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THE RIEMANN PROBLEM AND LINEAR SINGULAR INTEGRAL EQUATIONS WITH MEASURABLE COEFFICIENTS IN LEBESGUE TYPE SPACES WITH A VARIABLE EXPONENT

Abstract. In the present work the Riemann problem for analysis functions $\phi(z)$ is considered in a class of Cauchy type integrals with density from $L^{p(t)}$ and a singular integral equation

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau-t}\,d\tau=f(t)$$

in the space $\mathcal{L}^{p(t)}$ whose norm defined by the Lebesgue summation with a variable exponent. In both takes an integration curve is taken from a set containing non-smooth curves. The functions G and $(a-b)(a+b)^{-1}$ are take from a set of measurable functions $A(p(t),\Gamma)$ which is generalization of the class $A(p)$ of I. B. Simonenko. For the Riemann problem the necessary condition of solvability and the sufficient condition are pointed out, and solutions (if any) are constructed. For the singular integral equation the necessary Noetherity condition and one sufficient Noetherity condition are established; the index is calculated and solutions are constructed.

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Key words and phrases. Riemann's boundary value problem, measurable coefficient, factorization of functions, Lebesgue space with a variable exponent, Cauchy type integrals, Noetherian operator, Smirnov class of analytic functions with variable exponents, Cauchy singular integral equations.

რეზიუმე. ნაშრომში გამოკვლეულია ანალიზურ ფუნქციათა თეორიაში რიმანის სასაზღვრო ამოცანა $\phi^+(t)=G(t)\phi^-(t)+g(t)$ კოშის ტიპის ინტეგრალით წარმოდგენად იმ ფუნქციათა კლასში, რომელთა სიმკვრივე ეკუთვნის ლებეგის $L^{p(t)}$ სივრცეს და სინგულარული ინტეგრალური განტოლება

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau-t}\,d\tau=f(t)$$

ბანახის $\mathcal{L}^{p(t)}$ სივრცეში, რომელიც შემოღებულია ცვლადი მაჩვენებლით ინტეგრების მოშველიებით. G და $(a-b)(a+b)^{-1}$ ფუნქციები აიღება ზომად ფუნქციათა $A(p(t),\Gamma)$ კლასიდან, რომელიც წარმოადგენს ი. სიმონენკოს $A(p)$ კლასის განზოგადებას. რიმანის ამოცანისთვის დადგენილია ამოხსნადობის აუცილებელი და საკმარისი პირობა და აგრეთვე ზოგი საკმარისი პირობა. აგებულია ამონახსნები კვადრატურებში. სინგულარული განტოლებისათვის კი გამოკვლეულია ნეტერისეულობის საკითხები და დათვლილია ინდექსი. აგებულია ამონახსნები.

1. Introduction

The boundary value problems of the theory of analytic functions and tightly connected with them linear singular integral equations with Cauchy kernel are well-studied (see, e.g., [1]–[8]).

If the domain D^+ is bounded by a simple, rectifiable, closed curve Γ, $D^- = \mathbb{C} \setminus \overline{D}^+$, $G(t)$, $g(t)$ are the given on Γ functions and we seek for a function ϕ representable by the Cauchy type integral with density from $L^p(\Gamma)$ whose angular boundary values ϕ^+ from D^+ and ϕ^- from D^- satisfy almost everywhere on Γ the condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \tag{1}$$

then this problem is called the Riemann problem in the class $K^p(\Gamma)$.

When Γ is a Carleson curve, $\inf|G(t)| > 0$, $p > 1$, and

$$\phi(z) = (K_\Gamma\varphi)(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - z}\,d\tau, \quad \varphi \in L^{p(\,\cdot\,)}(\Gamma), \quad p > 1,$$

$$S = S_\Gamma : \varphi \to S_\Gamma\varphi, \quad (S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau,$$

then the problem (1) reduces equivalently to the equation

$$(1 - G(t))\varphi(t) + (1 + G(t))(S_\Gamma\varphi)(t) = g(t). \tag{2}$$

in $L^p(\Gamma)$ ([5, p. 134]).

Conversely, the considered in $L^p(\Gamma)$ equation

$$M\varphi := a(t)\varphi(t) + b(t)(S_\Gamma\varphi)(t) = f(t) \tag{3}$$

for

$$0 < \operatorname{ess\,inf}\left|a^2(t) - b^2(t)\right| \le \operatorname{ess\,sup}\left|a^2(t) - b^2(t)\right| < \infty$$

is equivalent to the problem

$$\phi^+(t) = \frac{a(t) - b(t)}{a(t) + b(t)}\,\phi^-(t) + \frac{f(t)}{a(t) + b(t)} \tag{4}$$

in $K^p(\Gamma)$.

The interest of researches in the Lebesgue spaces $L^{p(t)}(\Gamma)$ with a variable exponent and in their applications to the boundary value problems has appreciably increased in the recent years (see, e.g., [9]–[20]). A great number of problems of the theory of analytic functions have been investigated ([16]–[21]). Of importance are the works due to V. Kokilashvili and S. Samko in which they have revealed wide classes of curves for which the Cauchy singular operator is continuous in classes $L^{p(t)}(\Gamma)$, when $p(t)$ is Log–Hölder continuous and $\inf p(t) = \underline{p} > 1$. A more general result is presented in [10]. It is proved there that for the operator S to be continuous in $L^{p(t)}(\Gamma)$, it is necessary and sufficient that Γ is a Carleson curve. Further, in the case of the above-mentioned curves, it is stated that S is continuous in the space $L^{p(\cdot)}(\Gamma,\omega)$, $\omega=\prod\limits_{k=1}^{n}|t-t_k|^{\alpha_k}$, $t_k\in\Gamma$, $\alpha_k\in R$, if and only if

$$-[p(t_k)]^{-1}<\alpha_k<[q(t_k)]^{-1},\ \ q(t)=p(t)[p(t)-1]^{-1}.$$

When $p(t)=const>1$, the problem (1) in the class $K^p(\Gamma)$ is thoroughly studied (see, e.g. [5]). The case, in which G is a measurable, oscillating function, has been investigated by I. Simonenko ([22]). He has introduced a class of functions $A(p)$ and showed that when Γ is the Lyapunov curve and $G\in A(p)$, then a picture of solvability inherent in such curves remains the same for continuous G. In [23], this result has been generalized to wider classes of coefficients and boundary curves.

In Sections 3–7 of the present work we investigate the problem (1) in the class $K^{p(\cdot)}(\Gamma)$, when Γ belongs to a wide class of curves and $G(t)$ belongs to a class $A(p(t),\Gamma)$ introduced in Section 3. Sections 8–12 we consider equation (3) with measurable coefficients in the space $\mathcal{L}^{p(t)}(\gamma)$ which is defined in Section 9. The norm of the element φ in that space is defined by equality

$$\|\varphi\|_{\mathcal{L}^{p(\cdot)}}=\|\varphi\|_{p(\cdot)}+\|T\varphi\|_{p(\cdot)}+\Big\|\frac{\varphi_1}{G}\Big\|_{p(\cdot)}+\Big\|\frac{\varphi_2}{G}\Big\|_{p(\cdot)}, \tag{5}$$

where $T\varphi=X^+S\,\frac{\varphi}{X^+}$, $\varphi_1=\frac12\,(\varphi+T\varphi)$, $\varphi_2=\frac12\,(-\varphi+T\varphi)$, and X^+ is the function defined by means of G (see below (15)).

It should be noted that if Γ has singularities such, for example, as cusps, vorticities, or the coefficient G is "badly measurable", then all these facts should be taken into account on selecting the class of solutions. In [24], for instance, for a constant p, a space in which we are required to find a solution is chosen in such a way that the norm contains power weights of different growth on different sides from cusps. In our case, oscillation of the coefficient G has made a major contribution to that norm.

For investigation of the problem (1) we have used the method of factorization which this time met with an obstacle. The matter is that for the solvability of the problem (1) in $K^{p(t)}(\Gamma)$, it is necessary that the function Tg belong to $L^{p(t)}(\Gamma)$. When Γ has cusps and $G\in A(p(t),\Gamma)$, we have failed to prove or disprove that Tg satisfies this condition for any g from $L^{p(t)}(\Gamma)$. However, we have managed both to show that if $\operatorname{ind}G\geq 0$, then (1) has solutions from the set $\bigcap\limits_{0<\varepsilon<\underline{p}}K^{p(t)-\varepsilon}(\Gamma)$ and to construct all such solutions. If, in addition, $g\in\bigcup\limits_{\varepsilon>0}L^{p(t)+\varepsilon}(\Gamma)$, then the problem (1) is solvable in $K^{p(t)}(\Gamma)$, too. When $\operatorname{ind}G<0$, for the solvability of the problem there take place the conditions of orthogonality of the function g to solutions of a homogeneous conjugate problem (inherent in the problem (1) in classical assumptions).

We have succeeded in revealing such a picture of solvability (although not entirely complete, but rather informative) by reducing the problem (1) to a series of problems of the same type, but with a coefficient, different from a constant one in the neighborhood of some point. One of such methods, known for $p = const$ as the "local method" ([25]), or "local principle" ([4, pp. 353–363]) is valid for a variable p, as well (the proof is obtained by the method indicated in [4] with the use of results from [21]). Application of that method allows one in the best case to investigate the problem qualitatively, leaving the question of a solution construction in quadratures open.

Our approach is somewhat different from the "local method"; it provides us with opportunity to construct solutions (if any) in quadratures. But in this connection we have to require that $Tg \in L^{p(t)}(\Gamma)$. This circumstance did not allow us to get, on the basis of investigations of the Riemann problem, its traditional application, i.e., to prove the Noetherity of equation (3) in $L^{p(t)}(\Gamma)$.

However, our wish to possess Noether theorems for equation (3) is quite natural, if not in $L^{p(\,\cdot\,)}(\Gamma)$, but although for some space of type $L^{(p(t)}$, i.e., with the norm defined by the Lebesgue integration with a variable exponent.

Towards this end, we distinguish from $L^{p(t)}(\Gamma, \omega)$ a subset $\mathcal{L}^{p(t)}(\Gamma)$ and endow it with the norm (5) with respect to which this subset is the Banach space.

In the space $\mathcal{L}^{p(t)}(\Gamma)$, for equation (3) it is stated that: the operator M maps $\mathcal{L}^{p(t)}(\Gamma)$ into itself; the necessary and sufficient conditions of solvability are established; solutions (if any) are constructed; the space, conjugate to $\mathcal{L}^{(p(\,\cdot\,)}(\Gamma)$, is found; one necessary Noetherian condition is pointed out; the Noether theorems are proved and the index is calculated.

In this connection, of significance turned out to be the finding of properties of the operator T (in the spaces $L^{p(t)}(\Gamma)$ and $\mathcal{L}^{p(t)}(\Gamma)$).

In the final Section 13 we present a number of properties of the operator T which in the framework of the present paper are not applied, but have independent interest and will, in all probability, be applied to further investigations of the Riemann problem and singular integral equations of type (3).

2. Preliminaries

2.1. **Curves.** Throughout the paper, the use will be made of the following notation.

(a) C^1 is the set of Jordan smooth curves;

(b) $C^{1,L}$ is the set of the same Lyapunov curves;

(c) R is the set of regular (Carleson) simple, rectifiable, closed curves of Γ for which

$$\sup_{\rho>0,\ \zeta\in\Gamma} \rho^{-1}\ell(\zeta,\rho) < \infty,$$

where $\ell(\zeta,\rho)$ is a linear measure of some part of Γ falling into a circle with center ζ, of radius ρ;

(d) Λ is the set of Lavrent'ev curves , i.e., curves Γ for which $s(t_1,t_2)|t_1-t_2|^{-1} < M < \infty$ for any $t_1,t_2 \in \Gamma$, where $s(t_1,t_2)$ is the length of the smallest arc lying on Γ and connecting the points t_1 and t_2.

(e) J_0 is the set of curves with the equation $t = t(s)$, $0 \le s \le l$, such that there exists a smooth curve γ with the equation $\mu = \mu(s)$, $0 \le s \le l$, such that

$$\exp_{0\le s\le l}\left(\int\limits_0^l \Big|\frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{\mu'(\sigma)}{\mu(\sigma)-\mu(s)}\Big|\, d\sigma\right) < \infty.$$

.

(f) J^* is the set of those closed Jordan curves from Λ which are a union of a finite number of curves from J_0 having tangents at the ends.

(g) $C^1(A_1,\dots,A_n;\nu_1,\dots,\nu_n)$ is the set of piecewise-smooth curves Γ with angular points $A_1,\dots,A_n$ at which angle sizes, inner with respect to the domain bounded by Γ, are equal to $\pi\nu(A_k)$, $0 \le \nu(A_k) \le 2$;

(h) $C^{1,L}(A_1,\dots,A_n;\nu_1,\dots,\nu_n)$ is the set of piecewise-Lyapunov curves for which the condition of item (g) is fulfilled.

Obviously, $C^1 \subset J^*$. The class J^* contains curves of bounded variation (Radon's curves) ([6, pp. 20 and 146–7]), piecewise-smooth curves, free from cusps and, moreover, $J^* \subset R$ ([8, p. 23]).

2.2. **The class of functions $\mathcal{P}(\Gamma)$.** Let Γ be a simple rectifiable curve. We say that the given on Γ function $p = p(t)$ belongs to the class $\mathcal{P}(\Gamma)$ if:

(1) there exists a number $B(p)$ such that for any t_1 and t_2 from Γ we have

$$|p(t_1) - p(t_2)| < \frac{B(p)}{|\ln|t-t_0||}\ ;$$

(2) $1 < \underline{p} = \inf|p(t)| \le \sup|p(t)| = \overline{p} < \infty$.

2.3. **Lebesgue spaces with a variable exponent.**

2.3.1. By $L^{p(t)}(\Gamma;\omega)$ we denote the weight Banach space of measurable on Γ function f such that $\|f\|_{p(\,\cdot\,),\omega'} < \infty$, where

$$\|f\|_{p(\,\cdot\,),\omega} = \inf\left\{\lambda > 0 : \int\limits_0^l \Big|\frac{f(t(s))\omega(t(s))}{\lambda}\Big|^{p(t(s))} ds \le 1\right\}.$$

Here, $t = t(s)$, $0 \le s \le l$, is the equation of the curve Γ with respect to the arc abscissa s.

Assume $L^{p(t)}(\Gamma) := L^{p(t)}(\Gamma, 1)$.

2.3.2. For $p \in \mathcal{P}(\Gamma)$, a space, conjugate to $L^{p(\cdot)}(\Gamma;\omega)$, is $L^{q(t)}(\Gamma;\frac{1}{\omega})$, where $q(t)=\frac{p(t)}{p(t)-1}$. In particular,

$$[L^{p(t)}(\Gamma)]^* = L^{q(t)}(\Gamma)$$

(see [9]).

2.4. **Some properties of spaces** $L^{p(\cdot)}(\Gamma;\omega)$**.**

2.4.1. If $p \in \mathcal{P}(\Gamma)$, $u \in L^{p(\cdot)}(\Gamma;\omega)$, $v \in L^{q(\cdot)}(\Gamma;\frac{1}{\omega})$, then the inequality

$$\left|\int_\Gamma u(\tau)v(\tau)\,d\tau\right| \le K\|u\|_{p(\cdot),\omega}\|v\|_{q(\cdot),\frac{1}{\omega}}, \quad k = 1+\frac{1}{\underline{p}}+\frac{1}{\overline{p}} \tag{6}$$

is valid. Moreover,

$$\|f\|_{p(\cdot)} \sim \sup_{\|g\|_{q(\cdot)}\le 1}\left|\int_\Gamma f(t)g(t)\,dt\right|.$$

2.4.2. If $p(t)$ and $p_1(t)$ belong to $\mathcal{P}(\Gamma)$, and $p(t) \le p_1(t)$, then

$$\|f\|_{p(\cdot)} \le (1+\ell)\|f\|_{p_1(\cdot)}, \quad \ell = |\Gamma| = \operatorname{mes}\Gamma.. \tag{7}$$

2.4.3. If $p \in \mathcal{P}(\Gamma)$, then $L^{p(\cdot)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$.

(For the proofs of statements 2.3.2, 2.4.1 and 2.4.2 see, e.g., [9]).

2.5. **Classes of functions** $\widetilde{K}^{p(\cdot)}(\Gamma)$ **and** $K^{p(\cdot)}(\Gamma)$**.** Assume

$$\widetilde{K}^{p(\cdot)}(\Gamma,\omega) = \Big\{\phi(z) = (K_\Gamma\varphi)(z) + P_\phi(z) = \\ = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta + P_\phi(z),\ z \notin \Gamma,\ \varphi \in L^{p(\cdot)}(\Gamma;\omega)\Big\},$$

where P_ϕ is a polynomial;

$$K^{p(\cdot)}(\Gamma,\omega) = \Big\{\phi:\ \phi \in \widetilde{K}^{p(\cdot)}(\Gamma,\omega),\ P_\phi = 0\Big\}.$$

Denote

$$\widetilde{K}^{p(\cdot)}(\Gamma) := \widetilde{K}^{p(\cdot)}(\Gamma,1), \quad K^{p(\cdot)}(\Gamma) := K^{p(\cdot)}(\Gamma,1).$$

Since $L^{p(\cdot)}(\Gamma) \subseteq L^{\underline{p}}(\Gamma) \subset L^1(\Gamma)$, the Cauchy type integral $\phi = (K_\Gamma\varphi)(z)$, when $\varphi \in L^{p(\cdot)}(\Gamma)$, $p \in \mathcal{P}(\Gamma)$, almost for all $t \in \Gamma$ has angular boundary value $\phi^+(t)$ $(\phi^-(t))$, as the point z tends nontangentially to the point t, lying to the left (to the right) from the chosen on Γ positive direction (see, e.g., [26]), and the Plemelj–Sokhotskii's equalities

$$\phi^\pm(t) = \pm\frac{1}{2}\,\varphi(t) + \frac{1}{2}\,(S\varphi)(t) \tag{8}$$

are valid.

2.6. **Classes of functions $E^{p(t)}(D)$.** Let D be a simply-connected domain with the boundary Γ. By $z = z(w)$ we denote conformal mapping of the circle $U = \{w : \ |w| < 1\}$ onto D.

We say that an analytic in D function ϕ belongs to the class $E^{p(t)}(D)$ if

$$\sup_{0<r<1} \int_0^{2\pi} \big|\phi(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})| \, d\vartheta < \infty.$$

For $p = const$, this class coincides with Smirnov class $E^{p(t)}(D)$. Some properties of functions from $E^{p(t)}(D)$ can be found in [16] and [20] (see also [21, Ch. 3]).

For the constant p, the classes $E^{p(t)}(D)$ are defined for any $p > 0$. Their properties are treated in different books. We restrict ourselves to the reference [27].

If the operator S is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$, then the Cauchy type integral $(K_\Gamma\varphi)(z)$ belongs to $E^s(D)$ when $\varphi \in L^{p(\,\cdot\,)}(\Gamma)$ ([8, pp. 29–30]).

When $\Gamma \in R$, the operator S_Γ is continuous in the classes $L^p(\Gamma)$ for any $p \in (1, \infty)$ ([28]). Therefore, if $\Gamma \in R$, $\varphi \in L^p(\Gamma)$, $p > 1$, then $K_\Gamma\varphi \in E^p(D)$. Moreover, if $\varphi \in L^1(\Gamma)$, then $K_\Gamma\varphi \in \prod\limits_{\delta<1} E^\delta(D)$.

If $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, then $E^{p(t)}(D) \subset K^{p(t)}(D)$ ([16], [20]). If, however, Γ is a piecewise-smooth curve without cusps, then $E^{p(t)}(D) = K^{p(t)}(D)$ ([21, Ch. 3]).

3. Classes of Functions $A(p(t), \Gamma)$

3.1. **Definition of the classes $A(p(t), \Gamma)$.**

Definition 1. Let Γ be a simple, closed, rectifiable curve, and $p \in \mathcal{P}(\Gamma)$. We say that the given on Γ function G belongs to the class $A(p(t), \Gamma)$ if:

(i) $0 < m = \operatorname{ess\,inf} |G(t)| = \operatorname{ess\,sup} |G(t)| = M < \infty$;

(ii) for every point $\tau \in \Gamma$, there exists the arc $\Gamma_\tau \subset \Gamma$ containing the point r at which almost all values of the function G lie inside the angle with vertex at the origin of coordinates and opening

$$\alpha = 2\pi \Big[\sup_{t\in\Gamma_\tau} \max(p(t), q(t)) \Big]^{-1}.$$

It follows from the definition that

$$A(p(t), \Gamma) = A(q(t), \Gamma). \tag{9}$$

Let us consider the covering of the curve of Γ by the arcs Γ_τ. From that covering we can select a finite covering by the arcs $\Gamma_k = \Gamma_{\tau_k}$, $k = 1, \dots, \mu$. It follows from the definition of the class $A(p(t), \Gamma)$ that there exist numbers $\varepsilon_k > 0$ such that all values of $G(t)$ on Γ_k lie inside the angle of the opening

$$\alpha_{\varepsilon_k} = (2\pi - \varepsilon_k) \Big[\sup_{t\in\Gamma_l} \max(p(t), q(t)) \Big]^{-1}.$$

Without loss of generality, we may reckon that no arc of Γ_k is contained in the union of two adjacent arcs. Thus, $\Gamma = \bigcup\limits_{k=1}^{\mu} \Gamma_k$, and every arc of Γ_k intersects with two adjacent arcs. Suppose

$$\Gamma_k^{(1)} = \Gamma_k \cap \Gamma_{k-1}, \ \ \Gamma_k^{(3)} = \Gamma_k \cap \Gamma_{k+1}, \ \ \Gamma_k^{(2)} = \Gamma_k - (\Gamma_k^{(1)} \cup \Gamma_k^{(3)}),$$

then $\Gamma_k = \Gamma_k^{(1)} \cup \Gamma_k^{(2)} \cup \Gamma_k^{(3)}$. We renumerate the arcs $\Gamma_k^{(j)}$, denote them by $\gamma_1, \dots, \gamma_n$ and assume that they follow one after another. Let Γ_{j-1} and Γ_{j+1} be the arcs intersecting with γ_k; then there exists the number $m > 0$ such that if $\widetilde{\gamma}_k = \Gamma_{j-1} \cup \Gamma_{j+1}\Gamma_k$, then

$$\operatorname{dist}(\gamma_k, \Gamma \setminus \widetilde{\gamma}_k) > m > 0, \ \ k = 1, \dots, n. \tag{10}$$

Since every arc Γ_k is, in fact, a neighborhood of some point, therefore all values of $G(t)$ (on Γ_k) lie in the angle of size less than α_{ε_k}. Assume $\varepsilon = \min \varepsilon_k$. Then by this time, for every point $\tau \in \Gamma$, there exists the arc (denoted by Γ_τ) whose values $G(t)$ lie in the angle of size $\alpha_\varepsilon = (2\pi - \varepsilon)\big[\sup\limits_{t \in \varepsilon_\tau} \max(p(t), q(t))^{-1}\big]$. Thus, when defining the class $A(p(\,\cdot\,), \Gamma)$, we can replace α in condition (ii) by the number α_ε.

3.2. **One property of functions of the class $A(p(t), \Gamma)$.** From the statement proven in Subsection 3.1, from the continuity of $p(t)$ and equality (9) it easily follows that for every function $G \in A(p(t), \Gamma)$ there exists the number $\eta_\varepsilon > 0$ such that $G(t) \in A(p(t) + \eta_\varepsilon, \Gamma)$. Consequently,

$$A(p(t), \Gamma) \subset \bigcup_{\eta > 0} A(p(t) + \eta, \Gamma). \tag{11}$$

3.3. **The class $A(p(t), \gamma)$ for $\gamma \subset \Gamma$, and one its property.** Let γ be the arc lying on the closed curve Γ, $\overline{\gamma}$ be its closure and, moreover, let a and b be end points of γ.

If neighborhoods of the points a and b are, respectively, the sets of the type $[a, c]$ and $[c, b]$, $x \in \gamma$, then the class $A(p(t), \overline{\gamma})$ is defined analogously to $A(p(\,\cdot\,), \Gamma)$.

Suppose

$$\underline{p}_\gamma = \inf_{t \in \gamma} p(t), \quad \widetilde{p}_\gamma = \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big).$$

Theorem 1. *Let $\Gamma \in R$, $\gamma \subset \Gamma$, $p \in \mathcal{P}(\Gamma)$ and $G \in A(p(t), \gamma)$. For every point $\tau \in \gamma$, there exists the arc neighborhood $\gamma_\tau \subset \gamma$ such that all values of G on γ_τ lie in the angle of size $(2\pi - \varepsilon)\big[\max(\underline{p}_\gamma, (\underline{p}_\gamma)')\big]^{-1}$. Thus,*

$$A(p(\,\cdot\,), \gamma) \subseteq A(\widetilde{p}_\gamma), \quad \widetilde{p}_\gamma = \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big). \tag{12}$$

Proof. We consider the cases: 1) $p(\tau) > 2$, 2) $p(\tau) < 2$, 3) $p(\tau) = 2$.

1) $p(\tau) > 2$. Owing to the continuity of $p(t)$ on γ_τ, there is the neighborhood of the point τ at which $p(t) > 2$. Then

$$\sup_{t \in \gamma_\tau} \max(p(t), q(t)) = \sup_{t \in \gamma_\tau} p(t) \geq \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big) = \underline{p}_\gamma$$

and hence,

$$\alpha \le \frac{2\pi-\varepsilon}{\max(\underline{p}_\gamma, (\underline{p}_\gamma)')} = \frac{2\pi-\varepsilon}{\widetilde{p}_\gamma} = \alpha_\gamma.$$

2) $p(\tau) < 2$. In this case, $q(\tau) > 2$, and there exists the arc γ_τ in which $q(t) > 2$; therefore,

$$\max(p(t), q(t)) = \sup_{t\in\gamma_\tau} q(t) = (\underline{p}_\gamma)' = \widetilde{p}_\gamma.$$

Consequently, $\alpha < \alpha_\gamma$.

3) $(\tau) = 2$. Having some small number $\eta > 0$, we find neighborhood γ_τ in which values $p(t)$ lie on the segment $(2-\eta, 2+\eta)$. Then

$$\max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big) = \max\big(2+\eta, (2-\eta)'\big) = \max\Big(2+\eta, \frac{2-\eta}{1-\eta}\Big) =$$
$$= \max\Big(2+\eta, 2+\frac{\eta}{1-\eta}\Big) = 2+\frac{\eta}{1-\eta} = (\underline{p}_\gamma)' = \widetilde{p}_\gamma.$$

Hence, again, $\alpha < \alpha_\gamma$.

Thus, the point τ in all three cases possesses the neighborhood γ_τ with values $G(t)$ lying in the opening angle $\frac{2\pi-\varepsilon}{\widetilde{p}_\gamma}$. Since τ is arbitrary, this implies that the relations (12) are valid. □

3.4. **The index of the function of the class** $A(p(\,\cdot\,), \Gamma)$**. The class** $\widetilde{A}(p(\,\cdot\,), \Gamma)$**.** We choose the point $c \in \Gamma$ and fix the value of $G(c) = [\arg G(c)]^-$ from the interval $[0, 2\pi]$. Following along Γ, we can define a branch of the function $\arg G(t)$ so as to have $|\arg G(t_1) - \arg G(t_2)| < \alpha$ for $t_1, t_2 \in \gamma_k$. Going around Γ, we reach the arc, containing c, with a new value $\arg G(c) = [\arg G(c)]^+$. The difference $[\arg G(c)]^+ - [\arg G(c)]^-$ does not depend on the covering choice and on the point c. The integer

$$\operatorname{ind} G = \varkappa(G) = \varkappa = \frac{1}{2\pi}\big[(\arg G(c))^+ - (\arg G(c))^-\big]$$

is called an index of the function G in the class $K^{p(\,\cdot\,)}(\Gamma)$.

A subset of the functions G from $A(p(\,\cdot\,), \Gamma)$ for which $\sup|\arg G(t)| < \pi/2$ we denote by $\widetilde{A}(p(\,\cdot\,), \Gamma)$. Obviously, if $G \in \widetilde{A}(p(\,\cdot\,), \Gamma)$, then $\operatorname{ind} G = 0$.

4. On Factorization of the Function from $A(p(t), \Gamma)$ in the Class $K^{p(t)}(\Gamma)$

4.1. **Definition of factor-function.**

Definition 2. Let Γ the closed, rectifiable Jordan curve bounding the domains D^+ and D^- $(z = \infty \in D^-)$.

We say that the function $X_G(z) = X(z)$, analytic on the plane, cut along Γ, is a factor-function of the function G in the class $K^{p(t)}(\Gamma)$, if the following conditions are fulfilled:

(1) $X \in \widetilde{\mathcal{K}}^{p(t)}(\Gamma)$;

(2) $[X(z)]^{-1} \in \widetilde{\mathcal{K}}^{q(t)}(\Gamma)$;

(3) almost everywhere on Γ, $X^{+}(t)[X^{-}(t)]^{-1} = G(t)$;

(4) $X^{+} \in W^{p(t)}(\Gamma)$, i.e., the operator

$$T = T_G : g(t) \to (Tg)(t), \quad (Tg)(t) = \frac{X_G^{+}(t)}{\pi i} \int\limits_{\Gamma} \frac{g(\zeta)}{X_G^{+}(\zeta)} \, \frac{d\zeta}{\zeta - t}\,, \quad t \in \Gamma, \quad (13)$$

is continuous in $L^{p(t)}(\Gamma)$.

4.2. **Some properties of factor-functions.**

4.2.1. *The Case of Constant p.* If $\Gamma \in C^{1,L}$ and $G \in A(p, \Gamma)$, then G is factorable in $K^{p}(\Gamma)$ ([22]). The same result is valid when $\Gamma \in J^{*}$, and G is taken from a wider than $A(p, \Gamma)$ class $\widetilde{A}$ which, in particular, contains all admissible piecewise-continuous functions, not fallen in $A(p, \Gamma)$ ([8, p. 192]).

4.2.2. *The Case when $G \in \widetilde{A}(p(t), \Gamma)$ and is equal to the constant on $\Gamma \setminus \gamma$, where $\gamma \subset \Gamma$.* Let $G \in \widetilde{A}(p(t), \Gamma)$, $\tau \in \Gamma$, and $\gamma = \gamma_{ab} = \gamma_{\tau}$ be the arc mentioned in Theorem 1. Assuming $p \in \mathcal{P}(\Gamma)$, we put $\underline{p}_{\gamma} = \inf\limits_{t \in \gamma} p(t)$ and $\widetilde{p}_{\gamma} = \max(\underline{p}_{\gamma}, (\underline{p}_{\gamma})')$.

Consider the function

$$G_{\gamma}(t) = \begin{cases} G(t), & t \in \gamma, \\ G(a), & t \in \Gamma \setminus \gamma. \end{cases} \quad (14)$$

By virtue of Theorem 1 we can easily conclude that $G_{\gamma} \in A(\widetilde{p}_{\gamma}, \Gamma)$. Therefore, assuming $\ln G_{\gamma}(\tau) = \ln |G_{\gamma}(t)| + i \arg G(\tau)$ and

$$X(z) = X_{G_{\gamma}}(z) = \exp\left\{ \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\ln G_{\gamma}(\zeta)}{\zeta - z} \, d\zeta \right\}, \quad (15)$$

$[X(z)]^{\pm 1}$ belongs to $\widetilde{K}^{\widetilde{p}_{\gamma}}(\Gamma)$, and the operator $T = T_G$ is continuous in $L^{\widetilde{p}_{\gamma}}(\Gamma)$, i.e.,

$$\|T_{G_{\gamma}} f\|_{\widetilde{p}_{\gamma}, \Gamma} \le \|T_{G_{\gamma}}\|_{\widetilde{p}_{\gamma}} \|f\|_{\widetilde{p}_{\gamma}, \Gamma}$$

([22]).

In the sequel, frequently, if it does not give rise to misunderstanding, the subscript in our writings X_G, $X_{G_{\gamma}}$, T_G, $T_{G_{\gamma}}$ will be omitted and we write $A(p(\,\cdot\,))$ instead of $A(p(\,\cdot\,), \Gamma)$.

4.2.3. *The class of functions $B(p(\,\cdot\,), \Gamma)$.* By $B(p(\,\cdot\,), \Gamma)$ we denote a set of those functions $G(t)$ with a finite number of points of discontinuity t_k for which $\operatorname{ess\,inf} |G| > 0$ and

$$-\big[p(t_k)\big]^{-1} < \alpha_k \pmod{2\pi} < \big[q(t_k)\big]^{-1}.$$

The branch of $\arg G(t)$ and index for the functions from $B(p(\,\cdot\,), \Gamma)$ are defined in the same manner as in [8, pp. 92–93]. For $p = const$, this class covers all those piecewise-continuous functions which are admissible in the condition (1) when its solutions are sought in the class $K^p(\Gamma)$.

The functions of the class $B(p,\Gamma)$ for $p>1$ and $\Gamma\subset J^*$ are factorable in $K^p(\Gamma)$. Moreover, there exists the number $\delta>0$ such that the factor-function X_G of the function $G\in B(p,\Gamma)$ possesses the property

$$X_G^{\pm}\in \widetilde{K}_{\Gamma}^{\mu+\delta}(\Gamma),\ \ \mu=\max(p,q) \tag{16}$$

([8, p. 115]).

4.2.4. *On the factorization of the function $G_\gamma(t)$ in the classes $K^{p(\,\cdot\,)}(\Gamma)$.* Let $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$ and $\gamma=\gamma_{ab}$ be the arc mentioned in Theorem 1. Without loss of generality, we may assume that $G(t)$ is defined at the point a and $G(a)$ lies in the corresponding to the point a angle of size α. Suppose

$$G_\gamma(t)=\begin{cases}\dfrac{G(t)}{G(a)}\,, & t\in\gamma,\\ 1, & t\in\Gamma\setminus\gamma.\end{cases} \tag{17}$$

Theorem 2. *Let $\Gamma\in J^*$ be a closed, simple, restifiable curve bounding the domains D^+ and D^-, and $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$. Then the function G_γ defined by equality* (17) *is factorable in the class $K^{\widetilde{p}_\gamma}(\Gamma)$.*

Proof. Let us show that $G_\gamma\in A(\widetilde{p}_\gamma,\Gamma)$. By virtue of Theorem 1 and continuity of G_γ on $\Gamma\setminus\gamma$, only behavior of G_γ in the neighborhood of the points a and b needs testing. Let $\gamma_{1a}\subset\Gamma$ be the arc containing a point. By γ_{11} and γ_{12} we denote intersection of γ_{1a} with γ and $\Gamma\setminus\gamma$. Since γ_{11} lies on γ, all values of the function G_γ on it lie in the angle with vertex at the point $z=0$, of size $\beta=\frac{2\pi-\varepsilon}{\widetilde{p}_\gamma}$. As far as number 1 is in that angle, and $G_\gamma(t)$ on γ_{12} equals 1, therefore the values of G_γ on γ_{1a} lie in the above-mentioned angle.

Consider now the neighborhood of the point b. The point $G_\gamma(b)$ lies in the angle of size β together with the point $G_\gamma(a)=1$. Therefore the values of G_γ on the arc (c,a), where $e\subset\Gamma\setminus\gamma$, lie in the angle of size β. Thus it is proved that $G_\gamma\in A(\widetilde{p}_\gamma,\Gamma)$.

According to the statement in item 4.2.3, we can conclude that G_γ is factorable in $K^{\widetilde{p}_\gamma}(\Gamma)$, and its factor-function X_{G_γ} is given by the equality

$$X_{G_\gamma}(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\ln G_\gamma(\zeta)}{\zeta-z}\,d\zeta\right\}. \tag{18}$$

The theorem is proved. $\square$

Corollary. *If $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$, then the function*

$$\widetilde{G}(t)=\begin{cases}G(t), & t\in\gamma,\\ G(a), & t\in\Gamma\setminus\gamma,\end{cases}$$

where the arc γ defined in Theorem 1 is factorable in $K^{\widetilde{p}_\gamma}(\Gamma)$, and its factor-function is

$$\widetilde{X}(z)=\begin{cases}\dfrac{1}{M}\,X_{G_\gamma}(z), & z\in D^+,\\ X_{G_\gamma}(z), & z\in D^-,\end{cases}$$

where

$$M=\operatorname*{ess\,sup}_{t\in\Gamma}|G(t)|+\sup_{t\in\Gamma}|\arg G(t)|.$$

Proof. It suffices to show that $X^{\pm1}\in\widetilde{K}^{\widetilde{p}_\gamma}(\Gamma)$. In view of Subsection 4.2.2, we have $(X_{G_\gamma})^{\pm1}\in E^{\widetilde{p}_\gamma+\delta}(D^+)$, $(X_{G_\gamma})^{\pm1}\in\widetilde{E}^{\widetilde{p}_\gamma}(D^-)$, where $E^\mu(D^+)$ is Smirnov class in D^+, and $\widetilde{E}^\mu(D^-)=\{\phi:\ F+1,\ F\in E^\mu(D^-)\}$. Therefore $X_{G_\gamma}^{\pm1}$ and $X_{G_\gamma}^{\pm1}-1$ are representable by the Cauchy integral in D^+ and D^-, respectively. Consequently,

$$\widetilde{X}^{\pm1}(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{(\widetilde{X}^+)^{\pm1}-(\widetilde{X}^-)^{\pm1}}{t-z}\,dt+1. \qquad \square$$

4.2.5. *Auxiliary estimates.* Let $\Gamma\in J^*$, $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$, and let γ_k and $\widetilde{\gamma}_k$ be subsets of Γ defined in Subsection 3.1. Let, further, $\gamma_k=\gamma_{a_kb_k}$, $G_k(t)=G_{\gamma_k}(t)$ and

$$X_k(z)=\begin{cases}\dfrac{1}{G(a_k)}\,X_{G_k}(z), & z\in D^+,\\ X_{G_k}(z), & z\in D^-,\end{cases}, \tag{19}$$

where

$$X_{G_k}(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_\Gamma\frac{\ln G_{\gamma_k}(\zeta)}{\zeta-z}\,d\zeta\right\}.$$

Suppose

$$Y_k(t)=\prod_{j=1,\,j\neq k}^{n}X_j(t). \tag{20}$$

Lemma 1. *There exist the constants $c_j>0$, $j=1,2$, such that for all k, $k=1,2,\dots,n$, we have*

$$\sup_{t\in\gamma_k}|Y_k(t)|<c_1,\qquad \inf_{t\in\gamma_k}|Y_k(t)|>c_2. \tag{21}$$

Proof. We have

$$|Y_k(t)|\le\exp\left|\frac{1}{2\pi i}\int\limits_{\Gamma\setminus\gamma_k}\frac{\ln|G(\zeta)|+i\arg G(\zeta)}{\zeta-t}\,d\zeta\right|\le$$

$$\le\exp\frac{1}{2\pi}\int\limits_{\Gamma\setminus\gamma_k}\frac{\sup\ln|G|+\mu}{|\zeta-t|}\,|d\zeta|,\quad t\in\gamma_k,$$

where $\mu = \sup|\arg\zeta|$. At the last step here we have taken in account that $G(\zeta) = 1$ for $\zeta \in \widetilde{\gamma}_k \setminus \gamma_k$.

The closed sets $\overline{\gamma}_k$ and $\overline{\Gamma \setminus \widetilde{\gamma}_k}$ do not intersect, hence according to (10), we have $\operatorname{dist}(\gamma_k; \Gamma \setminus \gamma_k) = m_k > 0$, whence it follows that

$$|Y_k(t)| \leq \exp\Big(\frac{nM}{m}\,|\Gamma|\Big), \tag{22}$$

where

$$M = \sup_{\zeta\in\Gamma}\big|\ln|G(\zeta)|\big| + \mu, \quad m = \min_{k=1,2,\dots,n} m_k. \tag{23}$$

To estimate $|Y_k(t)|$, we note that if $Y_k(t) = \exp f_k(t)$, we have shown that $|\exp f_k| < \exp\frac{nM}{m}\,|\Gamma|$. But $|\exp f_k| \geq \exp(-\sup|f_k|)$, and therefore

$$|Y_k(t)| \geq \exp\Big(-\frac{nM}{m}\Big)\,|\Gamma|. \tag{24}$$

It follows from (22) and (24) that inequalities (21), where

$$c_1 = \exp\frac{nM}{m}\,|\Gamma|, \quad c_2 = \exp\Big(-\frac{nM}{m}\,|\Gamma|\Big)$$

are valid, and the numbers M and m in these equalities are defined according to (23). $\square$

5. Some Properties of the Function $X_G(z)$

As regards the data in the condition (1), we assume that either

$$\begin{gathered} \Gamma \in J^*, \ \ p \in \mathcal{P}(\Gamma), \ \ G \in A(p(\,\cdot\,),\Gamma), \\ \text{or} \\ \Gamma \text{ is a piecewise-smooth curve}, \ \ G \in B(p(\,\cdot\,),\Gamma), \ \ p \in \mathcal{P}(\Gamma). \end{gathered} \tag{25}$$

Let the conditions (25) are fulfilled, $\varkappa = \operatorname{ind} G(t)$ and $z_0 \in D^+$. Put

$$G_0(t) = (t-z_0)^{-\varkappa} G(t)$$

and

$$X(z) = \\ = \begin{cases} \exp\{K_\Gamma \ln G_0\}, & z \in D^+, \\ (z-z_0)^{-\varkappa}\exp(K_\Gamma \ln G_0)(z), & z \in D^-. \end{cases} \tag{26}$$

5.1. On the summability of the function $g|_{X^+}$.

Lemma 2. *If the conditions* (25) *are fulfilled, then there exists the number* $\eta > 0$ *such that*

$$g[X^+]^{-1} \in L^{1+\eta}(\Gamma), \ \ K_\Gamma\,\frac{g}{X^+} \in E^{1+\eta}(D^+), \ \ K_\Gamma\Big(\frac{g}{X^+}\Big) \in \widetilde{E}^{1+\eta}(D^-).$$

Proof. Let γ be that arc on Γ for which $G \in A(\widetilde{p}_\gamma, \gamma)$, then the function

$$G_\gamma(t) = \begin{cases} G(t), & t \in \gamma, \\ G(a), & t \in \Gamma \setminus \gamma, \end{cases}$$

belongs to $\widetilde{A}(\widetilde{p}_\gamma, \Gamma)$, and hence, $X_\gamma^{=\pm 1} \in L^{\widetilde{p}_\gamma}(\Gamma)$ (see item 4.2.3). Assuming

$$g_\gamma(t) = \begin{cases} g(t), t \in \gamma, \\ 0, \qquad\qquad t \in \Gamma \setminus \gamma, \end{cases}$$

we have $g_\gamma \in L^{\underline{p}_\gamma}(\Gamma)$, and hence, we obtain

$$\frac{g_\gamma}{X_\gamma^+} \in L^\alpha(\Gamma), \ \ \alpha = \widetilde{p}_\gamma(\widetilde{p}_\gamma + \delta)(\underline{p}_\gamma + \underline{p}_\gamma + \delta)^{-1}.$$

Let us consider two possible cases: 1) $\widetilde{p}_\gamma = \underline{P}_\gamma$, 2) $\widetilde{p}_\gamma = (\underline{P}_\gamma)'$.

1) $\widetilde{p}_\gamma = \underline{p}_\gamma$. This is possible when $p_\gamma \geq 2$. Denote $\lambda = \underline{p}_\gamma$, then consideration (16) we have

$$\alpha = \lambda(\lambda + \delta)(2\lambda + \sigma)^{-2} = \Big(\frac{\lambda}{2} + \frac{\delta}{2}\Big)\Big(1 + \frac{\delta}{2\lambda}\Big)^{-1}.$$

Since $\lambda \geq 2$, then $\alpha > 1$ and therefore

$$g_\gamma(X_\gamma^+)^{-1} \in L^{1+\eta}(\Gamma), \ \ \eta < \alpha < 1.$$

2) $\widetilde{p}_\gamma = (\underline{p}_\gamma)'$, then

$$\alpha = \lambda(\lambda' + \delta)(\lambda + \lambda' + \sigma)^{-1} = \Big(1 + \frac{\delta}{\lambda'}\Big)\Big(1 + \frac{\delta}{\lambda\lambda'}\Big)^{-1} > 1$$

and, hence, again $g/X^+ \in L^{1+\eta}(\Gamma)$.

Since $\Gamma = \cup\gamma_k$, and on γ_k we have $g_k/X^+ = g_k/(X_k^+ Y_k^+)$, $(g_k := g_{\gamma_k})$, taking into account Lemmas 1 and 2, we obtain

$$\int\limits_\Gamma \Big|\frac{g}{X^+}\Big|^{1+\eta} ds = \sum \int\limits_{\gamma_k} \Big|\frac{g_k}{X_k^+ Y_k^+}\Big|^{1+\eta} ds \leq$$
$$\leq \frac{1}{c_2^{1+\eta}} \sum \int\limits_{\gamma_k} \Big|\frac{g_k}{X_k^+}\Big|^{1+\eta} ds < \infty.$$

Statement of the lemma regarding $K_\Gamma \frac{g}{X^+}$ follows from the results given in Subsections 2.6 and in item 2.4.3. □

5.2. On the summability of the function X_G.

Theorem 3. *When the conditions* (25) *are fulfilled, we have* $X_G^+ \in L^{p(\cdot)}(\Gamma)$ *and* $(X_G^+)^{-1} \in L^{q(t)}(\Gamma)$.

Proof. Let γ be the arc mentioned in Theorem 1. Then $G \in A(\widetilde{p}_\gamma, \gamma)$, and the function G_γ belongs to $A(\widetilde{p}_\gamma, \Gamma)$. Since $\Gamma \in J^*$, therefore $X_G \in \widetilde{K}^{\widetilde{p}_\gamma + \delta}(\Gamma)$ ([8, p. 29]) and, hence $X_{G_\gamma}^+ \in L^{\widetilde{p}_\gamma}(\Gamma)$.

Represent now Γ in the form $\Gamma = \bigcup\limits_{k=1}^{n} \gamma_k$, where the curves γ_k satisfy the condition of Theorem 1. Then, according to the above-said,

$$X_k = X_{\gamma_k} \in L^{p_k+\delta}(\Gamma), \ \ p_k = \widetilde{p}_{\gamma_n}, \ \ X_k = \exp\big\{K_\Gamma(\ln(G_{\gamma_k})\big\}.$$

We have $X_G = \prod\limits_{k=1}^{n} X_k Y_k$. Then

$$\int\limits_\Gamma |X_G^+|^{p(t(s))}\, ds \le \sup_{t\in\gamma_k,\ k=1,2,\dots,n} \sum_{k=1}^{n} \int\limits_{\gamma_k} |X_k^+|^{p(t(s))}\, ds \le \\ \le c_1(1+\Gamma) \int\limits_\gamma |X^+|^{p(t(s))}\, ds. \tag{27}$$

On γ_k, we have $\underline{p}_{\gamma_k} \le p(t) \le \overline{p}_{\gamma_k}$, $k = 1,2,\dots,n$.

Due to the uniform continuity of $p(t)$ on Γ, there exists for $\delta > 0$ the number l_δ such that for any arc $\gamma_k \in \Gamma$ such that $|\gamma_k| < l_\delta$, we have

$$\overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta, \ \ (\overline{p}_{\gamma_k})' = \frac{\overline{p}_{\gamma_k}}{\overline{p}_{\gamma_k} - 1}\,. \tag{28}$$

For some γ_k, the condition $|\gamma_k| < l_\delta$ may violate. In this case we consider a new covering of Γ reducing the arcs γ_k to those of lesser length than l_δ. For the sake of simplicity, we denote again the arcs forming a new covering by γ_k. Then, according to (28), on γ_k we have $\overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta$. Moreover, on the above-mentioned arc,

$$\underline{p}_{\gamma_k} \le p(t) \le \overline{p}_{\gamma_k},$$

whence $p(t) - p_{\gamma_k} \le \overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta$, i.e., on γ_k, we have $p(t) < \underline{p}_{\gamma_k} < \delta$. By virtue of inequalities (8) and (27), we now obtain

$$\int\limits_\Gamma |X^+(t)|^{p(t)}\, ds \le c_3 \sum_{k=1}^{n} \int\limits_{\gamma_k} |X_k^+|^{\underline{p}_{\gamma_k}+\delta}\, ds < \infty.$$

Thus, the first statement of the theorem is proved.

The second statement follows from Lemma 2 according to which for an arbitrary function $g \in L^{p(\,\cdot\,)}(\Gamma)$, we have $g(t)\cdot \frac{1}{X^+(t)} \in L^1(\Gamma)$. This means that $\frac{1}{X^+}$ belongs to the class $L^{q(\,\cdot\,)}$. □

Corollary. *The function X_G in the conditions* (25) *belongs to* $L^{p(\,\cdot\,)+\delta}$ *for some $\delta > 0$. This follows from the inclusions* (12), (16) *and Theorem* 3.

6. On the Operator T_G for $G \in \widetilde{A}(p(t),\Gamma)$

6.1. The operator T_G acts from $L^{p(\,\cdot\,)}$ to L^λ for some $\lambda > 0$.

Lemma 3. *If the conditions* (25) *are fulfilled, then the operator T_G acts from $L^{p(\,\cdot\,)}$ to the space $L^{\lambda}(\Gamma)$, $\lambda \in (0, \frac{2+2\eta}{3+\eta})$, where η is the number defined in Lemma* 2.

Proof. From the condition $G \in \widetilde{A}(p(\,\cdot\,), \Gamma)$ it follows that $G(t) \in A(2, \Gamma)$, therefore $X_G^{\pm} \in E^2(D^{\pm})$ and, hence, $X^+ \in L^2$ (see Subsection 4.2). Assuming $0 < \lambda < 2$, we have

$$I = \int_{\Gamma} |Tg|^{\lambda} \, ds = \int_{\Gamma} |X^+|^{\lambda} \Big| S \, \frac{g}{X^+} \Big|^{\lambda} \, ds \le$$

$$\le \bigg(\int_{\Gamma} |X^+|^2 \, ds \bigg)^{\frac{\lambda}{2}} \bigg(\int_{\Gamma} \Big| S \, \frac{g}{X^+} \Big|^{\frac{2\lambda}{2-\lambda}} \, ds \bigg)^{\frac{2-\lambda}{2}}.$$

from which it can be easily seen that $I < \infty$, if $2\lambda(2-\lambda)^{-1} < 1 + \eta$, i.e., $\lambda < \frac{2+2\eta}{\zeta+\eta}$. □

6.2. **On the operator $T_G^2 = T_G(T_G)$.**

Theorem 4. *Under the conditions* (25)*, we have*

$$T^2 g = g. \tag{29}$$

Proof. We have

$$T(Tg) = X^+ S_{\Gamma} \Big(\frac{1}{X^+} \cdot X^+ S_{\Gamma} \, \frac{g}{X^+} \Big) = X^+ S_{\Gamma} \Big(S_{\Gamma} \, \frac{g}{X^+} \Big). \tag{30}$$

Since $\Gamma \in R$, the operator S_{Γ} is continuous in the Lebesgue spaces $L^{\lambda}(\Gamma)$, $\lambda > 1$. Consequently, since $\frac{g}{X^+} \in L^{1+\eta}(\gamma)$ (see Lemma 2), we have $S_{\Gamma} \frac{g}{X^+} \in L^{1+\eta}(\Gamma)$, whence $(K_{\Gamma} \frac{g}{X^+})(z) \in E^{1+\eta}(D^+) \subset E^1(D^+)$ (see Subsection 2.6). But if $K_{\Gamma}\varphi \in E^1(D^+)$, then $S_{\Gamma}(S_{\Gamma}\varphi) = \varphi$ ([8, p. 30]).

In the case under consideration, $\varphi = \frac{g}{X^+}$, and hence, $S_{\Gamma}(S_{\Gamma} \frac{g}{X^+}) = \frac{g}{X^+}$. Substituting this value into (30), we get equality (29). □

6.3. **The continuity of the operator T_G from $L^{p(\,\cdot\,)}(\Gamma)$ to the space of convergence in measure.**

Definition 3. By $M(\Gamma)$ we denote the space of measurable on Γ functions with metric

$$\rho(f, \varphi) = \int_{\Gamma} \frac{|f - \varphi|}{1 + |f - \varphi|} \, ds.$$

The convergence of the sequence $\{f_n\}$ to f_0 in the space $M(\Gamma)$ is equivalent to the convergence of $\{f_n\}$ in measure to f_0.

Lemma 4. *If $g_n \in L^{\lambda}(\Gamma)$, $0 < \lambda < 1$, and*

$$I_n = \int_{\Gamma} |g_n - g_0|^{\lambda} \, ds \to 0, \tag{31}$$

then g_n converges to g_0 in $M(\Gamma)$, as well.

It is not difficult to get the proof by estimating the integral I_n for large n on the set $\ell_{n,\sigma} = \{s : \ |g_n - g_0| > \sigma\}$.

Lemmas 3 and 4 lead to

Statement 1. *The operator T_G is continuous from $L^{p(\cdot)}(\Gamma)$ to $M(\Gamma)$.*

6.4. **Closure of the operator T_G from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$.** Remind the notion of a closed operator. Let A be the linear operator defined in the Banach space X (i.e., the operator defined on some lineal from X and is linear in it) with the domain of definition $D(A)$ and acting to the Banach space Y. The operator A is called closed from X to Y if it possesses the following property:

if $\|x_n - x_0\|_X \to 0$ and $\|Ax_n - y_0\|_Y \to 0$, then $x_0 \in D(A)$ and $Ax_0 = y_0$.

Theorem 5. *If the conditions* (25) *are fulfilled, the operator $T = T_G$ is closed from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$.*

Proof. The domain of definition of the operator $T = T_G$ will be assumed to be a linear set

$$D(T) = \Big\{g : \ g \in L^{p(\cdot)}(\Gamma), \ Tg \in L^{p(\cdot)}(\Gamma)\Big\}.$$

Let $g_n \in D(T)$, $n \in \mathbb{N}$, $\|g_n - g_0\|_{p(\cdot)} \to 0$, $\|Tg_n - f_0\|_{p(\cdot)} \to 0$. Then $g_0, f_0 \in L^{p(\cdot)}(\Gamma)$, and owing to Statement 1, $\|Tg_n - Tg_0\|_{M(\Gamma)} \to 0$. It follows from the condition $\|Tg_n - f_0\|_{p(\cdot)} \to 0$ that $\|Tg_n - f_0\|_{M(\Gamma)} \to 0$, whence we conclude that $f_0 = Tg_0$, by virtue of the limit uniqueness in measure. Thus, we have

$$g_0 \in L^{p(\cdot)}(\Gamma), \quad Tg_0 = f_0 \in L^{p(\cdot)}(\Gamma).$$

This implies that $g_0 \in D(T)$, and since $\|Tg_n - Tg_0\|_{p(\cdot)} \to 0$, the operator T is closed from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$. □

7. The Riemann Problem in the Class $K^{p(\cdot)}(\Gamma)$

7.1. **Statement of the problem.** Let Γ be the simple, rectifiable, closed curve, bounding the domains D^+ and D^- ($z = \infty \in D^-$), $g \in L^{p(\cdot)}(\Gamma)$ and the conditions (25) are fulfilled. We are required to find the functions $\phi \in K^{p(\cdot)}(\Gamma)$ whose angular boundary values $\phi^+(t)$ and $\phi^-(t)$ almost everywhere on Γ satisfy the boundary condition (1).

7.2. **Reduction of the problem** (1) **to the jump problem, when $\varkappa(G) = 0$.** Let

$$X_G(z) = X(z) = \exp\left\{\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\ln G(\zeta)}{\zeta - z}\, d\zeta\right\}.$$

By Theorem 3, we have $\frac{1}{X(z)} \in \widetilde{K}^{p(\,\cdot\,)}(\Gamma)$ and $X(\infty) = 1$. Since $G(t) = X^+(t)[X^-(t)]^{-1}$, the condition (1) can be written in the form

$$\Big(\frac{\phi}{X^+}\Big)^+ - \Big(\frac{\phi}{X^+}\Big)^- = \frac{g}{X^+}\,.$$

Putting $\phi_1(z) = \phi(z)[X(z)]^{-1}$, we get $\phi_1 \in K^1(\Gamma)$ and $\phi_1^+ - \phi_1^- \in g[X^+]^{-1}$. By Lemma 2, we have $g[X^+]^{-1} \in L^{1+\eta}(\Gamma)$, $\eta > 0$. Therefore, the solution of the last problem is unique, and $\phi_1(z) = (K_\Gamma \frac{g}{X^+})(z)$. Consequently, the solution of the problem (1) may be only the function

$$\phi(z) = X^+(z)\Big(K_\Gamma \,\frac{g}{X^+}\Big)(z), \tag{32}$$

and we have to elucidate the conditions under which this function belongs to the class $L^{p(\,\cdot\,)}$.

7.3. **Criterion of solvability of the problem** (1) **when** $G(t) \in A(p(\,\cdot\,),\Gamma)$ **and** $\operatorname{ind} G = 0$**.** If the conditions (25) are fulfilled, then $K[g(X^+)^{-1}] \in E^{1+\eta}(D^\pm)$ (see Lemma 2). Therefore the function ϕ given by equality (32) is representable by the Cauchy type integral with density $\varphi = \phi^+ - \phi^-$. Hence $\phi \in K^{p(\,\cdot\,)}(\Gamma)$, if and only if

$$\varphi(t) = \big[\phi^+(t) - \phi^-(t)\big] \in L^{p(\,\cdot\,)}. \tag{33}$$

Using formulas (8) and taking into account the fact that $G = \frac{X^+}{X^-}$, we obtain

$$\phi^+ = \frac{1}{2}\,(g + Tg), \quad \phi^- = \frac{1}{2G}\,(-g + Tg).$$

It now follows from (33) that

$$\varphi(t) = \frac{G+1}{2G}\,g(t) + \frac{G-1}{2G}\,(Tg)(t).$$

Obviously, if $G \equiv 1$, then $\varphi \in L^{p(\,\cdot\,)}(\Gamma)$. However, if $G \neq 1$, then for the condition (33) to be fulfilled, it is necessary and sufficient that the function Tg belong to $L^{p(\,\cdot\,)}(\Gamma)$.

Thus we have proved

Theorem 6. *If the conditions* (25) *are fulfilled and* $G(t) \equiv 1$, *then the problem* (1) *is uniquely solvable in the class* $K^{p(\,\cdot\,)}(\Gamma)$. *If, however,* $G \not\equiv 1$ *and* $\operatorname{ind} G = 0$, *then for its solvability it is necessary and sufficient that* $Tg \in L^{p(\,\cdot\,)}(\Gamma)$. *In case this condition is fulfilled, a solution is unique and given by the equality*

$$\phi(z) = K_\Gamma\Big[\frac{G+1}{2G}\,g + \frac{G-1}{2G}\,Tg\Big](z). \tag{34}$$

7.4. **Problem** (1) **in the class** $K^{p(\cdot)}(\Gamma)$ **when** $G \in A(p(\cdot),\Gamma)$ **and** $\varkappa(G)=\varkappa>0$**.** Let the conditions (25) be fulfilled and $Tg \in L^{p(\cdot)}(\Gamma)$.

As usually (see [5, p. 118]), we fix the point $z_0 \in D^+$ and write the condition (1) in the form

$$\phi^+(t)=\phi^-(t)(t-z_0)^{\varkappa}G(t)(t-z_0)^{-\varkappa}+g(t). \tag{35}$$

Assume

$$F(z)=\begin{cases}\phi(z), & z\in D^+,\\ \phi(z)(z-z_0)^{\varkappa}, & z\in D^-.\end{cases} \tag{36}$$

Then $F(z)$ has at the point $z=\infty$ the pole of order $\varkappa-1$. Hence, there is the polynomial $\Omega_{\varkappa-1}$ of order $\varkappa-1$ such that

$$\psi(z)=\big(F(z)-\Omega_{\varkappa-1}(z)\big]\in K^{p(\cdot)}(\Gamma). \tag{37}$$

The condition (35) yields

$$\psi^+(z)=G_0(t)\psi^-(t)+g_0(t), \tag{38}$$

where

$$\begin{aligned}G_0(t)&=|G(t)|e^{i[\arg G(t)-\varkappa\arg(t-z_0)]}|t-z_0|^{-\varkappa},\\ g_0(t)&=g(t)-\Omega_{\varkappa-1}(t)+G_0(t)\Omega_{\varkappa-1}(t).\end{aligned}$$

It can be easily shown that $\psi\in K^{p(\cdot)}(\Gamma)$, and $G_0\in\widetilde{A}(p(t),\Gamma)$. Using Theorem 6, we can conclude that the problem (38) is solvable if g_0 and Tg_0 belong to $L^{p(\cdot)}(\Gamma)$.

Since G_0 and $G_0\Omega_{\varkappa-1}$ are bounded functions, therefore $g_0\in L^{p(\cdot)}(\Gamma)$.

Let us show that $Tg_0=Tg-T\Omega_{\varkappa-1}+T(G_0\Omega_{\varkappa-1})$ belongs to $L^{p(\cdot)}(\Gamma)$. By our assumption, $Tg\in L^{p(\cdot)}(\Gamma)$. Putting $X_0(z)=X_{G_0}(z)$ for $T\Omega_{\varkappa-1}$, we have

$$T\Omega_{\varkappa-1}=X_0^+S_\Gamma\,\frac{\Omega_{\varkappa-1}}{X_0^+}\,,\quad G_0=\frac{X_0^+}{X_0^-}\,,\quad X_0(\infty)=a\neq0.$$

Since $\Omega_{\varkappa-1}$ is polynomial and $\frac{1}{X(z)}\in E^{1+\eta}(D^-)$, it follows that $\frac{\Omega_{\varkappa-1}(z)}{X_0(z)}\in E^1(D^+)$, and consequently, $S_\Gamma\,\frac{\Omega_{\varkappa-1}}{X_0^+}=\frac{\Omega_{\varkappa-1}}{X_0^+}$. Therefore $T\Omega_{\varkappa-1}=\Omega_{\varkappa-1}$.

Next,

$$T(G_0\Omega_{\varkappa-1})=X_0^+S_\Gamma\,\frac{\Omega_{\varkappa-1}G_0}{X_0^+}=X_0^+S\,\frac{\Omega_{\varkappa-1}}{X_0^-}\,.$$

The function $\Omega_{\varkappa-1}$ is constant if $\varkappa=1$; then assuming $\Omega_0=b$, we have

$$S_\Gamma\,\frac{\Omega_0}{X_0^-}=S_\Gamma\,\frac{b}{X_0^-}=S_\Gamma\Big(\frac{\Omega_0}{X_0^-}-\frac{b}{a}\Big)+S_\Gamma\,\frac{b}{a}=-\frac{b}{X_0^-}+\frac{2b}{a}\,,$$

that is, for $\varkappa=1$, we have $T\,\frac{\Omega_{\varkappa-1}}{X_0^+}=-bG_0+\frac{2b}{a}\,X_0^+$, and this function by Theorem 6 belongs to $L^{p(\cdot)}(\Gamma)$.

If $\varkappa - 1 \geq 1$, then there exists the polynomial $P_{\varkappa-2}$ of order $\varkappa - 2$ such that the function $\frac{\Omega_{\varkappa-1}}{\chi_0(z)} - P_{\varkappa-2}(z)$ in the domain D^- belongs to $E^1(D^-)$. Therefore

$$\begin{aligned}T(G_0\Omega_{\varkappa-1}) &= X_0^+ S\Big[\frac{\Omega_{\varkappa-1}}{X_0^-} - P_{\varkappa-2}\Big] + X_0^+ SP_{\varkappa-2} = \\ &= X_0^+\Big(-\Big(\frac{\Omega_{\varkappa-1}}{X_0^-} - P_{\varkappa-2}\Big)\Big) + X_0^+ P_{\varkappa-2} = \\ &= -G_0\Omega_{\varkappa-1} + X_0^+ P_{\varkappa-1} + X_0^+ P_{\varkappa-2} = -G_0\Omega_{\varkappa-1} + 2X_0^+ P_{\varkappa-2}.\end{aligned}$$

From the above, we can easily see that $T(G_0\Omega_{\varkappa-1})$ likewise belongs to $L^{p(\,\cdot\,)}(\Gamma)$. Thus g_0 and Tg_0 belong to $L^{p(\,\cdot\,)}(\Gamma)$, and the problem (38) is solvable in $K^{p(\,\cdot\,)}(\Gamma)$. Having solved it and getting back to $\phi(z)$, we successively get

$$\psi(z) = X_0(z) K_\Gamma\Big(\frac{g_0}{X_0^+}\Big)(z), \quad X_0(z) = \exp\Big\{K_\Gamma(\ln G_0)(z)\Big\},$$

$$K_\Gamma\,\frac{g_0}{X_0^+} = K_\Gamma\,\frac{g_0}{X_0^+} - \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^+(t)}\,\frac{dt}{t-z} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)}\,\frac{dt}{t-z}\,.$$

The last summands can be easily calculated:

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^+(t)}\,\frac{dt}{t-z} = \begin{cases}\dfrac{\Omega_{\varkappa-1}(z)}{X_0(z)}\,, z \in D^+, \\ 0, \qquad\qquad\qquad z \in D^-,\end{cases}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)}\,\frac{dt}{t-z} = \\ &= \frac{1}{2\pi i}\int\limits_\Gamma \Big[\frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)} - \Omega_{\varkappa-1}(t)\Big]\frac{dt}{t-z} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{t-z}\,dt = \\ &= \begin{cases}\Omega_{\varkappa-1}(z), & z \in D^+, \\ -\dfrac{\Omega_{\varkappa-1}(z)}{X_0(z)} + \Omega_{\varkappa-1}(z), & z \in D^-.\end{cases}\end{aligned}$$

Putting

$$X(z) = \begin{cases}X_0(z), & z \in D^+, \\ (z-z_0)^{-\varkappa} X_0(z), & z \in D^-,\end{cases} \quad X_0(z) = \exp(K_\Gamma \ln G_0)(z), \tag{39}$$

and take into (37) and (38), we obtain

$$\phi(z) = \frac{X(z)}{2\pi i}\int\limits_\Gamma \frac{g(t)}{X^+(t)}\,\frac{dt}{t-z} + X(z)\Omega_{\varkappa-1}(z).$$

7.5. **The case for $\varkappa < 0$.** In this case, the function $F(z)$ given by equality (36) belongs to $K^{p(\,\cdot\,)}(\Gamma)$, and $F^+ = G_0F^- + g$. Consequently, $F(z) = X_0(z)K_\Gamma(\frac{g}{X_0^+})(z)$. For the function $\phi(z) = (z - z_0)^{-\varkappa}F(z)$ in the domain D^- to belong to $E^1(D^-)$ (the fulfilment of this condition is necessary for $\phi(z) \in K^{p(\,\cdot\,)}(\Gamma)$), it is necessary that

$$\int\limits_\Gamma \frac{g(t)}{X_0^+(t)}\, t^k\, dt = 0, \quad k = 0, 1, \dots, |\varkappa| - 1. \tag{40}$$

If these conditions are fulfilled, then $\phi \in E^1(D^-)$, and since $\phi^- \in L^{p(\,\cdot\,)}(\Gamma)$, therefore

$$\phi(z) = -\frac{1}{2\pi i}\int\limits_\Gamma \frac{\phi^-}{t - z}\, dt, \quad z \in D^-.$$

Hence

$$\phi(z) = K_\Gamma(\phi^+ - \phi^-)(z) \in K^{p(\,\cdot\,)}(\Gamma).$$

Now we are ready to state the theorem on the solvability of the problem (1) in the class $K^{p(\,\cdot\,)}(\Gamma)$ when

$$g \in L^{p(\,\cdot\,)}(\Gamma), \quad Tg \in L^{p(\,\cdot\,)}(\Gamma). \tag{41}$$

But first we present one simple sufficient condition with respect to g which ensures belonging of the function Tg to the class $L^{p(\,\cdot\,)}(\Gamma)$.

Theorem 7. *Let the conditions* (25) *be fulfilled and* $\operatorname{ind} G = 0$. *If* $g \in \bigcup\limits_{\delta>0} L^{p(\,\cdot\,)+\delta}(\Gamma)$, *then* $Tg \in L^{p(\,\cdot\,)}(\Gamma)$.

Proof. Since $g \in \bigcup\limits_{\delta>0} L^{p(\,\cdot\,)+\delta}(\Gamma)$, there exists the number $\eta > 0$ such that $g \in L^{p(\,\cdot\,)+\eta}(\Gamma)$.

We divide Γ into the arcs γ_k so as to fulfil simultaneously the condition of the theorem and

$$\overline{p}_k - \underline{p}_k < \eta, \quad \text{where } \overline{p}_k = \sup_{t\in\gamma_k} p(t), \quad \underline{p}_k = \inf_{t\in\gamma_k} p(t).$$

Then for $t \in \gamma_k$ we have $\overline{p}_k < \underline{p}_k + \eta$, and hence,

$$p(t) + \eta > \underline{p}_k + \eta > \overline{p}_k.$$

Consequently, $g \in L^{\overline{p}_k}(\gamma_k)$. In addition, since

$$\sup_{t\in\gamma_k} \max(p(t), q(t)) \geq \sup_{t\in\gamma_k} \max p(t) = \overline{p}_k,$$

we find that $G \in A(\overline{p}_k, \gamma_k)$. Owing to this fact, the functions $X_k(z)$ given by equalities (19) belong to $L^{p(\,\cdot\,)}(\Gamma)$ (see Subsection 5.2) and moreover, $\operatorname{ind} G$ in $K^{\overline{p}_k}(\Gamma)$ equals zero. Consequently, the function $\phi(z)$ given by equality (32) belongs to classes $L^{p(\,\cdot\,)}(\gamma_k)$ from which it follows that $\phi \in K^{p(\,\cdot\,)}(\Gamma)$, that is, $\phi^+ \in L^{p(\,\cdot\,)}(\Gamma)$. But $\phi^+ = \frac{1}{2}(g + Tg)$. Hence, $Tg \in L^{p(\,\cdot\,)}(\Gamma)$. $\square$

From the above theorem follows

Statement 2. *If $g \in L^{p(\,\cdot\,)}(\Gamma)$ and the conditions* (25) *are fulfilled, then the function*

$$\phi(z) = X_G(z) \int\limits_{\Gamma} \frac{g(\tau)}{X_G^{+}(\tau)} \, \frac{d\tau}{\tau - z} \tag{42}$$

belongs to the class $K^{p(\,\cdot\,)-\delta}(\Gamma)$ for any $\delta \in (0, \underline{p})$.

To prove this, it suffices to notice that for $g \in L^{p(\,\cdot\,)}(\Gamma)$ we have $g \in L^{(p(\,\cdot\,)-\delta)+\delta}(\Gamma)$.

7.6. **The theorem below is a summation of results stated in Subsections 7.1–7.5.**

Theorem 8. *If the conditions* (25) *are fulfilled and $g \in L^{p(\,\cdot\,)}(\Gamma)$, then the Riemann problem has a solution ϕ* (*given by equality* (42)), *satisfying the condition $\phi \in \bigcap\limits_{\delta \in (0,\underline{p})} K^{p(\,\cdot\,)-\delta}(\Gamma)$.*

If, however, $G \in A(p(\,\cdot\,), \Gamma)$, then for the Riemann problem to be solvable in the class $K^{p(\,\cdot\,)}(\Gamma)$ for $\varkappa(G) \geq 0$, it is necessary and sufficient that the condition

$$Tg \in L^{p(\,\cdot\,)}(\Gamma) \tag{43}$$

is fulfilled.

When $\varkappa < 0$, for the solvability of the problem it is necessary and sufficient that the conditions (43) *and*

$$\int\limits_{\Gamma} \frac{g(t)}{X^{+}(t)} \, t^k \, dt = 0, \quad k = 0, 1 \ldots, |\varkappa - 1|$$

are fulfilled.

If the above-mentioned conditions are fulfilled, then the problem for $\varkappa \leq 0$ is uniquely solvable, but for $\varkappa > 0$ it is solvable unconditional. In all cases the solution is given by the equality

$$\phi(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(t)}{X^{+}(t)} \, \frac{dt}{t - z} + X(z)\Omega_{\varkappa-1}(z), \tag{44}$$

where $\Omega_{\varkappa-1}(z)$ is an arbitrary polynomial of order $\varkappa - 1$ ($\Omega_{\varkappa-1}(z) \equiv 0$ for $\varkappa \leq 1$), and $X(z)$ given by (26).

8. On the Noetherity of the Operator $M\varphi = a\varphi + bS_\Gamma\varphi$ in the Space $L^{p(\,\cdot\,)}(\Gamma)$

The results of Sections 3–7 do not allow us to establish Noetherity of the operator M in the space $L^{p(\,\cdot\,)}(\Gamma)$, when $G = (a-b)(a+b)^{-1} \in A(p(\,\cdot\,), \Gamma)$.

We intend to construct a space $\mathcal{L}^{p(t)}$ in which under sufficiently general assumptions with respect to Γ, p and G the operator M will be Noetherian.

As concerns the space $L^{p(\,\cdot\,)}(\Gamma)$, we can point out one necessary condition for the operator M to be Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$. This condition for $p \in \mathcal{P}(\Gamma)$ will be the same as for the constant p. We start with this result.

Theorem 9. *Let $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, $a(t)$, and $b(t)$ be measurable bounded on Γ functions. For the operator $M\varphi = a\varphi + bS_\Gamma\varphi$ to be Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$, it is necessary that the conditions*

$$\operatorname*{ess\,inf}_{t\in\Gamma} |a(t)+b(t)| > 0, \quad \operatorname*{ess\,inf}_{t\in\Gamma} |a(t)-b(t)| > 0 \tag{45}$$

are fulfilled.

Proof. Let us consider in $L^{p(\,\cdot\,)}(\Gamma)$ the equation

$$M\varphi = f, \quad f \in L^{p(\,\cdot\,)}(\Gamma). \tag{46}$$

Let $\phi(z) = (K_\Gamma\varphi)(z)$, where φ is a solution of equation (46). By the Sokhotskii–Plemelj formulas, $\varphi = \phi^+ - \phi^-$, $S_\Gamma\varphi = \phi^+ + \phi^-$. Therefore, (46) can be written in the form

$$(a+b)\phi^+ + (b-a)\phi^- = f.$$

Assuming $c = a+b$, $d = b-a$, we obtain

$$c\phi^+ + d\phi^- = f. \tag{47}$$

Assume now to the contrary that M is Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$ and, for example,

$$\operatorname{ess\,inf} |a+b| = \operatorname{ess\,inf} |c| = 0. \tag{48}$$

Since the operator under small perturbations preserves Noetherity ([4, p. 144]), there exists the number $\varepsilon > 0$ such that: if the operator $M_1\varphi = c_1\varphi + d_1S_\Gamma\varphi$ is Noetherian and $\|M - M_1\|_{p(\,\cdot\,)} < \varepsilon$, then M_1 is likewise Noetherian.

Let $\eta < \frac{\varepsilon}{1+\|S_\Gamma\|_{p(\,\cdot\,)}}$. Consider the functions

$$\begin{aligned} c_1(t) &= \begin{cases} c(t) & \text{if } |c(t)| \geq \eta, \\ 0 & \text{if } |c(t)| < \eta, \end{cases} \\ d_1(t) &= \begin{cases} d(t) & \text{if } |d(t)| \geq \eta, \\ 0 & \text{if } |c(t)| < \eta. \end{cases} \end{aligned} \tag{49}$$

Obviously,

$$\begin{aligned} \|M\varphi - M_1\varphi\|_{p(\,\cdot\,)} &\leq \eta\|\varphi\|_{p(\,\cdot\,)} + 2\eta\|S\varphi\|_{p(\,\cdot\,)} < \\ &< 2\eta\big(1 + \|S\|_{p(\,\cdot\,)}\big)\|\varphi\|_{p(\,\cdot\,)} < \varepsilon\|\varphi\|_{p(\,\cdot\,)}, \end{aligned}$$

therefore the operator M_1 is Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$. Let us show that the equation

$$M_1\varphi = 0 \tag{50}$$

has only a zero solution. Towards this end, we notice that $|d_1| > 0$ on Γ, and $c_1 = 0$ on the set e of positive measure, where $\operatorname{mes} e < \operatorname{mes} \Gamma$. Indeed, if $\operatorname{mes} e = \operatorname{mes}\Gamma$, then $d_1\phi \equiv 0$ on Γ, hence $\phi^- \equiv 0$ on Γ. Then any function of the type $\int\limits_\Gamma \frac{F^+(\tau)}{\tau - t}\,d\tau$, where $F \in E^1(D^+)$ with a boundary value $F^+ \in L^{p(\,\cdot\,)}(\Gamma)$ will be a solution of equation (50). Sets of such functions are of infinite dimension, hence M_1 is not Noetherian. Thus $\operatorname{mes} e < \operatorname{mes}\Gamma$, and hence $\operatorname{mes}(\Gamma \setminus e) > 0$.

On e, we now have $d_1\phi^- = 0$, and then $\phi^- = 0$ on e. By the theorem on the uniqueness of analytic functions (see, e.g., [27, p. 232]), $\phi^- = 0$ on Γ. Consequently, on $\Gamma \setminus e$ we have $c_1 \neq 0$ and $c_1\phi^+ = 0$. Again, by the uniqueness theorem, we conclude that $\phi^+ = 0$ on Γ. Finally, we obtain that on Γ both ϕ^- and ϕ^+ are equal to zero. This implies that $\varphi = \phi^+ - \phi^- = 0$. Thereby, equation (50) has only a zero solution. Hence $M_1\varphi = 0$ has only a zero solution and the operator M_1 is Noetherian one. Since $|d_1| > \eta > 0$, the operator $\widetilde{M} = c_1(d_1)^{-1}\phi^+ + \phi^-$ together with $c_1\phi_1^+ + d_1\phi^-$ is likewise Noetherian, and $\widetilde{M}\varphi$ has only a zero solution. In addition, the coefficient c_1/d_1 on e equals zero and is different from zero on $\Gamma \setminus e$; both sets are of positive measure. Therefore, also for the operator $(\widetilde{M})^*$ we have $\dim N((\widetilde{M})^*) = 0$ (this case for a variable $p(t)$ is proved in the same way as Lemma 4.1 in [4] on pages 292-3 for a constant p). Since the operators $\widetilde{M}$ and $\widetilde{M}^*$ are Noetherian, this implies that they are invertible. Owing to this fact, the equation $\frac{c_1}{d_1}\phi^+ + \phi^- = g$ should have a solution in $L^{p(\cdot)}(\Gamma)$ for any function $g \in L^{p(\cdot)}(\Gamma)$.

Let us show that this is not true.

Let $f = 1$, then $c_1\phi^+ + d_1\phi^- = d_1$, $t \in \Gamma$. But for $t \in e$, we get $0 + d_1\phi^- = d_1$, i.e., $\phi^-(t) \equiv 1$. If $F(z) = \phi(z) - 1$, then $F \in K^{p(\cdot)}(\Gamma)$. Hence $F(z)$ belongs to $E^1(D^-)$, and $F^-(t) = 0$, $t \in e$, whence it follows that $\phi(z) = 1$, $z \in D^-$, and $\phi(\infty) = 1$, as well. But this is impossible due to $\phi \in K^{p(\cdot)}(\Gamma)$, and for such functions we have $\phi(\infty) = 0$.

The obtained contradiction shows that the assumption (48) is invalid, hence $\operatorname{ess\,inf}\{a(t) + b(t)\} > 0$.

The validity of the second inequality in (48) can be proved analogously. □

As a conclusion, it should be noted that in proving the lemma we have followed the method suggested in [4, pp. 256-8].

9. The Space $\mathcal{L}^{p(\cdot)}$

9.1. **Definition of $\mathcal{L}^{p(\cdot)}$; its Banachity.** Let

$$\Gamma \in R, \ \ p \in \mathcal{P}(\Gamma), \ \ G \in A(p(\cdot)). \tag{51}$$

Assume

$$g \in L^{p(\cdot)}, \ \ Tg \in L^{p(\cdot)}, \ \ T\Big(g_k \frac{1}{G}\Big) \in L^{p(\cdot)}, \ \ k = 1, 2, \tag{52}$$

where

$$g_1 = \frac{1}{2}(g + Tg), \quad g_2 = \frac{1}{2}(-g + Tg). \tag{53}$$

It follows from (52) that

$$g_k \in L^{p(\cdot)}, \ \ k = 1, 2. \tag{54}$$

Let

$$\mathcal{L}^{p(\cdot)}=\Big\{g:\ g\in L^{p(\cdot)},\ Tg\in L^{p(\cdot)},\ T\Big(g_k\,\frac{1}{G}\Big)\in L^{p(\cdot)}\Big\}. \tag{55}$$

For the elements from $\mathcal{L}^{p(\cdot)}$ we introduce the norm as follows:

$$\|g\|_{\mathcal{L}^{p(\cdot)}}=\|g\|_{p(\cdot)}+\|Tg\|_{p(\cdot)}+\Big\|Tg_1\,\frac{1}{G}\Big\|_{p(\cdot)}+\Big\|Tg_2\,\frac{1}{G}\Big\|_{p(\cdot)}. \tag{56}$$

The set $\mathcal{L}^{p(\cdot)}$ together with the above-introduced norm, i.e.,

$$\mathcal{L}^{p(\cdot)}=\big\{g:\ \|g\|_{\mathcal{L}^{p(\cdot)}}<\infty\big\}$$

turns into a linear normalized space.

Lemma 5. *If the conditions* (51) *are fulfilled, than $\mathcal{L}^{p(\cdot)}$ is a complete space.*

Proof. Let $\{g^n\}$ be the fundamental sequence in $\mathcal{L}^{p(\cdot)}$, then it follows from (55) that the sequences $\{g^n\}$, $\{Tg^n\}$, $\big\{T(g_k^n\,\frac{1}{G})\big\}$, $k=1,2$, are fundamental in $L^{p(\cdot)}$. Let μ, λ, e, ψ be the functions from $L^{p(\cdot)}$ to which these sequences converge, respectively, i.e.,

$$\begin{gathered}\|g^n-\mu\|_{p(\cdot)}\to 0,\ \ \|Tg^n-\lambda\|_{p(\cdot)}\to 0,\\ \Big\|T\Big(g_1^n\,\frac{1}{G}\Big)-e\Big\|_{p(\cdot)}\to 0,\ \ \Big\|T\Big(g_2^n\,\frac{1}{G}\Big)-\psi\Big\|_{p(\cdot)}\to 0.\end{gathered} \tag{57}$$

Since T is continuous from $L^{p(\cdot)}$ to the space $M(\Gamma)$, Tg^n converges in measure to $T\mu$, and hence

$$\lambda=T\mu \tag{58}$$

Next, since $g_1^n=\frac{1}{2}\,(g^n+Tg^n)$, $\{g_1^n\}$ converges in $L^{p(\cdot)}$ and in measure to $\frac{1}{2}\,(\mu+\lambda)$, and owing to the fact that $\frac{1}{G}$ is bounded, we conclude that the sequences $\big\{g_k^n\,\frac{1}{G}\big\}$, $k=1,2$, converge in $L^{p(\cdot)}$, respectively, to $\frac{1}{2}\,(\mu+\lambda)\,\frac{1}{G}$ and to $\frac{1}{2}(-\mu+\lambda)\,\frac{1}{G}$. This implies that

$$e=\frac{1}{2}\,\bigg(\mu+T\mu+T\Big(\mu_1\,\frac{1}{G}\Big)+T\Big(\mu_2\,\frac{1}{G}\Big)\bigg), \tag{59}$$

$$\mu_1=\mu+\lambda,\ \ \mu_2=-\mu+\lambda,$$

$$\psi=\frac{1}{2}\,\bigg(\mu+T\mu-T\Big(\mu_1\,\frac{1}{G}\Big)+T\Big(\mu_2\,\frac{1}{G}\Big)\bigg), \tag{60}$$

and from (56)–(59) we conclude that

$$\|g^n-\mu\|_{\mathcal{L}^{p(\cdot)}}\to 0. \qquad\square$$

9.2. **The necessary condition for the operator M to be Noetherian in $\mathcal{L}^{p(\cdot)}$.** Let us show that the analogue of Theorem 9 is valid for the operator M to be Noetherian in $\mathcal{L}^{p(\cdot)}$.

Theorem 10. *Let $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, and let a and b be bounded measurable on Γ functions, then for the operator $M = a\varphi + bS\varphi$ to be Noetherian in $\mathcal{L}^{p(\,\cdot\,)}$, it is necessary that the conditions* (45) *or, what comes to the same thing, the condition*

$$\operatorname{ess\,inf} |a^2 - b^2| > 0$$

is fulfilled.

Proof. We proceed from the proof of Theorem 9. Tracing its proof, we conclude that we have used the following facts:

(1) $L^{p(\,\cdot\,)}$ is the Banach space;
(2) the set of Noetherian operators in the Banach space (and hence in $L^{p(\,\cdot\,)}(\Gamma)$), is open;
(3) equation (50) in $L^{p(\,\cdot\,)}$ has only a zero solution;
(4) if two analytic functions have in the domain G the same angular boundary values on the set of positive measure, then they are equal everywhere in G;
(5) the function $f \equiv 1$ belongs to $L^{p(\,\cdot\,)}$.

In the case under consideration:

(1′) $\mathcal{L}^{p(\,\cdot\,)}$ is the Banach space;
(2′) since $\mathcal{L}^{p(\,\cdot\,)}$ is the Banach space, the set of Noetherian operators is open;
(3′) equation (50) has in $\mathcal{L}^{p(\,\cdot\,)}$ only a zero solution, since in a wider space $L^{p(\,\cdot\,)}$ it has only a zero solution;
(4′) the theorem on the uniqueness of analytic functions is applicable;
(5′) the function $f \equiv 1$ belongs to $\mathcal{L}^{p(\,\cdot\,)}$;

By virtue of statements (1′)–(5′), repeating the same arguments as in proving Theorem 9, we find that Theorem 10 is likewise valid. ☐

10. Solution of Equation $M\varphi = f$ in the Space $\mathcal{L}^{p(\,\cdot\,)}$

10.1. **The case $\varkappa = 0$.** Assume that the conditions (51)–(52) with

$$G = \frac{a-b}{a+b} \in A(p(\,\cdot\,)) \tag{61}$$

and

$$\operatorname{ess\,inf} |a^2 - b^2| > 0 \tag{62}$$

are fulfilled, and the equation

$$M\varphi = a\varphi + bS\varphi = f, \quad f(a+b)^{-1} \in \mathcal{L}^{p(\,\cdot\,)}. \tag{63}$$

This equation is equivalent to the following Riemann problem:

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \quad g(t) = \frac{f(t)}{a(t)+b(t)} \tag{64}$$

in the class $K\mathcal{L}^{p(\,\cdot\,)}$, i.e., in the class of Cauchy type integrals with density from $\mathcal{L}^{p(\,\cdot\,)}$.

Indeed, if $\phi = K_\Gamma\varphi$ (where $\varphi \in \mathcal{L}^{p(\cdot)}$) is a solution of the problem (64), then it can be easily verified that φ is a solution of equation (63) of the class $\mathcal{L}^{p(\cdot)}$.

Conversely, if φ is a solution of equation (63) of the class $\mathcal{L}^{p(\cdot)}$, then $\phi = K_\Gamma\varphi \in K\mathcal{L}^{p(\cdot)}$, and it satisfies the condition (64).

Lemma 6. *If for Γ, p and G the conditions* (25) *are fulfilled and the functions g_1 and g_2 are defined by equalities* (53)*, then the equalities*

$$Tg_1 = g_1, \quad Tg_2 = -g_2 \tag{65}$$

are valid.

Proof. Follows immediately from the equality $T(Tg) = g$, valid due to the conditions (25) (see Theorem 4). □

Lemma 7. *If there take place the inclusions* (51)–(52) *and* $\operatorname{ind} G = \operatorname{ind} \frac{a-b}{a+b} = 0$*, then equation* (63) *is uniquely solvable in the class $\mathcal{L}^{p(\cdot)}$, and a solution is given by the equality*

$$\varphi = g_1 - \frac{g_2}{G}\,,$$

where

$$g_1 = \frac{1}{2}\,(g + Tg), \quad g_2 = \frac{1}{2}\,(-g + Tg), \quad g = \frac{f}{a+b}\,. \tag{66}$$

Proof. By virtue of Theorem 8, the problem (64) in $L^{p(\cdot)}$ has a unique solution

$$\phi(z) = X(z)\Big[K_\Gamma\Big(\frac{g}{X^+}\Big)\Big](z). \tag{67}$$

By the Sokhotskii–Plemelj formulas, we obtain

$$\phi^+ = \frac{1}{2}\,(g + Tg) = g_1, \quad \phi^- = \frac{1}{2G}\,(-g + Tg) = \frac{g_2}{G}\,. \tag{68}$$

Since $\operatorname{ind} G = 0$, therefore $\phi \in E^1(D^\pm)$, and hence

$$\phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\phi^+(t) - \phi^-(t)}{t-z}\,dt = \frac{1}{2\pi i}\int\limits_\Gamma \frac{g_1 - \frac{g_2}{G}}{t-z}\,dt. \tag{69}$$

Thereby, the only possible solution of equation (52) is the function

$$\varphi = g_1 - \frac{g_2}{G}\,. \tag{70}$$

Let us prove that $\varphi \in \mathcal{L}^{p(\cdot)}$, i.e., that

$$\varphi \in L^{p(\cdot)},\ \ Tg \in L^{p(\cdot)},\ \ T\Big(\frac{\varphi_k}{G}\Big) \in L^{p(\cdot)},\ \ k = 1, 2. \tag{71}$$

From the assumptions $g \in L^{p(\cdot)}$, $Tg \in L^{p(\cdot)}$, $\frac{1}{G} \in L^\infty$, it follows that

$$\varphi \in L^{p(\cdot)}. \tag{72}$$

Further, due to (65) and (70),

$$T\varphi = Tg_1 - T\,\frac{g_2}{G} = \Big(g_1 - T\,\frac{g_2}{G}\Big) \in L^{p(\cdot)}. \tag{73}$$

To study $T\,\frac{\varphi_1}{G}$, we first note that

$$\begin{aligned}\varphi_1 &= \frac{1}{2}\,(\varphi+T\varphi)=\frac{1}{2}\Big(g_1-\frac{g_2}{G}+Tg_1-T\,\frac{g_2}{G}\Big)=\\ &=\frac{1}{2}\,(g_1+Tg_1)-\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big)=g_1-\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big), \qquad (74)\\ \varphi_2 &= \frac{1}{2}\,(-\varphi+T\varphi)=\frac{1}{2}\Big(-g_1-\frac{g_2}{G}+Tg_1+T\,\frac{g_2}{G}\Big)=\\ &=\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big). \qquad (75)\end{aligned}$$

It follows from (52) and (70) that $\varphi_k\in L^{p(\,\cdot\,)}$.

Now, we have

$$\begin{aligned}T\,\frac{\varphi_1}{G} &= T\,\frac{g_1}{G}-\frac{1}{2}\bigg(T\,\frac{g_2}{G}\cdot\frac{1}{G}+T\Big(T\,\frac{g_2}{G}\Big)\,\frac{1}{G}\bigg)=\\ &=T\,\frac{g_1}{G}-\frac{1}{2}\Big(T\,\frac{g_1}{G}\cdot\frac{1}{G}+\frac{g_2}{G}\cdot\frac{1}{G}\Big), \qquad (76)\\ T\,\frac{\varphi_2}{G} &= \frac{1}{2}\bigg(T\,\frac{g_2}{G}\cdot\frac{1}{G}+\Big(T\,\frac{g_2}{G}\Big)\,\frac{1}{G}\bigg)=\\ &=\frac{1}{2}\Big(T\,\frac{g_2}{G}\cdot\frac{1}{G}+\frac{g_2}{G}\cdot\frac{1}{G}\Big). \qquad (77)\end{aligned}$$

Taking into account (70), relying on (76) and (77), we conclude that

$$T\,\frac{\varphi_1}{G},T\,\frac{\varphi_2}{G}\in L^{p(\,\cdot\,)}. \qquad (78)$$

The inclusions (72),(73) and (78) imply that the inclusion (71) is valid, and hence $\varphi\in\mathcal{L}^{p(\,\cdot\,)}$. □

10.2. **The case $\varkappa>0$.** Since $Tg\in L^{p(\,\cdot\,)}$, all possible solutions of the problem (64) lie in the set

$$\phi(z)=X(z)\Big(K_\Gamma\,\frac{g}{X^+}\Big)(z)+P_{\varkappa-1}(z)X(z)$$

(see item 7.4). The first summand here belongs to $K\mathcal{L}^{p(\,\cdot\,)}$ (see item 7.3). Let us show that the second summand likewise belongs to $K\mathcal{L}^{p(\,\cdot\,)}$.

Since $X(t)$ has at infinity zero of order $\varkappa$, $P_{\varkappa-1}(z)X(z)$ is representable by the Cauchy integral in the domains D^+ and D^-.

Lemma 8. *The function*

$$\varphi(t)=\big[X^+(t)-X^-\big]P_{\varkappa-1}(t)$$

satisfies the conditions (52), *and hence* $\varphi\in\mathcal{L}^{p(\,\cdot\,)}$.

Proof. Since $X^+,X^-\in L^p$ (see Theorem 3), $\varphi\in L^{p(\,\cdot\,)}$.

Further,

$$T\varphi = TX^+P - TX^-P = X^+S_\Gamma \frac{X^+P}{X^+} - X^+S_\Gamma \frac{X^-P}{X^+} =$$
$$= X^+P - X^+S_\Gamma \frac{P}{G}. \tag{79}$$

Here, for the multiplier $S_\Gamma \frac{P}{G}$, we have

$$S_\Gamma \frac{P}{G} = \int\limits_\Gamma \frac{P(\tau)}{G(\tau)} \frac{d\tau}{\tau - t} = \int\limits_\Gamma \frac{1}{G(\tau)} \frac{P(\tau) - P(t)}{\tau - t}\, d\tau + P(t) S \frac{1}{G}. \tag{80}$$

By virtue of the inclusion (11), we find that $X^+ \in L^{p(\,\cdot\,)+\eta}$ (see Corollary of Theorem 3). Next, the first summand in equality (80) is a bounded function; moreover, since $\Gamma \in R$ and $\frac{1}{G} \in L^\infty$, we have $S \frac{1}{G} \in \bigcap\limits_{s>1} L^s$. Then $PS \frac{1}{G} \in L^{p(\,\cdot\,)}$, and since $X^+ \in L^{p(\,\cdot\,)+\eta}$, therefore $X^+S_\Gamma \frac{1}{G} \in L^{p(\,\cdot\,)}$, as well. By virtue of (80), we can conclude from (79) that $T\varphi \in L^{p(\,\cdot\,)}$.

Further,

$$2\varphi_1 = \varphi + T\varphi, \quad 2\varphi_2 = -\varphi + T\varphi,$$

that is,

$$2\varphi_1 = (X^+ - X^-)P, \quad 2\varphi_2 = (-X^+ + X^-)P$$

and hence

$$\frac{2\varphi_1}{G} = \frac{X^+}{G}\Big(1 - \frac{X^-}{X^+}\Big)P = X^-\Big(1 - \frac{1}{G}\Big)P = X^-P\,\frac{G-1}{G},$$
$$\frac{2\varphi_2}{G} = \frac{X^+}{G}\Big(-1 + \frac{1}{G}\Big) = X^-P \frac{1-G}{G},$$

from which we get

$$2T\frac{\varphi_1}{G} = X^+S\,\frac{X^-P}{X^+}\,\frac{G-1}{G} = X^+S_\Gamma \frac{P}{G} - X^+S_\Gamma \frac{P}{G^2} =$$
$$= X^+ \int\limits_\Gamma \frac{P(\tau) - P(t)}{\tau - t}\, d\tau + X^+PS_\Gamma \frac{1}{G^2} =$$
$$= X^+ \int\limits_\Gamma \frac{P(\tau) - P(t)}{G(\tau - t)}\, d\tau + X^+PS_\Gamma \frac{1}{G} -$$
$$- X^+ \int\limits_\Gamma \frac{P(\tau) - P(t)}{G62(\tau - t)}\, d\tau + X^+PS_\Gamma \frac{1}{G^2} =$$
$$= X^+ \int\limits_\Gamma \Big(-\frac{1}{G^2} + \frac{1}{G}\Big) Q(\tau, t)\, d\tau + X^+P\Big(S_\Gamma \frac{1}{G^2} + S_\Gamma \frac{1}{G}\Big), \tag{81}$$
$$Q(\tau, t) = \frac{P(\tau) - P(t)}{\tau - t} = a_0\tau^{n-1} + a_1\tau^{n-2} + \cdots + a_n, \;\; n = \varkappa - 1.$$

It can be easily seen that $T\,\frac{\varphi}{G} \in L^{p(\,\cdot\,)}$ if the function $X^+ S_\Gamma\,\frac{1}{G^2}$ belongs to $L^{p(\,\cdot\,)}$. Since $\frac{1}{G^2} \in L^\infty$ and $\Gamma \in R$, we have $S_\Gamma\,\frac{1}{G^2} \in \bigcap\limits_{\nu>1} L^\nu$. Moreover, $X^+ \in L^{p(\,\cdot\,)+\varepsilon}$ (see Corollary of Theorem 3). These two fact allow us to conclude that

$$X^+ S_\Gamma \frac{1}{G^2} \in L^{p(\,\cdot\,)}. \tag{82}$$

From (81), it now follows that $T\,\frac{\varphi_1}{G} \in L^{p(\,\cdot\,)}$. Analogously, we can prove that $T\,\frac{\varphi_2}{G^2} \in L^{p(\,\cdot\,)}$.

Thus we have proved that for φ the conditions (52) are fulfilled, and hence $\varphi \in \mathcal{L}^{p(\,\cdot\,)}$. □

10.3. **The case $\varkappa < 0$.**

Lemma 9. *If the conditions* (45), (51)–(52) *are fulfilled, and* $\varkappa < 0$, *then for equation* (62) *to be solvable in the class* $\mathcal{L}^{p(\,\cdot\,)}$, *it is necessary and sufficient that*

$$\int\limits_\Gamma \frac{f(\tau)}{a(\tau)+b(\tau)}\,\frac{\tau^k}{X^+(\tau)}\,d\tau = 0, \quad k = 0,1,\dots,|-\varkappa|-1. \tag{83}$$

Proof. In the case under consideration, $X(z)$ has at infinity a pole of order $|\varkappa|$, therefore the only possible solution of equation (62) may be only the function $\varphi(t) = \phi^+(t) - \phi^-(t)$, where $\phi(z) = X(z)(K_\Gamma\,\frac{g}{X^+})(z)$, $g(t) = \frac{f(t)}{a(t)+b(t)}$. But the function $\varphi(t)$ belongs to $K\mathcal{L}^{p(\,\cdot\,)}$, if and only if $\phi(z) \in E^1(D^\pm)$, i.e., when the function $(K_\Gamma\,\frac{f}{a+b})$ at the point $z = \infty$ has zero of order $|\varkappa|$. Thus it is necessary and sufficient that equalities (83) are fulfilled. And if this condition is fulfilled, the solution is unique and given by the equality

$$\varphi = \frac{1}{2}\Big(\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) - \frac{1}{2G}\Big(-\frac{f}{a+b} + T\,\frac{f}{a+b}\Big). \tag{84}$$

□

10.4. **Summation of results stated in items 10.1–10.3.**

Theorem 11. *Let* Γ *be a simple, closed, rectifiable curve* $p \in \mathcal{P}(\Gamma)$, *and let* $a(t)$ *and* $b(t)$ *be bounded measurable on* Γ *functions such that*

$$\operatorname{ess\,inf}\big|a^2(t) - b^2(t)\big| > 0$$

and $G(t) = (a(t)-b(t))(a(t)+b(t))^{-1}$. *If for* Γ, p *and* G *the conditions* (25) *are fulfilled.*

Then the equation

$$M\varphi = a(t)\varphi(t) + b(t)(S_\Gamma\varphi)(t) = f(t), \quad \frac{f(t)}{a(t)+b(t)} \in \mathcal{L}^{p(\,\cdot\,)}$$

for $\varkappa = \varkappa(G) \ge 0$ *is solvable in the class* $\mathcal{L}^{p(\,\cdot\,)}(\Gamma)$; *for* $\varkappa = 0$, *it is unique and for* $\varkappa > 0$, *the homogeneous equation has* $\varkappa$ *linearly independent solutions.*

If $\varkappa < 0$, for the equation $M\varphi = f$ to be solvable in the class $\mathcal{L}^{p(\cdot)}(\Gamma)$, it is necessary and sufficient that the conditions (83) *are fulfilled.*

In all cases when a solution exists, it is given by the equality

$$\varphi(t) = \frac{1}{2}\Big(\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) - \frac{1}{2G}\Big(-\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) + \\ + (X^+ - X^-)P_{\varkappa-1} \tag{85}$$

($P_\nu \equiv 0$, if $\nu < 0$).

11. The Spaces $\widetilde{\mathcal{L}}^{p(t)}$ and $(\mathcal{L}^{p(t)})^*$

11.1. **Definition and some properties of the space $\widetilde{\mathcal{L}}^{p(\cdot)}$.** Let $\psi \in L^{p(\cdot)}$, and X be the function given by equality (44). Assume

$$\begin{gathered} \widetilde{T}\psi = \frac{1}{X^+}\,S(X^+\psi), \\ \widetilde{\mathcal{L}}^{p(\cdot)} = \big\{\psi:\ \psi \in L^{p(\cdot)},\ \widetilde{T}\psi \in L^{p(\cdot)}\big\}. \end{gathered} \tag{86}$$

For the functions $\psi \in \widetilde{\mathcal{L}}^{p(\cdot)}$ we introduce the norm

$$\|\psi\|_{\widetilde{\mathcal{L}}^{p(\cdot)}} = \|\psi\|_{p(\cdot)} + \|\widetilde{T}\psi\|_{p(\cdot)}. \tag{87}$$

Due to the continuity of the operator T from $\mathcal{L}^{p(\cdot)}$ to the space of convergence in measure, we can easily prove

Lemma 10. *If $\Gamma \in J^*$, $p \in \mathcal{P}(\Gamma)$, $G \in \widetilde{A}(p(\cdot))$, then the operator $\widetilde{T}$ is continuous from $\widetilde{\mathcal{L}}^{p(\cdot)}$ to the space of convergence in measure.*

Lemma 11. *$\widetilde{\mathcal{L}}^{p(\cdot)}$ is the complete, linear, normalized space.*

Proof runs in the same way as that of Lemma 5.

11.2. **The spaces ℓ_1 and ℓ_2.** Assume

$$\begin{aligned} \ell_1 &= \big\{\psi:\ \psi \in \mathcal{L}^{p(\cdot)},\ T\psi = \psi\big\},\ \ \|\psi\|^1 = \|\psi\|_{p(\cdot)}, \\ \ell_2 &= \big\{\psi:\ \psi \in \mathcal{L}^{p(\cdot)},\ T\psi = -\psi\big\},\ \ \|\psi\|^2 = \|\psi\|_{p(\cdot)}. \end{aligned} \tag{88}$$

Lemma 12. *ℓ_k, $k = 1, 2$, are closed subspaces of the space $L^{p(\cdot)}$.*

Proof. Let $\psi_n \in \ell_k$ and $\{\psi_n\}$ be the fundamental sequence in $L^{p(\cdot)}$, then there exists $\psi_0 \in L^{p(\cdot)}$ such that $\|\psi_n - \psi_0\|_{p(\cdot)} \to 0$. Let us prove that $\psi_0 \in \ell_k$.

Assuming for the definiteness that $k = 1$, then $T\psi_k = \psi_k$, and hence $\{T\psi_k\}$ converges in $L^{p(\cdot)}$ to ψ_0. By statement 1, $\{T\psi_k\}$ converges in measure to $T\psi_0$. Hence $\psi_0 = T\psi_0 \in \ell_1$. Consequently, ℓ_1 is closed in $L^{p(\cdot)}$.

The closure of ℓ_2 in $L^{p(\cdot)}$ is proved analogously. □

Lemma 13.

$$\mathcal{L}^{p(\cdot)} = \ell_1 \oplus \ell_2. \tag{89}$$

Proof. Let $\psi \in \mathcal{L}^{p(\cdot)}$; obviously,

$$\psi = \frac{1}{2}(\psi + T\psi) + \frac{1}{2}(\psi - T\psi) = \psi_1 + \psi_2, \tag{90}$$

where $\psi_1 = \frac{1}{2}(\psi + T\psi)$ and $\psi_2 = \frac{1}{2}(-\psi + T\psi)$. We have

$$T\psi_1 = \frac{1}{2}(T\psi + \psi) = \psi_1, \quad T\psi_2 = \frac{1}{2}(T\psi - \psi) = -\psi_2.$$

This implies that $\psi_k \in \ell_k$.

If $\psi = \mu_1 + \mu_1$, $\mu_k \in \ell_k$, then $\psi_1 - \mu_1 = \psi_2 - \mu_2$, where $\psi_k - \mu_k \in \ell_k$. Thereby, $(\psi_k - \mu_k) \subset \ell_1 \cap \ell_2$. But it can be easily verified that $\ell_1 \cap \ell_2 = \{0\}$. Indeed, if $\psi \in \ell_1 \cap \ell_2$, then $T\psi = \psi$ and $T\psi = -\psi$, i.e., $\psi = -\psi$, and hence $\psi = 0$.

Thus, for any $\psi \in \mathcal{L}^{p(\cdot)}$, the unique representation of type (90) with $\psi_k \in \ell_k$ is valid. This means that equality (89) is valid. □

11.3. **The space $(\mathcal{L}^{p(\cdot)})^*$.** Since $\mathcal{L}^{p(\cdot)} = \ell_1 \oplus \ell_2$, then following [30, p. 103], we have

$$(\mathcal{L}^{p(\cdot)})^* = \ell_1^* \oplus \ell_2^*.$$

Lemma 14. *Every linear continuous functional $\Lambda \in (\mathcal{L}^{p(\cdot)})^*$ generates the linear continuous functional $\widehat{\Lambda}$ from $(L^{p(\cdot)})^*$.*

Proof. We denote the narrowing of the functional Λ on ℓ_k by Λ_k (i.e., $\Lambda_k f = \Lambda f$, when $f \in \ell_k$).

Since ℓ_k is the closed subspace of the space $L^{p(\cdot)}$, there exists the linear, continuous functional Λ_k on $L^{p(\cdot)}$ such that $\widehat{\Lambda}_k f = \Lambda f$ when $f \in \ell_k$ (see e.g., [31, p. 72]).

Assume

$$\widehat{\Lambda} = \widehat{\Lambda}_1 + \widehat{\Lambda}_2.$$

By the continuity of functionals $\widehat{\Lambda}_k$, we conclude that $\widehat{\Lambda}$ is the linear, continuous functional on $L^{p(\cdot)}$.

If $f \in \mathcal{L}^{p(\cdot)}$, then $f = f_1 + f_2$, $f_k \in \ell_k$, therefore

$$\begin{aligned}\widehat{\Lambda} f = \widehat{\Lambda}_1 f + \widehat{\Lambda}_2 f = \widehat{\Lambda}_1(f_1 + f_2) + \widehat{\Lambda}_2(f_1 + f_2) = \\ = \widehat{\Lambda}_1 f + \widehat{\Lambda}_1 f_2 + \widehat{\Lambda}_2 f_1 + \widehat{\Lambda}_2 f_2.\end{aligned} \tag{91}$$

Before going further, we need the following

Lemma 15. *The equalities*

$$\widehat{\Lambda}_1 f_2 = 0, \quad f_2 \in \ell_2, \quad \widehat{\Lambda}_2 f_1 =, \quad f_1 \in \ell_1, \tag{92}$$

are valid.

Proof. Let $f = f_1 + f_2$, then $\widehat{\Lambda} f_1 = \widehat{\Lambda}_1 f_1 + \widehat{\Lambda}_2 f_1$, $\widehat{\Lambda} f_2 = \widehat{\Lambda}_1 f_2 + \widehat{\Lambda}_2 f_2$. By the definition of functionals Λ_k, we have $\widehat{\Lambda}_1 f_1 = \Lambda f_1$ and $\widehat{\Lambda}_2 f_2 = \Lambda f_2$. By virtue of the above-said, from the last equalities we arrive at equalities (92). □

We can now complete the proof of Lemma 14. Equalities (91) yield

$$\widehat{\Lambda} f=\widehat{\Lambda}_1 f_1+\widehat{\Lambda}_2 f_2=\Lambda f_1+\Lambda f_2=\Lambda(f_1+f_2)=\Lambda f,$$

i.e., $\widehat{\Lambda}$ is an extension of the functional Λ on $\mathcal{L}^{p(\cdot)}$ to the functional on $L^{p(\cdot)}$.

For the functional $\widehat{\Lambda}$ from Lemma 14, we have

$$\widehat{\Lambda} f=\int\limits_\Gamma f\mu\,dt, \tag{93}$$

where $\mu\in L^{p'(\cdot)}$ (since $(L^{p(\cdot)})^*=L^{p'(\cdot)}$, (see item 2.3.2). □

Lemma 16. *The function μ in equality* (93) *belongs to $L^{p'(\cdot)}$.*

Proof. We have

$$\widehat{\Lambda}\psi=\int\limits_\Gamma(\psi_1+\psi_2)\mu\,dt=\int\limits_\Gamma\psi_1\mu\,dt+\int\limits_\Gamma\psi_2\mu\,dt=I_1+I_2. \tag{94}$$

Here,

$$2I_1=\int\limits_\Gamma\psi_1\mu\,dt=\int\limits_\Gamma(\psi+T\psi)\mu\,dt=\int\limits_\Gamma\psi\mu\,dt+\int\limits_\Gamma T\psi\mu\,dt. \tag{95}$$

Transforming the second summand in (95) and applying the Riesz equalities

$$\int\limits_\Gamma fS_\Gamma g\,dt=-\int\limits_\Gamma gS_\Gamma f\,dt,\quad f\in L^{p(\cdot)},\quad g\in L^{p'(\cdot)} \tag{96}$$

([17]), we have

$$\int\limits_\Gamma T\psi\mu\,dt=\int\limits_\Gamma X^+S_\Gamma\,\frac{\psi}{X^+}\,\mu\,dt=$$
$$=\int\limits_\Gamma \mu X^+S_\Gamma\,\frac{\psi}{X^+}\,dt=-\int\limits_\Gamma\frac{\psi}{X^+}\,S_\Gamma X^+\,dt.$$

Assuming for the present that $\mu=\mu_n$ and $\psi=\psi_\nu$ are rational functions, we can apply formula (96). Thus we obtain

$$\int\limits_\Gamma T\psi_\nu\mu_n\,dt=-\int\limits_\Gamma\frac{\psi_\nu}{X^+}\,S_{\mu_n}X^+dt=-\int\limits_\Gamma\psi_\nu\widetilde{T}\mu_n\,dt. \tag{97}$$

For the fixed μ_n, in right-hand side of equality (97) we can pass to the limit with respect to ν. We get

$$\lim_{\nu\to\infty}\int\limits_\Gamma T\psi_\nu\mu_n\,dt=-\int\limits_\Gamma\psi\widetilde{T}\mu_n\,dt,\quad \psi\in L^{p(\cdot)}.$$

As far as $\{T\psi_\nu\}$ converges in measure to $T\psi$, we select a subsequence converging almost everywhere to $T\psi$ and, by the Fatou lemma, we find that

$$\int_\Gamma T\psi\mu_n\,dt = -\int_\Gamma \psi\widetilde{T}\mu_n\,dt.$$

In the above equality, we can pass to the limit in left-hand side and as a result, we have

$$\int_\Gamma T\psi\mu\,dt = \lim_{n\to\infty}\int_\Gamma \psi\widetilde{T}\mu_n\,dt.$$

According to Lemma 10, $\{\widetilde{T}\mu_n\}$ converges in measure to $\widetilde{T}\mu$. Just as above, we apply Fatou's lemma and obtain

$$\int_\Gamma T\psi\mu\,dt = -\int_\Gamma \psi\widetilde{T}\mu\,dt, \tag{98}$$

where $\mu \in L^{p'(\cdot)}$, $\psi \in L^{p(\cdot)}$. From (98) we can conclude that $\widetilde{T}\mu \in L^{p'(\cdot)}$. Consequently, $\mu \in L^{p'(\cdot)}$, $\widetilde{T}\mu \in L^{p'(\cdot)}$. □

It follows from equalities (93),(95) and (98) that if $\mu \in \mathcal{L}^{p'(\cdot)}$, then

$$\Lambda\psi = \int_\Gamma \psi\mu_1\,dt, \quad \mu_1 = -T\mu \in \widetilde{\mathcal{L}}^{p'(\cdot)}$$

is the linear continuous functional in $\mathcal{L}^{p(\cdot)}$. This and the statement of Lemma 14 allow us to conclude that the following theorem is valid.

Theorem 12. *If the conditions of Theorem* 3 *are fulfilled, then*

$$(\mathcal{L}^{p(\cdot)})^* = \widetilde{\mathcal{L}}^{q(\cdot)}, \quad q(t) = \frac{p(t)}{p(t)-1}\,.$$

12. On the Noetherity of Operator M in the Space $\mathcal{L}^{p(\cdot)}$

12.1. **The operator, conjugate to the operator M.** If the operator M acts from the Banach space X to Y, then the operator M^* acts from Y^* to X^* which to the linear functional Λ from Y^* to $\mathbb{C}$ puts into correspondence the functional Λ^* defined by the equality $\Lambda^*x = \Lambda(Mx)$, $x \in X$.

In the case under consideration, $X = Y = \mathcal{L}^{p(\cdot)}$ and $Y^* = X^* = \widetilde{\mathcal{L}}^{q(\cdot)}$.

Let $f \in \mathcal{L}^{p(\cdot)}$, then

$$\Lambda f = \int_\Gamma f\psi\,dt, \quad \psi \in \widetilde{\mathcal{L}}^{q(\cdot)},$$

$$\Lambda^* f = \int_\Gamma \psi Mf\,dt = \int_\Gamma \psi(t)\big(a(t)f(t) + b(t)(Sf)(t)\big)\,dt =$$

$$= \int_\Gamma a(t)\psi(t)f(t)\,dt + \int_\Gamma \psi(t)b(t)(Sf)(t)\,dt =$$

$$= \int_\Gamma a(t)\psi(t)f(t)\,dt - \int_\Gamma f(t)(Sb\psi)(t)\,dt =$$
$$= \int_\Gamma f(t)\big(a(t)\psi(t) - (Sb\psi)(t)\big)\,dt.$$

Consequently, the conjugate to the operator $M : \mathcal{L}^{p(\,\cdot\,)} \to \mathcal{L}^{p(\,\cdot\,)}$ is the operator $M^* : \widetilde{\mathcal{L}}^{q(\,\cdot\,)} \to \widetilde{\mathcal{L}}^{q(\,\cdot\,)}$ given by the equality

$$M^*\psi = a\psi - Sb\psi. \tag{99}$$

12.2. **About the equation $M^*\psi = \mu$.** The equation

$$M^*\psi = \mu. \tag{100}$$

considered in $\widetilde{\mathcal{L}}^{q(\,\cdot\,)}$ is equivalent to the problem of conjugation

$$\Psi^+ = \frac{1}{G}\,\Psi^- + \frac{\mu}{a-b}\,, \tag{101}$$

considered in the class $K\widetilde{\mathcal{L}}^{q(\,\cdot\,)}$. In addition,

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{b(\tau)\psi(\tau)}{\tau - z}\,d\tau. \tag{102}$$

Since

$$\Psi^+ = \frac{1}{2}\,(b\psi + Sb\psi), \quad \Psi^- = \frac{1}{2}\,(-b\psi + Sb\psi),$$

therefore

$$\Psi^+ - \Psi^- = b\psi, \quad \Psi^+ - \Psi^- = Sb\psi.$$

If $\mu = 0$, then $\psi = Sb\psi = \Psi^+ - \Psi^-$, and hence $a\psi = \Psi^+ + \Psi^-$, $b\psi = \Psi^+ - \Psi^-$. This implies that $(a+b)\psi = 2\Psi^+$, i.e.,

$$\psi(z) = \frac{2\Psi^+}{a+b}\,. \tag{103}$$

Since $\frac{1}{G} \in A(q(\,\cdot\,))$, for $\varkappa = \varkappa(G) \geq 0$ we have ind $\frac{1}{G} \leq 0$, therefore the equation

$$a\psi - Sb\psi = 0 \tag{104}$$

has only a zero solution.

When $\varkappa(G) < 0$, it is not difficult to verify that a general solution of the problem (101) for $\mu \equiv 0$ will have the form

$$\Psi = \frac{1}{2}\,X(z)P_{|\varkappa|-1}(z)$$

and from (73) we find that the set of functions

$$\Psi = \frac{P_{|\varkappa|-1}(z)}{X^+(a+b)}$$

provides us with a general solution of equation $M^*\psi = 0$. The base of a general solution for that equation is

$$\frac{1}{X^+(a+b)}, \frac{\tau}{X^+(a+b)}, \dots, \frac{\tau^{|\varkappa|-1}}{X^+(a+b)}.$$

12.3. **On the Noetherity of the operator M.** The conditions (54) designate that equation (62) for $\varkappa < 0$ has normal solvability.

If $\varkappa \geq 0$, then $N(M^*) = \{0\}$, and the equation $M\varphi = f$ is solvable for any $\frac{f}{a+b} \in \mathcal{L}^{p(\cdot)}$, i.e., the condition of normal solvability is fulfilled again.

This and the fact that $\ell = N(M) = \max(0, \varkappa)$ and $\ell' = N(M^*) = \max(0, -\varkappa)$ allow us to conclude that the theorem below is valid.

Theorem 13. *Let Γ be the simple, closed, rectifiable curve and let $a(t)$ and $b(t)$ be measurable bounded functions such that*

$$\operatorname{ess\,inf} \left| a^2(t) - b^2(t) \right| > 0,$$

and $G(t) = (a(t) - b(t))(a(t) + b(t))^{-1}$. If the conditions (25) *are fulfilled, then the equation*

$$M\varphi := a(t)\varphi(t) + b(t)(S\varphi)(t) = f(t)$$

is Noetherian in the space $\mathcal{L}^{p(\cdot)}$, where

$$M^*\psi = a\psi - Sb\psi, M^* : \widetilde{\mathcal{L}}^{q(\cdot)} \to \widetilde{\mathcal{L}}^{q(\cdot)},$$

$$\operatorname{ind}(M; \mathcal{L}^{p(\cdot)}) = \varkappa(G) = \varkappa = \operatorname{ind}\big((a-b)(a+b)^{-1}\big).$$

In all cases where a solution exists, it is given by equality (85).

Corollary. *If V is a compact operator from $\mathcal{L}^{p(\cdot)}$ to $\mathcal{L}^{p(\cdot)}$ and the conditions* (25) *are fulfilled, then the operator $M + V$ is Noetherian in $\mathcal{L}^{p(\cdot)}$, and* $\operatorname{ind}(M + V, \mathcal{L}^{p(\cdot)}) = \operatorname{ind} M = \operatorname{ind} \frac{a-b}{a+b}$.

This statement is a consequence of the result obtained in [29] according to which it follows that the addition of a compact operator to the Noetherian one does not change its Noetherity and index.

13. Some Properties of the Operator $T = T_G$, when $G \in A(p(\cdot))$

For the operator T_G, we frequently applied properties of the operator T_G proven in Section 6. Remind these properties.

(1) Under the assumptions (25), we have $T(Tg) = g$.
(2) The operator T is continuous from $L^{p(t)}$ to the space of convergence in measure.
(3) The operator T is closed from $L^{p(\cdot)}$ to $L^{p(\cdot)}$.

Moreover, when proving Lemma 6, we have used equality (66) which will be proved in Subsection 13.1.

Below, we will present some other properties of the operator T. We start with Lemma 17 which will be highly useful in establishing operator properties which will be treated in Subsections 13.3–13.5.

All curves considered in Section 13 are assumed (except requirements made by the theorem) to be simple, rectifiable and closed.

13.1. **Lemma about** $S(ab)$**.**

Lemma 17. *If* $\Gamma \in R$*,* $p \in \mathcal{P}(\Gamma)$*,* $a \in L^{p(\,\cdot\,)}$*,* $b \in L^{q(\,\cdot\,)}$*, then almost everywhere on* Γ *the equality*

$$S(ab) = bSa + aSb - S(Sa \cdot Sb) \tag{105}$$

is valid.

Proof. Assume that the point $z = 0$ lies in the inner domain bounded by Γ. Then rational functions of the type

$$\sum_{k=-m}^{-1} a_k t^k + \sum_{k=0}^{n} a_k t^k = m(t) + h(t)$$

form a complete set both in $L^{p(\,\cdot\,)}$ and in $L^{q(\,\cdot\,)}$. We denote it by Q.

Let us show that if $a(t) = m(t) + h(t)$, $b(t) = r(t) + S(t)$, then equality (105) is valid.

We have

$$\begin{aligned} S(ab) &= S\big((m+h)(r+s)\big) = S(mr + hr + ms + hs) = \\ &= S(mr + hs) + S(ms + hr) = mr - hs + S(ms + hs). \end{aligned} \tag{106}$$

Here we have used the equalities

$$(SP)(t) = P(t), \quad S\Big(P\Big(\frac{1}{t}\Big)\Big) = -P\Big(\frac{1}{t}\Big),$$

where P is the polynomial of its own argument.

Further,

$$\begin{gathered} bSa + aSb - S(Sa \cdot Sb) = \\ = (r+s)(m-h) + (m+h)(r-s) - S(mr - ms - hr + hs) = \\ = rm - rh + sm - sh + mr - ms + hr - hs - s(mr - ms - hr + hs) = \\ = 2rm - 2h - (mr - hs) - s(ms + hr) = \\ = mr - hs + s(ms + hr). \end{gathered} \tag{107}$$

From equalities (106) and (107) we obtain (105) in the form

$$S(R_n Q_m) = SR_n \cdot Q_m + R_n SQ_m - S(SR_n \cdot SQ_m), \tag{108}$$

where R_n and Q_m belong to Q.

Let now $a \in L^{p(\,\cdot\,)}$ and $b \in L^{q(\,\cdot\,)}$ be arbitrary functions, and let $\|R_n - a\|_{p(\,\cdot\,)} \to 0$, $\|Q_m - b\|_{q(\,\cdot\,)} \to 0$.

Since $\Gamma \in R$ and $p \in \mathcal{P}(\Gamma)$, by the boundedness of the operator S in $L^{p(\,\cdot\,)}$ ([10]), we admit in equality (108) the passage to the limit which allows us to conclude that equality (105) is valid in a general case. □

Corollary.

$$T(mn) = Tm \cdot n + m \cdot Sn - T(Tm \cdot Sn).$$

Proof. According to (75), we get

$$T(mn) = X^+ S\Big(\frac{m}{X^+}\, n\Big) = X^+\Big(nS\,\frac{m}{X^+} + \frac{m}{X^+}\, Sn\Big) - X^+ S\Big(S\,\frac{m}{X^+}\, Sn\Big) =$$
$$= Tm \cdot n + m \cdot Sn + X^+ - X^+ S\Big(\frac{1}{X^+}\, X^+ S\,\frac{m}{X^+}\, Sn\Big) =$$
$$= Tm \cdot n + m \cdot Sn - T(Tm \cdot Sn). \qquad \square$$

13.2. **Value of** $\sup \|T\|_\alpha$ **when** $\alpha \in [\underline{p}, \overline{p}]$**.**

Lemma 18. *If* $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, $\underline{p} = \inf\limits_{t\in\Gamma} p(t)$, $\overline{p} = \sup\limits_{t\in\Gamma} p(t)$, *and for any* $\alpha \in I = [\underline{p}, \overline{p}]$ *we have* $\|T\|_\alpha < \infty$, *then*

$$\sup_\alpha \|T\|_\alpha < \infty.$$

Proof. Assume the contrary; then there exists the sequence $\{\alpha_n\}$, $\alpha_n \in I$, such that

$$\|T\|_{\alpha_n} \to \infty.$$

Note that if p and p_1 belong to $\mathcal{P}(\Gamma)$, and $p(t) \le p_1(t)$, then

$$\|f\|_{p(\,\cdot\,)} \le (1 + \operatorname{mes}\Gamma)\|f\|_{p_1(\,\cdot\,)}$$

(see item 2.4.2).

Let $\alpha_0 = \sup \alpha_n$, then $\alpha_0 \in I$. Taking into account the last inequality, we obtain

$$\|T\|_{\alpha_0} = \sup_{\|\varphi\|_{\alpha_0}\le 1} \|T\varphi\|_{\alpha_0} \ge \sup_{\|\varphi\|_{\alpha_0}\le 1} \|T\varphi\|_{\alpha_n} \cdot \frac{1}{1 + \operatorname{mes}\Gamma}\,. \tag{109}$$

But $\|\varphi\|_{\alpha_0} \ge \frac{1}{1+\operatorname{mes}\Gamma}\, \|\varphi\|_{\alpha_n}$, hence $\|\varphi\|_{\alpha_n} \le (1 + \operatorname{mes}\Gamma)\|\varphi\|_{\alpha_0}$.

Consequently,

$$\sup_{\|\varphi\|_{\alpha_0}\le 1} \|T\varphi\|_{\alpha_n} = (1 + \operatorname{mes}\Gamma) \sup_{\|\varphi\|_{\alpha_n}\le 1} \|T\varphi\|_{\alpha} = (1 + \operatorname{mes}\Gamma)\|T\|_{\alpha_n}.$$

This together with the estimate (109) result in $\|T\|_{\alpha_0} = \infty$. But this contradicts the assumptions of the lemma by which $\|T\|_{\alpha_0}$ should be finite, since $\alpha_0 \in I$. $\square$

13.3. **On the operator** $T_{1/G}$**, when** $G \in A(p(\,\cdot\,))$**.**

Lemma 19. *If* $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$ *and the operator* T_G, $G \in A(p(\,\cdot\,))$, *is continuous in* $L^{p(\,\cdot\,)}$, *then the operator*

$$T_{1/G} : f \to T_{1/G} f, \quad (T_{1/G} f)(t) = \frac{1}{2\pi i X^+} \int\limits_\Gamma \frac{X^+(\tau) f(\tau)}{\tau - t}\, dt$$

is continuous in $L^{q(\,\cdot\,)}$.

Conversely, if $T_{1/G}$ is continuous in $L^{q(\,\cdot\,)}$, then T_G is continuous in $L^{p(\,\cdot\,)}$.

Moreover,

$$\|T_G\|_{p(\,\cdot\,)} \leq k\|T_{1/G}\|_{q(\,\cdot\,)}, \quad \|T_{1/G}\| \leq k\|T_G\|_{p(\,\cdot\,)}, \tag{110}$$

where $k = 1 + \frac{1}{\underline{p}} + \frac{1}{\overline{p}}$ is the constant from inequality (7).

Proof. We proceed from the relation

$$\|f\|_{p(\,\cdot\,)} \sim \sup_{\|g\|_{q(\,\cdot\,)}\leq 1} \bigg|\int\limits_{\Gamma} fg\,dt\bigg|$$

(see item 2.4.1).

Assuming for the present that f and g are rational functions of the class Q, we get

$$\|T_{1/G}g\|_{q(\,\cdot\,)} \sim \sup_{\|f\|_{p(\,\cdot\,)}\leq 1} \bigg|\int\limits_{\Gamma} fT_{1/G}\,dt\bigg| = \sup_{\|f\|_{p(\,\cdot\,)}\leq 1} \bigg|\int\limits_{\Gamma} f\,\frac{1}{X^+}\,SX^+g\,dt\bigg|.$$

Using the Riesz equality (see formula (66)), we obtain

$$\|T_{1/G}g\|_{q(\,\cdot\,)} = \sup_{\|f\|_{p(\,\cdot\,)}\leq 1} \bigg|\int\limits_{\Gamma} gX^+S\,\frac{f}{X^+}\,dt\bigg| = \sup \bigg|\int\limits_{\Gamma} gT_Gf\,dt\bigg| \leq$$
$$\leq k\cdot \sup_{\|f\|_{p(\,\cdot\,)}\leq 1} \|g\|_{q(\,\cdot\,)}\|T_Gf\|_{p(\,\cdot\,)} \leq k\|g\|_{q(\,\cdot\,)}\|T_G\|_{p(\,\cdot\,)}\|f\|_{p(\,\cdot\,)} =$$
$$= k\|T_G\|_{p(\,\cdot\,)}\|g\|_{q(\,\cdot\,)}.$$

Thus

$$\|T_{1/G}g\|_{q(\,\cdot\,)} \leq k\|T_G\|_{p(\,\cdot\,)}\|g\|_{q(\,\cdot\,)}, \quad f,g\in Q. \tag{111}$$

Analogously we can prove that

$$\|T_Gf\|_{p(\,\cdot\,)} \leq k\|T_{1/G}\|_{q(\,\cdot\,)}\|f\|_{p(\,\cdot\,)}. \tag{112}$$

By the passage to the limit (which is admissible due to $\Gamma\in R$), we find that inequalities (111) and (112) are valid for any $f\in L^{p(\,\cdot\,)}$, $g\in L^{q(\,\cdot\,)}$, i.e., inequalities (110) are valid in a general case. □

13.4. **On the operator ST.**

Theorem 14. *Let $\Gamma\in R$, $p\in\mathcal{P}(\Gamma)$, $G\in A(p(\,\cdot\,))$, $g\in L^{p(\,\cdot\,)}$, then*

$$S(Tg) = g + Tg - Sg.$$

Proof. First of all, we note that $Tg\in L^{p(\,\cdot\,)-\varepsilon}\in L^1$ (see Theorem 7).

Since $Tg\in L^1$, almost everywhere on Γ there exists the integral $S_\Gamma\,\frac{g}{X^+}$, and hence $g(X^+)^{-1}\in L^1$. This implies that $(K_\Gamma\,\frac{g}{X^+})(z)$ belongs to the set $\bigcap\limits_{\delta<1} E^\delta(D^+)$ (see Subsection 2.6). Since

$$\frac{1}{X(z)} = \frac{1}{X_G(z)} = \exp\bigg(-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\ln|G(\tau)|+i\arg G(t)}{\tau - z}\,d\tau\bigg).$$

$\Gamma \in R$ and $G \in \widetilde{A}(p(\,\cdot\,))$, and hence $\ln G$ is the bounded function, therefore $X(z)$ and $1/X(z)$ belong to $E^{\nu}(D^+)$ for some $\nu > 0$ ([8, pp. 96–98]). Thus the function

$$F(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau - z} = X(z)\Big(K_\Gamma\,\frac{g}{X^+}\Big)(z), \tag{113}$$

being a product of two Smirnov class functions, belongs to some class $E^{\eta}(D^+)$, $\eta > 0$. Moreover, $F^+ = \frac{1}{2}\,(g + Tg)$. Here, $g \in L^{p(\,\cdot\,)}$, while $Tg \in L(\Gamma)$. Thereby, $F^+ \in L(\Gamma)$. Thus, according to Smirnov's theorem (see, e.g., [27, p. 254]), we find that $F \in E^1(D^+)$. But then $\int\limits_{\Gamma} F^+ = F^+$. This results in

$$\frac{1}{2}\,(g + Tg) = \frac{1}{2}(Sg + STg)$$

from which we obtain the provable equality. □

13.5. **On the operator TS.** As it has been shown in proving Theorem 6, the function $F(z)$ given by equality (113) belongs to $E^1(D^+)$. This fact allows us to prove that the following theorem is valid.

Theorem 15. *In the assumptions of Theorem* 6*, the equality*

$$(TS)(g) = Sg + g - Tg \tag{114}$$

is valid.

Proof. Since $F(z) \in E^1(D^+)$, therefore

$$S_\Gamma\big[(K_\Gamma g)X_G^{-1}\big]^+ = (K_\Gamma g)(X_G^+)^{-1},$$

that is,

$$S_\Gamma\,\frac{g + S_\Gamma g}{X^+} = \frac{g + S_\Gamma g}{X^+}$$

from which we successively obtain

$$\begin{aligned} X^+ S_\Gamma\,\frac{g + Sg}{X^+} &= g + Sg,\\ T(g + Sg) &= g + Sg,\\ Tg + TSg &= g + Sg. \end{aligned}$$

Indeed, the last equalities show that equality (114) is valid. □

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