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MELLIN CONVOLUTION OPERATORS IN BESSEL POTENTIAL SPACES WITH ADMISSIBLE MEROMORPHIC KERNELS

Dedicated to the memory of Academician Victor Kupradze on the occasion of his 110-th birthday anniversary

Abstract. The paper is devoted to Mellin convolution operators with meromorphic kernels in Bessel potential spaces. We encounter such operators while investigating boundary value problems for elliptic equations in planar 2D domains with angular points on the boundary.

Our study is based upon two results. The first concerns commutants of Mellin convolution and Bessel potential operators: Bessel potentials alter essentially after commutation with Mellin convolutions depending on the poles of the kernel (in contrast to commutants with Fourier convolution operatiors.) The second basic ingredient is the results on the Banach algebra $\mathfrak{A}_p$ generated by Mellin convolution and Fourier convolution operators in weighted $\mathbb{L}_p$-spaces obtained by the author in 1970's and 1980's. These results are modified by adding Hankel operators. Examples of Mellin convolution operators are considered.

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Key words and phrases. Fourier convolution, Mellin convolution, Bessel potentials, meromorphic kernel, Banach algebra, symbol, fixed singularity, Fredholm property, index.

რეზიუმე. ნაშრომი მიძღვნილია მელინის კონვოლუციის ოპერატორებისადმი მერომორფული ბირთვებით, რომლებიც მოქმედებენ ბესელის პოტენციალთა სივრცეებში. ასეთი ოპერატორები გვხვდება სასაზღვრო ამოცანების გამოკვლევებში ელიფსური დიფერენციალური განტოლებებისათვის ბრტყელ 2-განზომილებიან არეებში კუთხოვანი საზღვრით.

ჩვენი გამოკვლევა ეყრდნობა ორ შედეგს. პირველი ეხება მელინის კონვოლუციების და ბესელის პოტენციალების კომუტანტს: ბესელის პოტენციალის ოპერატორი განიცდის არსებით ცვლილებას მელინის კონვოლუციის ოპერატორთან გადასმის შედეგად და ეს ცვლილება დამოკიდებულია მერომორფული ბირთვის პოლუსებზე (ფურიეს კონვოლუციის ოპერატორისაგან განსხვავებით, რომელთანაც გადასმის შედეგად ბესელის პოტენციალი არ იცვლება). მეორე მნიშვნელოვანი შედეგი, რომელსაც ვეყრდნობით, წარმოადგენს ოპერატორების გამოკვლევის შედეგებს ბანახის ალგებრიდან $\mathfrak{A}_p$, რომელიც წარმოქმნილია მელინის კონვოლუციის და ფურიეს კონვოლუციის ოპერატორების მიერ წონიან $\mathbb{L}_p$-სივრცეებში, რომლებიც მიღებულია 1980-იან წლებში სტატიის ავტორის მიერ. ამ შედეგებზე დამატებულია ახალი შედეგი, რომელიც ეხება ჰანკელის ოპერატორებს. განხილულია მელინის კონვულუციის ოპერატორების მაგალითები.

Introduction

It is well-known that various boundary value problems for PDE in planar domains with angular points on the boundary, e.g. Lamé systems in elasticity (cracks in elastic media, reinforced plates), Maxwell's system and Helmholtz equation in electromagnetic scattering, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory etc. can be studied with the help of the Mellin convolution equations of the form

$$\mathbf{A}\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau} = f(t), \tag{1}$$

with the kernel $\mathscr{K}$ satisfying the condition

$$\int_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty, \ \ 0 < \beta < 1, \tag{2}$$

which makes it a bounded operator in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$, provided $1 \leqslant p \leqslant \infty$, $-1 < \gamma < p-1$, $\beta := (1+\gamma)/p$ (cf. [17]).

In particular, integral equations with fixed singularities in the kernel

$$c_0(t)\varphi(t) + \frac{c_1(t)}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \\ + \sum_{k=0}^{n} \frac{c_{k+2}(t)t^{k-r}}{\pi i}\int_0^\infty \frac{\tau^r\varphi(\tau)\,d\tau}{(\tau+t)^{k+1}} = f(t), \ \ 0 \leqslant t \leqslant 1, \tag{3}$$

where $0 \leqslant r \leqslant k$ are of type (1) after localization, i.e. after "freezing" the coefficients.

The Fredholm theory and the unique solvability of equations (1) in the weighted Lebesgue spaces were accomplished in [17]. This investigation was based on the following observation: if $1 < p < \infty$, $-1 < \gamma < p-1$, $\beta := (1+\gamma)/p$, the following mutually invertible exponential transformations

$$\begin{gathered} Z_\beta : \mathbb{L}_p([0,1], t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \\ Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \ \ \xi \in \mathbb{R} := (-\infty, \infty), \\ Z_\beta^{-1} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p([0,1], t^\gamma), \\ Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \ \ t \in \mathbb{R}^+ := (0, \infty), \end{gathered} \tag{4}$$

transform the equation (1), treated in the weighted Lebesgue space $f, \varphi \in \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ into the Fourier convolution equation $W^0_{\mathscr{A}_\beta}\psi = g$, $\psi = Z_\beta\varphi, g = Z_\beta f \in \mathbb{L}_p(\mathbb{R})$ of the form

$$W^0_{\mathscr{A}_\beta}\psi(x) = c_0\psi(x) + \int_{-\infty}^{\infty} \mathscr{K}_1(x-y)\varphi(y)\,dy,$$

$$\mathscr{K}_1(x) = e^{-\beta x}\Big[\frac{c_1}{1-e^{-x}} + \mathscr{K}(e^{-x})\Big].$$

Note that the symbol of the operator $W^0_{\mathscr{A}_\beta}$, viz. the Fourier transform of the kernel

$$\begin{aligned}\mathscr{A}_\beta(\xi) &:= c_0 + \int\limits_{-\infty}^{\infty} e^{i\xi x}\mathscr{K}_1(x)\,dx \\ &:= c_0 - ic_1 \cot \pi(\beta - i\xi) + \int\limits_{-\infty}^{\infty} e^{(i\xi-\beta)x}\mathscr{K}(e^{-x})\,dx, \;\; \xi \in \mathbb{R} \qquad (5)\end{aligned}$$

is a piecewise continuous function. Let us recall that the theory of Fourier convolution operators with discontinuous symbols is well developed, cf. [13, 14, 15, 16, 42]. This allows one to investigate various properties of the operators (1), (3). In particular, Fredholm criteria, index formula and conditions of unique solvability of the equations (1) and (3) have been established in [17].

Similar integral operators with fixed singularities in kernel arise in the theory of singular integral equations with the complex conjugation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t}} = f(t), \;\; t \in \Gamma$$

and in more general R-linear equations

$$\begin{aligned}a(t)\varphi(t) + b(t)\overline{\varphi(t)} + \frac{c(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{d(t)}{\pi i}\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t} + \\ + \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t}} + \frac{g(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t}} = f(t), \;\; t \in \Gamma,\end{aligned}$$

if the contour Γ possesses corner points. Note that a complete theory of such equations is presented in [24, 25], whereas approximation methods have been studied in [10, 11].

Let $t_1, \dots, t_n \in \Gamma$ be the corner points of a piecewise-smooth contour Γ, and let $\mathbb{L}_p(\Gamma, \rho)$ denote the weighted $\mathbb{L}_p$-space with a power weight $\rho(t) := \prod\limits_{j=1}^{n} |t - t_j|^{\gamma_j}$. Assume that the parameters p and $\beta_j := (1 + \gamma_j)/p$ satisfy the conditions

$$1 < p < \infty, \;\; 0 < \beta_j < 1, \;\; j = 1, \dots, n.$$

If the coefficients of the above equations are piecewise-continuous matrix functions, one can construct a function $\mathscr{A}_{\vec{\beta}}(t, \xi)$, $t \in \Gamma$, $\xi \in \mathbb{R}$, $\vec{\beta} := (\beta_1, \dots, \beta_n)$, called the symbol of the equation (of the related operator). It is possible to express various properties of the equation in terms of $\mathscr{A}_{\vec{\beta}}$:

- The equation is Fredholm in $\mathbb{L}_p(\Gamma, \rho)$ if and only if its symbol is elliptic., i.e. iff $\inf_{(t,\xi)\in\Gamma\times\mathbb{R}} |\,\mathscr{A}_{\vec{\beta}}(t,\xi)| > 0$;
- To an elliptic symbol $\mathscr{A}_{\vec{\beta}}(t,\xi)$ there corresponds an integer valued index
  ind $\mathscr{A}_{\vec{\beta}}(t,\xi)$, the winding number, which coincides with the Fredholm index of the corresponding operator modulo a constant multiplier.

For more detailed survey of the theory and various applications to the problems of elasticity we refer the reader to [13, 14, 15, 17, 18, 19, 20, 21, 40].

Similar approach to boundary integral equations on curves with corner points based on Mellin transformation has been exploited by M. Costabel and E. Stephan [5, 6].

However, one of the main problems in boundary integral equations for elliptic partial differential equations is the absence of appropriate results for Mellin convolution operators in Bessel potential spaces, cf. [18, 20, 21] and recent publications on nano-photonics [1, 2, 32]. Such results are needed to obtain an equivalent reformulation of boundary value problems into boundary integral equations in Bessel potential spaces. Nevertheless, numerous works on Mellin convolution equations seem to pay almost no attention to the mentioned problem.

The first arising problem is the boundedness results for Mellin convolution operators in Bessel potential spaces. The conditions on kernels known so far are very restrictive. The following boundedness result for the Mellin convolution operator is proved in the yet unpublished paper by V. Didenko and R. Duduchava.

Proposition 0.1. *Let $1 < p < \infty$ and let $m = 1, 2, \dots$ be an integer. If a function $\mathscr{K}$ satisfies the condition*

$$\int_0^1 t^{\frac{1}{p}-m-1}|\mathscr{K}(t)|\,dt + \int_1^\infty t^{\frac{1}{p}-1}|\mathscr{K}(t)|\,dt < \infty, \tag{6}$$

then the Mellin convolution operator (see (1)*)*

$$\boldsymbol{A} = \mathfrak{M}^0_{\mathscr{A}_{1/p}} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{7}$$

with the symbol (*see* (5))

$$\mathscr{A}_{1/p}(\xi) := c_0 + c_1 \coth\pi\Big(\frac{i}{p} + \xi\Big) + \int_0^\infty t^{\frac{1}{p}-i\xi}\mathscr{K}(t)\,\frac{dt}{t}\,, \quad \xi\in\mathbb{R}, \tag{8}$$

is bounded for any $0 \leqslant s \leqslant m$.

Note that the condition

$$K_\beta := \int_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty \tag{9}$$

and the constraints (16) ensure that the operator

$$\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$$

is bounded and the norm of the Mellin convolution

$$\mathfrak{M}^0_{a_\beta}\varphi(t) := \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}\,. \tag{10}$$

admits the estimate $\|\mathfrak{M}^0_{a_\beta}\| \leqslant K_\beta$.

The above-formulated result has very restricted application. For example, the operators

$$\begin{aligned}
\boldsymbol{N}_\alpha\varphi(t) &:= \frac{\sin\alpha}{\pi}\int_0^\infty \frac{t\,\varphi(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}\,,\\
\boldsymbol{N}^*_\alpha\varphi(t) &:= \frac{\sin\alpha}{\pi}\int_0^\infty \frac{\tau\,\psi_j(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}\,,\\
\boldsymbol{M}_\alpha\varphi(t) &:= \frac{1}{2\pi}\int_{\mathbb{R}^+} \frac{[\tau\,\cos\alpha-t]\varphi(\tau)\,d\tau}{t^2+\tau^2-2t\,\tau\cos\alpha}\,, \quad -\pi<\alpha<\pi,
\end{aligned} \tag{11}$$

which we encounter in boundary integral equations for elliptic boundary value problems (see [4, 27]), as well as the operators

$$\boldsymbol{N}_{m,k}\varphi(t) := \frac{t^k}{\pi i}\int_0^\infty \frac{\tau^{m-k}\varphi(\tau)\,d\tau}{(\tau+t)^{m+1}}\,, \quad k=0,\dots,m, \tag{12}$$

represented in (3), do not satisfy the conditions (6). In particular, $\boldsymbol{N}_\alpha$ satisfies condition (6) only for $m=1$ and $\boldsymbol{N}_{m,k}$ only for $m=k$. Although, as we will see below in Theorem 2.5, all operators $\boldsymbol{N}_\alpha$, $\boldsymbol{N}^*_\alpha$ and $\boldsymbol{N}_{m,k}$ are bounded in Bessel potential spaces in the setting (17) for all $s\in\mathbb{R}$.

In the present paper we introduce *admissible kernels*, which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at the infinity

$$\begin{gathered}
\mathscr{K}(t) := \sum_{j=0}^{\ell}\frac{d_j}{t-c_j} + \sum_{j=\ell+1}^{\infty}\frac{d_j}{(t-c_j)^{m_j}}\,, \quad c_j\neq 0, \quad j=0,1,\dots,\\
c_0,\dots,c_\ell\in\mathbb{R}, \quad 0<\alpha_k:=|\arg c_k|\leqslant\pi, \quad k=\ell+1,\ell+2,\dots
\end{gathered} \tag{13}$$

having poles at $c_0,c_1,\ldots\in\mathbb{C}\setminus\{0\}$ and complex coefficients $d_j\in\mathbb{C}$. The Mellin convolution operator

$$\boldsymbol{K}^m_c\varphi(t) := \int_0^\infty \frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m}. \tag{14}$$

corresponding to the kernel

$$\mathscr{K}(t) := \frac{1}{(t-c)^m}, \quad c_j\neq 0$$

(see Definition 2.1) turns out to be bounded in the Bessel potential spaces (see Theorem 2.5).

In order to study Mellin convolution operators in Bessel potential spaces, we use the "lifting" procedure, performed with the help of the Bessel potential operators Λ^s_+ and Λ^{s-r}_-, which transform the initial operator $\mathfrak{M}^0_a$ into the lifted operator $\Lambda^{s-r}_-\mathfrak{M}^0_a\Lambda^{-s}_+$ acting already on a Lebesgue $\mathbb{L}_p$ spaces. However, the lifted operator is neither Mellin nor Fourier convolution and to describe its properties, one has to study the commutants of Bessel potential operators and Mellin convolutions with meromorphic kernels. It turns out that Bessel potentials alter after commutation with Mellin convolutions and the result depends essentially on poles of the meromorphic kernels. These results allows us to show that the lifted operator $\Lambda^{s-r}_-\mathfrak{M}_a\Lambda^{-s}_+$ belongs to the Banach algebra of operators generated by Mellin and Fourier convolution operators with discontinuous symbols. Since such algebras have been studied before [22], one can derive various information (Fredholm properties, index, the unique solvability) about the initial Mellin convolution equation $\mathfrak{M}^0_a\varphi = g$ in Bessel potential spaces in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+)$ and in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \mathbb{H}^{s-r}_p(\mathbb{R}^+)$.

The results of the present work will be applied to the investigation of some boundary value problems studied before by Lax–Milgram Lemma in [1, 2]. Note that the present approach is more flexible and provides better tools for analyzing the solvability of the boundary value problems and the asymptotic behavior of their solutions.

It is worth noting that the obtained results can also be used to study Schrödinger operator on combinatorial and quantum graphs. Such a problem has attracted a lot of attention recently, since the operator mentioned above possesses interesting properties and has various applications, in particular, in nano-structures (see [36, 37] and the references there). Another area for application of the present results are Mellin pseudodifferential operators on graphs. This problem has been studied in [39], but in the periodic case only. Moreover, some of the results can be applied in the study of stability of approximation methods for Mellin convolution equations in Bessel potential spaces.

The present paper is organized as follows. In the first section we observe Mellin and Fourier convolution operators with discontinuous symbols acting on Lebesgue spaces. Most of these results are well known and we recall them for convenience. In the second section we define Mellin convolutions with admissible meromorphic kernels and prove their boundedness in Bessel potential spaces. In Section 2 is proved the key result on commutants of the Mellin convolution operator (with admissible meromorphic kernel) and a Bessel potential. In Section 3 we enhance results on Banach algebra generated by Mellin and Fourier convolution operators by adding explicit definition of the symbol of a Hankel operator, which belong to this algebra. In Sections 4 the obtained results are applied to describe Fredholm properties and the index of Mellin convolution operators with admissible meromorphic kernels in Bessel potential spaces.

1. Mellin Convolution and Bessel Potential Operators

Let N be a positive integer. If there arises no confusion, we write $\mathfrak{A}$ for both scalar and matrix $N \times N$ algebras with entries from $\mathfrak{A}$. Similarly, the same notation $\mathfrak{B}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{B}$. It will be usually clear from the context what kind of space or algebra is considered.

The integral operator (1) is called Mellin convolution. More generally, if $a \in \mathbb{L}_\infty(\mathbb{R})$ is an essentially bounded measurable $N \times N$ matrix function, the Mellin convolution operator $\mathfrak{M}^0_a$ is defined by

$$\mathfrak{M}^0_a\varphi(t) := \mathscr{M}^{-1}_\beta a \mathscr{M}_\beta \varphi(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} a(\xi) \int\limits_0^\infty \Big(\frac{t}{\tau}\Big)^{i\xi-\beta} \varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi, \ \ \varphi \in \mathbb{S}(\mathbb{R}^+),$$

where $\mathbb{S}(\mathbb{R}^+)$ is the Schwartz space of fast decaying functions on $\mathbb{R}^+$, whereas $\mathscr{M}_\beta$ and $\mathscr{M}^{-1}_\beta$ are the Mellin transform and its inverse, i.e.

$$\mathscr{M}_\beta\psi(\xi) := \int\limits_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t}, \ \ \xi \in \mathbb{R},$$
$$\mathscr{M}^{-1}_\beta\varphi(t) := \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} t^{i\xi-\beta}\varphi(\xi)\,d\xi, \ \ t \in \mathbb{R}^+.$$

The function $a(\xi)$ is usually referred to as a symbol of the Mellin operator $\mathfrak{M}^0_a$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}^0_a$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ of N-vector functions endowed with the norm

$$\big\|\varphi \mid \mathbb{L}_p(\mathbb{R}^+, t^\gamma)\big\| := \Big[\int\limits_0^\infty t^\gamma |\varphi(t)|^p\,dt\Big]^{1/p},$$

then the symbol $a(\xi)$ is called an $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ *Mellin multiplier*. The transformations

$$\mathbf{Z}_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}), \quad \mathbf{Z}_\beta\varphi(\xi) := e^{-\beta t}\varphi(e^{-\xi}), \ \ \xi \in \mathbb{R},$$
$$\mathbf{Z}^{-1}_\beta : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \quad \mathbf{Z}^{-1}_\beta\psi(t) := t^{-\beta}\psi(-\ln t), \ \ t \in \mathbb{R}^+,$$

generate an isometrical isomorphism between the corresponding $\mathbb{L}_p$-spaces. Moreover, the relations

$$\begin{gathered}\mathscr{M}_\beta = \mathscr{F}\mathbf{Z}_\beta, \quad \mathscr{M}^{-1}_\beta = \mathbf{Z}^{-1}_\beta\mathscr{F}^{-1},\\ \mathfrak{M}^0_a = \mathscr{M}^{-1}_\beta a \mathscr{M}_\beta = \mathbf{Z}^{-1}_\beta\mathscr{F}^{-1}a\mathscr{F}\mathbf{Z}_\beta = \mathbf{Z}^{-1}_\beta W^0_a \mathbf{Z}_\beta,\end{gathered} \tag{15}$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and its inverse,

$$\mathscr{F}\varphi(\xi) := \int\limits_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathscr{F}^{-1}\psi(x) := \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \ \ x \in \mathbb{R},$$

show a close connection between Mellin $\mathfrak{M}^0_a$ and Fourier

$$W^0_a\varphi := \mathscr{F}^{-1}a\mathscr{F}\varphi, \ \ \varphi \in \mathbb{S}(\mathbb{R}),$$

convolution operators, as well as between the corresponding transforms. Here $\mathbb{S}(\mathbb{R})$ denotes the Schwartz class of infinitely smooth functions, decaying fast at the infinity.

An $N\times N$ matrix function $a(\xi)$, $\xi \in \mathbb{R}$ is called a *Fourier $\mathbb{L}_p$-multiplier* if the operator $W^0_a : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

From (15) immediately follows the following

Proposition 1.1. *The class $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$-multipliers coincides with the class of Mellin $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ multiplier.*

It is known, see, e.g. [17], that $\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra which contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with finite variation provided that

$$\beta := \frac{1+\gamma}{p}, \ \ 1 < p < \infty, \ \ -1 < \gamma < p-1. \tag{16}$$

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+). \tag{17}$$

The symbols of these operators are $N\times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only one possible jump at infinity. We commence with the definition of the Besseel potential spaces and Bessel potentials, arranging isometrical isomorphisms between these spaces and enabling the lifting procedure, writing a Fredholm equivalent operator (equation) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ for the operator $\mathfrak{M}^0_a$ in (17).

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is the subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of distributions having the finite norm

$$\big\|\varphi \mid \mathbb{H}^s_p(\mathbb{R})\big\| := \Bigg[\int\limits_{-\infty}^{\infty} \big|\mathscr{F}^{-1}\big(1+|\xi|^2\big)^{s/2}(\mathscr{F}\varphi)(t)\big|^p\,dt\Bigg]^{1/p} < \infty.$$

For an integer parameter $s = m = 1, 2, \dots$, the space $\mathbb{H}^s_p(\mathbb{R})$ coincides with the usual Sobolev space endowed with an equivalent norm

$$\big\|\varphi \mid \mathbb{W}^m_p(\mathbb{R})\big\| := \Bigg[\sum_{k=0}^{m}\int\limits_{-\infty}^{\infty} \Big|\frac{d^k\varphi(t)}{dt^k}\Big|^p\,dt\Bigg]^{1/p}. \tag{18}$$

If $s<0$, one gets the space of distributions. Moreover, $\mathbb{H}^{-s}_{p'}(\mathbb{R})$ is the dual to the space $\mathbb{H}^s_p(\mathbb{R}^+)$, provided $p':=\frac{p}{p-1}$, $1<p<\infty$. Note that $\mathbb{H}^s_2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_s=\int\limits_{\mathbb{R}}(\mathscr{F}\varphi)(\xi)\overline{(\mathscr{F}\psi)(\xi)}(1+\xi^2)^s\,d\xi,\quad \varphi,\psi\in\mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma\subset\mathbb{R}$. Thus $\mathbb{H}^s_p(\mathbb{R}^+)=r_+(\mathbb{H}^s_p(\mathbb{R}))$, and the norm in $\mathbb{H}^s_p(\mathbb{R}^+)$ is defined by

$$\big\|f\mid\mathbb{H}^s_p(\mathbb{R}^+)\big\|=\inf_{\ell}\big\|\ell f\mid\mathbb{H}^s_p(\mathbb{R})\big\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}^s_p(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}^s_p(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Notice that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$, and if $s\in(1/p-1,1/p)$, these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+):=\mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^-:=(-\infty,0)$.

Let $a\in\mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m\times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a:=\mathscr{F}^{-1}a\mathscr{F}. \tag{19}$$

If the operator

$$W^0_a:\mathbb{H}^s_p(\mathbb{R})\longrightarrow\mathbb{H}^{s-r}_p(\mathbb{R}) \tag{20}$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier (of order 0). The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

The Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ with the symbol a is defined by $W_a=r_+W^0_a$ where $r_+:=r_{\mathbb{R}^+}:\mathbb{H}^s_p(\mathbb{R})\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is the restriction operator.

Consider FCO

$$W_a=r_+W^0_a:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^{s-r}_p(\mathbb{R}^+), \tag{21}$$

and Hankel operators

$$H_a=r_+\boldsymbol{V}W^0_a:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^{s-r}_p(\mathbb{R}^+),\quad \boldsymbol{V}\psi(t):=\psi(-t), \tag{22}$$

where r_+ is the restriction operator to the semi-axes $\mathbb{R}^+$. Note that the generalized Hoermander's kernel of a FCO W_a depends on the difference of arguments $\mathscr{K}_a(t-\tau)$, while the Hoermander's kernel ä of a Hankel operator $r_+\boldsymbol{V}W^0_a$ depends of the sum of the arguments $\mathscr{K}_a(t+\tau)$.

If W_a in (22) is bounded, we say that W_a has order r and a is an $\mathbb{L}_p$ multiplier of order r. The set of all $\mathbb{L}_p$ multipliers of order r is denoted by $\mathfrak{M}^r_p(\mathbb{R})$. We did not use in the definition of the class of multipliers $\mathfrak{M}^r_p(\mathbb{R})$ the parameter $s\in\mathbb{R}$. This is due to the fact that $\mathfrak{M}^r_p(\mathbb{R})$ is independent of s: if the operator W_a in (22) is bounded for some $s\in\mathbb{R}$, it is bounded for all other values of s. Another definition of the multiplier class $\mathfrak{M}^r_p(\mathbb{R})$

is written as follows: $a \in \mathfrak{M}^r_p(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}^0_p(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of the following theorem.

Theorem 1.2. *Let $1 < p < \infty$. Then*

(1) *For any $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ the convolution operators (ΨDOs)*

$$\begin{gathered} \Lambda^r_\gamma = W^0_{\lambda^r_\gamma} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ \Lambda^r_{-\gamma} = r_+ W^0_{\lambda^r_{-\gamma}} \ell : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \\ \lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{gathered} \tag{23}$$

which arrange isomorphisms of the corresponding spaces (see [17, 28]*). Here $\ell : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R})$ is some extension operator, define an isomorphism between the corresponding spaces. The final result is independent of the choice of an extension ℓ. r_+ is the restriction from the axes $\mathbb{R}$ to the semi-axes $\mathbb{R}^+$.*

(2) *For any operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of the order r, the following diagram is commutative*

$$\begin{tikzcd} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \arrow[r, "\mathbf{A}"] & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \arrow[d, "\Lambda^{s-r}_-"] \\ \mathbb{L}_p(\mathbb{R}^+) \arrow[u, "\Lambda^{-s}_+"] \arrow[r, "\Lambda^{s-r}_-\mathbf{A}\Lambda^{-s}_+"'] & \mathbb{L}_p(\mathbb{R}^+) \end{tikzcd} . \tag{24}$$

The diagram (23) *provides an equivalent lifting of the operator $\mathbf{A}$ of order r to the operator $\Lambda^{s-r}_-\mathbf{A}\Lambda^{-s}_+ : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ of order* 0*.*

(3) *If $\mathbf{A} = W_a : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ is a bounded convolution operator of order r, then the lifted operator $\Lambda^{s-r}_-\mathbf{A}\Lambda^{-s}_+ : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is also a convolution operator W_{a_0}, with the symbol*

$$a_0(\xi) = \lambda^{s-r}_{-\gamma}(\xi)a(\xi)\lambda^{-s}_\gamma(\xi) = \left(\frac{\xi-\gamma}{\xi+\gamma}\right)^{s-r}\frac{a(\xi)}{(\xi+i)^r}\,.$$

Proof. For the proof we refer the reader to [17, Lemma 5.1] and [26, 28]. $\square$

Remark 1.3. The class of Fourier convolution operators is a subclass of pseudodifferential operators (ΨDOs). Moreover, for integer parameters $m = 1, 2, \ldots$ the Bessel potentials $\Lambda^m_\pm = W_{\lambda^m_{\pm\gamma}}$, which are the Fourier convolutions of order m, are ordinary differential operators of the same order m:

$$\Lambda^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \left(i\,\frac{d}{dt} \pm \gamma\right)^m = \sum_{k=0}^m \binom{m}{k} i^k(\pm\gamma)^{m-k}\,\frac{d^k}{dt^k}\,. \tag{25}$$

These potentials map both spaces (cf. (23))

$$\begin{aligned} \Lambda^m_{\pm\gamma} &: \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ &: \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-m}_p(\mathbb{R}^+), \end{aligned} \tag{26}$$

but the mappings are not isomorphisms because the inverses $\Lambda^{-m}_{\pm\gamma}$ do not map both spaces, only those indicated in (23).

2. Mellin Convolutions with Admissible Meromorphic Kernels

Now we consider kernels $\mathscr{K}(t)$, exposed in (13), (14), which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity, having poles at $c_0, c_1, \ldots \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 2.1. We call a kernel $\mathscr{K}(t)$ in (13) admissible iff:

(i) $\mathscr{K}(t)$ has only a finite number of poles $c_0, \ldots, c_\ell$ which belong to the positive semi-axes, i.e., $\arg c_0 = \cdots = \arg c_\ell = 0$;

(ii) The corresponding multiplicities are one $m_0 = \cdots = m_\ell = 1$;

(iii) The points $c_{\ell+1}, c_{\ell+2}, \ldots$ do not condense to the positive semi-axes except a finite number of points $c_0 > 0, \ldots, c_\ell > 0$ from conditions (i)–(ii) and their real parts are uniformly bounded

$$\varliminf_{j\longrightarrow\infty} c_j \notin [0,\infty), \qquad \sup_{j=\ell+1,\ell+2,\ldots} \operatorname{Re} c_j \leqslant K < \infty. \tag{27}$$

(iv) If $\mathscr{K}(t)$ emerges as a kernel of the operator, a superposition of finite number of operators with admissible kernels.

Example 2.2. The function

$$\mathscr{K}(t) = \exp\left(\frac{1}{t-c}\right), \ \ \operatorname{Re} c < 0 \ \text{ or } \ \operatorname{Im} c \neq 0$$

is an example of the admissible kernel which also satisfies the condition of the next Theorem 2.5. More trivial examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 2.5) are operators which we encounter in (3), in (11) and in (21) and, in general, any finite sum in (13).

Example 2.3. The function

$$\mathscr{K}(t) = \frac{\ln\tau - c_1c_2\ln t}{t - c_1c_2\tau}\,, \ \ \operatorname{Im} c_1 \neq 0, \ \ \operatorname{Im} c_2 \neq 0,$$

is another example of the admissible kernel, which is the composition of operators $c_2\boldsymbol{K}^1_{c_1}\boldsymbol{K}^1_{c_2}$ (see (14)) with admissible kernels which also satisfies the condition of the next Theorem 2.5. More trivial examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 2.5) are operators which we encounter in (3), in (11) and in (21) and, in general, any finite sum in (13).

Theorem 2.4. *Let conditions* (16) *hold,* $\mathscr{K}(t)$ *in* (13) *be an admissible kernel and*

$$K_\beta := \frac{\pi}{|\sin\pi\beta|}\sum_{j=0}^{\infty} 2^{m_j}|d_j|\,|c_j|^{\beta-m_j} < \infty. \tag{28}$$

Then the Mellin convolution $\mathfrak{M}^0_{a_\beta}$ *in* (10) *with the admissible meromorphic kernel* $\mathscr{K}(t)$ *in* (13) *is bounded in the Lebesgue space* $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ *and its norm is estimated by the constant* $\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\| \leqslant MK_\beta$ *with some* $M > 0$.

We can drop the constant M *and replace* 2^{m_j} *by* $2^{\frac{m_j}{2}}$ *in the estimate* (28) *provided* $\operatorname{Re} c_j < 0$ *for all* $j = 0, 1, \dots$.

Proof. The first $\ell + 1$ summands in the definition of the admissible kernel (13) correspond to the Cauchy operators

$$A_0\varphi(t) = \sum_{j=0}^{\ell} \frac{d_j}{\pi i} \int_0^\infty \frac{\varphi(\tau)\,d\tau}{t - c_j\tau}, \quad c_j > 0, \quad j = 0, 1, \dots, \ell,$$

and their boundedness property in the weighted Lebesgue space

$$A_0 : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \tag{29}$$

under constraints (16) is well known (see [35] and also [30]). Therefore we can ignore the first ℓ summands in the expansion of the kernel $\mathscr{K}(t)$ in (13). To the boundedness of the operator $\mathfrak{M}^0_{a^\ell_\beta}$ with the remainder kernel

$$\mathscr{K}^\ell(t) := \sum_{j=\ell+1}^{\infty} \frac{d_j}{(t - c_j)^{m_j}}\,, \quad c_j \neq 0, \quad j = 0, 1, \dots,$$
$$0 < \alpha_k := |\arg c_k| \leqslant \pi, \quad k = \ell + 1, \ell + 2, \dots$$

(see (13)), we apply the estimate (9)

$$\begin{aligned} &\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\right\| \leqslant \\ &\qquad \leqslant \int_0^\infty t^{\beta-1}|\mathscr{K}^\ell(t)|\,dt \leqslant \sum_{j=\ell+1}^{\infty} |d_j| \int_0^\infty \frac{t^{\beta-1}dt}{|t - c_j|^{m_j}}\,. \end{aligned} \tag{30}$$

Note now that

$$\begin{aligned} |t - c_j|^{-m_j} &= \left(t^2 + |c_j|^2 - 2\operatorname{Re} c_j t\right)^{-\frac{m_j}{2}} \leqslant \left(\frac{t^2 + |c_j|^2}{2}\right)^{-\frac{m_j}{2}} \leqslant \\ &\leqslant 2^{m_j}(t + |c_j|)^{-m_j} \quad \text{for all} \quad t \geqslant 2K = 2\sup|\operatorname{Re} c_j| > 0. \end{aligned}$$

due to the constraints (27). On the other hand,

$$|t - c_j|^{-m_j} \leqslant M(t + |c_j|)^{-m_j} \quad \text{for all} \quad 0 \leqslant t \leqslant 2K$$

and a certain constant $M > 0$. Therefore

$$|t - c_j|^{-m_j} \leqslant M 2^{m_j}(t + |c_j|)^{-m_j} \quad \text{for all} \quad 0 < t < \infty. \tag{31}$$

Next we recall the formula from [31, Formula 3.194.4]

$$\int_0^\infty \frac{t^{\beta-1}\,dt}{(t+c)^m} = (-1)^{m-1}\binom{\beta-1}{m-1}\frac{\pi c^{\,\beta-m}}{\sin\pi\beta}, \quad -\pi<\arg c<\pi, \quad \operatorname{Re}\beta<1, \tag{32}$$

$$\binom{\beta-1}{m-1} := \frac{(\beta-1)\cdots(\beta-m+1)}{(m-1)!}, \quad \binom{\beta-1}{0} := 1$$

to calculate the integrals. By inserting the estimate (31) into (30) and applying (32), we get

$$\begin{aligned}
&\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\right\| \leqslant \\
&\qquad \leqslant \sum_{j=\ell+1}^\infty |d_j| \int_0^\infty \frac{t^{\beta-1}dt}{|t-c_j|^{m_j}} \leqslant M \sum_{j=\ell+1}^\infty 2^{m_j}|d_j| \int_0^\infty \frac{t^{\beta-1}dt}{(t+|c_j|)^{m_j}} \leqslant \\
&\qquad\qquad \leqslant \frac{\pi M}{\sin\pi\beta} \sum_{j=\ell+1}^\infty 2^{m_j}|d_j| \left|\binom{\beta-1}{m_j-1}\right| c_j^{\beta-m_j} \leqslant \\
&\qquad\qquad\qquad \leqslant \frac{\pi M}{\sin\pi\beta} \sum_{j=\ell+1}^\infty 2^{m_j}|d_j| c_j^{\beta-m_j} = MK_\beta,
\end{aligned} \tag{33}$$

since (see (32))

$$\left|\binom{\beta-1}{m_j-1}\right| \leqslant 1,$$

where K_β is from (28). The boundedness (29) and the estimate (33) imply the claimed estimate

$$\left\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\right\| \leqslant MK_\beta.$$

If $\operatorname{Re} c_j < 0$ for all $j = 0, 1, \ldots$, we have

$$\begin{aligned}
\frac{1}{|t-c_j|^{m_j}} &= \left(t^2 + |c|^2 - 2\operatorname{Re} c_j t\right)^{-\frac{m_j}{2}} \leqslant \\
&\qquad \leqslant \left(t^2 + |c|^2\right)^{-\frac{m_j}{2}} \leqslant 2^{\frac{m_j}{2}}\left(t + |c_j|\right)^{-m_j}
\end{aligned}$$

valid for all $t > 0$ and a constant M does not emerge in the estimate. □

Let us find the symbol (the Mellin transform of the kernel) of the operator (14) for $0 < |\arg c| < \pi$, $m = 1, 2, \ldots$ (see (42), (14)). For this we apply formula (32):

$$\begin{aligned}
\mathscr{M}_\beta \mathscr{K}^m_c(\xi) &= \int_0^\infty t^{\beta-i\xi-1}\mathscr{K}^m_c(t)\,dt = \int_0^\infty \frac{t^{\beta-i\xi-1}}{(t+e^{\mp\pi i}c)^m}\,dt = \\
&= \binom{\beta-i\xi-1}{m-1}\frac{\pi(-1)^{m-1}e^{\mp\pi(\beta-i\xi-m)i}}{\sin\pi(\beta-i\xi)}\,c^{\beta-i\xi-m} =
\end{aligned}$$

$$= -\binom{\beta - i\xi - 1}{m-1} \frac{\pi e^{\mp\pi(\beta-i\xi)i}}{\sin\pi(\beta - i\xi)}\, c^{\beta-i\xi-m}, \quad 0 < \pm\arg c < \pi \tag{34}$$

and

$$\mathscr{M}_\beta \mathscr{K}^m_{-d}(\xi) = \int\limits_0^\infty \frac{t^{\beta-i\xi-1}dt}{(t+d)^m} = \binom{\beta - i\xi - 1}{m-1} \frac{(-1)^{m-1}\pi d^{\beta-i\xi-m}}{\sin\pi(\beta - i\xi)} \tag{35}$$

$$\text{for } \ 0 < |\arg d| < \pi, \quad \xi \in \mathbb{R}.$$

In particular,

$$\mathscr{M}_\beta \mathscr{K}^1_c(\xi) = -\frac{\pi e^{\mp\pi(\beta-i\xi)i} c^{\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,, \quad 0 < \pm\arg c < \pi, \tag{36}$$

$$\mathscr{M}_\beta \mathscr{K}^1_{-d}(\xi) = \frac{\pi d^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,, \quad 0 < |\arg d| < \pi, \tag{37}$$

$$\mathscr{M}_\beta \mathscr{K}^1_{-1}(\xi) = \frac{\pi}{\sin\pi(\beta-i\xi)}\,, \quad \xi \in \mathbb{R}. \tag{38}$$

Now let us find the symbol of the Cauchy singular integral operator $K^1_1 = -\pi i S_{\mathbb{R}^+}$ (see (43), (44)). For this we apply Plemeli formula and formula (32):

$$\begin{aligned}
\mathscr{M}_\beta \mathscr{K}^1_1(t) &:= \int\limits_0^\infty t^{\beta-i\xi-1} \mathscr{K}^1_1(t)\,dt = -\int\limits_0^\infty \frac{t^{\beta-i\xi-1}\,dt}{t-1} = \\
&= \lim_{\varepsilon\longrightarrow 0} \frac{1}{2} \int\limits_0^\infty \Big[\frac{t^{\beta-i\xi-1}}{t+e^{i(\pi-\varepsilon)}} + \frac{t^{\beta-i\xi-1}}{t+e^{-i(\pi-\varepsilon)}}\Big]\, dt = \\
&= \lim_{\varepsilon\longrightarrow 0} \pi\, \frac{e^{i(\pi-\varepsilon)(\beta-i\xi-1)} + e^{-i(\pi-\varepsilon)(\beta-i\xi-1)}}{2\sin\pi(\beta-i\xi)} = \\
&= \pi\cot\pi(\beta-i\xi).
\end{aligned} \tag{39}$$

For an admissible kernel with simple (non-multiple) poles $m_0 = m_1 = \dots = 1$ and $\arg c_0 = \arg c_\ell = 0$ and $0 < \pm\arg c_j < \pi$, $j = \ell+1, \dots$ we get

$$\begin{aligned}
\mathscr{M}_\beta \mathscr{K}(\xi) = \pi\cot\pi(\beta-i\xi) \sum_{j=0}^{\ell} d_j c_j^{\beta-i\xi-1} - \\
- \frac{\pi}{\sin\pi(\beta-i\xi)} \sum_{j=\ell+1}^{\infty} d_j \binom{\beta-i\xi-1}{m-1} \pi e^{\mp\pi(\beta-i\xi)i} c^{\beta-i\xi-m}.
\end{aligned} \tag{40}$$

Theorem 2.5. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. The Mellin convolution operator $\mathfrak{M}^0_{a_\beta}$ in (10) with an admissible kernel $\mathscr{K}$ (see (13)) is bounded in Bessel potential spaces*

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{41}$$

provided the condition (28) holds and $m^0 := \sup\limits_{j=0,1,\dots} m_j < \infty$.

The condition on the parameter p can be relaxed to $1 \leqslant p \leqslant \infty$*, provided the admissible kernel* $\mathscr{K}$ *in* (13) *has no poles on positive semi-axes* $\alpha_j = \arg c_j \neq 0$ *for all* $j = 0, 1, \dots$ *.*

Proof. Due to the representation (13), we have to prove the theorem only for a model kernel

$$\mathscr{K}_c^m(t) := \frac{1}{(t-c)^m}, \quad c \neq 0, \quad 0 \leqslant |\arg c| < \pi, \quad m = 1, 2, \dots . \tag{42}$$

The corresponding Mellin convolution operator $\boldsymbol{K}_c^m$ (see (14)) is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 \leqslant p \leqslant \infty$ for arbitrary $0 < |\arg c| < \pi$ (cf. (2)).

For $\arg c = 0$ (i.e., $c \in (0, \infty)$,) by the definition of an admissible kernel $m = 1$ and the corresponding operator coincides with the Cauchy singular integral operator $S_{\mathbb{R}^+}$

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i} \int_0^\infty \frac{\varphi(\tau)\, d\tau}{\tau - t} \tag{43}$$

modulo compact multiplier

$$\boldsymbol{K}_c^1\varphi(t) := \int_0^\infty \frac{\varphi(\tau) d\tau}{t - c\,\tau} = -\frac{\pi i}{c}\,(S_{\mathbb{R}^+}\varphi)\Big(\frac{t}{c}\Big) \tag{44}$$

and is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 < p < \infty$ (cf., e.g., [17, 30]).

Now let $0 < \arg c < 2\pi$ and $m = 1$. Then, if $\varphi \in C_0^\infty(\mathbb{R}^+)$ is a smooth function with compact support and $k = 1, 2, \dots$, integrating by parts we get

$$\begin{aligned} \frac{d^k}{dt^k}\, \boldsymbol{K}_c^1\varphi(t) &= \int_0^\infty \frac{d^k}{dt^k}\, \frac{1}{t - c\,\tau}\varphi(\tau)\, d\tau = (-c)^{-k} \int_0^\infty \frac{d^k}{d\tau^k}\, \frac{1}{t - c\,\tau}\varphi(\tau)\, d\tau = \\ &= c^{-k} \int_0^\infty \frac{1}{t - c\,\tau}\, \frac{d^k\varphi(\tau)}{d\tau^k}\, d\tau = c^{-k}\Big(\boldsymbol{K}_c^1\, \frac{d^k}{dt^k}\varphi\Big)(t). \end{aligned} \tag{45}$$

For $m = 2, 3, \dots$, we similarly get

$$\begin{aligned} \frac{d}{dt}\, \boldsymbol{K}_c^m\varphi(t) &= \int_0^\infty \frac{d}{dt}\, \frac{\tau^{m-1}}{(t - c\,\tau)^m}\, \varphi(\tau)\, d\tau = \\ &= \sum_{j=0}^{m-1} (-c)^{-1-j} \int_0^\infty \frac{d}{d\tau}\, \frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\, \varphi(\tau)\, d\tau = \\ &= -\sum_{j=0}^{m-1} (-c)^{-1-j} \int_0^\infty \frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\, \frac{d}{d\tau}\, \varphi(\tau)\, d\tau = \\ &= -\sum_{j=0}^{m-1} (-c)^{-1-j}\Big(\boldsymbol{K}_c^{m-j}\, \frac{d}{dt}\, \varphi\Big)(t) \end{aligned}$$

and, recurrently,

$$\frac{d^k}{dt^k}\,\boldsymbol{K}^m_c\varphi(t)=(-1)^k\sum_{j=0}^{m-1}(-c)^{-k-j}\gamma^k_j\Big(\boldsymbol{K}^{m-j}_c\,\frac{d^k}{dt^k}\varphi\Big)(t),\quad k=1,2,\dots,\tag{46}$$

$$\gamma^1_j=j+1,\quad \gamma^k_0=1,\quad \gamma^k_j:=\sum_{r=0}^{j}\gamma^{k-1}_r,\quad j=0,1,\dots,m,\quad k=1,2,\dots.$$

Recall now that for an integer $s=n$ the spaces $\mathbb{H}^n_p(\mathbb{R}^+)$, $\widetilde{\mathbb{H}}^n_p(\mathbb{R}^+)$ coincide with the Sobolev spaces $\mathbb{W}^n_p(\mathbb{R}^+)$, $\widetilde{\mathbb{W}}^n_p(\mathbb{R}^+)$, respectively (these spaces are isomorphic and the norms are equivalent) and $C^\infty_0(\mathbb{R}^+)$ is a dense subset in $\widetilde{\mathbb{W}}^n_p(\mathbb{R}^+)=\widetilde{\mathbb{H}}^n_p(\mathbb{R}^+)$. Then, using the equalities (45), (46) and the boundedness results of the operators $\boldsymbol{K}^{m-j}_c$ (see (14) and (43)), we proceed as follows:

$$\begin{aligned}\big\|\boldsymbol{K}^m_c\varphi\mid\mathbb{H}^n_p(\mathbb{R}^+)\big\|&=\sum_{k=0}^{n}\Big\|\frac{d^k}{dt^k}\,\boldsymbol{K}^m_\alpha\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|=\\&=\sum_{k=0}^{m}\sum_{j=0}^{m-1}|c|^{-k-j}\gamma^k_j\Big\|\boldsymbol{K}^{m-j}_c\,\frac{d^k}{dt^k}\,\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|\leqslant\\&\leqslant M\sum_{k=0}^{m}\Big\|\frac{d^k}{dt^k}\,\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|=M\big\|\varphi\mid\mathbb{H}^m_p(\mathbb{R}^+)\big\|,\end{aligned}\tag{47}$$

where $M>0$ is a constant, and there follows the boundedness result (41) for $s=0,1,2,\dots$. The case of an arbitrary $s>0$ follows by the interpolation between the spaces $\mathbb{H}^m_p(\mathbb{R}^+)$ and $\mathbb{H}^0_p(\mathbb{R}^+)=\mathbb{L}_p(\mathbb{R}^+)$, also between the spaces $\widetilde{\mathbb{H}}^m_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^0_p(\mathbb{R}^+)=\mathbb{L}_p(\mathbb{R}^+)$.

The boundedness result (41) for $s<0$ follows by duality: the adjoint operator to $\boldsymbol{K}^m_c$ is

$$\boldsymbol{K}^{m,*}_c\varphi(t):=\int_0^\infty\frac{t^{m-1}\varphi(\tau)\,d\tau}{(\tau-c\,t)^m}=\sum_{j=1}^{m}\omega_j\boldsymbol{K}^j_{c^{-1}}\varphi(t),\tag{48}$$

for some constant coefficients $\omega_1,\dots,\omega_m$. The operator $\boldsymbol{K}^{m,*}_c$ has the admissible kernel and, due to the proved part of the theorem is bounded in the space setting $\boldsymbol{K}^{m,*}_c:\widetilde{\mathbb{H}}^{-s}_{p'}(\mathbb{R}^+)\longrightarrow\mathbb{H}^{-s}_{p'}(\mathbb{R}^+)$, $p':=p/(p-1)$, since $-s>0$. The initial operator $\boldsymbol{K}^m_c:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is dual to $\boldsymbol{K}^{m,*}_c$ and, therefore, is bounded as well $\square$

Corollary 2.6. *Let* $1<p<\infty$ *and* $s\in\mathbb{R}$. *A Mellin convolution operator* $\mathfrak{M}^0_a$ *with an admissible kernel described in Definition* 2.1 (*also see Example* 2.3) *and Theorem* 2.5 *is bounded in Bessel potential spaces, see* (41).

With the help of formulae (25) and (45) for an integer $m = 1, 2\dots$ and arbitrary complex parameters $\gamma, c \in \mathbb{C}$ it follows that

$$\begin{aligned}
\Lambda^m_{-\gamma}\boldsymbol{K}^1_c\varphi &= \Big(i\,\frac{d}{dt}\pm\gamma\Big)^m\boldsymbol{K}^1_c\varphi = \sum_{k=0}^m\binom{m}{k}i^k(\pm\gamma)^{m-k}\,\frac{d^k}{dt^k}\,\boldsymbol{K}^1_c\varphi = \\
&= \sum_{k=0}^m\binom{m}{k}i^k(\pm\gamma)^{m-k}c^{-k}\Big(\boldsymbol{K}^1_c\,\frac{d^k}{dt^k}\,\varphi\Big)(t) = \\
&= c^{-m}\boldsymbol{K}^1_c\Big(\sum_{k=0}^m\binom{m}{k}i^k(\pm c\,\gamma)^{m-k}\,\frac{d^k}{dt^k}\,\varphi\Big)(t) = \\
&= c^{-m}\boldsymbol{K}^1_c\Lambda^m_{-c\,\gamma}\varphi,\ \ \varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),\ \ 0<|\arg\gamma|<\pi. \qquad (49)
\end{aligned}$$

Next, we will generalize formula (49).

Theorem 2.7. *Let* $0 < |\arg c| < \pi$, $0 < |\arg\gamma| < \pi$, $0 < |\arg(c\,\gamma)| < \pi$, $r, s \in \mathbb{R}$, $m = 1, 2, \dots$, $1 < p < \infty$. *Then*

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\varphi = \\
= \begin{cases} e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c\Lambda^s_{-c\,\gamma}\varphi & \text{if } -\pi<\arg c\,\gamma<0,\\ e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c\Lambda^s_{-c\,\gamma}\varphi & \text{if } 0<\arg c\,\gamma<\pi,\ \ \varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),\end{cases} \qquad (50)$$

where

$$\sigma(c,\gamma) := \begin{cases} 0 & \text{if } 0<\arg c<\pi,\\ \operatorname{sign}\arg(c\,\gamma)-\operatorname{sign}\arg\gamma & \text{if } -\pi<\arg c<0,\end{cases} \qquad (51)$$

$$\widetilde{\boldsymbol{K}}^m_c\psi(t) = \boldsymbol{K}^m_c\psi_+(t)+(-1)^{m-1}\boldsymbol{K}^m_{-c}\psi_-(t),\ \psi\in\mathbb{L}_p(\mathbb{R}),\ \psi_\pm\in\mathbb{L}_p(\mathbb{R}^+), \qquad (52)$$

$\psi_\pm(t) := r_+\psi(\pm t)$ *and* r_+ *is the restriction from* $\mathbb{R}$ *to* $\mathbb{R}^+$.

Proof. First we consider the case $m = 1$ (a simple pole). Let $\Lambda^s_{-\gamma,t}\psi(t,\tau)$ denote the action of the Bessel potential operator $\Lambda^s_{-\gamma}$ (see (23)) on a function $\psi(t,\tau)$ with respect to the variable t (see (14)):

$$\begin{aligned}
\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) &:= r_+\int_0^\infty\Big[\Lambda^s_{-\gamma,t}\,\frac{1}{t-c\,\tau}\Big]\varphi(\tau)\,d\tau = \\
&= \frac{1}{2\pi}\,r_+\int_0^\infty\varphi(\tau)\int_{-\infty}^\infty e^{-i\xi t}(\xi-\gamma)^s\int_{-\infty}^\infty\frac{e^{i\xi y}}{y-c\tau}\,dy\,d\xi\,d\tau, \qquad (53)
\end{aligned}$$

where r_+ is the restriction to $\mathbb{R}^+$. The integrand in the last integral in (53) is a meromorphic function with a single pole at $c\,\tau$ and the function vanishes as $|y| \longrightarrow \infty$, provided $\xi < 0$ for $0 < \arg c < \pi$ and for $\xi > 0$ for $-\pi < \arg c < 0$, respectively. Therefore, by the Cauchy theorem, the integral vanishes for $\xi < 0$ in the first and for $\xi > 0$ in the second case, respectively. Since $\tau > 0$, the integral is found with the help of the residue theorem:

$$\int_{-\infty}^{\infty} \frac{e^{i\xi y}}{y - c\tau}\, dy = \begin{cases} 0 & \text{for } \ \xi \arg c < 0, \\ 2\pi i e^{ic\xi\tau} & \text{for } \ \xi > 0 \ \text{ and } \ 0 < \arg c < \pi, \\ -2\pi i e^{ic\xi\tau} & \text{for } \ \xi < 0 \ \text{ and } \ -\pi < \arg c < 0. \end{cases} \tag{54}$$

Then

$$\Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) = i r_+ \int_0^{\infty} \varphi(\tau) \int_0^{\infty} e^{-i\xi(t-c\tau)} (\xi - \gamma)^s \, d\xi \, d\tau, \quad 0 < \arg c < \pi, \tag{55a}$$

$$\begin{aligned} \Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) &= -i r_+ \int_0^{\infty} \varphi(\tau) \int_{-\infty}^{0} e^{-i\xi(t-c\tau)} (\xi - \gamma)^s \, d\xi \, d\tau = \\ &= -i e^{-\sigma(\gamma)\pi s i} r_+ \int_0^{\infty} \varphi(\tau) \int_0^{\infty} e^{i\xi(t-c\tau)} (\xi + \gamma)^s \, d\xi \, d\tau \qquad (55\text{b}) \\ &\qquad \text{for } \ \sigma(\gamma) := \operatorname{sign} \arg \gamma, \quad -\pi < \arg c < 0 \end{aligned}$$

because $\arg(-\xi - \gamma) = \arg(\xi + \gamma) \pm \pi \in (-\pi, \pi)$ for $0 < \mp \arg \gamma < \pi$. To (55a) and (55b) we apply the formula (see [31, Formula 3.382.4])

$$\int_0^{\infty} e^{-\mu\xi} (\xi + \nu)^s \, d\xi = \mu^{-s-1} e^{\nu\mu} \Gamma(s + 1, \nu\mu), \tag{56}$$

$$s \in \mathbb{R}, \quad -\pi < \arg \nu < \pi, \quad \operatorname{Re} \mu > 0.$$

To comply with the constraint $-\pi < \arg \nu < \pi$ for $\nu = -\gamma$, we choose $\arg(-\gamma) = \arg \gamma \pm \pi$ for $0 < \mp \arg \gamma < \pi$. From $0 < \arg c < \pi$ follows the constraint $\operatorname{Re} \mu > 0$ for $\mu = i(t - c\tau)$ and from (55a) with the help of (56) we get

$$\begin{aligned} \Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) &= i r_+ \int_0^{\infty} (it - ic\tau)^{-s-1} e^{-i\gamma(t-c\tau)} \Gamma(s + 1, -i\gamma(t - c\tau)) \varphi(\tau) \, d\tau = \\ &= e^{-\frac{\pi}{2} si} r_+ \int_0^{\infty} \frac{e^{-i\gamma(t-c\tau)} \Gamma(s + 1, -i\gamma(t - c\tau))}{(t - c\tau)^{s+1}} \varphi(\tau) \, d\tau, \qquad (57\text{a}) \end{aligned}$$

since $\arg(it - ic\tau) = \arg(t - c\tau) + \pi/2 \in (-\pi, \pi)$ and, therefore, $i(it - ic\tau)^{-s-1} = e^{-\frac{\pi}{2} si} (t - c\tau)^{-s-1}$.

Similarly, from $-\pi < \arg c < 0$ follows the constraint $\operatorname{Re} \mu > 0$ for $\mu = -i(t - c\tau)$ and from (55b) with the help of (56) we get

$$\Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) =$$

$$= -i e^{-\sigma(\gamma)\pi s i} r_+ \int_0^{\infty} (-it + ic\tau)^{-s-1} e^{-i\gamma(t-c\tau)} \Gamma(s + 1, -i\gamma(t - c\tau)) \varphi(\tau) \, d\tau =$$

$$= e^{(-\sigma(\gamma)\pi+\frac{\pi}{2})si}\, r_+\int\limits_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(s+1,-i\gamma(t-c\tau))}{(t-c\tau)^{s+1}}\,\varphi(\tau)\,d\tau, \tag{57b}$$

$$\text{for}\ \ \sigma(\gamma):=\operatorname{sign}\arg\gamma,\quad -\pi<\arg c<0,$$

since $\arg(-it+ic\tau)=\arg(t-c\tau)-\pi/2\in(-\pi,\pi)$ and, therefore, $i(-it+ic\tau)^{-s-1}=-e^{\frac{\pi}{2}\,si}\,(t-c\tau)^{-s-1}$.

Next, we check what are the results if the Bessel potential $\Lambda^s_{c\,\gamma,\tau}$ is applied to the kernel $\frac{1}{t-c\tau}$ of the operator $\boldsymbol{K}^1_c$ with respect to the variable τ:

$$\begin{aligned}
\boldsymbol{A}_\gamma\varphi(t) &:= r_+\int\limits_0^\infty\Big[\Lambda^s_{c\,\gamma,y}\,\frac{1}{t-c\,y}\Big]\varphi(\tau)\,d\tau=\\
&= \frac{1}{2\pi}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_{-\infty}^\infty e^{-i\xi\tau}(\xi+c\,\gamma)^s\int\limits_{-\infty}^\infty\frac{e^{i\xi y}\,dy}{t-cy}\,d\xi\,d\tau=\\
&= -\frac{1}{2\pi c}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_{-\infty}^\infty e^{-i\xi\tau}(\xi+c\,\gamma)^s\int\limits_{-\infty}^\infty\frac{e^{i\xi y}\,dy}{y-c^{-1}t}\,d\xi\,d\tau.
\end{aligned}\tag{58}$$

The last integral in (58) is found with the help of the residue theorem, by taking into account that $\tau>0$ (cf. (54)):

$$\int\limits_{-\infty}^\infty\frac{e^{i\xi y}}{y-c^{-1}t}\,dy=\begin{cases}0 & \text{for}\ \ \xi\arg c>0,\\ -2\pi i e^{ic^{-1}\xi\,t} & \text{for}\ \ \xi<0\ \ \text{and}\ \ 0<\arg c<\pi,\\ 2\pi i e^{ic^{-1}\xi\,t} & \text{for}\ \ \xi>0\ \ \text{and}\ \ -\pi<\arg c<0.\end{cases}\tag{59}$$

Applying formula (59), we proceed as follows:

$$\begin{aligned}
\boldsymbol{A}_\gamma\varphi(t) &= \frac{i}{c}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_{-\infty}^0 e^{-i\xi(\tau-c^{-1}t)}(\xi+c\,\gamma)^s\,d\xi\,d\tau=\\
&= \frac{ie^{\sigma(c\gamma)\pi si}}{c}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_0^\infty e^{-ic^{-1}\xi(t-c\,\tau)}(\xi-c\,\gamma)^s\,d\xi\,d\tau,
\end{aligned}\tag{60a}$$

$$\sigma(\gamma):=\operatorname{sign}\arg\gamma\ \ \text{for}\ \ 0<\arg c<\pi,$$

because $\arg(-\xi+c\,\gamma)=\arg(\xi-c\,\gamma)\pm\pi\in(-\pi,\pi)$. Similarly,

$$\boldsymbol{A}_\gamma\varphi(t) = -\frac{i}{c}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_0^\infty e^{-i\xi(\tau-c^{-1}t)}(\xi+c\,\gamma)^s\,d\xi\,d\tau=$$

$$= -\frac{i}{c}\,r_+\int\limits_0^\infty\varphi(\tau)\int\limits_0^\infty e^{ic^{-1}\xi(t-c\,\tau)}(\xi+c\,\gamma)^s\,d\xi\,d\tau,\quad -\pi<\arg c<0. \tag{60b}$$

To (60a) and (60b) we apply the formula (56) with $\mu=\pm ic^{-1}(t-c\tau)$ and $\nu=\mp c\,\gamma$, which yields $\nu\mu=-i\gamma(t-c\tau)$. The constraint $0<|\arg(c\,\gamma)|<\pi$, imposed in the theorem, allows us to comply with the condition $-\pi < \nu < \pi$ by choosing $\arg(-c\,\gamma) = \arg(c\,\gamma) \mp \pi$ for $\pm \arg(c\,\gamma) > 0$. Another constraint $0 < |\arg c| < \pi$ allows to comply with the condition $\mathrm{Re}\,\mu > 0$ in (56): $\mathrm{Re}(\pm i c^{-1} t \mp i\tau) = \mp \mathrm{Im}\, c^{-1} t = \pm t\, \frac{\mathrm{Im}\, c}{|c|^2} > 0$ for $0 < \pm \arg c < \pi$. We get the following:

$$\boldsymbol{A}_\gamma \varphi(t) = \frac{i e^{-\sigma(c\gamma)\pi s i}}{c}\, r_+ \times$$
$$\times \int_0^\infty (i c^{-1})^{-s-1} (t - c\,\tau)^{-s-1} e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau)) \varphi(\tau)\, d\tau =$$
$$= c^s e^{(\sigma(c\gamma)\pi - \frac{\pi}{2}) s i} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\tau))}{(t - c\,\tau)^{s+1}}\, \varphi(\tau)\, d\tau \tag{61a}$$
$$\text{for } \ \sigma(c\,\gamma) := \operatorname{sign} \arg(c\gamma), \quad 0 < \arg c < \pi,$$

since $i^{-s-1} = i^{-1} e^{-\frac{\pi}{2}\, si}$, and

$$\boldsymbol{A}_\gamma \varphi(t) = -\frac{i}{c}\, r_+ \times$$
$$\times \int_0^\infty (-i c^{-1})^{-s-1} (t - c\,\tau)^{-s-1} e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau)) \varphi(\tau)\, d\tau =$$
$$= c^s e^{\frac{\pi}{2}\, si} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau))}{(t - c\,\tau)^{s+1}}\, \varphi(\tau)\, d\tau \tag{61b}$$

for $-\pi < \arg c < 0$, since $(-i)^{-s-1} = i e^{\frac{\pi}{2}\, si}$.

From (57a)–(57b), (58) and (61)–(61) we derive the following equality:

$$\Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) = \int_0^\infty \Big[\Lambda^s_{-\gamma,t}\, \frac{1}{t - c\tau}\Big] \varphi(\tau)\, d\tau =$$
$$= c^{-s} e^{-\sigma_0(c,\gamma)\pi\, si} \int_{-\infty}^\infty \Big[\Lambda^s_{c\,\gamma,\tau}\, \frac{1}{t - c\tau}\Big] \varphi_0(\tau)\, d\tau, \tag{62}$$

where

$$\sigma_0(c, \gamma) := \begin{cases} \sigma(c\,\gamma) & \text{if } \ 0 < \arg c < \pi, \\ \sigma(\gamma) & \text{if } \ -\pi < \arg c < 0 \end{cases} \tag{63}$$

and $\varphi_0 \in \mathbb{H}^1_2(\mathbb{R})$ is the extension of $\varphi_0 \in \widetilde{\mathbb{H}}^1_2(\mathbb{R}^+)$ by 0 to the semi-axes $\mathbb{R}^- := \mathbb{R} \setminus \overline{\mathbb{R}^+}$. Now note, that the operator $\Lambda^s_{c\,\gamma,\tau}$ is the dual (adjoint) to the operator $e^{\sigma(c\,\gamma)\pi si}\Lambda^s_{-c\,\gamma,\tau}$ i.e,

$$\int\limits_{-\infty}^{\infty}(\Lambda^s_{c\,\gamma,\tau}u)(\tau)v(\tau)\,d\tau=e^{\sigma(c\,\gamma)\pi si}\int\limits_{-\infty}^{\infty}u(\tau)(\Lambda^s_{-c\,\gamma,\tau}v)(\tau)\,d\tau,\qquad \forall\,u,v\in C^\infty_0(\mathbb{R}),$$

the equality can easily be verified by changing the orders of integration and change the Fourier transform variable ξ to $-\xi$. We continue the equality (62) as follows:

$$\begin{aligned}\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=c^{-s}e^{-\sigma_0(c,\gamma)\pi\,si}\int\limits_{-\infty}^{\infty}\Big[\Lambda^s_{c\,\gamma,\tau}\frac{1}{t-c\tau}\Big]\varphi_0(\tau)\,d\tau=\\&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\int\limits_{-\infty}^{\infty}\frac{\Lambda^s_{-c\,\gamma}\varphi(\tau)\,d\tau}{t-c\tau},\end{aligned}$$

where $\sigma(c,\gamma)$ is defined in (51). By the properties of the Bessel potential $\Lambda^m_{-c\,\gamma}$, it maintains the support of a function $\operatorname{supp}\varphi\subset\overline{\mathbb{R}^+}$ for $-\pi<\arg c\,\gamma<0$ but not for $0<\arg c\,\gamma<\pi$. Therefore,

$$\begin{aligned}\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\boldsymbol{K}^1_c\Lambda^s_{-c\,\gamma}\varphi(t)\ \text{ for }\ -\pi<\arg c\,\gamma<0,\\\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\widetilde{\boldsymbol{K}}^1_c\Lambda^s_{-c\,\gamma}\varphi(t)\ \text{ for }\ 0<\arg c\,\gamma<\pi.\end{aligned}\tag{64}$$

Formula (64) accomplishes the proof of formula (50) for an operator $\boldsymbol{K}^1_c$ (case $m=1$) and under the additional constraint $\arg c\neq 0$. For an operator $\boldsymbol{K}^1_c$ (case $m=1$) but $\arg c=0$ and a case of an operator $\boldsymbol{K}^m_c$, $m=2,3,\ldots$ we can deal with a perturbation:

$$\begin{gathered}\frac{1}{(t-c)^m}=\lim_{ve\longrightarrow 0}\mathscr{K}_\varepsilon(t),\\\mathscr{K}_{c_{1,\varepsilon},\ldots,c_{m,\varepsilon}}(t):=\frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{m,\varepsilon})}=\sum_{j=1}^m\frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}},\\c_{j,\varepsilon}=c(1+\varepsilon e^{i\omega_j}),\quad\omega_j\in(-\pi,\pi),\quad\arg c_{j,\varepsilon},\quad\arg c_{j,\varepsilon}\,\gamma_j\neq 0,\quad j=1,\ldots,m.\end{gathered}\tag{65}$$

the points and $\omega_1,\ldots,\omega_m\in(-\pi,\pi]$ are distinct $\omega_j\neq\omega_k$ for $j\neq k$. The case $\arg c=0$ is covered for $m=1$. By equating the numerators in the formula (65)

$$\begin{aligned}&\sum_{j=1}^m d_j(\varepsilon)t^m-(m-1)\sum_{j=1}^m d_j(\varepsilon)c_{j,\varepsilon}t^{m-1}+\cdots=\\&\qquad=\sum_{j=1}^m d_j(\varepsilon)(t^m-ct^{m-1})-(m-1)\varepsilon\sum_{j=1}^m e^{\omega_j}d_j(\varepsilon)t^{m-1}+\mathscr{O}(\varepsilon)=1,\end{aligned}$$

we derive the last two equalities

$$d_j(\varepsilon)=\mathscr{O}(\varepsilon^{-1}),\quad \sum_{j=1}^m d_j(\varepsilon)=0,\quad \sum_{j=1}^m e^{\omega_j}d_j(\varepsilon)=0, \tag{66}$$

while the first one is well known. The claimed equality (64) holds for each operator $\boldsymbol{K}^1_{c_{j,\varepsilon}}$ and

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi=\sum_{j=1}^m d_j(\varepsilon)\Lambda^s_{-\gamma}\boldsymbol{K}^1_{c_{j,\varepsilon}}\varphi=\sum_{j=1}^m c^s_{j,\varepsilon}d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi, \tag{67}$$

where

$$\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi(t)=\int_0^\infty \mathscr{K}_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}=\int_0^\infty\frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c_{1,\varepsilon}\tau)\cdots(t-c_{m,\varepsilon}\tau)}\,.$$

Further, we assume that $-\pi<\arg c\,\gamma<0$. The case $0<\arg c\,\gamma<\pi$ is considered similarly and we drop its proof.

Using the Bessel potentials (see (23)), we get

$$\begin{aligned}
&\Lambda^{-s}_{-c\,\gamma}\big[\Lambda^s_{-c\,\gamma_{j,\varepsilon}}-\Lambda^s_{-c\,\gamma}\big]=W_{a_{j,\varepsilon}}-I=W_{a_{j,\varepsilon}-1},\quad \sigma:=\sigma(c,\gamma)=\sigma(c,\gamma),\\
&\qquad a_{j,\varepsilon}(\xi)-1=\Big(\frac{\xi-c\,\gamma_{j,\varepsilon}}{\xi-c\,\gamma}\Big)^s-1=\Big(1-\frac{\varepsilon e^{i\omega_j}}{\frac{\xi}{c\,\gamma}-1}\Big)^s-1=\\
&\quad=-\frac{se^{i\omega_j}}{\frac{\xi}{c\,\gamma}-1}\varepsilon+a^0_{j,\varepsilon}(\xi)\varepsilon^2=\frac{s\,c\,\gamma e^{i\omega_j}}{\xi-c\,\gamma}\,\varepsilon+a^0_{j,\varepsilon}(\xi)\varepsilon^2,\ \ a^0_{j,\varepsilon}=\mathscr{O}(1), && (68)\\
&\quad c^{-s}_{j,\varepsilon}=c^{-s}(1+\varepsilon e^{i\omega_j})^{-s}=c^{-s}-c^{-s}se^{i\omega_j}\varepsilon+b_{j,\varepsilon}\varepsilon^2,\ \ b_{j,\varepsilon}=\mathscr{O}(1) && (69)
\end{aligned}$$

as $\varepsilon\longrightarrow 0$. For ε sufficiently small, the value $\sigma(c_{j,\varepsilon},\gamma)$ becomes independent of $j=1,\dots,m$ and ε, and we use the notation $\sigma(c,\gamma):=\sigma(c_{j,\varepsilon},\gamma)$. Then, by virtue of the equality (66) and asymptotic (68), (69), we get the following equalities:

$$\begin{aligned}
\Lambda^s_{-\gamma}\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi &:=\sum_{j=1}^m d_j(\varepsilon)\Lambda^s_{-\gamma}\boldsymbol{K}^1_{c_{j,\varepsilon}}\varphi=\\
&=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}c^{-s}_{j,\varepsilon}d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=\\
&=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}\Big[c^{-s}-c^{-s}se^{i\omega_j}\varepsilon+b_{j,\varepsilon}\varepsilon^2\Big]d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=\\
&=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}\big[c^{-s}+b_{j,\varepsilon}\varepsilon^2\big]d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=\\
&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\sum_{j=1}^m d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma}\varphi+
\end{aligned}$$

$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma} \Lambda^{-s}_{-c\,\gamma} \Big[\Lambda^s_{-c\,\gamma_{j,\varepsilon}} - \Lambda^s_{-c\,\gamma} \Big] \varphi +$$

$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) b_{j,\varepsilon} \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi =$$

$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma} \varphi +$$

$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} \boldsymbol{K}^1_{c_{j,\varepsilon}} d_j(\varepsilon) \Big[- s\, c\, \gamma e^{i\omega_j} W_{\frac{1}{\xi + c\,\gamma}} \varepsilon + W_{a^0_{j,\varepsilon}(\xi)} \varepsilon^2 \Big] \Lambda^s_{-c\,\gamma} \varphi +$$

$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \varepsilon^2 \sum_{j=1}^{m} d_j(\varepsilon) b_{j\varepsilon} \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-\gamma\, c_{j,\varepsilon}} \varphi =$$

$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma} \varphi +$$

$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \big[b_{j\varepsilon} + W_{a^0_{j,\varepsilon}} \big] \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi. \quad (70)$$

By using the boundedness result proved in Theorem 2.5, we get

$$\lim_{\varepsilon \longrightarrow 0} \big\| \boldsymbol{K}^m_c - \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \varphi \mid \mathbb{H}^\nu_2(\mathbb{R}^+) \big\| \leqslant$$

$$\leqslant \lim_{\varepsilon \longrightarrow 0} \varepsilon \sum_{j=1}^{m} \big\| \boldsymbol{K}^m_{c,\dots,c,c_{j,\varepsilon},\dots,c_{m,\varepsilon}} \varphi \mid \mathbb{H}^\nu_2(\mathbb{R}^+) \big\| = 0. \quad (71)$$

Further, invoking the well known formula for the norm of a convolution operator in the Hilbert-Bessel spaces $\mathbb{L}_p(\mathbb{R}^+)$

$$\big\| W_a \mid \mathscr{L}(\mathbb{H}^\mu_2(\mathbb{R}^+)) \big\| = \big\| W_a \mid \mathscr{L}(\mathbb{L}_2(\mathbb{R}^+)) \big\| = \sup_{\xi \in \mathbb{R}} |a(\xi)| \quad (72)$$

(cf., e.g., [17]) and using the property $\lim\limits_{\varepsilon \longrightarrow 0} \varepsilon^2 d_j(\varepsilon) = 0$ (see (66)), from (70)–(72) we derive

$$\Lambda^s_{-\gamma} \boldsymbol{K}^m_c \varphi = \lim_{\varepsilon \longrightarrow 0} \Lambda^s_{-\gamma} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \varphi =$$

$$= \lim_{\varepsilon \longrightarrow 0} \Bigg[e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma} \varphi +$$

$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \big[b_{j\varepsilon} + W_{a^0_{j,\varepsilon}} \big] \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi \Bigg] =$$

$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \lim_{\varepsilon \longrightarrow 0} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma} \varphi =$$

$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_c \Lambda^s_{-c\,\gamma} \varphi$$

which accomplishes the proof. $\square$

3. Algebra Generated by Mellin and Fourier Convolution Operators

Unlike the operators W^0_a and $\mathfrak{M}^0_a$ (see Section 1), possessing the property

$$W^0_a W^0_b = W^0_{ab}, \ \ \mathfrak{M}^0_a\mathfrak{M}^0_b = \mathfrak{M}^0_{ab} \ \text{ for all } \ a,b \in \mathfrak{M}_p(\mathbb{R}), \tag{73}$$

the composition of the convolution operators on the semi-axes W_a and W_b (see (73)) cannot be computed by the rules similar to (73). Nevertheless, the following propositions hold.

Proposition 3.1 ([17, Section 2])**.** *Assume that $1 < p < \infty$, and let $[W_a, W_b] := W_aW_b - W_bW_a$ be the commutant of the operators W_a and W_b. If $a, b \in \mathfrak{M}_p(\overline{\mathbb{R}}^+) \cap PC(\dot{\mathbb{R}})$ are piecewise-continuous scalar $\mathbb{L}_p$-multipliers, then the commutant $[W_a, W_b] : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ is compact.*

Moreover, if, in addition, the symbols $a(\xi)$ and $b(\xi)$ of the operators W_a and W_b have no common discontinuity points, i.e., if

$$\big[a(\xi + 0) - a(\xi + 0)\big]\big[b(\xi + 0) - b(\xi + 0)\big] = 0 \ \textit{ for all } \ \xi \in \dot{\mathbb{R}},$$

then $T = W_aW_b - W_{ab}$ is a compact operator in $\mathbb{L}_p(\mathbb{R}^+)$.

Note that the algebra of $N \times N$ matrix multipliers $\mathfrak{M}_2(\mathbb{R})$ coincides with the algebra of $N \times N$ matrix functions essentially bounded on $\mathbb{R}$. For $p \neq 2$, the algebra $\mathfrak{M}_p(\mathbb{R})$ is rather complicated. There are multipliers $g \in \mathfrak{M}_p(\mathbb{R})$ which are elliptic, i.e. $\operatorname{ess\,inf} |g(x)| > 0$, but $1/g \notin \mathfrak{M}_p(\mathbb{R})$. In connection with this, let us consider the subalgebra $PC\mathfrak{M}_p(\mathbb{R})$ which is the closure of the algebra of piecewise-constant functions on $\mathbb{R}$ in the norm of multipliers $\mathfrak{M}_p(\mathbb{R})$

$$\big\|a \mid \mathfrak{M}_p(\mathbb{R})\big\| := \big\|W^0_a \mid \mathbb{L}_p(\mathbb{R})\big\|.$$

Note that any function $g \in PC\mathfrak{M}_p(\mathbb{R}) \subset PC(\mathbb{R})$ has limits $g(x \pm 0)$ for all $x \in \overline{\mathbb{R}}$, including the infinity. Let

$$C\mathfrak{M}_p(\overline{\mathbb{R}}) := C(\overline{\mathbb{R}}) \cap PC\mathfrak{M}^0_p(\mathbb{R}), \quad C\mathfrak{M}^0_p(\dot{\mathbb{R}}) := C(\dot{\mathbb{R}}) \cap PC\mathfrak{M}_p(\mathbb{R}),$$

where functions $g \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ (functions $h \in C(\dot{\mathbb{R}})$) might have jump only at the infinity $g(-\infty) \neq g(+\infty)$ (are continuous at the infinity $h(-\infty) = h(+\infty)$).

$PC\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra and contains all functions of bounded variation as a subset for all $1 < p < \infty$ (Stechkin's theorem, see [17, Section 2]). Therefore, $\coth \pi(i\beta + \xi) \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ for all $p \in (1, \infty)$.

Proposition 3.2 ([17, Section 2])**.** *If $g \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ is an $N \times N$ matrix multiplier, then its inverse $g^{-1} \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ if and only if it is elliptic, i.e. $\det g(x \pm 0) \neq 0$ for all $x \in \overline{\mathbb{R}}$. If this is the case, the corresponding Mellin convolution operator $\mathfrak{M}^0_g : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ is invertible and $(\mathfrak{M}^0_g)^{-1} = \mathfrak{M}^0_{g^{-1}}$.*

Moreover, any $N\times N$ matrix multiplier $b\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ can be approximated by polynomials

$$r_n(\xi):=\sum_{j=-m}^{m} c_m\Big(\frac{\xi-i}{\xi+i}\Big)^m,\quad r_m\in C\mathfrak{M}^0_p(\overline{\mathbb{R}}),$$

with constant $N\times N$ matrix coefficients, whereas any $N\times N$ matrix multiplier $g\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$ having a jump discontinuity at infinity can be approximated by $N\times N$ matrix functions $d\coth\pi(i\beta+\xi)+r_m(\xi)$, $0<\beta<1$.

Due to the connection between the Fourier and Mellin convolution operators (see Introduction, (4)), the following is a direct consequence of Proposition 3.2.

Corollary 3.3. *The Mellin convolution operator*

$$\boldsymbol{A}=\mathfrak{M}^0_{\mathscr{A}_\beta}:\mathbb{L}_p(\mathbb{R},t^\gamma),$$

in (1) with the symbol $\mathscr{A}_\beta(\xi)$ in (5) is invertible if and only if the symbol is elliptic,

$$\inf_{\xi\in\mathbb{R}}\big|\det\mathscr{A}_\beta(\xi)\big|>0 \tag{74}$$

and the inverse is then written as $\mathbf{A}^{-1}=\mathfrak{M}^0_{\mathscr{A}^{-1}_{1/p}}$.

The Hilbert transform on the semi-axis

$$S_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int_0^\infty\frac{\varphi(y)\,dy}{y-x} \tag{75}$$

is the Fourier convolution $S_{\mathbb{R}^+}=W_{-\mathrm{sign}}$ on the semi-axis $\mathbb{R}^+$ with the discontinuous symbol $-\operatorname{sign}\xi$ (see [17, Lemma 1.35]), and it is also the Mellin convolution

$$S_{\mathbb{R}^+}=\mathfrak{M}^0_{s_\beta}=\mathbf{Z}_\beta W^0_{s_\beta}\mathbf{Z}^{-1}_\beta, \tag{76}$$

$$s_\beta(\xi):=\coth\pi(i\beta+\xi)=\frac{e^{\pi(i\beta+\xi)}+e^{-\pi(i\beta+\xi)}}{e^{\pi(i\beta+\xi)}-e^{-\pi(i\beta+\xi)}}=-i\cot\pi(\beta-i\xi),\ \xi\in\mathbb{R}$$

(cf. (5) and (8)). Indeed, to verify (76) rewrite $S_{\mathbb{R}^+}$ in the following form

$$S_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int_0^\infty\frac{\varphi(y)}{1-\frac{x}{y}}\,\frac{dy}{y}=\int_0^\infty K\Big(\frac{x}{y}\Big)\varphi(y)\,\frac{dy}{y}\,,$$

where $K(t):=(1/\pi i)(1-t)^{-1}$. Further, using the formula

$$\int_0^\infty\frac{t^{z-1}}{1-t}\,dt=\pi\cot\pi z,\quad \operatorname{Re}z<1,$$

cf. [31, formula 3.241.3], one shows that the Mellin transform $\mathscr{M}_\beta K(\xi)$ coincides with the function $s_\beta(\xi)$ from (76).

For our aim we will need certain results concerning the compactness of Mellin and Fourier convolutions in $\mathbb{L}_p$-spaces. These results are scattered in literature. For the convenience of the reader, we reformulate them here as Propositions 3.4–3.8. For more details, the reader can consult [8, 17, 22].

Proposition 3.4 ([22, Proposition 1.6])**.** *Let* $1 < p < \infty$*,* $a \in C(\dot{\overline{\mathbb{R}}}^+)$*,* $b \in C\mathfrak{M}_p^0(\dot{\mathbb{R}})$ *and* $a(0) = b(\infty) = 0$*. Then the operators* $a\mathfrak{M}_b^0, \mathfrak{M}_b^0 aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 3.5 ([17, Lemma 7.1] and [22, Proposition 1.2])**.** *Let* $1 < p < \infty$*,* $a \in C(\dot{\overline{\mathbb{R}}}^+)$*,* $b \in C\mathfrak{M}_p^0(\dot{\mathbb{R}})$ *and* $a(\infty) = b(\infty) = 0$*. Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 3.6 ([22, Lemma 2.5, Lemma 2.6] and [8])**.** *Assume that* $1 < p < \infty$*. Then*

(1) *If* $g \in C\mathfrak{M}_p^0(\dot{\mathbb{R}})$ *and* $g(\infty) = 0$*, the Hankel operator* $H_g : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is compact;*

(2) *If the functions* $a \in C(\dot{\mathbb{R}})$*,* $b \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$*,* $c \in C(\overline{\mathbb{R}}^+)$ *and satisfy at least one of the conditions*

  (i) $c(0) = b(+\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi > 0$*,*

  (ii) $c(0) = b(-\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi < 0$*,*

then the operators $cW_a\mathfrak{M}_b^0$*,* $c\mathfrak{M}_b^0 W_a$*,* $W_a\mathfrak{M}_b^0\, cI$*,* $\mathfrak{M}_b^0 W_a\, cI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proof. Let us comment only on item 2 in Proposition 3.6, which is not proved in [22], although is well known. The kernel $k(x+y)$ of the operator H_a is approximated by the Laguerre polynomials $k_m(x+y) = e^{-x-y}p_m(x+y)$, $m = 1, 2, \dots$, where $p_m(x+y)$ are polynomials of order m so that the corresponding Hankel operators converge in norm $\|H_a - H_{a_m} \,|\, \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\| \longrightarrow 0$, where $a_m = \mathscr{F}k_m$ are the Fourier transforms of the Laguerre polynomials (see, e.g. [29]). Since

$$|k_m(x+y)| = \left|e^{-x-y}p_m(x+y)\right| \leqslant C_m e^{-x}e^{-y}x^m y^m, \quad m = 1, 2, \dots ,$$

for some constant C_m, the condition on the kernel

$$\int_0^\infty \left[\int_0^\infty |k_m(x+y)|^{p'}\, dy\right]^{p/p'} dx < \infty, \quad p' := \frac{p}{p-1}\, ,$$

holds and ensures the compactness of the operator $H_{a_m} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$. Then the limit operator $H_a = \lim\limits_{m\longrightarrow\infty} H_{a_m}$ is compact as well. $\square$

Proposition 3.7 ([17, Lemma 7.4] and [22, Lemma 1.2])**.** *Let* $1 < p < \infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a \in C(\overline{\mathbb{R}}^+)$*,* $b \in \mathfrak{M}_p^0(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$*,*

(ii) $a \in PC(\overline{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$.

Then the commutants $[aI, W_b]$ *and* $[aI, \mathfrak{M}^0_b]$ *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Proposition 3.8 ([22]). *The Banach algebra, generated by the Cauchy singular integral operator* $S_{\mathbb{R}^+}$ *and by the identity operator* I *on the semi-axis* $\mathbb{R}^+$*, contains all Mellin and Fourier convolution operators on the semi-axis with symbols from* $C\mathfrak{M}^0_p(\overline{\mathbb{R}})$*, having discontinuity of the jump type only at the infinity.*

Moreover, the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ *generated by the Cauchy singular integral operators with "shifts"*

$$S^c_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int\limits_0^\infty \frac{e^{-ic(x-y)}\varphi(y)\,dy}{y-x} = W_{-\operatorname{sign}(\xi-c)}\varphi(x) \ \textit{ for all } \ c \in \mathbb{R}$$

and by the identity operator I *on the semi-axis* $\mathbb{R}^+$ *over the field of* $N\times N$ *complex valued matrices coincides with the Banach algebra generated by Fourier convolution operators with piecewise-constant* $N\times N$ *matrix symbols contains all Fourier convolution* W_a *and hankel* H_b *operators with* $N\times N$ *matrix symbols* (*multipliers*) $a, b \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$

$$\mathbf{A} := \sum_{j=1}^m \mathfrak{M}^0_{a_j} W_{b_j}, \tag{77}$$

where $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N\times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, W_{b_j} are Fourier convolution operators with $N\times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$ in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, x^\alpha)$. The algebra of $N\times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\})$ consists of those piecewise-continuous $N\times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

This and more general algebras (see Remark 3.14) were studied in [22] and also in earlier works [12, 21, 42].

In order to keep the exposition self-contained, to improve formulations from [22] and to add Hankel operators as generators of the algebra, the results concerning the Banach algebra generated by the operators (77) are presented here with some modification and the proofs.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a acting on the space $\mathbb{L}_p(\mathbb{R}^+)$ and having piecewise-constant symbols $a(\xi)$, cf. Proposition 3.8. Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$, the following is true.

Corollary 3.9. *The quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$.*

To describe the symbol of the operator (77), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\},\ \ \Gamma_2^{\pm} := \{\pm\infty\} \times \overline{\mathbb{R}}^+,\ \ \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

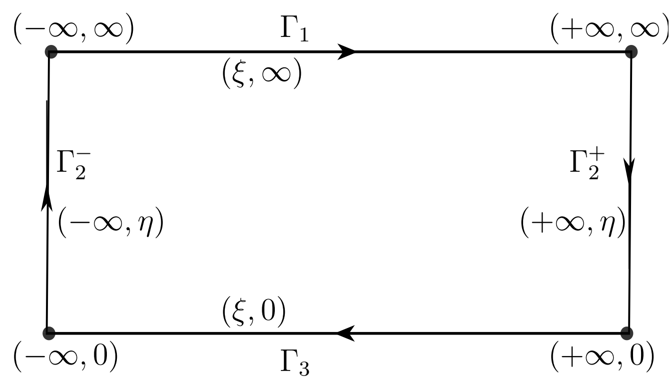

FIGURE 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}_p(\xi, \eta)$.

The symbol $\mathscr{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (77) is a function on the set $\mathfrak{R}$, viz.

$$\mathscr{A}_p(\omega) := \begin{cases} \sum\limits_{j=1}^{m} a_j(\xi)(b_j)_p(\infty, \xi), & \omega = (\xi, \infty) \in \overline{\Gamma_1}, \\ \sum\limits_{j=1}^{m} a_j(+\infty)b_j(-\eta), & \omega = (+\infty, \eta) \in \Gamma_2^+, \\ \sum\limits_{j=1}^{m} a_j(-\infty)b_j(\eta), & \omega = (-\infty, \eta) \in \Gamma_2^-, \\ \sum\limits_{j=1}^{m} a_j(\xi)(b_j)_p(0, \xi), & \omega = (\xi, 0) \in \overline{\Gamma_3}. \end{cases} \tag{78}$$

In (78) for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ we use the notation

$$\begin{aligned} g_p(\infty, \xi) &:= \frac{1}{2}\left[g(+\infty) + g(-\infty)\right] - \\ &\qquad - \frac{1}{2}\left[g(+\infty) - g(-\infty)\right] \cot \pi\Big(\frac{1}{p} - i\xi\Big), \\ g_p(t, \xi) &:= \frac{1}{2}\left[g(t+0) + g(t-0)\right] - \\ &\qquad - \frac{1}{2}\left[g(t+0) - g(t-0)\right] \coth \pi\Big(\frac{1}{p} - i\xi\Big), \end{aligned} \tag{79}$$

where $t, \xi \in \mathbb{R}$.

To make the symbol $\mathscr{A}_p(\omega)$ continuous, we endow the rectangle $\mathfrak{R}$ with a special topology. Thus let us define the distance on the curves Γ_1, $\Gamma_2^{\pm}$, Γ_3 and on $\overline{\mathbb{R}}$ by

$$\rho(x,y) := \left|\arg\frac{x-i}{x+i} - \arg\frac{y-i}{y+i}\right| \quad \text{for arbitrary} \quad x,y\in\overline{\mathbb{R}}.$$

In this topology, the length $|\mathfrak{R}|$ of $\mathfrak{R}$ is 6π, and the symbol $\mathscr{A}_p(\omega)$ is continuous everywhere on $\mathfrak{R}$. The image of the function $\det\mathscr{A}_p(\omega)$, $\omega\in\mathfrak{R}$ ($\det\mathscr{B}_p(\omega)$) is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits coincide. Thus

$$\mathscr{A}_p(\pm\infty,\infty) = \sum_{j=1}^{m}[a_j(\pm\infty)b_j(\mp\infty),$$
$$\mathscr{A}_p(\pm\infty,0) = \sum_{j=1}^{m}[a_j(\pm\infty)b_j(0\mp 0).$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega\in\mathfrak{R}}\left|\det\mathscr{A}_p(\omega)\right| > 0, \tag{80}$$

the increment of the argument $(1/2\pi)\arg\mathscr{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the positive direction is an integer, is called the winding number or the index and it is denoted by $\operatorname{ind}\det\mathscr{A}_p$.

Theorem 3.10. *Let $1<p<\infty$ and let $\mathbf{A}$ be defined by* (77)*. The operator $\mathbf{A}:\mathbb{L}_p(\mathbb{R}^+)\longrightarrow\mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator has the value*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathscr{A}_p. \tag{81}$$

Proof. Note that our study is based on a localization technique. For more details concerning this approach we refer the reader to [17, 19, 9, 30, 41].

Let us apply the Gohberg–Krupnik local principle to the operator $\mathbf{A}$ in (79), "freezing" the symbol of $\mathbf{A}$ at a point $x\in\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$. For $x\in\mathbb{R}$ and $\ell\in\mathbb{N}$, $\ell\geq 1$, let $C^\ell_x(\overline{\mathbb{R}})$ denote the set of all ℓ-times differentiable non-negative functions which are supported in a neighborhood of $x\in\mathbb{R}$ and are identically one everywhere in a smaller neighborhood of x. For $x\in\{-\infty\}\cup\{+\infty\}\cup\{\infty\}$, the functions from the corresponding classes $C^\ell_{+\infty}(\overline{\mathbb{R}})$ and $C^\ell_{-\infty}(\overline{\mathbb{R}})$ vanish on semi-infinite intervals $[-\infty,c)$ and $(-c,\infty]$, respectively, for certain $c>0$ and are identically one in smaller neighborhoods. It is easily seen that the system of localizing classes $\{C^\ell_x(\overline{\mathbb{R}})\}_{x\in\overline{\mathbb{R}}}$ is covering in the algebras $C(\overline{\mathbb{R}})$, $\mathfrak{M}_p(\overline{\mathbb{R}})$, respectively (cf. [17, 19, 9, 30]).

Let us now consider a system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. These localizing classes depend on two variables, viz. on $\omega\in\mathfrak{R}$ and $x\in\overline{\mathbb{R}^+}$. In particular, the class $\mathfrak{L}_{\omega,x}$

contains the operator $\Lambda_{\omega,x}$,

$$\Lambda_{\omega,x} := \begin{cases} \big[h_0\mathfrak{M}^0_{v_\xi} W_{g_\infty}\big] = \big[h_0\mathfrak{M}^0_{v_\xi}\big] \\ \qquad\qquad \text{if } \ \omega = (\xi,\infty) \in \Gamma_1, \ \ x = 0; \\ \big[h_x\mathfrak{M}^0_{v_{\pm\infty}} W_{g_\infty}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\mp\infty}}\big] \\ \qquad\qquad \text{if } \ \omega = (\pm\infty,\infty) \in \Gamma^{\pm}_2 \cap \Gamma_1, \ \ x \in \mathbb{R}^+; \\ \big[h_\infty\mathfrak{M}^0_{v_{\pm\infty}} W_{g_\eta}\big] = \big[h_\infty\mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\mp\,\eta}}\big] \\ \qquad\qquad \text{if } \ \omega = (\pm\infty,\eta) \in \Gamma^{\pm}_2, \ \ x = \infty; \\ \big[h_\infty\mathfrak{M}^0_{v_\xi} W_{g_0}\big] = \big[\mathfrak{M}^0_{v_\xi} W_{g_0}\big] \\ \qquad\qquad \text{if } \ \omega = (\xi,0) \in \overline{\Gamma}_3, \ \ x = \infty, \end{cases} \tag{82}$$

where $h_x \in C^1_x(\overline{\mathbb{R}}^+)$, $v_\xi \in C^1_\xi(\overline{\mathbb{R}}^+)$, $g_\eta \in C^1_\eta(\overline{\mathbb{R}}^+)$, and $[\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denotes the coset containing the operator $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$.

To verify the equalities in (82), one has to show that the difference between the operators in the square brackets is compact.

Consider the first equality in (82): The operator

$$h_0 W_{g_\infty} - h_0 I = h_0 W_{(g_\infty - 1)} = h_0 W_{g_0}$$

is compact, since both functions h_0 and $1 - g_\infty = g_0$ have compact supports, so Proposition 3.4 applies.

To check the second equality in (82), let us note that $h_x(0) = 0$, $v_{\pm\infty}(\mp\infty) = 0$ and $g_{\pm\infty}(\xi) = 0$ for all $\mp\,\xi > 0$. From the fourth part of Proposition 3.6 we derive that for any $x \in \mathbb{R}^+$ the operator $h_x\mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\pm\infty}}$ is compact. This leads to the claimed equality since

$$\big[h_x\mathfrak{M}^0_{v_{\pm\infty}} W_{g_\infty}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}}\{W_{g_{-\infty}} + W_{g_{+\infty}}\}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}} W_{g_{\mp\infty}}\big].$$

The third identity in (82) can be verified analogously. As far as the fourth identity in (82) is concerned, one can replace h_∞ by 1 because the difference $h_\infty W_{g_0} - W_{g_0} = (1 - h_\infty) W_{g_0} = h_0 W_{g_0}$ is compact due to Proposition 3.4.

Consider now other properties of the system $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$. Propositions 3.4–3.6 imply that

$$\big[h_x\mathfrak{M}^0_{v_\xi} W_{g_\infty}\big] = 0 \ \text{ for all } \ (\xi,\eta,x) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus \mathfrak{R} \times \overline{\mathbb{R}}^+.$$

Therefore, the system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ is covering: for a given system $\{\Lambda_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ of localizing operators one can select a finite number of points $(\omega_1, x_1) = (\xi_1, \eta_1, x_1), \dots, (\omega_s, x_s) = (\xi_s, \eta_s, x_s) \in \mathfrak{R}$ and add appropriately chosen terms $[h_{x_{s+j}}\mathfrak{M}^0_{v_{\xi_{s+j}}} W_{g_{s+j}}] = 0$ with $(\xi_{s+j}, \eta_{s+j}, x_{s+j})) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus (\mathfrak{R} \times \overline{\mathbb{R}}^+)$, $j = 1, 2, \dots, r$ so, that the equality

$$\sum_{j=1}^{r}\sum_{k=1}^{s} \big[c_{x_j}\mathfrak{M}^0_{a_{\xi_j}} W_{b_{\eta_k}}\big] = \big[c\mathfrak{M}^0_a W_b\big] \tag{83}$$

holds and the functions $c \in C(\overline{\mathbb{R}}^+)$, $a \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, $b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ are all elliptic. This implies the invertibility of the coset $[c\mathfrak{M}_a^0 W_b]$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the inverse coset is $[c\mathfrak{M}_a^0 W_b]^{-1} = [c^{-1}\mathfrak{M}^0_{a^{-1}} W_{b^{-1}}]$.

Note that the choice of a finite number of terms in (83) is possible due to Borel–Lebesgue lemma and the compactness of the sets $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ (two point and one point compactification of $\mathbb{R}$ and of $\mathbb{R}^+$, respectively).

Moreover, localization in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ leads to the following local representatives of the cosets containing Mellin and Fourier convolution operators with symbols $a, b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$:

$$[\mathfrak{M}_a^0] \overset{\mathfrak{M}^0_{v_{\xi_0}}}{\sim} [\mathfrak{M}^0_{a(\xi_0)}] = [a(\xi_0)I] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}}, \tag{84a}$$

$$[\mathfrak{M}_a^0] \overset{v_{x_0}I}{\sim} [\mathfrak{M}^0_{a^\infty}] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}^+}, \ \ x_0 \neq 0, \tag{84b}$$

$$[\mathfrak{M}_a^0] \overset{v_0 I}{\sim} [\mathfrak{M}_a^0] \ \text{ if } \ \xi_0 = 0, \tag{84c}$$

$$[W_b] \overset{W_{b_{\eta_0}}}{\sim} [W_{b(\eta_0)}] = [b(\eta_0)I] \ \text{ if } \ \eta_0 \in \mathbb{R} \setminus \{0\}, \tag{84d}$$

$$[W_b] \overset{W_{b_0}}{\sim} [W_{b^0}] = [\mathfrak{M}^0_{b_p(0,\cdot)}] \ \text{ if } \ \eta = 0, \tag{84e}$$

$$[W_b] \overset{W_{g^\infty}}{\sim} [W_{b^\infty(\infty,\cdot)}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ \eta_0 = \pm\infty, \tag{84f}$$

$$[W_b] \overset{v_{x_0}I}{\sim} [W_{b^\infty}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ x_0 \in \mathbb{R}^+, \tag{84g}$$

$$[W_b] \overset{v_\infty I}{\sim} [W_b] \ \text{ if } \ x_0 = \infty, \tag{84h}$$

where

$$\begin{aligned} g^\infty(\xi) &:= \frac{1}{2}\big[g(+\infty) + g(-\infty)\big] + \frac{1}{2}\big[g(+\infty) - g(-\infty)\big] \operatorname{sign}\xi = \\ &\qquad = g(-\infty)\chi_-(\xi) + g(+\infty)\chi_+(\xi), \\ g^0(\xi) &:= \frac{1}{2}\big[g(0+0) + g(0-0)\big] + \frac{1}{2}\big[g(0+0) + g(0-0)\big] \operatorname{sign}\xi = \\ &\qquad = g(0-0)\chi_-(\xi) + g(0+0)\chi_+(\xi), \end{aligned} \tag{85}$$

and $\chi_\pm(\xi) := (1/2)(1 \pm \operatorname{sign}\xi)$. Note that in the equivalency relations (84e)–(84g) we used the identities, cf. (75) and (79),

$$\begin{aligned} W_{g^\infty} &= \frac{1}{2}\big[g(-\infty) - g(+\infty)\big] - \frac{1}{2}\big[g(-\infty) - g(+\infty)\big] S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(\infty,\cdot)}, \\ W_{g^0} &= \frac{1}{2}\big[g(0+0) + g(0-0)\big] - \frac{1}{2}\big[g(0+0) - g(0-0)\big] S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(0,\cdot)}, \end{aligned}$$

which means that the Fourier convolution operators with homogeneous of order 0 symbols $g^\infty(\xi)$ and $g^0(\xi)$ are, simultaneously, Mellin convolutions with the symbols $g_p(\infty, \xi)$, $g_p(0, \xi)$.

Using the equivalence relations (84a)–(84h) and the compactness of the corresponding operators, cf. Propositions 3.4–3.6, one finds easily the following local representatives of the operator (coset) $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}\mathbb{L}_p(\mathbb{R}^+)$

(see (79) for the operator $\mathbf{A}$):

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)} W_{(b_j)\infty}\Big] =$$

$$= \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\cdot)}\Big] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\xi_0)}\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,\infty)I\big] \ \text{ if } \ \omega = (\xi_0,\infty) \in \Gamma_1, \ \ x_0 = 0, \tag{86a}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{(b_j)\infty}\Big] = \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)(b_j)_p(\infty,\cdot)}\Big] =$$

$$= \big[\mathfrak{M}^0_{\mathscr{A}_p(\pm\infty,\cdot)}\big] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \big[\mathscr{A}_p(\pm\infty,\infty)I\big] \tag{86b}$$

$$\text{if } \ \omega = (\pm\infty,\infty) \in \overline{\Gamma_2^{\pm}} \cap \overline{\Gamma_1}, \ \ 0 < x_0 < \infty;$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\mp\eta_0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(\mp\eta_0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(\mp\eta_0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,\mp\eta_0)I\big] \ \text{ if } \ \eta_0 > 0, \ \omega = (\pm\infty,\mp\eta_0) \in \Gamma_2^{\pm}, \ x_0 = \infty; \tag{86c}$$

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j} W_{b_j^0}\Big] =$$

$$= \Big[\sum_{j=1}^{m} a_j(\xi_0) \mathfrak{M}_{(b_j)_p(0,\cdot)}\Big] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} a_j(\xi_0)(b_j)_p(0,\xi_0)\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,0)I\big] \ \text{ if } \ \omega = (\xi_0,0) \in \overline{\Gamma}_3, \ \ x_0 = \infty; \tag{86d}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\eta),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,0)I\big] \ \text{ if } \ \omega = (\pm\infty,0) \in \overline{\Gamma}_3, \ \ x_0 = \infty. \tag{86e}$$

It is remarkable that the local representatives (86a)–(86e) are just the quotient classes of multiplication operators by constant $N \times N$ matrices $[\mathscr{A}_p(\xi_0,\eta_0)I]$. If $\det \mathscr{A}_p(\xi_0,\eta_0) = 0$, these representatives are not invertible, both locally and globally. On the other hand, they are globally invertible if $\det \mathscr{A}_p(\xi_0,\eta_0) \neq 0$. Thus, the conditions of the local invertibility for all points $\omega_0 = (\xi_0,\eta_0) \in \mathfrak{R}$ and the global invertibility of the operators under consideration coincide with the ellipticity condition for the symbol $\inf\limits_{(\xi_0,\eta_0)\in\mathfrak{R}} \det \mathscr{A}_p(\xi_0,\eta_0) \neq 0$.

The index $\operatorname{Ind} \mathbf{A}$ is a continuous integer-valued multiplicative function $\operatorname{Ind} \mathbf{AB} = \operatorname{Ind} \mathbf{A} + \operatorname{Ind} \mathbf{B}$ defined on the group of Fredholm operators of $\mathfrak{A}_p(\mathbb{R}^+)$. On the other hand, the index function $\operatorname{ind}\det \mathscr{A}_p$ defined on L_p-symbols $\mathscr{A}_p$ possesses the same property $\operatorname{ind}\det \mathscr{A}_p\mathscr{B}_p = \operatorname{ind}\det \mathscr{A}_p + \operatorname{ind}\det \mathscr{B}_p$, see explanations after (80). Moreover, the set of operators (79) is dense in the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ and the corresponding set of their symbols is dense in the algebra $C(\mathfrak{R})$ of all continuous functions on $\mathfrak{R}$. For $p = 2$ these algebras even coincide. Therefore, there is an algebraic homeomorphism between the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the algebra of their symbols which is a dense subalgebra of $C(\mathfrak{R})$. Hence, two various index functions can be only connected by the relation $\operatorname{Ind}\mathbf{A} = M_0 \operatorname{ind}\det\mathscr{A}_p$ with an integer constant M_0 independent of $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Since for any Fourier convolution operator $\mathbf{A} = W_a$ the index formula is $\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathscr{A}_p$ [12, 13, 17], the constant $M_0 = -1$, and the index formula (81) is proved. □

Remark 3.11. Let us emphasize that the formula (81) does not contradict the invertibility of "pure Mellin convolution" operators $\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ with an elliptic matrix symbol $a \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf_{\xi\in\mathbb{R}} |a(\xi)| > 0$, stated in Proposition 0.1, even if $\operatorname{ind} a \neq 0$.

In fact, computing the symbol of $\mathfrak{M}^0_a$ by formula (78), one obtains

$$(\mathfrak{M}^0_a)_p(\omega) := \begin{cases} a(\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ a(+\infty), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ a(-\infty), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ a(\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases}$$

Noting that on the sets Γ_1 and Γ_3 the variable ω runs in opposite direction, the increment of the argument $[\arg\det(\mathfrak{M}^0_a)_p(\omega)]_{\mathfrak{R}} = 0$ is zero, implying $\operatorname{Ind}\mathfrak{M}^0_a = 0$.

In contrast to the above, the pure Fourier convolution operators $W_b : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ with elliptic matrix symbol $b \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf_{\xi\in\mathbb{R}} |b_p(\xi,\eta)| > 0$ can possess non-zero indices. Since

$$b_p(\omega) := \begin{cases} b_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ b(-\eta), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ b(\eta), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ b(0), & \omega = (\xi,0) \in \overline{\Gamma_3}, \end{cases}$$

one arrives at the well-known formula

$$\operatorname{Ind} W_b = -\operatorname{ind} b_p.$$

Moreover, in the case where the symbol $b(-\infty) = b(+\infty)$ is continuous, one has $b_p(\xi,\eta) = b(\xi)$. Thus the ellipticity of the corresponding operator leads to the formula

$$\operatorname{ind} b_p = \operatorname{ind}\det b.$$

If $\mathscr{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ of (77), the set $\mathscr{R}(\mathscr{A}_p) := \{\mathscr{A}_p(\omega) \in \mathbb{C} : \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A} - \lambda I]$ is not invertible in the quotient algebra

$\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach theorem, the essential norm $|||\mathbf{A}|||$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega\in\omega}|\mathscr{A}_p(\omega)| \leqslant |||\mathbf{A}||| := \inf_{T\in\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))}\left\|(\mathbf{A}+T) \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\right\|. \tag{87}$$

The inequality (87) enables one to extend continuously the symbol map (78)

$$[\mathbf{A}] \longrightarrow \mathscr{A}_p(\omega), \quad [\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+)) \tag{88}$$

on the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, using Theorem 3.10 and conventional methods, cf. [22, Theorem 3.2], one can derive the following result.

Corollary 3.12. *Let $1 < p < \infty$ and $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if it's symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\mathscr{A}_p.$$

Theorem 3.10 and Corollary 3.12 lead to the assertion.

Corollary 3.13. *The set of maximal ideals of the commutative Banach quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ generated by scalar $N = 1$ operators in* (77)*, is homeomorphic to $\mathfrak{R}$, and the symbol map in* (78), (88) *is a Gelfand homeomorphism of the corresponding Banach algebras.*

The proof of this result is similar to [22, Theorem 3.1] and is left to the reader.

Remark 3.14. All the above results are valid in a more general setting viz., for the Banach algebra $\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$ generated in the weighted Lebesgue space of N-vector-functions $\mathbb{L}^N_p(\mathbb{R}^+, x^\alpha)$ by the operators

$$\mathbf{A} := \sum_{j=1}^{m}\left[d^1_j\mathfrak{M}^0_{a^1_j}W_{b^1_j} + d^2_j\mathfrak{M}^0_{a^2_j}H_{c^1_j} + d^3_jW^0_{b^2_j}H_{c^2_j}\right] \tag{89}$$

when coefficients $d^1_j, d^2_j, d^3_j \in PC^{N\times N}(\overline{\mathbb{R}})$ are piecewise-continuous $N\times N$ matrix functions, symbols of Mellin convolution operators $\mathfrak{M}^0_{a^1_j}$, $\mathfrak{M}^0_{a^2_j}$, Winer–Hopf (Fourier convolution) operators $W_{b^1_j}$, $W_{b^2_j}$ and Hankel operators $H_{c^1_j}$, $H_{c^2_j}$ are $N\times N$ piecewise-continuous matrix $\mathbb{L}_p$-multipliers $a^k_j, b^k_j, c^k_j \in PC^{N\times N}\mathfrak{M}_p(\mathbb{R})$.

The spectral set $\Sigma(\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+))$ of such Banach algebra (viz., the set where the symbols are defined, e.g. $\mathfrak{R}$ for the Banach algebra $\mathfrak{A}^{N\times N}_p(\mathbb{R}^+)$ investigated above) is more sophisticated and described in the papers [15, 16, 22, 42]. Let $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ be the sub-algebra of $\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+) = \mathfrak{P}\mathfrak{A}^{1\times 1}_{p,\alpha}(\mathbb{R}^+)$ generated by scalar operators (89) with continuous coefficients $c_j, h_j \in C(\overline{\mathbb{R}})$ and scalar piecewise-continuous $\mathbb{L}_p$-multipliers) $a_j, b_j, d_j, g_j \in PC\mathfrak{M}_p(\mathbb{R})$. The quotient-algebra $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ with respect to the ideal of all compact operators is a commutative algebra and the spectral set $\Sigma(\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+))$ is homeomorphic to the set of maximal ideals.

We drop further details about the Banach algebra $\mathfrak{PA}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$, because the result formulated above are sufficient for the purpose of this and subsequent papers dealing with the BVPs in domains with corners at the boundary.

4. Mellin Convolution Operators in Bessel Potential Spaces

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+). \tag{90}$$

The symbols of these operators are $N\times N$ matrix functions $a\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 4.1. *Let $0<|\arg\gamma|<\pi$, $0<|\arg c|<\pi$, $0<|\arg(c\,\gamma)|<\pi$, $r,s\in\mathbb{R}$, $m=1,2,\dots$, $1<p<\infty$. Then the operator $K^m_c:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is lifted equivalently to the operator*

$$\boldsymbol{A}^{m,s}_c := \Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{91a}$$

where

$$\boldsymbol{A}^{m,s}_c = \begin{cases} e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} \\ \qquad\qquad \text{if } -\pi<\arg c\,\gamma<0, \\ e^{\sigma(c,\gamma)\pi si}c^{-s}\Big[\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} + (-1)^{m-1}\boldsymbol{K}^m_{-c}H_{g^s_{\gamma,c}}\Big] \\ \qquad\qquad \text{if } 0<\arg c\,\gamma<\pi, \end{cases} \tag{91b}$$

$$H_{g^s_{\gamma,c}} = \begin{cases} I+T \\ \qquad\qquad \text{if } \sigma(c,\gamma)\neq 0, \\ H_{g^s_\infty}+T = e^{\sigma(\gamma)\pi si}\Big[\cos\pi s I - \sigma(\gamma)\,\dfrac{\sin\pi s}{\pi}\boldsymbol{K}^1_{-1}\Big]+T \\ \qquad\qquad \text{if } \sigma(c,\gamma)=0, \end{cases} \tag{91c}$$

$$\begin{aligned} g^s_{\gamma,c}(\xi) &:= \Big(\frac{\xi-c\,\gamma}{\xi+\gamma}\Big)^s, \\ g^s_\infty(\xi) &:= \frac{1}{2}\big[e^{\sigma(\gamma)2\pi si}+1\big]+\frac{1}{2}\big[e^{\sigma(\gamma)2\pi si}-1\big]\operatorname{sign}\xi, \end{aligned} \tag{91d}$$

T is a compact operator in $\mathbb{L}_p(\mathbb{R}^+)$, $\sigma(\gamma):=\operatorname{sign}\arg\gamma$ and $\sigma(c,\gamma)$ is defined in (51)

$$\sigma(c,\gamma) := \begin{cases} 0 & \text{if } 0<\arg c<\pi, \\ \operatorname{sign}\arg(c\,\gamma)-\operatorname{sign}\arg\gamma & \text{if } -\pi<\arg c<0. \end{cases}$$

Proof. Let $a_\pm\in\mathbb{L}_\infty(\mathbb{R})$ be $\mathbb{L}_p$-multipliers, which have analytic extensions $a_-(\xi)$ in the lower $\operatorname{Im}\xi<0$ and $a_+(\xi)$ in the upper $\operatorname{Im}\xi>0$ complex half planes. Then

$$W_{a_-}W_gW_{a_+} = W_{a_-ga_+}, \quad \forall\, g\in\mathbb{L}_\infty(\mathbb{R}) \tag{92}$$

(cf., e.g., [17]).

Let $-\pi < \arg c\,\gamma < 0$. Theorem 2.7 and the property 92 yield the equalities

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_\gamma = e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c\Lambda^s_{-c\,\gamma}\Lambda^{-s}_\gamma =$$
$$= e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c W_{\lambda^s_{-c\,\gamma}}W_{\lambda^{-s}_\gamma} = e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}}.$$

For $0 < \arg c\,\gamma < \pi$ we have similarly to (92)

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_\gamma = e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c\Lambda^s_{-c\,\gamma}\Lambda^{-s}_\gamma =$$
$$= e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{\lambda^s_{-c\,\gamma}}W^0_{\lambda^{-s}_\gamma} = e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}}. \quad (93)$$

On the other hand,

$$\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}}\varphi(t) = \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + \int_{-\infty}^{0}\frac{\tau^{m-1}W^0_{g^s_{\gamma,c}}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m}\,\varphi(t) =$$
$$= \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + \int_0^{\infty}\frac{(-\tau)^{m-1}r_+\boldsymbol{V}W^0_{g^s_{\gamma,c}}\varphi(\tau)\,d\tau}{(t+c\,\tau)^m}\,\varphi(t) =$$
$$= \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + (-1)^{m-1}\boldsymbol{K}^m_{-c}r_+\boldsymbol{V}W^0_{g^s_{\gamma,c}}\varphi(t) =$$
$$= \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + (-1)^{m-1}\boldsymbol{K}^m_{-c}H_{g^s_{\gamma,c}}\varphi(t). \quad (94)$$

The proved equalities justify formula (91b) for $\boldsymbol{A}^{m,s}_c$.

To justify the remainder formulae (91c) and (91d) note that if $\sigma(c,\gamma)\neq 0$, the meromorphic function $g_{\gamma,c}(\xi)$ in (91d) has one pole and one zero in the same half-plane $\operatorname{Im}\xi < 0$ or $\operatorname{Im}\xi > 0$ and, therefore, has equal limits at the infinity: $\lim\limits_{\xi\pm\infty} g^s_{\gamma,c}(\xi) = 1$. Then $g^s_{\gamma,c}(\xi) := 1 + g^s_0(\xi)$ where $g^s_0(\xi)$ is continuous (is $C^\infty(\mathbb{R})$-smooth) and vanishes at the infinity: $g^s_0(\pm\infty) = 0$. By virtue of Proposition 3.6 the operator $T := H_{g^s_0}$ is compact in $\mathbb{L}_p(\mathbb{R}^+)$.

In contrast to the foregoing case, where $\sigma(c,\gamma) = 0$, the meromorphic function $g_{\gamma,c}(\xi)$ in (91c) has the pole and the zero in different half-planes and, therefore, the function has different limits at the infinity:

$$g^s_{\gamma,c}(-\infty) = \lim_{\xi\longrightarrow-\infty} g^s_{\gamma,c}(\xi) = 1,$$
$$g^s_{\gamma,c}(+\infty) = \lim_{\xi\longrightarrow+\infty} g_{\gamma,c}(\xi) = e^{\sigma(\gamma)2\pi si},$$

where $\sigma(\gamma) = \sigma(c\,\gamma) = \operatorname{sign}\arg\gamma = \operatorname{sign}\operatorname{Im}\gamma$. Consider the representation

$$g^s_{\gamma,c}(\xi) := g^s_\infty(\xi) + g^s_0(\xi), \quad (95)$$

where $g^s_\infty(\xi)$ is defined in (91c) and the function h^s_0 is, as above, continuous and $g^s_0(\pm\infty) = 0$. The operator $T := H_{g^s_0}$ is compact in $\mathbb{L}_p(\mathbb{R}^+)$.

On the other hand,

$$\begin{aligned} H_{g^s_\infty} &= \frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}+1\right]I-\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}-1\right]H_{-\,\mathrm{sign}}=\\ &=\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}+1\right]I-\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}-1\right]r_+\boldsymbol{V}S_{\mathbb{R}^+}=\\ &=e^{\sigma(\gamma)\pi si}\Big[\cos\pi sI-\sigma(\gamma)\,\frac{\sin\pi s}{\pi}\,\boldsymbol{K}^1_{-1}\Big]. \end{aligned}\tag{96}$$

From (94)–(96) follows the representation (91b), (91d) in the case $0 < \arg c\,\gamma < \pi$, and the proof is complete. □

Let us consider a combined convolution operator

$$\mathbf{A} := d_0I + W_a + \sum_{j=1}^n d_j\boldsymbol{K}^{m_j}_{c_j},\quad c_1,\dots,c_n\in\mathbb{C},\quad a\in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\})\tag{97}$$

with constant coefficients $d_0, d_1, \dots, d_n\in\mathbb{C}$ in Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$.

For a complex number $\gamma\in\mathbb{C}$, with the positive imaginary part $0 < \arg\gamma < \pi$, we assume the following:

$$\begin{gathered} -\pi < \arg c_j\gamma < 0\ \text{ for }\ j=1,\dots,m,\\ 0<\arg c_j\gamma<\pi\ \text{ for }\ j=m+1,\dots,n. \end{gathered}\tag{98}$$

Then, due to the imposed constraint (97), the lifting property (91b) of the Mellin convolution operator and the lifting property (24) of the Fourier convolution operator, the lifted operator

$$\mathbf{A}^s := \Lambda^s_{-\gamma}\boldsymbol{A}\Lambda^{-s}_\gamma : \mathbb{L}_p(\mathbb{R}^+)\longrightarrow\mathbb{L}_p(\mathbb{R}^+)\tag{99}$$

has the form

$$\begin{aligned} \mathbf{A}^s &:= W_{d_0g^s_\gamma}+W_{ag^s_\gamma}+\sum_{j=1}^m d_jc_j^{-s}\boldsymbol{K}^{m_j}_{c_j}W_{g^s_{\gamma,c_j}}+\\ &+\sum_{j=m+1}^n d_je^{\sigma(c_j,\gamma)\pi si}c_j^{-s}\Big[\boldsymbol{K}^{m_j}_{c_j}W_{g^s_{\gamma,c_j}}-(-1)^{m_j}\boldsymbol{K}^{m_j}_{-c_j}H_{g^s_{\gamma,c_j}}\Big]+T, \end{aligned}\tag{100}$$

where (see (51))

$$\sigma(c_j,\gamma):=\begin{cases}0 & \text{if }\ 0<\arg c_j<\pi,\\ 0 & \text{if }\ -\pi<\arg c_j<0,\ \ 0<\arg c_j\gamma<\pi,\\ -2 & \text{if }\ -\pi<\arg c_j<0,\ \ -\pi<\arg c_j\gamma<0,\end{cases}\tag{101}$$

the functions $g^s_{\gamma,c_j}\in C(\dot{\mathbb{R}})$ are defined in (91d) and, due to the conditions (98), have the following limits at the infinity:

$$\begin{gathered} g^s_{\gamma,c_j}(-\infty)=1,\ \ g^s_{\gamma,c_j}(0)=e^{-\sigma(c_j)\pi si}c_j^s,\ \ g^s_{\gamma,c_j}(+\infty)=1,\ \ j=1,\dots,m,\\ g^s_{\gamma,c_j}(-\infty)=1,\ \ g^s_\gamma(0)=e^{-\sigma(c_j)\pi si}c_j^s,\ \ g^s_{\gamma,c_j}(+\infty)=e^{2\pi si},\ \ j=m+1,\dots,n,\\ \sigma(c_j):=\operatorname{sign}\arg c_j. \end{gathered}$$

The function $g^s_\gamma \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at the infinity

$$g^s_\gamma(-\infty) = 1, \quad g^s_\gamma(+\infty) = e^{2\pi s i}.$$

And, finally, the symbols

$$\mathscr{K}^{m_j}_{c_j,p}(\xi) := \mathscr{M}_{1/p}\mathscr{K}^{m_j}_{c_j}(\xi), \quad \mathscr{K}^1_{-1,p}(\xi) := \mathscr{M}_{1/p}\mathscr{K}^1_{-1}(\xi)$$

of the operators $\boldsymbol{K}^{m_j}_{c_j}$ and $\boldsymbol{K}^1_{-1} = \pi i S_{\mathbb{R}^+}$ are defined in (34)–(38) and have the following limits at the infinity

$$\mathscr{K}^{m_j}_{c_j}(\pm\infty) = 0, \;\; j = 1, \ldots, n, \quad \mathscr{K}^1_{-1,p}(\pm\infty) = \pm 1.$$

Using the equality (100), we announce the symbol $\mathscr{A}^s_p(\omega)$, $\omega \in \mathfrak{R}$, of the lifted operator $\boldsymbol{A}^s$ in $\mathbb{L}_p(\mathbb{R}^+)$ as the symbol of $\boldsymbol{A}$ in Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$ (cf. the definition (78))

$$\mathscr{A}^s_p(\omega) :=$$

$$:= \begin{cases} d_0 g^s_p(\xi) + a^s_p(\infty,\xi) + \\ \quad + \displaystyle\sum_{j=1}^{m} d_j c_j^{-s} \mathscr{K}^{m_j}_{c_j,p}(\xi) + \sum_{j=m+1}^{n} d_j e^{\sigma(c_j,\gamma)\pi s i} c_j^{-s} \times \\ \quad \times \Big[\mathscr{K}^{m_j}_{c_j,p}(\xi)\mathscr{W}_{g^s_{\gamma,c_j,p}}(\infty,\xi) - (-1)^{m_j}\mathscr{K}^{m_j}_{-c_j,p}(\xi)\mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi)\Big], \\ \qquad\qquad \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ \big\{d_0 + a(-\eta)\big\}\Big(\dfrac{\eta+\gamma}{\eta-\gamma}\Big)^s, \;\; \omega = (+\infty,\eta) \in \Gamma^+_2, \\ \big\{d_0 + a(\eta)\big\}\Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^s \omega = (-\infty,\eta) \in \Gamma^-_2, \\ e^{\pi s i}\big\{d_0 + a_p(0,\xi)\big\} + \\ \quad + \displaystyle\sum_{j=1}^{m} d_j e^{-\sigma(c_j)\pi s i} \mathscr{K}^{m_j}_{c_j,p}(\xi) + \sum_{j=m+1}^{n} d_j e^{\sigma(c_j,\gamma)\pi s i} \times \\ \quad \times \Big[e^{-\sigma(c_j)\pi s i}\mathscr{K}^{m_j}_{c_j,p}(\xi) - (-1)^{m_j} c_j^{-s}\mathscr{K}^{m_j}_{-c_j,p}(\xi)\mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi)\Big], \\ \qquad\qquad \omega = (\xi,0) \in \overline{\Gamma_3}, \end{cases} \tag{102}$$

where, since $\sigma(\gamma) = \operatorname{sign}\arg\gamma = 1$,

$$\mathscr{W}_{g^s_{\gamma,c_j,p}}(\infty,\xi) := e^{\pi s i}\Big[\cos\pi s - \sin\pi s\, \cot\pi\Big(\frac{1}{p} - i\xi\Big)\Big], \tag{103}$$

$$\mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi) := e^{\pi s i}\Big[\cos\pi s - \frac{\sin\pi s}{\sin\pi(1/p - i\xi)}\Big], \;\; j = m+1, \ldots, n, \tag{104}$$

$$\begin{aligned}a^s_p(\infty,\xi) &:= \frac{1}{2}\left[e^{2\pi si}a(+\infty)+a(-\infty)\right]-\\&\qquad -\frac{1}{2}\left[e^{2\pi si}a(+\infty)-a(-\infty)\right]\cot\pi\Big(\frac{1}{p}-i\xi\Big),\\a_p(t,\xi) &:= \frac{1}{2}\left[a(t+0)+a(t-0)\right]-\\&\qquad -\frac{1}{2}\left[a(t+0)-a(t-0)\right]\cot\pi\Big(\frac{1}{p}-i\xi\Big).\end{aligned}$$

Theorem 4.2. *Let $1<p<\infty$, $s\in\mathbb{R}$ and let $\mathbf{A}$ be defined by* (97). *The operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}^s_p(\omega)$, defined in* (102)*, is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator has the value*

$$\operatorname{Ind}\mathbf{A}=-\operatorname{ind}\det\mathscr{A}^s_p. \tag{105}$$

Proof. The proof follows if we apply to the lifted operator $\boldsymbol{A}^s$ (see (99)) having the form (100), Theorem 3.10. □

For the definition of the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p(\Omega)=\mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega)=\widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ we for arbitrary domain $\Omega\subset\mathbb{R}^n$, including the half axes $\mathbb{R}^+$ refer, e.g., to the monograph [43].

Corollary 4.3. *Let $1<p<\infty$, $s\in\mathbb{R}$ and let $\mathbf{A}$ be defined by* (87). *If the operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm* (*is invertible*) *for all $a\in(s_0,s_1)$ and $p\in(p_0,p_1)$, where $-\infty<s_0<s_1<\infty$, $1<p_o<p_1<\infty$, then*

$$\mathbf{A}:\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{W}^s_p(\mathbb{R}^+),\quad s\in(s_0,s_1),\ \ p\in(p_0,p_1) \tag{106}$$

is Fredholm and has the equal index

$$\operatorname{Ind}\mathbf{A}=-\operatorname{ind}\det\mathscr{A}^s_p. \tag{107}$$

(*is invertible, respectively*) *in the Sobolev–Slobodeckij* (*Besov*) *spaces $\mathbb{W}^s_p=\mathbb{B}^s_{p,p}$.*

Proof. First of all recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p=\mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between Bessel potential spaces

$$\begin{gathered}\big(\mathbb{H}^{s_0}_{p_0}(\Omega),\mathbb{H}^{s_1}_{p_1}(\Omega)\big)_{\theta,p}=\mathbb{W}^s_p(\Omega),\ \ s:=s_0(1-\theta)+s_1\theta,\\\big(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega),\widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\big)_{\theta,p}=\widetilde{\mathbb{W}}^s_p(\Omega),\ \ p:=\frac{1}{p_0}\,(1-\theta)+\frac{1}{p_1}\,\theta,\ \ 0<\theta<1.\end{gathered} \tag{108}$$

If $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (or is invertible) for all $s\in(s_0,s_1)$ and $p\in(p_0,p_1)$, it has a regularizer $\mathbf{R}$ (has the inverse $\mathbf{A}^{-1}=\mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R}:\mathbb{W}^s_p(\mathbb{R}^+)\longrightarrow\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$$

due to the interpolation (108) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$, respectively ($\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible).

Due to the Krasnoselskij interpolation theorem (see [43]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (106) is Fredholm (is invertible, respectively).

The index formulae (107) follows from the embedding properties of the Sobolev–Slobodeckij and Bessel potential spaces by standard well-known arguments. $\square$

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