-1)}(t)} = \pm\infty \quad (k = 1, \dots, n). \tag{2.7}$$

Along with this lemma, we will also need the next result on the existence of vanishing at infinity solutions of a system of quasi-linear differential equations

$$\begin{cases} v_k' = \beta_0 \Big[f_k(\tau, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ki} v_i + V_k(v_1, \dots, v_n) \Big] & (k = 1, \dots, n-1), \\ v_n' = H(\tau) \Big[f_n(\tau, v_1, \dots, v_n) + \sum\limits_{i=1}^{n} c_{ni} v_i + V_n(v_1, \dots, v_n) \Big], \end{cases} \tag{2.8}$$

in which $\beta_0 \in \mathbb{R} \setminus \{0\}$, $c_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$, $H : [\tau_0, +\infty[\to \mathbb{R} \setminus \{0\}$ is a continuous function, $f_k : [\tau_0, +\infty[\times \mathbb{R}^n_{\frac{1}{2}}$ $(k = 1, \dots, n)$ are continuous functions satisfying the conditions

$$\lim_{t \uparrow \omega} f_k(\tau, v_1, \dots, v_n) = 0 \quad \text{uniformly in} \quad (v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}, \tag{2.9}$$

where

$$\mathbb{R}^n_{\frac{1}{2}} = \Big\{ (v_1, \dots, v_n) \in \mathbb{R}^n : \ |v_i| \le \frac{1}{2} \ (i = 1, \dots, n) \Big\},$$

and $V_k : \mathbb{R}^n_{\frac{1}{2}} \to \mathbb{R}$ $(k = 1, \dots, n)$ are continuously differentiable functions such that

$$V_k(0, \dots, 0) = 0 \quad (k = 1, \dots, n), \qquad \frac{\partial V_k(0, \dots, 0)}{\partial v_i} = 0 \quad (i, k = 1, \dots, n). \tag{2.10}$$

By Theorem 2.6 of [8], for a system of the differential equations (2.8), we have the following

Lemma 2.2. *Let the function $H : [\tau_0, +\infty[\, [\mathbb{R} \setminus \{0\}$ be continuously differentiable and satisfy the following conditions:*

$$\lim_{\tau \to +\infty} H(\tau) = 0, \qquad \lim_{\tau \to +\infty} \frac{H'(\tau)}{H(\tau)} = 0, \qquad \int\limits_{\tau_0}^{+\infty} H(\tau)\, d\tau = \pm\infty, \tag{2.11}$$

and the matrices $C_n = (c_{ki})_{k,i=1}^n$ and $C_{n-1} = (c_{ki})_{k,i=1}^{n-1}$ are such that $\det C_n \ne 0$, and C_{n-1} has no eigenvalues with a zero real part. Then the system of differential equations (2.8) *has at least one solution $(v_k)_{k=1}^n : [\tau_1, +\infty[\to \mathbb{R}^n_{\frac{1}{2}}$ $(\tau_1 \ge \tau_0)$, which tends to zero as $t \to +\infty$. Moreover, if among the eigenvalues of the matrix C_{n-1} there are m eigenvalues* (*taking into account multiplicity*)*, the real parts of which have opposite sign to β_0, then the system* (2.8) *has m-parametric family of solutions if $H(\tau)(\det C_n)(\det C_{n-1}) > 0$, and $m+1$-parametric family of solutions if the inequality holds in opposite direction.*

3. Proof of the Main Theorem and the Corollary to a Linear Differential Equation

Proof of Theorem 2.1. *Necessity.* Let $y : [t_y, \omega[\to \mathbb{R} \setminus \{0\}$ be an arbitrary $P_\omega(1)$ solution of (1.1). Then, according to Lemma 2.1, the conditions (2.6) and (2.7) are satisfied. In view of (2.7), in a left neighborhood of ω,

$$\operatorname{sign}\Big(\frac{y'(t)}{y(t)}\Big) = \mu, \quad \text{where} \quad \mu \in \{-1; 1\}. \tag{3.1}$$

Since from (1.1)

$$\frac{y^{(n)}(t)}{y(t)} = \alpha_0 p(t) || \ln |y(t)||^{\sigma}$$

and by (2.6)

$$\frac{y^{(n)}(t)}{y(t)} = \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \cdot \frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} \cdots \frac{y'(t)}{y(t)} \sim \Big(\frac{y'(t)}{y(t)}\Big)^n \ \text{ as } \ t \uparrow \omega,$$

then

$$\Big(\frac{y'(t)}{y(t)}\Big)^n = \alpha_0 p(t) |\ln |y(t)||^{\sigma}[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Hence, in view of (3.1), it is clear that the inequality (2.1) holds, and so we have the asymptotic relation

$$\frac{y'(t)}{y(t)|\ln |y(t)||^{\frac{\sigma}{n}}} = \mu p^{\frac{1}{n}}(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.2}$$

Since $\sigma \neq n$, therefore, integrating this relation from t_y to t and taking into account the definition of $P_\omega(1)$-solution, we find that

$$|\ln |y(t)||^{\frac{n-\sigma}{n}} \operatorname{sign}\big(\ln |y(t)|\big) = \frac{\mu(n-\sigma)}{n}\, J_B(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Thus (2.3) holds. Taking into account (2.3), from (3.2) we obtain the representation

$$\frac{y'(t)}{y(t)} = \mu p^{\frac{1}{n}}(t) \Big|\frac{n-\sigma}{n}\, J_B(t)\Big|^{\frac{\sigma}{n-\sigma}}[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

from which, by (2.6) and (2.7), it follows that the condition (2.2) holds and we have the asymptotic representation (2.4).

Sufficiency. Let $p : [a,\omega[\to]0,+\infty[$ be continuously differentiable function for which there is a finite or equal to $\pm\infty$ limit (2.5). We show that in this case, if the conditions (2.1) and (2.2) are satisfied, then the equation (1.1) has solutions defined in the left neighborhood of ω and admits as $t \uparrow \omega$ the asymptotic representations (2.3) and (2.4).

We choose arbitrary $a_0 \in]a,\omega[$. By (2.2) we get

$$\int\limits_{a_0}^{\omega} p^{\frac{1}{n}}(t) |J_B(t)|^{\frac{\sigma}{\sigma-n}}\, dt = +\infty,$$

hence, taking into account the form of the function J_B, it follows that

$$\lim_{t\uparrow\omega} |J_B(t)|^{\frac{n}{n-\sigma}} = +\infty. \tag{3.3}$$

Next, we establish that the limit (2.5) is equal to zero. Assume the contrary. Then, by virtue of its existence,

$$\lim_{t\uparrow\omega} Q(t) = \begin{cases} \text{either} & const \neq 0, \\ \text{or} & \pm\infty, \end{cases} \tag{3.4}$$

where

$$Q(t) = \frac{(p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t)|J_B(t)|^{\frac{2\sigma}{n-\sigma}}}.$$

Integrating the function Q from a_0 to t, we obtain

$$\int\limits_{a_0}^{t} Q(\tau)\, d\tau = -\frac{1}{p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}}} + C, \tag{3.5}$$

where C is a constant. If $\omega = +\infty$, then $\pi_\omega(t) = t$, and in this case, by (2.2), we have

$$\lim_{t\to+\infty} \frac{\int_{a_0}^t Q(\tau)\,d\tau}{t} = 0.$$

However, this is impossible since by the de L'Hospital's rule and (3.4),

$$\lim_{t\to+\infty} \frac{\int_{a_0}^t Q(\tau)\,d\tau}{t} = \lim_{t\to+\infty} Q(t) \neq 0.$$

If $\omega < \infty$, then $\pi_\omega(t) = t - \omega$ and by (2.2)

$$\lim_{t\uparrow\omega} p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}} = +\infty.$$

Therefore, from (3.5) it follows that

$$\lim_{t\uparrow\omega} \int_{a_0}^{t} Q(\tau)\,d\tau = C.$$

Due to this condition, the equation (3.5) can be rewritten as

$$\int_{\omega}^{t} Q(\tau)\,d\tau = -\frac{1}{p^{\frac{1}{n}}(t)|J_B(t)|^{\frac{\sigma}{n-\sigma}}}\,.$$

Dividing this relation by $\pi_\omega(t)$, taking then the limit as $t \uparrow \omega$ and using (2.2) we obtain

$$\lim_{t\uparrow\omega} \frac{\int_{\omega}^t Q(\tau)\,d\tau}{t-\omega} = 0.$$

However, the last equality is impossible because the limit owing to the de L'Hospital's rule and (3.4), is nonzero. Therefore, the assumption that the limit (2.5) is not equal to zero was incorrect.

Now, applying to the equation (1.1) the transformation

$$\begin{aligned} \frac{y^{(k)}(t)}{y^{(k-1)}(t)} &= \mu p^{\frac{1}{n}}(t)\Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{\sigma}{n-\sigma}}[1+v_k(\tau)] \quad (k = 1,\dots,n-1), \\ \ln|y(t)| &= \nu\Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{n}{n-\sigma}}[1+v_n(\tau)], \quad \tau = \Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{\frac{n}{n-\sigma}}, \end{aligned} \tag{3.6}$$

we obtain the following system of differential equations:

$$\begin{cases} v_k' = \mu\,(1+v_k)[v_{k+1} - v_k - \mu h(\tau)] \quad (k = 1,\dots,n-2), \\ v_{n-1}' = \mu\Big[\dfrac{|1+v_n|^\sigma}{(1+v_1)\cdots(1+v_{n-2})} - (1+v_{n-1})^2 - \mu h(\tau)(1+v_{n-1})\Big], \\ v_n' = g(\tau)(v_1 - v_n), \end{cases} \tag{3.7}$$

in which

$$g(\tau(t)) = \Big|\frac{n-\sigma}{n}\,J_B(t)\Big|^{-\frac{n}{n-\sigma}}, \quad h(\tau(t)) = \frac{(p^{\frac{1}{n}}(t)|\frac{n-\sigma}{n}\,J_B(t)|^{\frac{\sigma}{n-\sigma}})'}{p^{\frac{2}{n}}(t)|\frac{n-\sigma}{n}\,J_B(t)|^{\frac{2\sigma}{n-\sigma}}}\,.$$

We will consider this system on the set $[\tau_0, +\infty[\,\times \mathbb{R}^n_{\frac{1}{2}}$, where

$$\tau_0 = \Big|\frac{n-\sigma}{n}\,J_B(a_0)\Big|^{\frac{n}{n-\sigma}}, \quad \mathbb{R}^n_{\frac{1}{2}} = \Big\{(v_1,\dots,v_n) \in \mathbb{R}^n:\ |v_k| \le \frac{1}{2}\ (k = 1,\dots,n)\Big\}.$$

By (3.3) and the fact that the limit (2.5) is equal to zero as established above, we have

$$\lim_{\tau\to+\infty} g(\tau) = \lim_{t\uparrow\omega} g(\tau(t)) = 0, \qquad \lim_{\tau\to+\infty} h(\tau) = \lim_{t\uparrow\omega} h(\tau(t)) = 0. \tag{3.8}$$

Moreover,

$$\int_{\tau_0}^{+\infty} g(\tau)\,d\tau = \frac{n}{n-\sigma}\int_{a_0}^{\omega} \frac{p^{\frac{1}{n}}(s)\,ds}{J_B(s)} = \frac{n}{n-\sigma}\,\ln|J_B(s)|\Big|_{a_0}^{\omega} = \pm\infty. \tag{3.9}$$

By separating linear parts in the equations of the system (3.7), we obtain a system of differential equations (2.8) in which

$$\beta_0=\mu,\quad H(\tau)=g(\tau),\quad f_k(\tau,v_1,\dots,n)=-\mu(1+v_k)h(\tau)\ \ (k=1,\dots,n-1),$$
$$f_n(\tau,v_1,\dots,n)\equiv 0,\quad V_k(v_1,\dots,v_n)=v_kv_{k+1}-v_k^2\ \ (k=1,\dots,n-2),$$
$$V_{n-1}(v_1,\dots,v_n)=\frac{|1+v_n|^{\sigma}}{(1+v_1)\cdots(1+v_{n-2})}+\sum_{i=1}^{n-2}v_i-v_{n-1}^2-\sigma v_n,\quad V_n(v_1,\dots,v_n)\equiv 0,$$
$$c_{kk}=-1,\quad c_{kk+1}=1,\quad c_{ki}=0\ \text{ for }\ i\neq k,k+1\ \ (k=1,\dots,n-2),$$
$$c_{n-1i}=-1\ \ (i=1,\dots,n-2),\quad c_{n-1n-1}=-2,\quad c_{n-1n}=\sigma,$$
$$c_{n1}=1,\quad c_{ni}=0\ \ (i=2,\dots,n-1),\quad c_{nn}=-1.$$

Here the functions V_k $(k=1,\dots,n)$ satisfy (2.10) and by (3.8) and (3.9) the conditions (2.9) and (2.11) hold. Furthermore, for the matrices $C_{n-1}=(c_{ki})_{k,i=1}^{n-1}$ and $C_n=(c_{ki})_{k,i=1}^{n}$, we find

$$\det C_n=(-1)^{n+1}[\sigma-n],\quad \det[C_{n-1}-\rho E]=(-1)^{n+1}\sum_{k=1}^{n}(1+\rho)^{k-1}.$$

Therefore, $(\det C_n)(\det C_{n-1})=n(\sigma-n)$ and the characteristic equation of the matrix C_{n-1} has the form

$$\sum_{k=1}^{n}(1+\rho)^{k-1}=0.$$

The roots of this equation differ from the roots of $(1+\rho)^n=1$. Clearly, all such roots have negative real parts.

Hence, taking into account the condition $\sigma\neq n$, it is clear that the system of differential equations (3.7) satisfy all the conditions of Lemma 2.2. On the basis of this lemma, the given system of differential equations has at least one solution $(v_k)_{k=1}^n:[\tau_1,+\infty[\,\to\mathbb{R}^n$ $(\tau_1\geq\tau_0)$, which tends to zero as $\tau\to+\infty$. Moreover, if $\mu=1$ and $\sigma>n$, there exist $(n-1)$-parametric family of such solutions and n-parametric family in case $\mu=1$ and $\sigma<n$. If $\mu=-1$ and $\sigma<n$, there exists one-parametric family of solutions. Each such solution of the system (3.7) by virtue of the substitutions (3.6) corresponds to y-solution of the differential equation (1.1), which admits the asymptotic representations (2.3), (2.4) as $t\uparrow\omega$. It is not difficult to see that using the conditions (2.1) and (2.2) any of these solutions is a $P_\omega(1)$-solution. □

From this theorem we get the following corollary for the linear differential equation (1.3).

Corollary 3.1. *For the existence of* $P_\omega(1)$*-solution of the differential equation* (1.3), *it is necessary, and if the function* $p:[a,\omega[\,\to]0,+\infty[$ *is continuously differentiable and* $\lim_{t\uparrow\omega}p'(t)p^{-\frac{n+1}{n}}(t)$ *is finite or equal to* $\pm\infty$, *then it is sufficient that for some* $\mu\in\{-1;1\}$, *the inequality* (2.1) *holds and the condition*

$$\lim_{t\uparrow\omega}p(t)|\pi_\omega(t)|^n=+\infty \tag{3.10}$$

is fulfilled.

Moreover, for each of these solutions there take place the following asymptotic representations as $t \uparrow \omega$:

$$\ln|y(t)| = \mu\, J_B(t)[1+o(1)],$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \mu p^{\frac{1}{n}}(t)[1+o(1)] \ \ (k=1,\dots,n-1),$$

whereas, for $\mu = 1$, there exists an n-parametric family of $P_\omega(1)$-solutions for this representation, and for $\mu = -1$, there exists one-parametric family of solutions.

This corollary complements the results given in [9, Chapter 1, § 6] on the asymptotic behavior of solutions of linear differential equations. In view of (3.10), it does not refer to the cases where the differential equation (1.3) is asymptotically close to the Euler equation and the equation with almost constant coefficients.

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