Mem. Differential Equations Math. Phys. 59 (2013), 105–112

Short Communications

Malkhaz Ashordia

ON CONDITIONS FOR THE WELL-POSEDNESS OF NONLOCAL PROBLEMS FOR A CLASS OF SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Dedicated to the blessed memory of Professor Levan Magnaradze

Abstract. The conditions for the so called conditionally well-posedness of a class of a linear generalized boundary value problems are given in the case when the generalized differential system has singularities.

რეზიუმე. ერთი კლასის სინგულარობებიანი განზოგადოებული დიფერენციალური სისტემებისათვის მოყვანილია ზოგადი წრფივი სასაზღვრო ამოცანის ე. წ. პირობითი კორექტულობის პირობები.

2000 Mathematics Subject Classification: 34K10.
Key words and phrases: Systems of linear generalized ordinary differential equations, the Lebesgue–Stiltjes integral, boundary value problems, singularities, conditionally well-posedness.

In the paper we give conditions for the well-posedness for the following linear system of generalized ordinary differential equations with singularities

$$\begin{aligned} &dx_i(t) = x_{i+1}da_i(t) \ \text{ for } \ t \in [a,b] \ (i = 1,\dots,n-1),\\ &dx_n(t) = \sum_{i=1}^{n} h_i(t)x_i(t)db_i(t) + df(t) \ \text{ for } \ t \in [a,b], \end{aligned} \tag{1}$$

with the nonlocal boundary value condition

$$\ell_i(x_1,\dots,x_n) = 0 \ \ (i = 1,\dots,n), \tag{2}$$

where n is a natural number, $a_i \in \mathrm{BV}([a,b],\mathbb{R})$ $(i = 1,\dots,n-1)$, $f \in \mathrm{BV}([a,b],\mathbb{R})$, $b_i \in \mathrm{BV}([a,b],\mathbb{R})$ $(i = 1,\dots,n)$, $h_i : [a,b] \to \mathbb{R}$ is a function measurable with respect to the measures $\mu(b_{i1})$ and $\mu(b_{i2})$, corresponding, respectively, to the nondecreasing functions $b_{i1}(t) \equiv \bigvee\limits_a^t(b_i)$ and $b_{i2}(t) \equiv$

$b_i(t) - b_{i1}(t)$ for every $i \in \{1, \dots, n\}$, and $\ell_i : \mathrm{BV}_v([a,b], \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ are linear bounded functionals.

The general differential system of the form (1) represents in a defined sense the analogy of the nth order linear ordinary differential equation of the form

$$u^{(n)} = \sum_{i=1}^{n} h_i(t) u^{(i-1)} + h(t) \quad \text{for} \quad t \in [a,b]. \tag{3}$$

Note that the ordinary differential equation of the form (3) is a particular case of the system (1), where $a_i(t) \equiv t$ $(i = 1, \dots, n)$, and $f(t) \equiv \int\limits_a^t h(\tau)\, d\tau$.

It is well known that in the regular case, where the coefficients of the system (1) are Lebesgue–Stieltjes integrable on $[a,b]$ with respect the corresponding measures, problem (1), (2) has the Fredholm property in the defined conditions, and the unique solvability of that problem ensures its well-posedness (see [3]–[6], [13], [14], [22], [25]).

We are interested in the case, where the system (1) is singular, i.e., when some of the coefficients h_i $(i = 1, \dots, n)$ are not, in general, Lebesgue–Stieltjes integrable on $[a,b]$ with respect to the corresponding measures, having singularities at some boundary or interior points of the interval $[a,b]$. Some questions dealing with the singular boundary value problems of the form (1), (2), e.g., those dealing with the Fredholm property and the solvability have been investigated in [8]–[10]. As we know, in this case, the question on the well-posedness of the generalized problem (1), (2) remains still unstudied. In the present paper, an attempt is made to fill up the existing gaps.

As for the question of the solvability and well-posedness for singular boundary value problems for ordinary differential equations, i.e., for the singular (3), (2) problem, it is investigated in [19] for the general case, and in [1], [15–17], [20], [21], [23] for some important particular cases. Note that the questions for the regular case of the ordinary differential equations are investigated sufficiently well for the linear and nonlinear cases (see, e.g., [2], [11], [12], [17], [18] and the references therein).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.g., [7], [24], [25] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a,b]$ $(a, b \in \mathbb{R})$ is a closed segment.

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$, i.e., the sum of total variations of the latter components x_{ij} $(i = 1,\dots,n;$ $j = 1,\dots,m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \bigvee\limits_a^t(x_{ij})$ for $a < t \le b$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a,b]\big\}, \quad \|X\|_v = \|x(a)\| + \bigvee_a^b(X);$$

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < +\infty$);

$\mathrm{BV}_s([a,b],\mathbb{R}^n)$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^n), \|\cdot\|_s)$; $\mathrm{BV}_v([a,b],\mathbb{R}^n)$ is the Banach space $(\mathrm{BV}([a,b],\mathbb{R}^n), \|\cdot\|_v)$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for} \quad a < t \le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for} \quad t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) =$$

$$= \int_{]s,t[} x(\tau)\,dS_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int\limits_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$; if $a = b$, then we assume $\int\limits_a^b x(t)\,dg(t) = 0$;

$L([a,b],\mathbb{R};g)$ is a set of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measure $\mu(g)$.

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \quad \text{for} \quad s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],\mathbb{R};g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\right)_{i,j=1}^{l,m} \quad \text{for} \ \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \quad (j = 0,1,2).$$

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \quad \text{for} \ \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b],D;G) = \bigcap_{j=1}^{2} L([a,b],D;G_j),$$

$$K([a,b]\times D_1,D_2;G) = \bigcap_{j=1}^{2} K([a,b]\times D_1,D_2;G_j).$$

A vector-function $x = (x_i)_{i=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system (1) if the function $h_i x_i$ belongs to $L([a,b],b_{i1}) \cap L([a,b],b_{i2})$ and

$$x_i(t) = x_i(s) + \int_s^t x_i(\tau)\,da_i(\tau) \quad \text{for} \ \ a \le s \le t \le b \ \ (i = 1,\dots,n-1),$$

$$x_n(t) = x_n(s) + \sum_{i=1}^n \int_s^t h_i(\tau)x_i(\tau)\,db_i(\tau) \quad \text{for} \ \ a \le s \le t \le b.$$

A solution of the system (1) satisfying the boundary conditions (2) is called a solution of the problem (1),(2).

Along with the system (1), we will need to consider, respectively, the corresponding homogeneous and perturbed systems

$$\begin{aligned} dx_i(t) &= x_{i+1}da_i(t) \ \ \text{for} \ \ t \in [a,b] \ (i = 1,\dots,n-1),\\ dx_n(t) &= \sum_{i=1}^n h_i(t)x_i(t)db_i(t) \ \ \text{for} \ \ t \in [a,b], \end{aligned} \tag{1_0}$$

and

$$dx_i(t) = x_{i+1} da_i(t) \ \text{ for } \ t \in [a,b] \ (i = 1, \dots, n-1),$$
$$dx_n(t) = \sum_{i=1}^{n} h_{ki}(t) x_i(t) db_i(t) + d\widetilde{f}(t) \ \text{ for } \ t \in [a,b], \tag{4}$$

with the inhomogeneous boundary conditions

$$\ell_i(x_1, \dots, x_n) = c_i \ \ (i = 1, \dots, n), \tag{5}$$

where $\widetilde{f} \in \mathrm{BV}([a,b], \mathbb{R})$, and $c_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

Definition 1. The problem (1), (2) is said to be *well-posed* if for an arbitrary $\widetilde{f} \in \mathrm{BV}([a,b], \mathbb{R})$ and $c_i \in \mathbb{R}$ $(i = 1, \dots, n)$ the problem (4), (5) is uniquely solvable, and there exists a positive constant r independent of $\widetilde{f}$ and c such that

$$\|\widetilde{x} - x\|_s \le r \Big(\sum_{i=1}^{n} \|c_i\| + \|\widetilde{f} - f\| \Big),$$

where $x = (x_i)_{i=1}^n$ and $\widetilde{x} = (\widetilde{x}_i)_{i=1}^n$ are, respectively, the solutions of the problems (1), (2) and (4), (5).

Definition 2. The problem (1), (2) is said to be *conditionally well-posed* if for an arbitrary $\widetilde{f} \in \mathrm{BV}([a,b], \mathbb{R})$ the problem (1_0), (2) is uniquely solvable and there exists a positive constant r, independent of $\widetilde{f}$ and c, such that

$$\|\widetilde{x} - x\|_s \le r \|\widetilde{f} - f\|,$$

where $x = (x_i)_{i=1}^n$ and $\widetilde{x} = (\widetilde{x}_i)_{i=1}^n$ are, respectively, the solutions of the problems (1), (2) and (1_0), (2).

Note that if the coefficients of the system (1_0) are integrable on $[a,b]$ with corresponding measures, then the conditional well-posedness of the problem (1), (2) implies its well-posedness. If, however,

$$\sum_{i=1}^{n} \int_a^b |h_i(t)| \, dv(b_i)(t) = +\infty,$$

then the conditional well-posedness of the problem (1), (2) does not guarantee its well-posedness.

Definition 3. Let $\ell_i : \mathrm{BV}_v([a,b], \mathbb{R}^n) \to \mathbb{R}$ $(i = 1, \dots, n)$ be linear bounded functionals. We say that the vector function $(\varphi_1, \dots, \varphi_n) : [a,b] \to \mathbb{R}^n$ belongs to the set $\mathcal{E}_{\ell_1, \dots, \ell_n}$ if:

(i) for an arbitrary $i \in \{1, \dots, n\}$, the function $\varphi_i : [a,b] \to \mathbb{R}$ is continuous and $\varphi_i(t) > 0$ for $\mu(v(b_i))$-almost all $t \in [a,b]$;

(ii) an arbitrary vector function $(x_i)_{i=1}^n \in \mathrm{BV}([a,b], \mathbb{R}^n)$, satisfying the boundary conditions (2), admits the estimate

$$|x_i(t)| \le \bigvee_a^t (x_n) \cdot \varphi_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n).$$

Note that the set $\mathcal{E}_{\ell_1,\dots,\ell_n}$ is nonempty if and only if the system

$$dx_i(t) = x_{i+1} da_i(t) \ \ (i = 1, \dots, n-1), \ \ dx_n(t) = 0 \ \text{ for } \ t \in [a, b]$$

under the condition (2) has only the trivial solution.

Theorem 1. *Let there exist a vector function* $(\varphi_1, \dots, \varphi_n) : [a, b] \to \mathbb{R}^n$ *such that*

$$(\varphi_1, \dots, \varphi_n) \in \mathcal{E}_{\ell_1,\dots,\ell_n} \tag{6}$$

and

$$\sum_{i=1}^{n} \int_a^b \varphi_i(t)|h_i(t)|\, dv(b_i)(t) < +\infty \ \ (i = 1, \dots, n).$$

Then the problem (1), (2) *is conditionally well-posed if and only if the corresponding homogeneous problem* (1_0), (2) *has only the trivial solution.*

Theorem 2. *Let there exist a vector function* $(\varphi_1, \dots, \varphi_n) : [a, b] \to \mathbb{R}^n$ *such that conditions* (6) *and*

$$\sum_{i=1}^{n} \int_a^b \varphi_i(t)|h_i(t)|\, dv(b_i)(t) < 1 \ \ (i = 1, \dots, n) \tag{7}$$

hold. Then the problem (1), (2) *is conditionally well-posed.*

Theorem 3. *Let there exist a vector function* $(\varphi_1, \dots, \varphi_n) : [a, b] \to \mathbb{R}^n$ *such that the conditions* (6) *and* (7) *hold and*

$$\int_a^b \varphi_i(t)|h_1(t)|\, dv(b_1)(t) = +\infty.$$

Then the problem (1), (2) *is conditionally well-posed but not well-posed.*

Basing on the above results, we can establish the effective conditions for the problem (1),(2) to have the *well-posed* and *conditionally well-posed* properties for some concrete type of linear bounded functionals ℓ_i $(i = 1, \dots, n)$.

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Project No. GNSF/ST09_175_3-101).

References

1. R. P. Agarwal and I. Kiguradze, On multi-point boundary value problems for linear ordinary differential equations with singularities. *J. Math. Anal. Appl.* **297** (2004), No. 1, 131–151.
2. M. Ashordia, On the stability of solutions of linear boundary value problems for the system of ordinary differential equations. *Georgian Math. J.* **1** (1994), No. 2, 115–126.

3. M. Ashordia, On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* **1** (1994), No. 4, 343–351.
4. M. Ashordia, On the correctness of nonlinear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* **3** (1996), No. 6, 501–524.
5. M. Ashordia, On the stability of solutions of the multipoint boundary value problem for the system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **6** (1995), 1–57.
6. M. Ashordia, Criteria of correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Czechoslovak Math. J.* **46(121)** (1996), No. 3, 385–404.
7. M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems. *Mem. Differential Equations Math. Phys.* **36** (2005), 1–80.
8. M. T. Ashordia, On boundary value problems for systems of linear generalized ordinary differential equations with singularities. (Russian) *Differ. Uravn.* **42** (2006), No. 3, 291–301; English transl.: *Differ. Equ.* **42** (2006), No. 3, 307–319.
9. M. T. Ashordia, On some boundary value problems for linear generalized differential systems with singularities. (Russian) *Differ. Uravn.* **46** (2010), No. 2, 163–177; English transl.: *Differ. Equ.* **46** (2010), No. 2, 167–181.
10. M. Ashordia, On two-point singular boundary value problems for systems of linear generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **56** (2012), 9–35.
11. R. Conti, Problèmes linéaires pour les équations différentielles ordinaires. *Math. Nachr.* **23** (1961), 161–178.
12. P. Hartman, Ordinary differential equations. *John Wiley & Sons, Inc., New York–London–Sydney*, 1964.
13. Z. Halas, G. A. Monteiro, and M. Tvrdý, Emphatic convergence and sequential solutions of generalized linear differential equations. *Mem. Differential Equations Math. Phys.* **54** (2011), 27–49.
14. Z. Halas and M. Tvrdý, Continuous dependence of solutions of generalized linear differential equations on a parameter. *Funct. Differ. Equ.* **16** (2009), No. 2, 299–313.
15. I. T. Kiguradze, On a singular problem of Cauchy–Nicoletti. *Ann. Mat. Pura Appl. (4)* **104** (1975), 151–175.
16. I. T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1975.
17. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *"Metsniereba", Tbilisi*, 1997.
18. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* 43 (1988), no. 2, 2259–2339. *Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results*, Vol. 30 (Russian), 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.
19. I. Kiguradze and T. Kiguradze, Conditions for the well-posedness of nonlocal problems for higher order linear differential equations with singularities. *Georgian Math. J.* **18** (2011), No. 4, 735–760.
20. T. Kiguradze, On some nonlocal boundary value problems for linear singular differential equations of higher order. *Mem. Differential Equations Math. Phys.* **47** (2009), 169–173.
21. T. I. Kiguradze, On conditions for the well-posedness of linear singular boundary value problems. (Russian) *Differ. Uravn.* **46** (2010), No. 2, 183–190; English transl.: *Differ. Equ.* **46** (2010), No. 2, 187–194.

22. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. (Russian) *Czechoslovak Math. J.* **7 (82)** (1957), 418–449.
23. N. Partsvania, On solvability and well-posedness of two-point weighted singular boundary value problems. *Mem. Differential Equations Math. Phys.* **54** (2011), 139–146.
24. Š. Schwabik, Generalized ordinary differential equations. *Series in Real Analysis*, 5. *World Scientific Publishing Co., Inc., River Edge, NJ*, 1992.
25. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London*, 1979.

(Received 27.03.2012)

Author's addresses:

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Sukhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge