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Short Communications

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ON A TWO-POINT SINGULAR BOUNDARY VALUE PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The two-point boundary value problem is considered for the system of nonlinear generalized ordinary differential equations with singularities on a non-closed interval. Singularity is understood in a sense of the vector-function corresponding to the system which belongs to the local Carathéodory class with respect to the matrix-function corresponding to the system.

The general sufficient conditions are established for the unique solvability of this problem. Relying on these results, the effective conditions are established for the unique solvability of the problem.

რეზიუმე. სინგულარობებიან განზოგადოებულ ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისათვის განხილულია ორწერტილოვანი სასაზღვრო ამოცანა არაჩაკეტილ ინტერვალზე. სინგულარობა გაიგება იმ აზრით, რომ სისტემის შესაბამისი ვექტორული ფუნქცია მიეკუთვნება ლოკალურ კარათეოდორის კლასს სისტემის შესაბამისი მატრიცული ფუნქციის მიმართ.

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1. Statement of the Problem and Basic Notation

In the present paper, for a system of linear generalized ordinary differential equations with singularities

$$dx_i = f_i(t, x_1, \dots, x_n)\,da_i(t) \;\text{ for }\; t \in [a, b] \;\; (i = 1, \dots, n) \tag{1.1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \;\; (i = 1, \dots, n_0), \quad x_i(b-) = 0 \;\; (i = n_0 + 1, \dots, n), \tag{1.2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of a desired solution x, $a_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are nondecreasing functions, and $f_i :]a, b[\, \times \mathbb{R}^n \to \mathbb{R}$ is a function belonging to the local Carathéodory class $\mathrm{Car}_{loc}(]a, b[\, \times \mathbb{R}^n, \mathbb{R}; a_i)$ corresponding to the function a_i for every $i \in \{1, \dots, n\}$.

We investigate the question of solvability of the problem (1.1), (1.2), when the system (1.1) has singularities. Singularity is understood in a sense that the components of the vector-function f may have non-integrable components at the boundary points a and b, in general. We present a general theorem for the solvability of this problem. On the basis of this theorem we obtain the effective criteria for the solvability of the problem.

Analogous and related questions are investigated in [13]–[18] (see also references therein) for the singular two-point and multipoint boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [1]–[7] (see also references therein) for regular two-point and multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As for the two-point and multipoint singular boundary value problems for generalized differential systems, they are little studied and, despite some results given in [8–10] for two-point and multipoint singular boundary value problem, their theory is rather far from completion even in the linear case. Therefore, the problem under consideration is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1]–[12], [19]–[22] and references therein).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $R_+ = [0, +\infty[$; $[a, b]$, $]a, b[$ and $]a, b]$, $[a, b[$ are, respectively, closed, open and half-open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i,l=1}^{n,m} |x_{il}|.$$

$\mathbb{R}_+^{n\times n} = \big\{(x_{il})_{i,l=1}^{n,m} :\ x_{il} \geq 0\ (i = 1, \dots, n;\ l = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

$\bigvee\limits_{c}^{d}(X)$, where $a < c < d < b$, is the variation of the matrix-function $X :]a, b[\to \mathbb{R}^{n\times m}$ on the closed interval $[c, d]$, i.e., the sum of total variations of the latter components x_{il} $(i = 1, \dots, n;\ l = 1, \dots, m)$ on this interval; if $d < c$, then $\bigvee\limits_{c}^{d}(X) = -\bigvee\limits_{d}^{c}(X)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0) = 0$, $v(x_{il})(t) = \bigvee\limits_{c_0}^{t}(x_{il})$ for $a < t < b$, and $c_0 = (a + b)/2$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X :]a, b[\to \mathbb{R}^{n\times m}$ at the point $t \in]a, b[$ (we assume $X(t) = X(a+)$ for $t \le a$ and $X(t) = X(b-)$ for $t \ge b$, if necessary).

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a, b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_{a}^{b}(X) < +\infty$).

$\mathrm{BV}_{loc}(]a, b[, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X :]a, b[\to \mathbb{R}^{n\times m}$ such that $\bigvee\limits_{c}^{d}(X) < +\infty$ for every $a < c < d < b$.

If $X \in \mathrm{BV}_{loc}(]a, b[, \mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_jX(t)) \ne 0$ for $t \in]a, b[$ $(j = 1, 2)$, and $Y \in \mathrm{BV}_{loc}(]a, b[, \mathbb{R}^{n\times m})$, then $\mathcal{A}(X, Y)(t) \equiv \mathcal{B}(X, Y)(c_0, t)$, where $\mathcal{B}$ is the operator defined as follows:

$$\mathcal{B}(X, Y)(t, t) = O_{n\times m} \ \text{ for } \ t \in]a, b[\,,$$
$$\mathcal{B}(X, Y)(s, t) = Y(t) - Y(s) + \sum_{s<\tau\le t} d_1X(\tau) \cdot \big(I_n - d_1X(\tau)\big)^{-1} d_1Y(\tau) -$$
$$- \sum_{s\le\tau<t} d_2X(\tau) \cdot \big(I_n + d_2X(\tau)\big)^{-1} d_2Y(\tau) \ \text{ for } \ a < s < t < b$$

and

$$\mathcal{B}(X, Y)(s, t) = -\mathcal{B}(X, Y)(t, s) \ \text{ for } \ a < t < s < b.$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $\alpha : [a, b] \to \mathbb{R}$ is a nondecreasing function, then $D_\alpha = \{t \in [a, b] : d_1\alpha(t) + d_2\alpha(t) \ne 0\}$.

If $\alpha \in \mathrm{BV}([a, b], \mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1, 2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points from $[a, b]$ for which $d_m\alpha(t) \ne 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : \ t \in D_{\alpha m}\}$ $(m = 1, 2)$.

If $\beta \in \mathrm{BV}([a, b], \mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\Big\{ d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m} \Big\}$$

$(j, m = 1, 2)$; here $t_{\alpha 20} = a - 1$, $t_{\alpha 1 n_{\alpha 1}+1} = b + 1$.

$s_1, s_2, s_c : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \ \text{ and } \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$; if $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$; thus, $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [19], [20], [22]). Moreover, we put

$$\int_{s+}^t x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_{s+\varepsilon}^t x(\tau)\,dg(\tau)$$

and

$$\int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L^p([a,b],\mathbb{R};g)$ $(1 \le p < +\infty)$ is the space of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<\tau\le b} |x(t)|^\mu d_1g(\tau) + \sum_{a\le\tau<b} |x(t)|^\mu d_2g(t) < +\infty,$$

with the norm

$$\|x\|_{p,g} = \left(\int_a^b |x(t)|^p\,dg(t)\right)^{\frac{1}{p}}.$$

$L^{+\infty}([a,b],\mathbb{R};g)$ is the space of all $\mu(s_0(g))$-measurable and $\mu(s_0(g))$-essentially bounded functions $x : [a,b] \to \mathbb{R}$ such that $\sup\{|x(t)| : t \in D_\alpha\} < +\infty$, with the norm

$$\|x\|_{+\infty,g} = \inf\Big\{r > 0 : |x(t)| \le r \ \text{ for } \ \mu(s_0(g))\text{-almost all } \ t \in [a,b] \ \text{ and for } \ t \in D_\alpha\Big\}.$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ ($i = 1,\dots,l$; $k = 1,\dots,n$; $j = 1,\dots,m$);

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \ \text{ for } \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 0,1,2).$$

The inequalities between the vectors and between the matrices are understood componentwise.

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}$, then $\mathrm{Car}([a,b]\times D_1, D_2; g)$ is the Carathéodory class, i.e., the set of all mappings $f : [a,b]\times D_1 \to D_2$ such that:

(i) the function $f(\cdot\,,x) : [a,b] \to D_2$ is $\mu(g)$-measurable for every $x \in D_1$;

(ii) the function $f(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g)$-almost all $t \in [a,b]$, and

$$\sup\big\{|f(\cdot\,,x)| : \ x \in D_0\big\} \in L([a,b],R;g)$$

for every compact $D_0 \subset D_1$.

$\mathrm{Car}_{loc}(]a,b[\times D_1, D_2; g)$ is the set of all mappings $f :]a,b[\times D_1 \to D_2$ the restriction of which on every closed interval $[c,d]$ of $]a,b[$ belongs to $\mathrm{Car}([c,d]\times D_1, D_2; g)$. Analogously are defined the sets $\mathrm{Car}_{loc}(]a,b]\times D_1, D_2; G)$ and $\mathrm{Car}_{loc}([a,b[\times D_1, D_2; G)$.

We assume that $a_i : [a,b] \to \mathbb{R}$ ($i = 1,\dots,n$) are nondecreasing functions and $f_i \in \mathrm{Car}(]a,b[\times\mathbb{R}^n, \mathbb{R}^n; a_i)$ ($i = 1,\dots,n$). A vector-function $x = (x_i)_{i=1}^n$ is said to be a solution of the system (1.1) if $x_i \in \mathrm{BV}_{loc}(]a,b],\mathbb{R})$ ($i = 1,\dots,n_0$), $x_i \in \mathrm{BV}_{loc}([a,b[\,,\mathbb{R})$ ($i = n_0+1,\dots,n$) and

$$x_i(t) = x_i(s) + \sum_{l=1}^n \int_s^t f_l(\tau, x_1(\tau),\dots,x_n(\tau))\,da_{il}(\tau)$$

for $a<s\le t\le b$ if $i\in\{1,\dots,n_0\}$ and for $a\le s<t<b$ if $i\in\{n_0+1,\dots,n\}$.

Under the solution of the problem (1.1), (1.2) we mean a solution $x(t) = (x_i(t))_{i=1}^n$ of the system (1.1) such that the one-sided limits $x_i(a+)$ ($i = 1,\dots,n_0$) and $x_i(b-)$ ($i = n_0+1,\dots,n$) exist and the equalities (1.2) are fulfilled. We assume $x_i(a) = 0$ ($i = 1,\dots,n_0$) and $x_i(b) = 0$ ($i = n_0 + 1,\dots,n$), if necessary.

A vector-function $x = (x_i)_{i=1}^n$, $x \in \mathrm{BV}(]a,b[\,,\mathbb{R})$, is said to be a solution of the system of generalized differential inequalities

$$dx_i(t) \le \sum_{l=1}^n x_l(t)\, db_{il}(t)\ (\ge)\ \text{ for }\ t \in\,]a,b[\ \ (i=1,\dots,n),$$

where $b_{il} : [a,b] \to \mathbb{R}$ $(i,l=1,\dots,n)$ are nondecreasing functions, if

$$x_i(t) - x_i(s) \le \sum_{l=1}^n \int\limits_s^t x_l(\tau)\, db_{il}(\tau)\ (\ge)\ \text{ for }\ a < s \le t < b\ \ (i=1,\dots,n).$$

Without loss of generality, we assume that $a_i(a) = O_{n\times n}$ $(i=1,\dots,n)$. Moreover, we assume

$$\det\big(I_n + (-1)^j d_j a_i(t)\big) \ne 0\ \text{ for }\ t \in\,]a,b[\ \ (j=1,2; i=1,\dots,n). \tag{1.3}$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (see [22, Theorem III.1.4]).

If $s \in\,]a,b[$ and $\alpha \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R})$ are such that

$$1 + (-1)^j d_j \beta(t) \ne 0\ \text{ for }\ (-1)^j(t-s) < 0\ \ (j=1,2),$$

then by $\gamma_\beta(\cdot\,,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t) d\beta(t),\ \ \gamma(s) = 1.$$

It is known (see [11], [12]) that

$$\gamma_\alpha(t,s) = \begin{cases} \exp\big(s_0(\beta)(t) - s_0(\beta)(s)\big)\times \\ \qquad \times \prod\limits_{s<\tau\le t} (1-d_1\alpha(\tau))^{-1} \prod\limits_{s\le\tau<t} (1+d_2\beta(\tau)) & \text{for }\ t > s, \\ \exp\big(s_0(\beta(t) - s_0(\beta(s)\big)\times \\ \qquad \times \prod\limits_{t<\tau\le s} (1-d_1\beta(\tau)) \prod\limits_{t\le\tau<s} (1+d_2\beta(\tau))^{-1} & \text{for }\ t < s, \\ 1 & \text{for }\ t = s. \end{cases} \tag{1.4}$$

It is evident that if the last inequalities are fulfilled on the whole interval $[a,b]$, then $\gamma_\alpha^{-1}(t)$ exists for every $t \in [a,b]$.

Definition 1.1. Let $a_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ be nondecreasing functions and $n_0 \in \{1,\dots,n\}$. We say that the matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}_+^{n\times n})$ belongs to the set $\mathcal{U}(a+,b-;a_1,\dots,a_n;n_0)$, if the system

$$\operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)\, dx_i(t) \le \\ \le \sum_{l=1}^n c_{il}(t) x_l(t)\, da_i(t)\ \text{ for }\ t \in [a,b]\ \ (i=1,\dots,n) \tag{1.5}$$

has no nontrivial nonnegative solution satisfying the condition (1.2).

Definition 1.2. We say that a vector-function $g : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $g(t,x) = (g_i(t,x_1,\dots,x_n))_{i=1}^n$, is nondecreasing outside of the diagonal elements (or quasi-nondecreasing) with respect to nondecreasing vector-function $\alpha = (\alpha_i)_{i=1}^n$ if from the condition

$$x_1 \le y_1, \dots, x_{i-1} \le y_{i-1}, x_{i+1} \le y_{i+1}, \dots, x_n \le y_n$$

follows

$$g_i(t,x_1,\dots,x_{i-1},x_i,\dots,x_n) \le g_i(t,y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n)$$

for $\mu(a_i)$-almost all t $(i=1,\dots,n)$.

Definition 1.3. Let $a_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ be nondecreasing functions and $n_0 \in \{1,\dots,n\}$. We say that a vector-function $g(t,x) = (g_i(t,x_1,\dots,x_n))_{i=1}^n, g_i \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R};a_i)$ $(i=1,\dots,n)$, belongs to the set $\mathcal{U}_0(a+,b-;a_1,\dots,a_n;n_0)$ if it is nonnegative, quasi-nondecreasing and there exists a positive number $r \in \mathbb{R}_+$ such that

$$0 \le x(t) \le r \ \text{ for } \ t \in [a,b]$$

for every nonnegative solution $x = (x_i)_{i=1}^n$ of the system

$$\operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)\, dx_i(t) \le \\ \le g_i(t,x_1,\dots,x_n(t))da_i(t) \ \text{ for } \ t \in [a,b] \ \ (i=1,\dots,n) \quad (1.6)$$

under the boundary condition (1.2).

The similar definition of the sets $\mathcal{U}_0$ and $\mathcal{U}$ has been introduced by I. Kiguradze for ordinary differential equations (see [13]–[15]).

Theorem 1.1. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i = 1,\dots,n)$ *be such that*

$$f_i(t,x_1,\dots,x_n)\operatorname{sgn}\Big(\Big(n_0+\frac{1}{2}-i\Big)x_i\Big) \le -b_i(t)|x_i| + g_i(t,|x_1|,\dots,|x_n|)$$
$$\text{for } \mu(s_c(a_i))\text{-almost all } t \in [a,b] \text{ and for every } t \in D_{a_i},$$
$$(x_k)_{k=1}^n \in \mathbb{R}^n \ \ (i=1,\dots,n),$$

$$f_i(t,x_1,\dots,x_n)d_2a_i(t)\operatorname{sgn}\big(x_i + f_i(t,x_1,\dots,x_n)d_2a_i(t)\big) \le \\ \le -b_i(t)|x_i| + g_i\big(t,|x_1|,\dots,|x_n|\big)$$
$$\text{for } t \in [a,b] \text{ and } (x_k)_{k=1}^n \in \mathbb{R}^n \ \ (i=1,\dots,n_0)$$

and

$$f_i(t,x_1,\dots,x_n)d_1a_i(t)\operatorname{sgn}\big(x_i - f_i(t,x_1,\dots,x_n)d_1a_i(t)\big) \ge \\ \ge b_i(t)|x_i| - g_i(t,|x_1|,\dots,|x_n|)$$
$$\text{for } t \in [a,b] \text{ and } (x_k)_{k=1}^n \in \mathbb{R}^n \ \ (i=n_0+1,\dots,n),$$

where $g_i \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$, *the functions* $b_i \in L_{loc}(]a,b],\mathbb{R};a_i)$ *for* $(i=1,\dots,n_0)$ *and* $b_i \in L_{loc}([a,b[\,,\mathbb{R};a_i)$ *for* $(i=n_0+1,\dots,n)$ *are nonnegative. Let, moreover,*

$$\begin{gathered} g=(g_i)_{i=1}^n \in \mathcal{U}_0(a+,b-;a_1,\dots,a_n;n_0),\\ \lim_{t\to a+} b_i(t)d_2a_i(t)<1 \ \ (i=1,\dots,n_0),\\ \lim_{t\to b-} b_i(t)d_1a_i(t)<1 \ \ (i=n_0+1,\dots,n) \end{gathered} \tag{1.7}$$

and

$$\begin{gathered} \lim_{t\to a+}\ \lim_{k\to\infty}\sup\gamma_{\alpha_i}(t,a+1/k)=0 \ \ (i=1,\dots,n_0),\\ \lim_{t\to b-}\ \lim_{k\to\infty}\sup\gamma_{\alpha_i}(t,b-1/k)=0 \ \ (i=n_0+1,\dots,n), \end{gathered} \tag{1.8}$$

where $\alpha_i(t)\equiv\int\limits_{c_0}^{t} b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, *and* γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4). *Then the problem* (1.1), (1.2) *is solvable.*

Theorem 1.2. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i=1,\dots,n)$ *be such that*

$$\begin{gathered} f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}\Big(\Big(n_0+\frac{1}{2}-i\Big)x_i\Big)\le\\ \le -b_i(t)|x_i|+\sum_{l=1}^n\eta_{il}(t)|x_l|+q_i\Big(t,\sum_{l=1}^n|x_l|\Big)\\ \text{for } \mu(s_c(a_i))\text{-almost all } t\in[a,b] \text{ and for every } t\in D_{a_i},\\ (x_k)_{k=1}^n\in\mathbb{R}^n \ \ (i=1,\dots,n), \end{gathered} \tag{1.9}$$

$$\begin{gathered} f_i(t,x_1,\dots,x_n)d_2a_i(t)\,\mathrm{sgn}\big(x_i+f_i(t,x_1,\dots,x_n)d_2a_i(t)\big)\le\\ \le -b_i(t)|x_i|+\sum_{l=1}^n\eta_{il}(t)|x_l|+q_i\Big(t,\sum_{l=1}^n|x_l|\Big) \text{ for } t\in[a,b] \ \ (i=1,\dots,n_0) \end{gathered} \tag{1.10}$$

and

$$\begin{gathered} f_i(t,x_1,\dots,x_n)d_1a_i(t)\,\mathrm{sgn}\big(x_i-f_i(t,x_1,\dots,x_n)d_1a_i(t)\big)\ge\\ \ge b_i(t)|x_i|-\sum_{l=1}^n\eta_{il}(t)|x_l|-q_i\Big(t,\sum_{l=1}^n|x_l|\Big) \text{ for } t\in[a,b] \ \ (i=n_0+1,\dots,n) \end{gathered} \tag{1.11}$$

where $\eta_{il}\in L([a,b],\mathbb{R};a_i)$ $(i,l=1,\dots,n)$, *the functions* $b_i\in L_{loc}(]a,b],\mathbb{R};a_i)$ $(i=1,\dots,n_0)$ *and* $b_i\in L_{loc}([a,b[,\mathbb{R};a_i)$ $(i=n_0+1,\dots,n)$ *are nonnegative, and* $q_i\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$ *are nondecreasing functions in the second variable. Let, moreover, the conditions* (1.7), (1.8),

$$C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a+,b-;a_1,\dots,a_n;n_0),$$

and

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int\limits_a^b q_i(t,\rho)\,da_i(t)=0 \quad (i=1,\dots,n) \tag{1.12}$$

be valid, where $\alpha_i(t)\equiv\int\limits_{c_0}^t b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, $c_{il}(t)\equiv\int\limits_a^t \eta_{il}(\tau)\,da_i(\tau)$ $(i,l=1,\dots,n)$, *and* γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4). *Then the problem* (1.1), (1.2) *is solvable.*

Corollary 1.1. *Let the functions* $f_i\in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_i)$ $(i=1,\dots,n)$ *be such that the conditions* (1.9)–(1.12) *hold, where the functions* a_i $(i=1,\dots,n)$ *have not more than a finite number of points of discontinuity, the functions* $b_i\in L_{loc}(]a,b],\mathbb{R};a_i)$ $(i=1,\dots,n_0)$ *and* $b_i\in L_{loc}([a,b[,\mathbb{R};a_i)$ $(i=n_0+1,\dots,n)$ *are nonnegative,* $q_i\in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_i)$ $(i=1,\dots,n)$ *are nondecreasing functions in the second variable,* $\alpha_i(t)\equiv\int\limits_{c_0}^t b_i(\tau)\,da_i(\tau)$ $(i=1,\dots,n)$, $c_0=(a+b)/2$, γ_{α_i} $(i=1,\dots,n)$ *are the functions defined according to* (1.4),

$$\int\limits_a^t \eta_{il}(\tau)\,da_i(\tau)\equiv\int\limits_c^t h_{il}(\tau)d\beta_l(\tau) \quad (i,l=1,\dots,n),$$

β_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$, $h_{ii}\in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i\neq l;\ i,l=1,\dots,n)$, $1\le\mu\le+\infty$. *Let, moreover,*

$$r(\mathcal{H})<1, \tag{1.13}$$

where the $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1}=\Big(\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)}\Big)_{i,k=1}^n \quad (j,m=0,1,2),$$

$$\xi_{ij}=\big(s_j(\beta_i)(b)-s_j(\beta_i)(a)\big)^{\frac{1}{\nu}} \quad (j=0,1,2,;\ \ i=1,\dots,n);$$

$$\lambda_{k0i0}=\begin{cases}\Big(\dfrac{4}{\pi^2}\Big)^{\frac{1}{\nu}}\xi_{k0}^2 & \text{if } s_0(\beta_i)(t)\equiv s_0(\beta_k)(t),\\ \xi_{k0}\xi_{i0} & \text{if } s_0(\beta_i)(t)\not\equiv s_0(\beta_k)(t) \quad (i,k=1,\dots,n);\end{cases}$$

$$\lambda_{kmij}=\xi_{km}\xi_{ij} \quad \text{if } m^2+j^2>0,\ \ mj=0 \quad (j,m=0,1,2;\ \ i,k=1,\dots,n),$$

$$\lambda_{kmij}=\Big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\Big)^{\frac{1}{\nu}} \quad (j,m=1,2;\ i,k=1,\dots,n),$$

and $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1.1), (1.2) *is solvable.*

Remark 1.1. The $3n \times 3n$-matrix $\mathcal{H}$, appearing in Corollary 1.1 can be replaced by the $n \times n$-matrix

$$\bigg(\max\Big\{ \sum_{j=0}^{2} \lambda_{kmij} \|h_{ik}\|_{\mu,S_m(\alpha_k)} :\ m = 0,1,2 \Big\} \bigg)_{i,k=1}^{n}.$$

Remark 1.2. If $a_i(t) \equiv a_0(t)\,(i = 1,\dots,n)$, where the function a_0 has not more than a finite number of points of discontinuity, then we can assume that $h_{il}(t) \equiv \eta_{il}(t)$ and $\beta_l(t) \equiv a_0(t)\,(i,l = 1,\dots,n)$.

By Remark 1.1, Corollary 1.1 has the following form for $a_i(t) \equiv a_0(t)$, $b_i(t) \equiv b_0(t)$, $\eta_{il}(t) \equiv \eta_{il} = const$, $q_i(t,x) \equiv q(t,x)$ $(i,l = 1,\dots,n)$ and $\mu = +\infty$ since, by the choice of $h_{il}(t) \equiv \eta_{il}(t) = \eta_{il}$ $(i,l = 1,\dots,n)$, we have $\beta_l(t) \equiv a_0(t)$ $(l = 1,\dots,n)$ in this case.

Corollary 1.2. *Let the functions $f_i \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n,\mathbb{R}^n;a_0)$ be such that the conditions* (1.12),

$$f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}\Big(\Big(n_0+\frac{1}{2}-i\Big)x_i\Big) \le$$
$$\le -b_0(t)|x_i| + \sum_{l=1}^{n}\eta_{il}|x_l| + q_i\Big(t,\sum_{l=1}^{n}|x_l|\Big)$$
$$\textit{for } \mu(s_c(a_0))\textit{-almost all } t\in[a,b] \textit{ and for every } t \in D_a,$$
$$(x_k)_{k=1}^n \in \mathbb{R}^n \ \ (i=1,\dots,n),$$

$$f_i(t,x_1,\dots,x_n)d_2a_i(t)\,\mathrm{sgn}\big(x_i + f_i(t,x_1,\dots,x_n)d_2a_i(t)\big) \le$$
$$\le -b_0(t)|x_i| + \sum_{l=1}^{n}\eta_{il}|x_l| + q_i\Big(t,\sum_{l=1}^{n}|x_l|\Big) \ \textit{for } t\in[a,b] \ \ (i=1,\dots,n_0),$$
$$f_i(t,x_1,\dots,x_n)d_1a_i(t)\,\mathrm{sgn}\big(x_i - f_i(t,x_1,\dots,x_n)d_1a_i(t)\big) \ge$$
$$\ge b_0(t)|x_i| - \sum_{l=1}^{n}\eta_{il}|x_l| - q_i\Big(t,\sum_{l=1}^{n}|x_l|\Big) \ \textit{for } t\in[a,b] \ \ (i=n_0+1,\dots,n)$$

and

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int_a^b q(t,\rho)\,da_0(t) = 0$$

hold, where a_0 is a nondecreasing function on $[a,b]$ having no more than a finite number of points of discontinuity, $b_0 \in L([a,b],\mathbb{R}_+;a_0)$, $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+;a_0)$ is a nondecreasing function in the second variable, the function $\alpha(t) \equiv \int_{c_0}^{t} b(\tau)\,da(\tau)$, $c_0 = (a+b)/2$, satisfies the conditions (1.7) *and* (1.8), *γ_α is the function defined according to* (1.4), *$\eta_{ii} \in \mathbb{R}$, $\eta_{il} \in \mathbb{R}_+$ $(i \ne l;$ $i,l = 1,\dots,n)$. Let, moreover,*

$$\rho_0\, r(\mathcal{H}) < 1,$$

where

$$\mathcal{H} = (\eta_{ik})_{i,k=1}^{n}, \quad \rho_0 = \max\Big\{\sum_{j=0}^{2} \lambda_{mj} : \ m = 0, 1, 2\Big\},$$

$$\lambda_{00} = \frac{2}{\pi}\,\big(s_0(a_0)(b) - s_0(a_0)(a)\big),$$

$$\lambda_{0j} = \lambda_{j0} = \big(s_0(a_0)(b) - s_0(a_0)(a)\big)^{\frac{1}{2}} \big(s_j(a_0(b) - s_j(a_0(a)\big)^{\frac{1}{2}} \ \ (j = 1, 2),$$

$$\lambda_{mj} = \frac{1}{2}\,\big(\mu_{\alpha m}\nu_{\alpha m\alpha j}\big)^{\frac{1}{2}} \sin^{-1} \frac{\pi}{4n_{\alpha m} + 2} \ \ (m, j = 1, 2).$$

Then the problem (1.1), (1.2) *is solvable.*

Theorem 1.3. *Let the functions* $f_i \in \mathrm{Car}_{loc}(]a, b[\times\mathbb{R}^n, \mathbb{R}^n; a_i)$ $(i = 1, \dots, n)$ *be such that the conditions* (1.7)–(1.12),

$$d_2\beta_i(a) \le 0 \ \ \textit{and} \ \ 0 \le d_1\beta_i(t) < |\eta_i|^{-1} \ \ \textit{for} \ \ a < t \le b \ \ (i = 1, \dots, n_0),$$

$$d_1\beta_i(b) \le 0 \ \ \textit{and} \ \ 0 \le d_2\beta_i(t) < |\eta_i|^{-1} \ \ \textit{for} \ \ a \le t < b \ \ (i = n_0 + 1, \dots, n)$$

and

$$\int_c^t \eta_{il}(\tau)\, da_(\tau) = h_{il}\beta_i(t) + \beta_{il}(t) \ \ \textit{for} \ \ t \in [q, b] \ \ (i, l = 1, \dots, n)$$

are fulfilled, where $\eta_{il} \in L([a, b], \mathbb{R}; a_i)$ $(i, l = 1, \dots, n)$*, the functions* $b_i \in L_{loc}(]a, b], \mathbb{R}; a_i)$ $(i = 1, \dots, n_0)$ *and* $b_i \in L_{loc}([a, b[, \mathbb{R}; a_i)$ $(i = n_0 + 1, \dots, n)$ *are nonnegative, and* $q_i \in \mathrm{Car}([a, b] \times \mathbb{R}_+, \mathbb{R}_+; a_i)$ $(i = 1, \dots, n)$ *are nondecreasing functions in the second variable,* $\alpha_i(t) \equiv \int_{c_0}^{t} b_i(\tau)\, da_i(\tau)$ $(i = 1, \dots, n)$, $c_0 = (a + b)/2$*, and* γ_{α_i} $(i = 1, \dots, n)$ *are the functions defined according to* (1.4)*,* $h_{ii} < 0$*,* $h_{il} \ge 0$*,* $\eta_i < 0$ $(i \ne l;\ i, l = 1, \dots, n)$*,* β_{ii} $(i = 1, \dots, n)$ *are the functions nondecreasing on* $[a, b]$*;* $\beta_{il}, \beta_i \in \mathrm{BV}([a, b], \mathbb{R})$ $(i \ne l;\ i, l = 1, \dots, n)$ *are the functions nondecreasing on the interval* $]a, b]$ *for* $i \in \{1, \dots, n_0\}$ *and on the interval* $[a, b[$ *for* $i \in \{n_0 + 1, \dots, n\}$*. Let, moreover, the condition* (1.16) *hold, where* $\mathcal{H} = (\xi_{il})_{i,l=1}^{n}$,

$$\xi_{ii} = \lambda_i, \ \ \xi_{il} = \frac{h_{il}}{|h_{ii}|} \ \ (i \ne l; \ \ i, l = 1, \dots, n),$$

$$\lambda_i = V\big(\mathcal{A}(\zeta_i, \gamma_i)\big)(b) - V\big(\mathcal{A}(\zeta_i, \gamma_i)\big)(a+) \ \ \textit{for} \ \ i \in \{1, \dots, n_0\},$$

$$\lambda_i = V\big(\mathcal{A}(\zeta_i, \gamma_i)\big)(b-) - V\big(\mathcal{A}(\zeta_i, \gamma_i)\big)(a) \ \ \textit{for} \ \ i \in \{n_0 + 1, \dots, n\};$$

$$\zeta_i(t) \equiv \sum_{k=l}^{n} \beta_{il}(t) \ \ (i = 1, \dots, n);$$

and

$$\gamma_i(t) \equiv \big(\beta_i(t) - \beta_i(a+)\big)h_{ii} \ \ \textit{for} \ \ a < t \le b \ \ (i = 1, \dots, n_0),$$

$$\gamma_i(t) \equiv \big(\beta_i(b-) - \beta_i(t)\big)h_{ii} \ \ \textit{for} \ \ a \le t < b \ \ (i = n_0 + 1, \dots, n).$$

Then the problem (1.1), (1.2) *is solvable.*

Remark 1.3. If
$$\lambda_i < 1 \ \ (i = 1, \dots, n), \tag{1.14}$$
then, in Theorem 1.2, we can assume that
$$\xi_{ii} = 0, \ \ \xi_{il} = \frac{h_{il}}{(1-\lambda_i)|h_{ii}|} \ \ (i \neq l;\ i, l = 1, \dots, n). \tag{1.15}$$

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