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Akihito Shibuya

ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN THE FRAMEWORK OF REGULAR VARIATION

Dedicated to the 80th birthday anniversary
of Professor Takaŝi Kusano

Abstract. This paper is devoted to the asymptotic analysis of positive solutions of a class of second order functional differential equations in the framework of regular variation. It is shown that precise asymptotic behavior of intermediate positive solutions of the equations under consideration can be established by means of Karamata's integration theorem combined with fixed point techniques.

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1. Introduction

This paper is devoted to the study of the existence and asymptotic behavior of positive solutions of second order Emden–Fowler type functional differential equations of the form

$$x''(t) + q(t)|x(g(t))|^{\gamma}\operatorname{sgn} x(g(t)) = 0, \tag{A}$$

where

(a) γ is a positive constant less than 1,

(b) $q : [a, \infty) \to (0, \infty)$ is a continuous function, $a > 0$,

(c) $g : [a, \infty) \to (0, \infty)$ is a continuous increasing function such that
$$g(t) < t \ \text{ and } \ \lim_{t\to\infty} g(t) = \infty.$$

This equation (A) is called *sublinear*. Equation (A) with $\gamma > 1$ is said to be *superlinear*.

By a *proper solution* of equation (A) we mean a function $x(t)$ which is defined in a neighborhood of infinity and is nontrivial in the sense that

$$\sup\big\{|x(t)| : \ t \geqq T\big\} > 0 \ \text{ for any sufficiently large } \ T > a.$$

A proper solution of (A) is said to be *oscillatory* if it has an infinite sequence of zeros clustering at infinity and *nonoscillatory* otherwise. Thus a nonoscillatory solution is eventually positive or negative.

We are interested in the existence and asymptotic behavior of possible nonoscillatory solutions of (A). If $x(t)$ is a solution of (A), then so is $-x(t)$, and hence in studying nonoscillatory solutions it suffices to restrict our consideration to *positive solutions*. It is known that any positive solution $x(t)$ falls into one of the following three types:

(I) $\lim\limits_{t\to\infty} x(t) = const > 0$,

(II) $\lim\limits_{t\to\infty} x(t) = \infty$, $\lim\limits_{t\to\infty} \dfrac{x(t)}{t} = 0$,

(III) $\lim\limits_{t\to\infty} \dfrac{x(t)}{t} = const > 0$.

Our primary concern in this paper will be with type (II)-solutions, which are referred to as *intermediate solutions of* (A), because the other two types of solutions are fully understood as the following statements show:

(i) (A) has solutions of type (I) if and only if $\int\limits_a^\infty tq(t)\,dt < \infty$;

(ii) (A) has solutions of type (III) if and only if $\int\limits_a^\infty g(t)^{\gamma} q(t)\,dt < \infty$.

It seems to be very difficult to obtain detailed information about the existence of intermediate solutions of (A) having precise asymptotic behavior at infinity in the case of general positive continuous $q(t)$, and hence we limit ourselves to the case where the coefficient $q(t)$ is a regularly varying function (in the sense of Karamata) and focus our attention on regularly varying solutions of (A). Analyzing equation (A) in the framework of regular variation was motivated by a recent interesting paper [2] in which complete analysis has been made of positive regularly varying solutions of type (II) of the sublinear Emden–Folwer equation

$$x'' + q(t)|x|^{\gamma}\operatorname{sgn} x = 0,$$

under the assumption that $q(t)$ is regularly varying.

It is natural to obtain the desired solutions of (A) by solving the integral equation

$$x(t) = x_0 + \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \geqq T_0, \tag{B}$$

where $x_0 > 0$ and $T_0 > a$. Note that any type (II)-solution of (A) satisfies (B) for some x_0 and T_0. In view of the difficulty in the analysis of (B) for general retarded argument $g(t)$ we confine our attention to the class of $g(t)$ such that

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1. \tag{1.1}$$

Associated with (B) is the following integral asymptotic relation

$$x(t) \sim \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \to \infty, \tag{C}$$

which is regarded as an approximation at infinity of (B). Here and throughout, the symbol $\sim$ is used to mean the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

It is shown that if $q(t)$ is regularly varying and $g(t)$ satisfies (1.1), then one can acquire full knowledge of the structure of all possible regularly varying solutions of (C), and that the results for (C) thus obtained play a central role in establishing the existence of intermediate solutions with accurate asymptotic behavior at infinity for equation (A).

Our main results are presented in Section 3 consisting of three subsections. The first subsection is devoted to the analysis of relation (C) with regularly varying $q(t)$ by means of regular variation under condition (1.1), and three types of its regularly varying solutions are shown to exist. These three types of solutions are effectively used in the second subsection to construct three kinds of intermediate solutions for equation (A) with the help of fixed point techniques. In the third subsection two kinds of intermediate solutions thus constructed will be verified to be regularly varying. The definition and some basic properties of regularly varying functions will be summarized in Section 2 of preliminary nature.

2. Regularly Varying Functions

We state here the definition and some basic properties of regularly varying functions which will be needed in developing our main results in the next section.

Definition 2.1. A measurable function $f : [0, \infty) \to (0, \infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \ \text{ for all } \ \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV to denote $\mathrm{RV}(0)$, and call members of SV *slowly varying functions.* Any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem.*

Definition 2.2. $f(t) \in \mathrm{RV}(\rho)$ *if and only if* $f(t)$ *is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \right\}, \quad t \geqq t_0,$$

for some $t_0 > 0$ *and for some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \ \text{ and } \ \lim_{t\to\infty} \delta(t) = \rho.$$

If $c(t) \equiv c_0$, then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ, and is denoted by $f(t) \in \text{n-RV}(\rho)$.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \right\}, \quad \beta_n \in (0, 1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function $L(t) = \exp\left\{ (\log t)^{\frac{1}{3}} \cos\, (\log t)^{\frac{1}{3}} \right\}$ is a slowly varying function which is oscillating in the sense that $\limsup\limits_{t\to\infty} L(t) = \infty$ and $\liminf\limits_{t\to\infty} L(t) = 0$.

The following result concerns operations which preserve slow variation.

Proposition 2.1. *Let* $L(t)$, $L_1(t)$, $L_2(t)$ *be slowly varying. Then,* $L(t)^\alpha$ *for any* $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ *and* $L_1(L_2(t))$ (*if* $L_2(t) \to \infty$) *are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 2.2. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition 2.3. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho.$$

The following result which is called Karamata's integration theorem is useful in handling slowly and regularly varying functions analytically.

Proposition 2.4. *Let $L(t) \in \mathrm{SV}$. Then,*

(i) *if $\alpha > -1$,*

$$\int_a^t s^{\alpha} L(s)\, ds \sim \frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty.$$

(ii) *if $\alpha < -1$,*

$$\int_t^{\infty} s^{\alpha} L(s)\, ds \sim -\frac{1}{\alpha+1}\, t^{\alpha+1} L(t), \quad t \to \infty.$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s}\, ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^{\infty} \frac{L(s)}{s}\, ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

Definition 2.3. A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^{\rho} L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) = \mathit{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) denotes the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions)

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels [1]. See also Seneta [7]. A comprehensive survey of results up to 2000 on the asymptotic analysis of ordinary differential equations by means of regular variation can be found in the monograph of Marić [6].

3. Existence of Intermediate Solutions of Equation (A)

Intermediate solutions of (A), that is, positive solutions $x(t)$ such that

$$\lim_{t\to\infty} x(t) = \infty \ \text{ and } \ \lim_{t\to\infty} \frac{x(t)}{t} = 0, \tag{3.1}$$

are constructed as solutions of the integral equation (B) under the assumption that $q(t) \in \mathrm{RV}(\sigma)$ ($\sigma \in \mathbb{R}$) and $g(t)$ satisfy (1.1). For this purpose an essential role is played by the fact that regularly varying solutions of the integral asymptotic relation (C) satisfying (3.1) can be thoroughly analyzed in the framework of regular variation. Throughout this section, the use is made of the following expression for $q(t)$

$$q(t) = t^{\sigma} l(t), \ \ l(t) \in \mathrm{SV}. \tag{3.2}$$

3.1. **Regularly varying solutions of asymptotic relation (C).** Let $x(t) = t^{\rho}\xi(t)$, $\xi(t) \in \mathrm{SV}$, be a regularly varying solution of (C) satisfying (3.1). We see that ρ must satisfy $\rho \in [0,1]$, and that $\xi(t) \to \infty$, $t \to \infty$, if $\rho = 0$ and $\xi(t) \to 0$, $t \to \infty$, if $\rho = 1$, which means that $x(t)$ must be in one of the following three classes of regularly varying functions:

$$\mathrm{ntr}-\mathrm{SV}, \ \ \mathrm{RV}(\rho) \ \text{ with } \ \rho \in (0,1), \ \ \mathrm{ntr}-\mathrm{RV}(1). \tag{3.3}$$

One can establish the existence of these three kinds of regularly varying solutions of (C) as the following theorems demonstrate.

Theorem 3.1. *Relation* (C) *has nontrivial slowly varying solutions if and only if $\sigma = -2$ and*

$$\int_a^{\infty} t q(t)\, dt = \infty, \tag{3.4}$$

in which case any such solution $x(t)$ has one and the same asymptotic behavior

$$x(t) \sim \left[(1-\gamma)\int_a^t s q(s)\, ds\right]^{\frac{1}{1-\gamma}}, \ \ t \to \infty. \tag{3.5}$$

Theorem 3.2. *Relation* (C) *has regularly varying solutions of index $\rho \in (0,1)$ if and only if $\sigma \in (-2, -\gamma-1)$, in which case ρ is given by*

$$\rho = \frac{\sigma+2}{1-\gamma} \tag{3.6}$$

and any such solution $x(t)$ has one and the same asymptotic behavior

$$x(t) \sim \left[\frac{t^2 q(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\gamma}}, \ \ t \to \infty. \tag{3.7}$$

Theorem 3.3. *Relation* (C) *has nontrivial regularly varying solutions of index* 1 *if and only if* $\sigma = -\gamma - 1$ *and*

$$\int_a^\infty t^\gamma q(t)\,dt < \infty, \tag{3.8}$$

in which case any such solution $x(t)$ *has one and the same asymptotic behavior*

$$x(t) \sim t\left[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.9}$$

Lemma 3.1. *If* $f(t)$ *is regularly varying and* $g(t)$ *satisfies* (1.1), *then* $f(g(t)) \sim f(t)$ *as* $t \to \infty$.

Proof. Suppose that $f(t) \in \mathrm{RV}(\rho)$. Then by Proposition 2.1 it is expressed as

$$f(t) = c(t)\exp\left\{\int_{t_0}^t \frac{\delta(s)}{s}\,ds\right\}, \quad t \geqq t_0,$$

for some constant $t_0 > 0$ and some functions $c(t)$ and $\delta(t)$ such that $c(t) \to c_0 > 0$ and $\delta(t) \to \rho$ as $t \to \infty$. Then, we have

$$\frac{f(g(t))}{f(t)} = \frac{c(g(t))}{c(t)}\,\exp\left\{-\int_{g(t)}^t \frac{\delta(s)}{s}\,ds\right\}, \quad t \geqq t_0. \tag{3.10}$$

Noting that $|\delta(t)| \leqq k$, $t \geqq t_0$, for some constant $k > 0$, we see because of (1.1) that

$$\left|\int_{g(t)}^t \frac{\delta(s)}{s}\,ds\right| \leqq k\left|\int_{g(t)}^t \frac{ds}{s}\right| \leqq k\log\left|\frac{t}{g(t)}\right| \longrightarrow 0, \quad t \to \infty,$$

which, combined with (3.10), implies that $f(g(t))/f(t) \to 1$ or $f(g(t)) \sim f(t)$ as $t \to \infty$. This completes the proof. □

Proof of the "only if" parts of Theorems 3.1, 3.2 *and* 3.3. Let $x(t) = t^\rho\xi(t)$, $\xi(t) \in \mathrm{SV}$, be a solution of (C) satisfying (3.1). Using (3.2) and Lemma 3.1, we have

$$\int_t^\infty q(s)x(g(s))^\gamma\,ds \sim \int_t^\infty q(s)x(s)^\gamma\,ds = \int_t^\infty s^{\sigma+\rho\gamma}l(s)\xi(s)^\gamma\,ds, \quad t \to \infty. \tag{3.11}$$

The convergence of the last integral in (3.11) implies $\sigma + \rho\gamma \leqq -1$.

(i) We first consider the case where $\sigma + \rho\gamma = -1$. Then, since

$$\int_t^\infty s^{-1}l(s)\xi(s)^\gamma\,ds \in \mathrm{SV},$$

we have by Karamata's integration theorem ((i) of Proposition 2.5)

$$\int_{T_0}^{t} \int_{s}^{\infty} r^{-1} l(r) \xi(r)^{\gamma}\, dr\, ds \sim t \int_{t}^{\infty} s^{-1} l(s) \xi(s)^{\gamma}\, ds,$$

and hence by (C)

$$x(t) \sim t \int_{t}^{\infty} s^{-1} l(s) \xi(s)^{\gamma}\, ds \in \mathrm{RV}(1), \quad t \to \infty. \tag{3.12}$$

This means that $\rho = 1$, so that $\sigma = -\gamma - 1$. From (3.12) we see that

$$\xi(t) \sim \int_{t}^{\infty} s^{-1} l(s) \xi(s)^{\gamma}\, ds, \quad t \to \infty. \tag{3.13}$$

Let $\eta(t)$ denote the right-hand side of (3.13). Then, we obtain the following differential asymptotic relation for $\eta(t)$:

$$-\eta(t)^{-\gamma} \eta'(t) \sim t^{-1} l(t) = t^{\gamma} q(t), \quad t \to \infty. \tag{3.14}$$

Since the left-hand side of (3.14) is integrable on $[T_0, \infty)$, so is $t^{\gamma} q(t)$, which shows that (3.8) is satisfied, and integrating (3.14) from t to ∞, we obtain

$$\xi(t) \sim \eta(t) \sim \left[(1-\gamma) \int_{t}^{\infty} s^{\gamma} q(s)\, ds \right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

which, in view of (3.13), leads to

$$x(t) \sim t \left[(1-\gamma) \int_{t}^{\infty} s^{\gamma} q(s)\, ds \right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

implying that $x(t)$ satisfies (3.9).

(ii) Next, we consider the case where $\sigma + \rho\gamma < -1$. Then, applying Karamata's integration theorem ((ii) of Proposition 2.5) to (3.11), we have

$$\int_{t}^{\infty} q(s) x(s)^{\gamma}\, ds \sim \frac{t^{\sigma+\rho\gamma+1} l(t) \xi(t)^{\gamma}}{-(\sigma+\rho\gamma+1)}, \quad t \to \infty. \tag{3.15}$$

We distinguish the three cases:

(a) $\sigma + \rho\gamma + 2 > 0$,

(b) $\sigma + \rho\gamma + 2 = 0$,

(c) $\sigma + \rho\gamma + 2 < 0$.

If (a) holds, then applying Karamata's integration theorem to (3.15), we find that

$$x(t) \sim \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(r)^{\gamma}\, dr\, ds \sim$$
$$\sim \frac{t^{\sigma+\rho\gamma+2} l(t)\xi(t)^{\gamma}}{[-(\sigma+\rho\gamma+1)](\sigma+\rho\gamma+2)}, \quad t \to \infty, \tag{3.16}$$

which shows that $x(t) \in \mathrm{RV}(\sigma+\rho\gamma+2)$, where $\sigma+\rho\gamma+2 \in (0,1)$. This means that $\rho = \sigma+\rho\gamma+2$ or $\rho = (\sigma+2)/(1-\gamma)$, that is, ρ is given by (3.6). From $\rho \in (0,1)$ the range of σ is determined to be $\sigma \in (-2, -\gamma-1)$. Note that (3.16) is rewritten as

$$x(t) \sim \frac{t^{\sigma+2} l(t) x(t)^{\gamma}}{\rho(1-\rho)} = \frac{t^2 q(t) x(t)^{\gamma}}{\rho(1-\rho)},$$

from which it follows that

$$x(t) \sim \left[\frac{t^2 q(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty.$$

This shows that $x(t)$ satisfies (3.7).

If (b) holds, then (3.15) takes the form $\int_t^{\infty} q(s)x(s)^{\gamma}\, ds \sim t^{-1} l(t)\xi(t)^{\gamma}$ and we have

$$x(t) \sim \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(r)^{\gamma}\, dr\, ds \sim \int_{T_0}^{t} s^{-1} l(s)\xi(s)^{\gamma}\, ds \in \mathrm{SV}, \quad t \to \infty, \tag{3.17}$$

which implies that $\rho = 0$, so that $x(t) = \xi(t)$ and $\sigma = -2$. Denoting the right-hand side of (3.17) by $y(t)$, we obtain from (3.17)

$$y(t)^{-\gamma} y'(t) \sim t^{-1} l(t) = t q(t), \quad t \to \infty. \tag{3.18}$$

Noting that the left-hand side of (3.18) and hence $tq(t)$ is not integrable on $[T_0, \infty)$ because $y(t) \to \infty$ as $t \to \infty$, we see that (3.4) holds and integrating (3.18) on $[T_0, t]$ yields

$$x(t) \sim y(t) \sim \left[(1-\gamma)\int_{T_0}^{t} sq(s)\, ds\right]^{\frac{1}{1-\gamma}} \sim \left[(1-\gamma)\int_{a}^{t} sq(s)\, ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

showing that $x(t)$ satisfies (3.5).

Finally, we note that case (c) is impossible. In fact, if (c) would hold, then the last integral in (3.15) would be integrable over $[T_0, \infty)$, which would imply that $x(t)$ tends to a constant as $t \to \infty$, that is, $x(t) \in \mathrm{ntr}-\mathrm{SV}$, an impossibility.

Let us now suppose that relation (C) admits a regularly varying solution $x(t)$ belonging to one of the three classes in (3.3). If $x(t) \in \mathrm{ntr}-\mathrm{SV}$ and $x(t) \to \infty$, $t \to \infty$, then from the above observations it is clear that $x(t)$

must fall into case (b) of (ii), which means that $\sigma = -2$ and (3.4) holds and that the asymptotic behavior of $x(t)$ is given by (3.5). Next, let (C) have a solution $x(t) \in \mathrm{RV}(\rho)$ with $\rho \in (0,1)$. Then, only case (a) of (ii) is admissible, showing that $\sigma \in (-2, -\gamma-1)$ and $x(t)$ must satisfy (3.7) with ρ defined by (3.6). Finally, if $x(t) \in \mathrm{ntr}-\mathrm{RV}(1)$ and its slowly varying part $\xi(t)$ tends to 0 as $t \to \infty$, then case (i) necessarily fits $x(t)$, so that $\sigma = -\gamma - 1$, (3.8) holds and the asymptotic behavior of $x(t)$ is governed by the formula (3.9). $\square$

Proof of the "if" parts of Theorems 3.1, 3.2 *and* 3.3. Let $X(t)$ denote any one of the functions $X_i(t)$, $i = 1,2,3$, defined on $[a,\infty)$ as follows:

$$X_1(t) = \left[(1-\gamma)\int_a^t sq(s)\,ds\right]^{\frac{1}{1-\gamma}} \in \mathrm{SV}, \tag{3.19}$$

$$\text{if } \sigma = -2 \text{ and (3.4) holds,}$$

$$X_2(t) = \left[\frac{t^2 q(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\gamma}} \in \mathrm{RV}(\rho), \tag{3.20}$$

$$\text{if } \sigma \in (-2, -\gamma-1), \text{ where } \rho = \frac{\sigma+2}{1-\gamma} \in (0,1),$$

$$X_3(t) = t\left[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\right]^{\frac{1}{1-\gamma}} \in \mathrm{RV}(1), \tag{3.21}$$

$$\text{if } \sigma = -\gamma - 1 \text{ and (3.8) holds.}$$

It suffices to verify that $X(t)$ satisfies the asymptotic relation

$$X(t) \sim \int_T^t \int_s^\infty q(r)X(g(r))^\gamma\,dr\,ds \sim \int_T^t \int_s^\infty q(r)X(r)^\gamma\,dr\,ds, \quad t \to \infty, \tag{3.22}$$

for any $T > a$ such that $g(t) \geqq a$ for $t \geqq T$, where the last relation follows from Lemma 3.1 ensuring that $X(g(t)) \sim X(t)$ as $t \to \infty$.

Suppose that $\sigma = -2$ and (3.4) holds. Then, $X_1(t)$ satisfies

$$\int_t^\infty q(s)X_1(s)^\gamma\,ds \sim t^{-1}l(t)\left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{\gamma}{1-\gamma}}$$

and hence

$$\int_T^t \int_s^\infty q(r)X_1(r)^\gamma\,dr\,ds \sim \int_T^t s^{-1}l(s)\left[(1-\gamma)\int_a^s r^{-1}l(r)\,dr\right]^{\frac{\gamma}{1-\gamma}} ds \sim$$

$$\sim \left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}} = \left[(1-\gamma)\int_a^t sq(s)\,ds\right]^{\frac{1}{1-\gamma}} = X_1(t), \quad t \to \infty.$$

Suppose next that $\sigma \in (-2, -\gamma-1)$. Rewriting $X_2(t)$ as $X_2(t) = t^\rho (l(t)/\rho(1-\rho))^{\frac{1}{1-\gamma}}$ and applying Karamata's integration theorem twice, we see that

$$\int_t^\infty q(s) X_2(s)^\gamma \, ds = \frac{\int_t^\infty s^{\rho-2} l(s)^{\frac{1}{1-\gamma}} \, ds}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}} \sim \frac{t^{\rho-1} l(t)^{\frac{1}{1-\gamma}}}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}(1-\rho)},$$

and

$$\int_T^t \int_s^\infty q(r) X_2(r) \, dr \, ds \sim \frac{t^\rho l(t)^{\frac{1}{1-\gamma}}}{(\rho(1-\rho))^{\frac{\gamma}{1-\gamma}}(1-\rho)\rho} = X_2(t), \quad t \to \infty.$$

Suppose finally that $\sigma = -\gamma - 1$ and (3.8) holds. Then, using

$$\int_t^\infty q(s) X_3(s)^\gamma \, ds = \left[(1-\gamma) \int_t^\infty s^\gamma q(s) \, ds\right]^{\frac{1}{1-\gamma}},$$

we conclude via Karamata's integration theorem that

$$\int_T^t \int_s^\infty q(r) X_3(r)^\gamma \, dr \, ds \sim t\left[(1-\gamma) \int_t^\infty s^\gamma q(s) \, ds\right]^{\frac{1}{1-\gamma}} = X_3(t), \quad t \to \infty.$$

This completes the proof of Theorems 3.1, 3.2 and 3.3. □

3.2. **Construction of Intermediate Solutions of Equation (A).** The purpose of this subsection is to prove the existence of three kinds of intermediate solutions for equation (A) with regularly varying coefficient $q(t)$ and retarded argument $g(t)$ satisfying (1.1), and furthermore to verify that two kinds of them are really regularly varying solutions. Our discussions here essentially depend on the results on regularly varying solutions of the asymptotic relation (C) developed in the first subsection. We use the following notation.

Notation 3.1. Let $f(t)$ and $g(t)$ be positive functions defined on $[t_0, \infty)$. We write $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants m and M such that $mg(t) \leqq f(t) \leqq Mg(t)$ for $t \geqq t_0$. Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. If $f(t) \asymp g(t)$, $t \to \infty$, and $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$. Our main results follow.

Theorem 3.4. *Suppose that* $q(t) \in \mathrm{RV}(-2)$ *satisfies* (3.4) *and* $g(t)$ *satisfies* (1.1). *Then equation* (A) *possesses an intermediate solution* $x(t)$ *such that*

$$x(t) \asymp \left[(1-\gamma) \int_a^t sq(s) \, ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.23}$$

Theorem 3.5. *Suppose that $q(t) \in \mathrm{RV}(\sigma)$ with $\sigma \in (-2, -\gamma-1)$ and $g(t)$ satisfies* (1.1). *Then equation* (A) *possesses an intermediate solution $x(t)$ such that*

$$x(t) \asymp \Big[\frac{t^2 q(t)}{\rho(1-\rho)}\Big]^{\frac{1}{1-\gamma}}, \quad t \to \infty, \tag{3.24}$$

where ρ is given by (3.6).

Theorem 3.6. *Suppose that $q(t) \in \mathrm{RV}(-\gamma-1)$ satisfies* (3.8) *and $g(t)$ satisfies* (1.1). *Then, equation* (A) *possesses an intermediate solution $x(t)$ such that*

$$x(t) \asymp t\Bigg[(1-\gamma)\int_t^\infty s^\gamma q(s)\,ds\Bigg]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.25}$$

Proof of Theorems 3.4, 3.5 *and* 3.6. Under the assumptions of these theorems one can define the functions $X_i(t)$, $i = 1, 2, 3$, by (3.19), (3.20) or (3.21). Let $X(t)$ denote one of $X_i(t)$, $i = 1, 2, 3$, depending on the indicated values of σ. Since $X(t)$ satisfies (3.22), there exists $T_0 > a$ such that $g(t) \geqq a$ for $t \geqq T_0$ and

$$\int_{T_0}^t \int_s^\infty q(r) X(g(r))^\gamma \,dr\,ds \leqq 2X(t), \quad t \geqq T_0. \tag{3.26}$$

We may assume that $X(t)$ is increasing for $t \geqq g(T_0)$. Using (3.21) again, one can choose $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty q(r) X(g(r))^\gamma \,dr\,ds \geqq \frac{1}{2}\,X(t), \quad t \geqq T_1. \tag{3.27}$$

Furthermore, choose positive constants $k < 1$ and $K > 1$ satisfying

$$k^{1-\gamma} \leqq \frac{1}{2}, \quad K^{1-\gamma} \geqq 4, \quad kX(T_1) \leqq \frac{1}{2}\,KX(g(T_0)), \tag{3.28}$$

and define the set $\mathcal{X}$ and the mapping $\mathcal{F} : \mathcal{X} \to C[g(T_0), \infty)$ as follows:

$$\mathcal{X} = \Big\{x(t) \in C[g(T_0), \infty) : \ kX(t) \leqq x(t) \leqq KX(t), \ \ t \geqq g(T_0)\Big\}, \tag{3.29}$$

$$\begin{cases} \mathcal{F}x(t) = x_0 + \displaystyle\int_{T_0}^t \int_s^\infty q(r) x(g(t))^\gamma \,dr\,ds, & t \geqq T_0, \\ \mathcal{F}x(t) = x_0, & g(T_0) \leqq t \leqq T_0, \end{cases} \tag{3.30}$$

where x_0 is a constant such that

$$kX(T_1) \leqq x_0 \leqq \frac{1}{2}\,KX(g(T_0)). \tag{3.31}$$

It can be shown that $\mathcal{F}$ is a continuous self-map of $\mathcal{X}$ which sends $\mathcal{X}$ into a relatively compact subset of $C[g(T_0), \infty)$.

(i) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$. This follows from the following calculations in which (3.26)–(3.31) are used:

$$\mathcal{F}x(t) \geqq x_0 \geqq kX(T_1) \geqq kX(t) \quad \text{for } g(T_0) \leqq t \leqq T_1,$$

$$\mathcal{F}x(t) \geqq \int_{T_0}^{t} \int_{s}^{\infty} q(r)(kX(g(r)))^{\gamma}\, dr\, ds \geqq \frac{1}{2}\, k^{\gamma} X(t) \geqq kX(t) \quad \text{for } t \geqq T_1,$$

$$\mathcal{F}x(t) \leqq \frac{1}{2}\, KX(g(T_0)) \leqq \frac{1}{2}\, KX(t) \leqq KX(t) \quad \text{for } g(T_0) \leqq t \leqq T_0,$$

$$\mathcal{F}x(t) \leqq \frac{1}{2}\, KX(T_0) + \int_{T_0}^{t} \int_{s}^{\infty} q(r)(KX(g(r)))^{\gamma}\, dr\, ds$$

$$\leqq \frac{1}{2}\, KX(t) + 2K^{\gamma} X(t) \leqq \frac{1}{2}\, KX(t) + \frac{1}{2}\, KX(t) = KX(t) \quad \text{for } t \geqq T_0.$$

(ii) $\mathcal{F}(\mathcal{X})$ is relatively compact. The set $\mathcal{F}(\mathcal{X})$ is locally uniformly bounded on $[g(T_0), \infty)$, since it is a subset of $\mathcal{X}$. The inequality $0 \leqq (\mathcal{F}x)'(t) \leqq K^{\gamma} \int_t^{\infty} q(s)X(g(s))^{\gamma}\, ds$, $t \geqq T_0$, holding for all $x(t) \in \mathcal{X}$ guarantees that $\mathcal{F}(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$ and hence on $[g(T_0), \infty)$. The desired relative compactness then follows from Arzela–Ascoli's lemma.

(iii) $\mathcal{F}$ is continuous. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging as $n \to \infty$ to $x(t) \in \mathcal{X}$ uniformly on every compact subinterval of $[g(T_0), \infty)$. Naturally, we need only to study the convergence on $[T_0, \infty)$. Our aim is to prove that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ as $n \to \infty$ uniformly on compact subintervals of $[T_0, \infty)$. But this follows immediately from the Lebesgue dominated convergence theorem applied to the inner integral of the right-hand side of the inequality

$$\left|\mathcal{F}x_n(t) - \mathcal{F}x(t)\right| \leqq \int_{T_0}^{t} \int_{s}^{\infty} q(r)\left|x_n(g(r))^{\gamma} - x(g(r))^{\gamma}\right|\, dr\, ds, \quad t \geqq T_0.$$

Therefore, all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled and so there exists $x(t) \in \mathcal{X}$ such that $x(t) = \mathcal{F}x(t)$ for $t \geqq g(T_0)$, which implies in particular that

$$x(t) = x_0 + \int_{T_0}^{t} \int_{s}^{\infty} q(r)x(g(r))^{\gamma}\, dr\, ds, \quad t \geqq T_0.$$

This implies that $x(t)$ is a solution of (A) on $[T_0, \infty)$. Since $x(t) \in \mathcal{X}$, i.e., $x(t) \asymp X(t)$, $t \to \infty$, $x(t)$ is an intermediate solution of (A). This completes the simultaneous proof of Theorems 3.4, 3.5 and 3.6. □

3.3. **Regularity of Intermediate Solutions.** It is shown that the two kinds of intermediate solutions of (A) obtained in Theorems 3.4 and 3.6 are actually regularly varying of indices 0 and 1, respectively. Combining this fact with Theorems 3.1 and 3.3 on the asymptotic relation (C), one can characterize completely the situation in which the sublinear equation (A) with regularly varying $q(t)$ possesses nontrivial regularly varying solutions of indices 0 and 1.

Theorem 3.7. *Let* $q(t) \in \mathrm{RV}(\sigma)$ *and suppose that* $g(t)$ *satisfies* (1.1). *Equation* (A) *possesses nontrivial slowly varying solutions if and only if* $\sigma = -2$ *and* (3.4) *holds, in which case the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* (3.5).

Proof. (The "if" part) Suppose that $\sigma = -2$ and (3.4) holds. Then $q(t) = t^{-2}l(t)$ and (3.4) is expressed as $\int_a^\infty s^{-1}l(s)\,ds = \infty$. Let $x(t)$ be an intermediate solution of (A) constructed in Theorem 3.4 as a solution of the integral equation (B). It is known that

$$x(t) \asymp X_1(t) = \left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{3.32}$$

Using (B), (3.32) and one of the properties of $X_1(t)$ mentioned in the proof of the "if" part of Theorem 3.1, we find that

$$x'(t) = \int_t^\infty q(s)x(g(s))^\gamma\,ds \asymp \int_t^\infty q(s)X_1(g(s))^\gamma\,ds \sim$$

$$\sim \int_t^\infty q(s)X_1(s)^\gamma\,ds \sim t^{-1}l(t)\left[(1-\gamma)\int_a^t s^{-1}l(s)\,ds\right]^{\frac{\gamma}{1-\gamma}}, \quad t \to \infty. \tag{3.33}$$

We combine (3.32) and (3.33) to obtain

$$t\frac{x'(t)}{x(t)} \asymp \frac{l(t)}{(1-\gamma)\int_a^t s^{-1}l(s)\,ds}, \quad t \to \infty,$$

from which, noting that the right-hand side of the above tends to 0 as $t \to \infty$ by (iii) of Proposition 2.5, we conclude that $\lim_{t\to\infty} tx'(t)/x(t) = 0$. From Proposition 2.4 it follows that $x(t)$ is a nontrivial slowly varying function.

(The "only if" part) If $x(t)$ is a nontrivial slowly varying solution of (A), then it clearly satisfies relation (C) and hence from the "only if" part of Theorem 3.1 it follows that $\sigma = -2$ and (3.4) holds and, moreover, that the asymptotic behavior of $x(t)$ is given by (3.5). This completes the proof of Theorem 3.7. $\square$

Theorem 3.8. *Let* $q(t) \in \mathrm{RV}(\sigma)$ *and suppose that* $g(t)$ *satisfies* (1.1). *Equation* (A) *possesses nontrivial regularly varying solutions of index* 1 *if and only if* $\sigma = -\gamma-1$ *and* (3.8) *holds, in which case the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* (3.9).

Proof. (The "if" part) Suppose that $\sigma = -\gamma - 1$ and (3.8) holds. Then, $q(t) = t^{-\gamma-1}l(t)$ and (3.8) is expressed as $\int_a^\infty s^{-1}l(s)\,ds < \infty$. Let $x(t)$ be an intermediate solution of (A) obtained in Theorem 3.4 as a solution of the integral equation (B). It satisfies

$$x(t) \asymp X_3(t) = t\left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

which implies that

$$\begin{aligned} -x''(t) &= q(t)x(g(t))^\gamma \asymp q(t)X_3(g(t))^\gamma \sim \\ &\sim q(t)X_3(t)^\gamma = t^{-\gamma-1}l(t)\left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{\gamma}{1-\gamma}}, \quad t \to \infty. \end{aligned} \tag{3.34}$$

On the other hand, taking the proof of the "if" part of Theorem 3.3, we see that $x'(t)$ satisfies

$$\begin{aligned} x'(t) &= \int_t^\infty q(s)x(g(s))^\gamma\,ds \asymp \int_t^\infty q(s)X_3(g(s))^\gamma\,ds \sim \\ &\sim \int_t^\infty q(s)X_3(s)^\gamma\,ds = \left[(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \end{aligned} \tag{3.35}$$

Using (3.34) and (3.35), we obtain

$$-t\frac{x''(t)}{x'(t)} \asymp \frac{l(t)}{(1-\gamma)\int_t^\infty s^{-1}l(s)\,ds} \to 0, \quad t \to \infty,$$

where (iii) of Proposition 2.5 has been used. This means by Proposition 2.4 that $x'(t)$ is slowly varying, and from (i) of Proposition 2.5 we conclude that

$$x(t) \sim \int_{T_0}^t x'(s)\,ds \sim tx'(t) \in \mathrm{RV}(1), \quad t \to \infty,$$

which implies that $x(t)$ is a nontrivial regularly varying solution of index 1.

(The "only if" part) Let $x(t)$ be a nontrivial RV(1)-solution of (A). Then, since it satisfies relation (C), from the "only if" part of Theorem 3.3 it follows that $\sigma = -\gamma - 1$ and (3.8) holds and, moreover, that the asymptotic behavior of $x(t)$ is given by (3.9). This completes the proof of Theorem 3.8. □

Remark 3.1. It is impossible for us to prove that the solution obtained in Theorem 3.5 is regularly varying of index $\rho \in (0, 1)$. A more powerful criterion than Proposition 2.4 seems to be necessary.

Example 3.1. Consider equation (A) with $g(t)$ satisfying (1.1). Suppose that $q(t)$ satisfies

$$q(t) \sim \frac{c}{t^2 \log t (\log\log t)^{\gamma}}, \quad t \to \infty,$$

for some positive constant $c > 0$. It is clear that $q(t) \in \mathrm{RV}(-2)$ and (3.4) is satisfied, and that

$$\left[(1-\gamma)\int_a^t sq(s)\,ds\right]^{\frac{1}{1-\gamma}} \sim c^{\frac{1}{1-\gamma}} \log\log t, \quad t \to \infty.$$

By Theorem 3.7, we see that equation (A) possesses nontrivial SV-solutions $x(t)$, all of which have one and the same asymptotic behavior $x(t) \sim c^{\frac{1}{1-\gamma}} \log\log t$, $t \to \infty$, for any retarded argument $g(t)$. If, in particular,

$$q(t) = \frac{1}{t^2 \log t (\log\log g(t))^{\gamma}}\Big(1 + \frac{1}{\log t}\Big),$$

then equation (A) has an exact solution $x_0(t) = \log\log t \in \mathrm{ntr}-\mathrm{SV}$.

Example 3.2. Consider equation (A) with $g(t)$ satisfying (1.1). Suppose that $q(t)$ satisfies

$$q(t) \sim \frac{c}{t^{\gamma+1} \log t (\log\log t)^{2-\gamma}} \in \mathrm{RV}(-\gamma-1), \quad t \to \infty,$$

for some constant $c > 0$. As is easily checked, (3.8) is satisfied and

$$\left[(1-\gamma)\int_t^{\infty} s^{\gamma} q(s)\,ds\right]^{\frac{1}{1-\gamma}} \sim \frac{c^{\frac{1}{1-\gamma}}}{\log\log t}, \quad t \to \infty,$$

and hence by Theorem 3.8, equation (A) possesses nontrivial RV(1)-solutions $x(t)$, all of which have one and the same asymptotic behavior $x(t) \sim \frac{c^{\frac{1}{1-\gamma}} t}{\log\log t}$, $t \to \infty$, for any retarded argument $g(t)$. If, in particular,

$$q(t) = \frac{(\log\log g(t))^{\gamma}}{tg(t)^{\gamma} \log t (\log\log t)^2}\Big(1 - \frac{1}{\log t} - \frac{2}{\log t \cdot \log\log t}\Big),$$

then equation (A) has an exact solution $x_1(t) = t/\log\log t$.

Example 3.3. Consider the equation

$$x''(t) + t^{-\frac{3}{2}}\big(2 + \sin(\log\log t)\big)^2 x(g(t))^{\frac{1}{3}} = 0, \tag{3.36}$$

which is a special case of (A) in which

$$\gamma = \frac{1}{3} \text{ and } q(t) = t^{-\frac{3}{2}}\Big(2 + \sin(\log\log t)\Big)^2 \in \mathrm{RV}\Big(-\frac{3}{2}\Big).$$

Since $\sigma = -\frac{3}{2}$ satisfies $-2 < \sigma < -\gamma - 1 = -\frac{4}{3}$, Theorem 3.5 is applicable to (3.35) and ensures the existence of its intermediate solution $x(t)$ such that

$$x(t) \asymp \Big(\frac{16}{3}\Big)^{\frac{3}{2}} t^{\frac{3}{4}}\big(2 + \sin(\log\log t)\big)^3, \quad t \to \infty.$$

It is impossible to decide whether or not this solution is regularly varying of index $\frac{3}{4}$.

Remark 3.2. A question naturally arises: what will happen if condition (1.1) on $g(t)$ is not required? The problem of investigating the accurate asymptotic behavior of positive solutions of (A) for general retarded argument is much more difficult to handle as the following example indicates. It is to be noted that very little is known about regularly varying solutions of functional differential equations, linear or nonlinear, with general deviating arguments. See e.g. the papers [3]–[5].

Example 3.4. Consider the equation

$$x''(t) + q(t)x(\log t)^{\gamma} = 0, \quad 0 < \gamma < 1, \tag{3.37}$$

where $q(t)$ is given by

$$q(t) = \frac{(\log\log\log t)^{\gamma}}{t(\log t)^{\gamma+1}(\log\log t)^2}\Big(1 - \frac{1}{\log t} - \frac{2}{\log t \cdot \log\log t}\Big) \in \mathrm{RV}(-1).$$

As is easily checked, equation (3.37) has a nontrivial RV(1)-solution $x(t) = t/\log\log t$ in marked contrast to Theorem 3.6 or Theorem 3.8.

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Authors' addresses:

Department of Mathematics, Graduate School of Science and Technology, Kumamoto University, 2 – 39 – 1 Kurokami, Kumamoto, 860 – 8555, Japan.

e-mail: akihito.shibuya@gmail.com