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AN ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS OF THOMAS–FERMI TYPE SUBLINEAR DIFFERENTIAL EQUATIONS

Dedicated to Professor Kusano Takaŝi
on the occasion of his 80th birthday

Abstract. The set of positive solutions of Thomas–Fermi type differential equation

$$x'' = q(t)\phi(x)$$

is studied under the assumptions that q, ϕ are regularly varying functions in the sense of Karamata. It is shown that such solutions exist and their accurate asymptotic behavior at infinity is determined.

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$$x'' = q(t)\phi(x)$$

დიფერენციალური განტოლების დადებით ამონახსნთა სიმრავლე იმ შემთხვევაში, როცა q და ϕ კარამატას აზრით რეგულარულად ცვალებადი ფუნქციებია. სახელდობრ, ნაჩვენებია ასეთი ამონახსნების არსებობა და დადგენილია უსასრულობაში მათი ასიმპტოტური ყოფაქცევა.

1. Introduction

The present paper is devoted to the existence and the asymptotic analysis of positive solutions of nonlinear ordinary differential equations of *Thomas–Fermi type*

$$x'' = q(t)\phi(x) \tag{A}$$

assuming that $q : [a, \infty) \to (0, \infty)$, $a > 0$, is a continuous function which is regularly varying at infinity of index $\sigma \in \mathbb{R}$ and $\phi(x)$ is a positive, continuous function which is regularly varying at zero or at ∞ of index $\gamma \in (0, 1)$.

We begin by stating some obvious but important facts valid for all positive solutions of equation (A): Let $x(t)$ be a positive solution of (A) on $[a, \infty)$, $a \geq 0$. Since all positive solutions are convex, it follows that $x'(t)$ is increasing, and hence either $x'(t) < 0$ on $[a, \infty)$ or $x'(t) > 0$ on $[t_0, \infty)$ for some $t_0 > a$. In the former case, $x'(t)$ tends to 0 as $t \to \infty$. In fact, if $x'(t)$ tends to some negative constant w_1, we have $x(t) \leq w_1 t$, for $t \geq t_1 \geq t_0$, which contradicts positivity of $x(t)$. Moreover, $x(t)$ is positive and decreasing, so that it tends either to a positive constant or to 0 as $t \to \infty$. In the latter case, $x'(t)$ is positive and increasing, so it tends either to ∞ or to some positive constant as $t \to \infty$. Thus, $x'(t) \geq k$ for some positive constant k and for $t \geq t_1 \geq t_0$. Accordingly, by integration we get $x(t) \geq x(t_1) + k(t - t_1)$ which implies that $x(t) \to \infty$ as $t \to \infty$.

On the basis of the above observations all possible *positive decreasing solutions* of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t) = const > 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.1}$$

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.2}$$

while all possible *positive increasing solutions* of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t} = const > 0, \tag{1.3}$$

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = \infty. \tag{1.4}$$

In our analysis we shall extensively use the class of regularly varying functions introduced by J. Karamata in 1930 by the following

Definition 1.1. A measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$, is said to be *regularly varying at infinity of index $\rho \in \mathbb{R}$* if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

A measurable function $f : (0, a) \to (0, \infty)$ is said to be *regularly varying at zero of index $\rho \in \mathbb{R}$* if $f(\frac{1}{t})$ is regularly varying at ∞ i.e. if

$$\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0. \tag{1.5}$$

By $\mathrm{RV}(\rho)$ and $\mathcal{RV}(\rho)$ we denote, respectively, the set of regularly varying functions of index ρ at infinity and at zero. If, in particular, $\rho = 0$, the function f is called *slowly varying* at infinity or at zero. By SV and $\mathcal{SV}$ we denote, respectively, the set of slowly varying functions at infinity and at zero. Saying only regularly or slowly varying function, we mean regularity at infinity.

It follows from Definition 1.1 that any function $f(t) \in \mathrm{RV}(\rho)$ is written as

$$f(t) = t^\rho g(t) \quad \text{with} \quad g(t) \in \mathrm{SV}. \tag{1.6}$$

If, in particular, the function $g(t) \to k > 0$ as $t \to \infty$, it is called a *trivial slowly varying* one denoted by $g(t) \in tr$-SV, the function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial regularly varying of index* ρ, denoted by $f(t) \in tr$-$\mathrm{RV}(\rho)$. Otherwise $g(t)$ is called a *nontrivial slowly varying* function denoted by $g(t) \in ntr$-SV and $f(t)$ is called a *nontrivial* $\mathrm{RV}(\rho)$ *function*, denoted by $f(t) \in ntr$-$\mathrm{RV}(\rho)$. Similarly for the set $\mathcal{RV}(\rho)$.

Comprehensive treatises on regular variation are given in N. H. Bingham et al. [2] and by E. Seneta [15]. To help the reader, we present here a fundamental result which will be used throughout the paper.

Proposition 1.1 (Karamata's integration theorem)**.** *Let* $L(t) \in \mathrm{SV}$. *Then*

(i) *if* $\alpha > -1$,

$$\int_a^t s^\alpha L(s)\, ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if* $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)\, ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if* $\alpha = -1$,

$$m_1(t) = \int_a^t \frac{L(s)}{s}\, ds \in \mathrm{SV}, \quad m_2(t) = \int_t^\infty \frac{L(s)}{s}\, ds$$

and

$$\lim_{t\to\infty} \frac{L(t)}{m_i(t)} = 0, \quad i = 1, 2.$$

The symbol $\sim$ denotes the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \Longleftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Also, $f(t) \asymp g(t)$ means that there exist constants $0 < m < M$ such that

$$m g(t) \le f(t) \le M g(t), \quad t \ge t_0.$$

Throughout the text, "$t \geq t_0$" means that t is sufficiently large, so that t_0 need not to be the same at each occurrence.

We shall also use the following results:

Proposition 1.2. *Let* $q_1(t) \in \mathrm{RV}(\sigma_1)$, $q_2(t) \in \mathrm{RV}(\sigma_1)$, $q_3(t) \in \mathcal{RV}(\sigma_3)$. *Then*

(i) $g_1(t) + g_2(t) \in \mathrm{RV}(\sigma)$, $\sigma = \max(\sigma_1, \sigma_2)$;

(ii) $g_1(t)g_2(t) \in \mathrm{RV}(\sigma_1 + \sigma_2)$, $(g_1(t))^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$;

(iii) $q_1(q_2(t)) \in \mathrm{RV}(\sigma_1\sigma_2)$ *if* $q_2(t) \to \infty$, *as* $t \to \infty$;
$q_3(q_2(t)) \in \mathrm{RV}(\sigma_3\sigma_2)$ *if* $q_2(t) \to 0$, *as* $t \to \infty$;

(iv) *for any* $\varepsilon > 0$ *and* $L(t) \in \mathrm{SV}$, *one has* $t^\varepsilon L(t) \to \infty$, $t^{-\varepsilon} L(t) \to 0$, *as* $t \to \infty$.

Proposition 1.3. *If* $f(t) \sim t^\alpha l(t)$ *as* $t \to \infty$ *with* $l(t) \in \mathrm{SV}$, *then* $f(t)$ *is a regularly varying function of index* α *i.e.* $f(t) = t^\alpha l^\star(t)$, $l^\star(t) \in \mathrm{SV}$, *where, in general,* $l^\star(t) \neq l(t)$, *but* $l^\star(t) \sim l(t)$ *as* $t \to \infty$.

Proposition 1.4. *A positive measurable function* $f(t)$ *belongs to* SV *if and only if for every* $\alpha > 0$, *there exist a non-decreasing function* Ψ *and a non-increasing function* ψ *with*

$$t^\alpha f(t) \sim \Psi(t), \quad \text{and} \quad t^{-\alpha} f(t) \sim \psi(t), \quad t \to \infty.$$

Proposition 1.5. *For the function* $f(t) \in \mathrm{RV}(\alpha)$, $\alpha > 0$, *there exists* $g(t) \in \mathrm{RV}(1/\alpha)$ *such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as} \quad t \to \infty.$$

Here, g is an *asymptotic inverse of* f (and it is determined uniquely to within asymptotic equivalence).

Note, the same result holds for $t \to 0$ i.e. when $f(t) \in \mathcal{RV}(\alpha)$, $\alpha > 0$:

Proposition 1.6. *For the function* $f(t) \in \mathcal{RV}(\alpha)$, $\alpha > 0$, *there exists* $f(t) \in \mathcal{RV}(1/\alpha)$ *such that*

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as} \quad t \to 0.$$

This follows from Proposition 1.5, since by Definition 1.1 the assumption is equivalent to the saying that $f(1/t) \in \mathrm{RV}(-\alpha)$. Thus, one applies Proposition 1.5 to the function $1/f(1/t) \in \mathrm{RV}(\alpha)$.

The assumptions on q and ϕ, using notation (1.6), imply that equation (A) can be written in the form

$$x''(t) = t^\sigma l(t)\, x^\gamma L(x), \quad l(t) \in \mathrm{SV}, \quad L(x) \in \mathrm{SV} \ \text{ or } \ L(x) \in \mathcal{SV}. \qquad (1.7)$$

If in (1.7), $\gamma \in (0, 1)$ or $\gamma > 1$, equation is called *sublinear* or *superlinear*, respectively.

The study of nonlinear differential equations of the form (A) in the framework of regular variation was initiated by Avakumović [1] (as the very first attempt of the kind in the theory of differential equations), followed by

Marić and Tomić [12]–[14] and some more recent results [4], [5], [7], [8], [10]. See also Marić [11, Chapter3]. These papers and some closely related ones [16], [17] are concerned exclusively with decreasing positive solutions of *superlinear* Thomas–Fermi type equations. No analysis from the viewpoint of regular variation, until recently in [9], seems to have been made of positive solutions of *sublinear* type of equations. There positive increasing solutions of the both types (1.3), (1.4) of the equation (A) (or (1.7)) with $\gamma \in (0,1)$ were analyzed. Very recently a paper [6] by Evtukhov and Samoilenko appeared. A more general equation $x^{(n)} = \alpha q(t)x(t)$ is studied and the existence and the asymptotics of solutions is obtained covering a subclass of regularly varying solutions. Here α may be $+1$ (Thomas–Fermi type), or -1 (Emden–Fowler one).

Our purpose here is to proceed further in studying positive solutions of sublinear equation (A) by establishing the sharp conditions for the existence and constructing the precise asymptotic forms of these. Besides regular variation, the main tools employed in the proof of our main results are the Schauder–Tychonoff fixed point theorem in locally convex spaces and the following generalized L'Hospital's rule (see [3]):

Lemma 1.1. *Let* $f, g \in C^1[T,\infty)$ *and*

$$\lim_{t\to\infty} g(t) = \infty \ \text{ and } \ g'(t) > 0 \quad \text{for all large } t$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \ \text{ and } \ g'(t) < 0 \quad \text{for all large } t.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)} .$$

2. Results

To avoid repetitions, we state here basic conditions imposed on the functions q and ϕ in all theorems which follows:

$$q(t) \in \mathrm{RV}(\sigma), \quad \sigma \in \mathbb{R}, \tag{2.1}$$

$$\begin{aligned} &\text{a)} \quad \phi(x) \in \mathcal{RV}(\gamma), \quad \gamma \in (0,1);\\ &\text{b)} \quad \phi(x) \in \mathrm{RV}(\gamma), \quad \gamma \in (0,1). \end{aligned} \tag{2.2}$$

First, observe that in either of two cases a) or b) in (2.2), by Propositions 1.5 and 1.6 there exists an asymptotic inverse $\varphi(x)$ of the function $x/\phi(x)$.

In addition, in some of the theorems it is required that either

$$\begin{gathered} \phi(x) \in \mathcal{RV}(\gamma) \ \text{ satisfies } \ \phi(t^\lambda u(t)) \sim \phi(t^\lambda)u(t)^\gamma, \ \text{ as } \ t \to \infty,\\ \text{for each } \ \lambda \in \mathbb{R}^- \ \text{ and } \ u(t) \in \mathcal{SV} \cap C^1(\mathbb{R}), \end{gathered} \tag{2.3}$$

or

$$\begin{gathered} \phi(x) \in \mathrm{RV}(\gamma) \ \text{ satisfies } \ \phi(t^\lambda u(t)) \sim \phi(t^\lambda)u(t)^\gamma, \ \ t \to \infty,\\ \text{for each } \ \lambda \in \mathbb{R}^+ \ \text{ and } \ u(t) \in \mathrm{SV} \cap C^1(\mathbb{R}); \end{gathered} \tag{2.4}$$

In other words, the slowly varying part $L(x)$ of $\phi(x)$ must satisfy $L(t^\lambda u(t)) \sim L(t^\lambda)$, $t \to \infty$, for each slowly varying $u(t) \in C^1(\mathbb{R})$. It is easy to check that this is satisfied by for e.g.

$$L(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \quad \alpha_k \in \mathbb{R},$$

but not by

$$L(t) = \exp\Big(\prod_{k=1}^{N} (\log_k t)^{\beta_k}\Big), \quad \beta_k \in (0,1),$$

where $\log_k t = \log\log_{k-1} t$.

For the future analysis we need the following preparatory

Lemma 2.1. *Put*

$$Y_0(t) = \varphi\Big(\frac{t^2 q(t)}{\rho(\rho-1)}\Big), \tag{2.5}$$

and

$$I(t) = \int_t^\infty \int_s^\infty q(r)\phi(Y_0(r))\, dr\, ds, \tag{2.6}$$

where $\varphi(x)$ is an asymptotic inverse of the function $x/\phi(x)$ and ρ is given by

$$\rho = \frac{\sigma+2}{1-\gamma}\,. \tag{2.7}$$

If (2.2) a) *and* (2.1) *with $\sigma < -2$ hold, then as $t \to \infty$*

(i) $Y_0(t) \in \mathrm{RV}(\frac{\sigma+2}{1-\gamma})$ *and* $Y_0(t) \to 0$*;*

(ii) $I(t) \sim Y_0(t)$.

Proof. Since $t^2 q(t) \to 0$, $t \to \infty$, by Proposition 1.2-(iii), we conclude that $Y_0(t) \in \mathrm{RV}\,(\rho)$, with ρ given by (2.7). Thus, $Y_0(t)$ is expressed as $Y_0(t) = t^\rho \eta(t)$, $\eta(t) \in \mathrm{SV}$ and $Y_0(t) \to 0$, $t \to \infty$, because $\rho < 0$. Moreover, in view of (2.5), there follows

$$\frac{Y_0(t)}{\phi(Y_0(t))} \sim \frac{t^2 q(t)}{\rho(\rho-1)}, \quad t \to \infty\,. \tag{2.8}$$

Hence, by writing $I(t)$ in the form

$$I(t) = \int_t^\infty \int_s^\infty q(r)\,\frac{\phi(Y_0(r))}{Y_0(r)}\,Y_0(r)\, dr\, ds \sim$$

$$\sim \rho(\rho-1)\int_t^\infty \int_s^\infty r^{\rho-2}\eta(r)\, dr\, ds, \quad t \to \infty,$$

and applying Karamata's theorem twice on the last integral (Proposition 1.1-(ii)), one obtains the desired result. □

To prove the existence and determine the exact asymptotic behavior of solutions $x(t) \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$ we shall consider the following three cases separately:

(i) $\rho < 0$ or $\rho > 1$,

(ii) $\rho = 0$,

(iii) $\rho = 1$.

Note, the case $\rho \in (0,1)$ does not exist due to (1.1)–(1.4).

(i) Regularly varying solution of index $\rho < 0$ or $\rho > 1$.

Theorem 2.1. *Suppose that* (2.1), (2.2) a) *and* (2.3) *hold. Then equation* (A) *possesses a decreasing regularly varying solution $x(t)$ of index $\rho < 0$ if and only if*

$$\sigma < -2. \tag{2.9}$$

Also, $x(t)$ satisfies (1.2).

If, on the other hand, (2.1), (2.2) b) *and* (2.4) *hold, then equation* (A) *possesses an increasing regularly varying solution $x(t)$ of index $\rho > 1$ if and only if*

$$\sigma > -\gamma - 1. \tag{2.10}$$

Also, $x(t)$ satisfies (1.4).

In either case any such solution $x(t)$ has for $t \to \infty$ the exact asymptotic behavior

$$x(t) \sim \varphi\Big(\frac{t^2 q(t)}{\rho(\rho-1)}\Big), \tag{2.11}$$

where φ and ρ are as in Lemma 2.1.

Proof. We begin with the proof of the first part of Theorem 2.1, where $\rho < 0$. Let (2.1), (2.2) a) and (2.3) hold.

The "only if" part: Let $x(t) \in \mathrm{RV}(\rho)$, $\rho < 0$, be a decreasing solution of (A) on $[t_0, \infty)$. We express it as $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$. To avoid ambiguity, notice that $\rho \in \mathbb{R}$ and has to be determined. Due to Proposition 1.2-(iv) $x(t) \to 0$ as $t \to \infty$, and as is pointed out in the Introduction, $x'(t) \to 0$ as $t \to \infty$. Integrating (A) over (t, ∞) and using (1.7), we get for $t \ge t_0$

$$-x'(t) = \int_t^\infty q(s)\phi(x(s))\,ds = \int_t^\infty s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds. \tag{2.12}$$

The convergence of the last integral implies that $\sigma + \rho\gamma \le -1$. However, the possibility $\sigma + \rho\gamma = -1$ is excluded. In fact, if this were the case, then (2.12) reduces to

$$-x'(t) = \int_t^\infty s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds,$$

and since due to Proposition 1.1-(iii) the last integral is slowly varying, an integration over $[t, \infty)$ gives

$$x(t) \sim t \int_t^\infty s^{-1} l(s) \xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{RV}(1), \quad t \to \infty,$$

contradicting $\rho < 0$. Thus, we have $\sigma + \rho\gamma < -1$. Then, by Karamata's integration theorem from (2.12), we obtain

$$-x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t) \xi(t)^\gamma L(t^\rho \xi(t))}{-(\sigma + \rho\gamma + 1)}\,, \quad t \to \infty. \tag{2.13}$$

Since $x(t) \to 0$ as $t \to \infty$, by integration we further get

$$\int^\infty \frac{t^{\sigma+\rho\gamma+1} l(t) \xi(t)^\gamma L(t^\rho \xi(t))}{-(\sigma + \rho\gamma + 1)}\, dt < \infty,$$

and hence $\sigma + \rho\gamma + 1 \leq -1$ i.e. $\sigma + \rho\gamma \leq -2$. If $\sigma + \rho\gamma = -2$, then (2.13) reduces to

$$x'(t) \sim -t^{-1} l(t) \xi(t)^\gamma L(t^\rho \xi(t)), \quad t \to \infty,$$

and integration over $[t, \infty)$ yields

$$x(t) \sim \int_t^\infty s^{-1} l(s) \xi(s)^\gamma L(s^\rho \xi(s))\, ds \in \mathrm{SV}, \quad t \to \infty,$$

which leads to an impossibility that $\rho = 0$. Therefore, we must have $\sigma + \rho\gamma < -2$, in which case, integrating (2.13) over $[t, \infty)$, we get for $t \to \infty$

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t) \xi(t)^\gamma L(t^\rho \xi(t))}{[-(\sigma + \rho\gamma + 1)]\,[-(\sigma + \rho\gamma + 2)]} = \\ = \frac{t^2 q(t) \phi(t^\rho \xi(t))}{[-(\sigma + \rho\gamma + 1)]\,[-(\sigma + \rho\gamma + 2)]} \tag{2.14}$$

implying, in view of Proposition 1.3, that the regularity index of $x(t)$ is $\rho = \sigma + \rho\gamma + 2$, i.e. $\rho = \frac{\sigma+2}{1-\gamma}$. Then, since $\rho < 0$, we conclude that $\sigma < -2$. Since, $(\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2) = \rho(\rho - 1)$, (2.14), due to (2.8), becomes

$$\frac{x(t)}{\phi(x(t))} \sim \frac{t^2 q(t)}{\rho(\rho - 1)} \sim \frac{Y_0(t)}{\phi(Y_0(t))}\,, \quad t \to \infty. \tag{2.15}$$

Because $Y_0(t) \to 0$ and $x(t) \to 0$ as $t \to \infty$, (2.15) is, in view of Proposition 1.6, equivalent to (2.11).

The "if" part: Note that any solution $x(t)$ of the integral equation

$$x(t) = \int_t^\infty \int_s^\infty q(r) \phi(x(r))\, dr\, ds, \tag{2.16}$$

(if it exists) satisfies (A) and is obviously positive, decreasing and (1.2) holds. We shall prove that it indeed exists and possesses the properties stated in the Theorem.

Applying Proposition 1.4 to the function $\phi(x) \in \mathcal{RV}(\gamma)$ with $\gamma > 0$, we see that there exists a constant $A > 1$ such that

$$\phi(x) \leq A\,\phi(y) \;\text{ for each }\; a > y \geq x > 0. \tag{2.17}$$

Due to Lemma 2.1, there exists $t_0 > a$ so that

$$\frac{Y_0(t)}{2} \leq I(t) \leq 2Y_0(t), \;\; t \geq t_0. \tag{2.18}$$

In addition, since $Y_0(t) \to 0$ as $t \to \infty$ and (1.5) holds uniformly on each compact λ-set on $(0, \infty)$ ([2, Theorem 1.2.1]) there exists $t_0 > a$ such that

$$\frac{\lambda^\gamma}{2}\,\phi(Y_0(t)) \leq \phi(\lambda Y_0(t)) \leq 2\lambda^\gamma \phi(Y_0(t)) \;\text{ for }\; t \geq t_0. \tag{2.19}$$

Choose $0 < k < 1$ and $K > 1$ such that

$$k^{1-\gamma} \leq \frac{1}{4A} \;\text{ and }\; K^{1-\gamma} \geq 4A, \tag{2.20}$$

which is possible due to $0 < \gamma < 1$.

Now we choose t_0 such that (2.18) and (2.19) both hold and define the set $\mathcal{X}$ to be the set of continuous functions $x(t)$ on $[t_0, \infty)$ satisfying

$$kY_0(t) \leq x(t) \leq KY_0(t) \;\text{ for }\; t \geq t_0. \tag{2.21}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[t_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. We shall show that the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = \int_t^\infty \int_s^\infty q(r)\phi(x(r))\,dr\,ds, \;\; t \geq t_0,$$

is a continuous self-map on $\mathcal{X}$ and that $\mathcal{F}(\mathcal{X})$ is a relatively compact subset of $C[t_0, \infty)$ and then apply the Schauder–Tychonoff fixed point theorem. Notice that, in view of Lemma 2.1, the above integral converges on the set $\mathcal{X}$ under consideration.

Let $x(t) \in \mathcal{X}$. By using successively (2.17), (2.19) with $\lambda = K$ and $\lambda = k$, (2.20) and (2.18), one obtains

$$\begin{aligned}\mathcal{F}x(t) &\leq A \int_t^\infty \int_s^\infty q(r)\phi(KY_0(r))\,dr\,ds \leq \\ &\leq 2A\,K^\gamma \int_t^\infty \int_s^\infty q(r)\phi(Y_0(r))\,dr\,ds \leq \\ &\leq 4A\,K^\gamma Y_0(t) \leq K\,Y_0(t), \quad t \geq t_0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}x(t) &\geq \frac{1}{A}\int_t^\infty\int_s^\infty q(r)\phi(kY_0(r))\,dr\,ds \geq \\ &\geq \frac{k^\gamma}{2A}\int_t^\infty\int_s^\infty q(r)\phi(Y_0(r))\,dr\,ds \geq \\ &\geq \frac{k^\gamma}{4A}\,Y_0(t) \geq kY_0(t), \quad t \geq t_0.\end{aligned}$$

Therefore, $\mathcal{F}x(t) \in \mathcal{X}$, that is, $\mathcal{F}$ maps $\mathcal{X}$ into itself.

Furthermore, it can be verified that $\mathcal{F}$ is a continuous map and $\mathcal{F}(\mathcal{X})$ is relatively compact in $C[t_0, \infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, there exists a fixed point $x(t)$ of $\mathcal{F}$ which satisfies the integral equation (2.16) and hence equation (A).

Now we prove that any such solution $x(t)$ has the asymptotic behavior (2.11). Because of (2.21), $x(t)$ satisfies

$$0 < \liminf_{t\to\infty} \frac{x(t)}{Y_0(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{Y_0(t)} < \infty,$$

or in view of Lemma 2.1, we have

$$0 < \liminf_{t\to\infty} \frac{x(t)}{I(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{I(t)} < \infty.$$

Put $Y_0(t) = t^\rho \eta(t)$, $\eta(t) \in \mathrm{SV}$. An application of Lemma 1.1, in view of assumption (2.3), yields

$$\begin{aligned}L &= \limsup_{t\to\infty} \frac{x(t)}{I(t)} \leq \limsup_{t\to\infty} \frac{x''(t)}{I''(t)} = \limsup_{t\to\infty} \frac{q(t)\phi(x(t))}{q(t)\phi(Y_0(t))} = \\ &= \limsup_{t\to\infty} \frac{\phi(t^\rho\xi(t))}{\phi(t^\rho\eta(t))} = \limsup_{t\to\infty} \frac{\xi(t)^\gamma\phi(t^\rho)}{\eta(t)^\gamma\phi(t^\rho)} = \limsup_{t\to\infty} \frac{(x(t)/t^\rho)^\gamma}{(Y_0(t)/t^\rho)^\gamma} = \\ &= \Big(\limsup_{t\to\infty} \frac{x(t)}{Y_0(t)}\Big)^\gamma = \Big(\limsup_{t\to\infty} \frac{x(t)}{I(t)}\Big)^\gamma = L^\gamma.\end{aligned}$$

Since $\gamma < 1$, from the above we conclude that

$$0 < L \leq 1. \tag{2.22}$$

Similarly, we can see that $l = \liminf_{t\to\infty} \dfrac{x(t)}{I(t)}$ satisfies

$$1 \leq l < \infty. \tag{2.23}$$

From (2.22) and (2.23) we obtain that $l = L = 1$, which means that $x(t) \sim I(t) \sim Y_0(t)$, $t \to \infty$, i.e. (2.11) holds. This also shows, due to Proposition 1.3, that $x(t)$ is a regularly varying solution of (A) with the requested regularity index.

We now turn our attention to the second part of Theorem 2.1, where $\rho > 1$. Let (2.1), (2.2) b) and (2.4) hold.

The "only if" part: Suppose that (A) has solution of the form $x(t) = t^\rho \xi(t)$ on $[t_0, \infty)$ with $\rho > 1$ and $\xi(t) \in \mathrm{SV}$. Note that $x'(t) \to \infty$ and $x(t) \to \infty$ as $t \to \infty$. Integrating (A) on $[t_0, t]$, we have

$$x'(t) \sim \int_{t_0}^{t} q(s)\phi(x(s))\,ds = \int_{t_0}^{t} s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds, \quad t \to \infty. \tag{2.24}$$

The divergence of the last integral as $t \to \infty$ means that $\sigma + \rho\gamma \geq -1$. But the possibility $\sigma + \rho\gamma = -1$ is precluded, because if this was the case, then

$$\int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds \in \mathrm{SV},$$

and hence integration of (2.24) on $[t_0, t]$ shows that

$$x(t) \sim t \int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))\,ds \in \mathrm{RV}(1),$$

which contradicts the condition $\rho > 1$. Thus, $\sigma + \rho\gamma > -1$. In this case, applying Karamata's integration theorem to the last integral in (2.24), we have

$$x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{\sigma + \rho\gamma + 1}, \quad t \to \infty,$$

and integrating the above relation on $[t_0, t]$, we obtain

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{(\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2)} \in \mathrm{RV}(\sigma + \rho\gamma + 2), \quad t \to \infty, \tag{2.25}$$

which, in view of Proposition 1.3, shows that the regularity index of $x(t)$ is $\rho = \frac{\sigma+2}{1-\gamma}$. From the requirement $\rho > 1$ it follows that $\sigma > -\gamma - 1$. Exactly as when $\rho < 0$, (2.25) leads to the asymptotic formula (2.11).

The "if" part: It is proved in [9, Lemma 2.1, Theorem 2.1] that if the regularity index σ of $q(t)$ satisfies $\sigma > -\gamma - 1$, then the function $Y_0(t) \in \mathrm{RV}(\rho)$ satisfies the relation

$$Y_0(t) \sim \int_a^t \int_a^s q(r)\phi(Y_0(r))\,dr\,ds, \quad t \to \infty,$$

and there exists a positive increasing solution $x(t)$ of equation (A) which satisfies (1.4) and (2.21). Then, proceeding exactly as when $\rho < 0$, with application of Lemma 1.1 and using (2.4), we conclude that $x(t) \sim Y_0(t)$ as $t \to \infty$. This implies $x(t) \in \mathrm{RV}(\rho)$, with ρ given by (2.7), as before. □

(ii) Regularly varying solutions of index $\rho = 0$.

We distinguish two subcases: $x(t) \in tr\text{-SV}$ and $x(t) \in ntr\text{-SV}$.

Observe that slowly varying solutions must decrease. For otherwise (1.3) and (1.4) would hold contradicting Proposition 1.2-(iv).

Theorem 2.2. *Suppose that* (2.1) *and* (2.2) a) *hold. Equation* (A) *possesses a* (*decreasing*) *trivial slowly varying solution if and only if*

$$\int_{t_0}^{\infty} sq(s)\,ds < \infty. \tag{2.26}$$

Proof. The "only if" part: Suppose that (A) has a decreasing tr-SV-solution $x(t)$ on $[t_0,\infty)$ i.e. satisfying $x(t)\to c$, $t\to\infty$, $c>0$. Integrating (A) over $[t,\infty)$ and observing (1.1), one gets

$$-x'(t) = \int_{t}^{\infty} s^{\sigma} l(s)\phi(x(s))\,ds, \quad t \ge t_0, \tag{2.27}$$

implying $\sigma \le -1$. But the case $\sigma = -1$ is impossible since then, by Proposition 1.1-(iii), the integral in (2.27) is an SV function, and another integration on $[t,\infty)$ would give $\rho = 1$. Thus $\rho < -1$ and by Karamata's theorem, (2.27) leads to

$$-x'(t) \sim \frac{t^{\sigma+1} l(t)\phi(x(t))}{-(\sigma+1)}\,, \quad t\to\infty, \tag{2.28}$$

which together with $x(t)\to c$, $t\to\infty$ yields

$$\int_{t_0}^{\infty} \frac{t^{\sigma+1} l(t)\phi(x(t))}{-(\sigma+1)} < \infty,$$

implying (2.26).

The "if" part: Suppose that (2.26) holds. Then there exists $t_0 \ge a$ such that

$$\int_{t_0}^{\infty} tq(t)\,dt \le \frac{c}{2A\phi(c)}\,, \quad t\ge t_0, \tag{2.29}$$

where $A>1$ is a constant such that (2.17) holds. Let us now define the integral operator

$$\mathcal{F}x(t) = \frac{c}{2} + \int_{t}^{\infty}\int_{s}^{\infty} q(r)\phi(x(r))\,dr\,ds, \quad t\ge t_0,$$

and the set

$$\mathcal{X} = \left\{ x(t)\in C[t_0,\infty) : \ \frac{c}{2} \le x(t) \le c, \ \ t\ge t_0 \right\}.$$

If $x(t) \in \mathcal{X}$, then clearly, $\mathcal{F}x(t) \geq c/2$. Also, due to (2.29), we obtain

$$\int_t^\infty \int_s^\infty q(r)\phi(x(r))\,dr\,ds \leq A\phi(c) \int_t^\infty \int_s^\infty q(r)\,dr\,ds =$$

$$= A\phi(c) \int_t^\infty (r-t)q(r)dr \leq \frac{c}{2}, \quad t \geq t_0,$$

and hence $\mathcal{F}x(t) \leq c$ for $t \geq t_0$. This shows that $\mathcal{F}x(t) \in \mathcal{X}$, and hence $\mathcal{F}$ is a self-map of the closed convex set $\mathcal{X}$. Moreover, we can verify that $\mathcal{F}$ is continuous and $\mathcal{F}(\mathcal{X})$ is relatively compact in the topology of the locally convex space $C[t_0,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, $\mathcal{F}$ has a fixed point $x_0(t) \in \mathcal{X}$, which gives birth to a solution of equation (A) tending to a positive constant as $t \to \infty$. □

Remark 2.1. It is clear that (2.26) implies $\sigma < -2$, or $\sigma = -2$ and $\int_t^\infty \frac{l(s)}{s}\,ds < \infty$.

Theorem 2.3. *Suppose that* (2.1) *and* (2.2) a) *hold. Equation* (A) *possesses a* (*decreasing*) *nontrivial slowly varying solution if and only if*

$$\sigma = -2 \ \ \text{and} \ \ \int_t^\infty tq(t)\,dt < \infty, \tag{2.30}$$

and any such solution $x(t)$ has the exact asymptotic behavior

$$x(t) \sim \Phi^{-1}(Q(t)), \ \ t \to \infty, \tag{2.31}$$

where

$$Q(t) = \int_t^\infty sq(s)\,ds, \ \ t \geq a, \ \ \text{and} \ \ \Phi(x) = \int_0^x \frac{dv}{\phi(v)}, \ \ x > 0. \tag{2.32}$$

Proof. The "only if" part: Suppose that (A) has a nontrivial SV-solution $x(t)$ on $[t_0,\infty)$, so it has to satisfy (1.2). Then, as in the proof of Theorem 2.2, we get (2.28) and conclude that σ must satisfy $\sigma + 1 \leq -1$. If $\sigma < -2$, integrating (2.28) over $[t,\infty)$ and applying Karamata's integration theorem, we obtain

$$x(t) \sim \frac{t^{\sigma+2} l(t)\phi(x(t))}{(\sigma+1)(\sigma+2)} \in \mathrm{RV}(\sigma+2), \ \ t \to \infty,$$

which is impossible because for the regularity index of $x(t)$ we would get $\rho = \sigma + 2 < 0$. Thus, one has $\sigma = -2$ and so, integration of (2.28) over $[t,\infty)$ gives

$$x(t) \sim \int_t^\infty s^{-1} l(s)\phi(x(s))\,ds, \ \ t \to \infty. \tag{2.33}$$

Let the integral in (2.33) be denoted by $\chi(t)$. Then, $\chi(t) \to 0$, $t \to \infty$ and satisfies

$$\chi'(t) = -t^{-1}l(t)\phi(x(t)) \sim -t^{-1}l(t)\phi(\chi(t)), \;\; t \to \infty,$$

that is

$$\frac{\chi'(t)}{\phi(\chi(t))} \sim -tq(t), \;\; t \to \infty.$$

An integration of the last relation over $[t, \infty)$ results in

$$\int_0^{\chi(t)} \frac{du}{\phi(u)} = \Phi(\chi(t)) \sim \int_t^\infty sq(s)\,ds = Q(t), \;\; t \to \infty, \tag{2.34}$$

or

$$\chi(t) \sim \Phi^{-1}(Q(t)), \;\; t \to \infty,$$

which is equivalent to (2.31) since by (2.33), $x(t) \sim \chi(t)$ as $t \to \infty$.

Observe that because of (2.2) a) and Proposition 1.2-(iv), the left-hand side integral in (2.34) converges at 0 and the same holds for the right-hand side one at ∞. Thus, the second condition in (2.30) also holds. In addition, since Φ is continuous and increasing and $\phi(x) \in \mathcal{RV}(1-\gamma)$, its inverse function exists and

$$\Phi^{-1}(x) \in \mathrm{RV}\Big(\frac{1}{1-\gamma}\Big). \tag{2.35}$$

The "if" part: Suppose that (2.30) holds, so that $q(t) = t^{-2}l(t)$, $l(t) \in$ SV. We show that $Y_1(t)$ defined by

$$Y_1(t) = \Phi^{-1}\Big(\int_t^\infty sq(s)\,ds\Big), \;\; t \geq a,$$

satisfies the integral asymptotic relation

$$\int_t^\infty \int_s^\infty q(r)\phi(Y_1(r))\,dr\,ds \sim Y_1(t), \;\; t \to \infty.$$

Notice that, in view of (2.30), $Q(t) \in$ SV and $Q(t) \to 0$, $t \to \infty$, so that Proposition 1.2-(iii) and (2.35) show that $Y_1(t) \in$ SV. Also, $Y_1(t) \to 0$ as $t \to \infty$, so that $\phi(Y_1(t)) \in$ SV. Since $\Phi(Y_1(t)) = Q(t)$, we get

$$tq(t) = -\Phi'(Y_1(t))Y_1'(t) = -\frac{Y_1'(t)}{\phi(Y_1(t))}\,,$$

implying that $Y_1(t)$ is a solution of the differential equation

$$Y_1'(t) + tq(t)\phi(Y_1(t)) = 0.$$

Thus, applying Karamata's integration theorem, we have, due to the preceding differential equation,

$$\int_t^\infty \int_s^\infty q(r)\phi(Y_1(r))\,dr\,ds =$$

$$= \int_t^\infty \int_s^\infty r^{-2}l(r)\phi(Y_1(r))\,dr\,ds \sim \int_t^\infty s^{-1}l(s)\phi(Y_1(s))\,ds =$$

$$= \int_t^\infty sq(s)\phi(Y_1(s))\,ds = -\int_t^\infty Y_1'(s)\,ds = Y_1(t), \quad t\to\infty.$$

Then, by replacing in the proof of Theorem 2.1 the function $Y_0(t)$ by $Y_1(t)$, an application of the Schauder–Tychonoff fixed point theorem provides the existence of a decreasing solution $x(t)$ of equation (A) satisfying

$$x(t) \asymp Y_1(t). \tag{2.36}$$

We show that the obtained solution $x(t)$ of (A) is slowly varying and hence satisfies (2.31). Using (2.36) and (2.17), from equation (A) we get

$$x''(t) \asymp q(t)\phi(Y_1(t)) = t^{-2}l(t)\phi(Y_1(t)).$$

Integrating over $[t,\infty)$, we get

$$x'(t) \asymp t^{-1}l(t)\phi(Y_1(t)), \quad x(t) \asymp \int_t^\infty s^{-1}l(s)\phi(Y_1(s))\,ds.$$

Then

$$t\,\frac{x'(t)}{x(t)} \asymp l(t)\phi(Y_1(t))\Big[\int_t^\infty s^{-1}l(s)\phi(Y_1(s))\,ds\Big]^{-1}. \tag{2.37}$$

Application of Karamata's integration theorem gives

$$\lim_{t\to\infty} l(t)\phi(Y_1(t))\Big[\int_t^\infty s^{-1}l(s)\phi(Y_1(s))\,ds\Big]^{-1} = 0,$$

which implies with (2.37) that $tx'(t)/x(t)\to 0$ as $t\to\infty$. Therefore, by [11, Proposition 10], $x(t)$ is slowly varying and so enjoys the precise asymptotic behavior (2.31). This completes the proof of Theorem 2.3. □

Remark 2.2. If specially $\phi(x)=x^\gamma$, then formulas (2.11) and (2.31) read, respectively,

$$x(t) \sim \Big(\frac{t^2q(t)}{\rho(\rho-1)}\Big)^{\frac{1}{1-\gamma}}, \quad x(t) \sim \Big(\int_t^\infty sq(s)\,ds\Big)^{\frac{1}{1-\gamma}}, \quad t\to\infty.$$

(iii) Regularly varying solutions of index $\rho = 1$.

This case is completely resolved by Theorems 3.2 and 3.3 in [9] and we present it here for the sake of completeness.

Theorem 2.4. *Suppose that* (2.1) *and* (2.2) b) *hold. Equation* (A) *possesses a trivial* RV(1) *solution if and only if*

$$\sigma < \gamma - 1, \quad or \ \sigma = -\gamma - 1 \ \ and \ \int_{t_0}^{\infty} q(t)\phi(t)\,dt < \infty.$$

If, in addition, (2.4) *holds for* $\lambda = 1$, *equation* (A) *possesses a nontrivial* RV(1) *solution if and only if*

$$\sigma = -\gamma - 1 \ \ and \ \int_{t_0}^{\infty} q(t)\phi(t)\,dt = \infty,$$

and any such solution has the exact asymptotic behavior

$$x(t) \sim t\left[(1-\gamma)\int_{a}^{t} q(s)\phi(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty.$$

Remark 2.3. It is worthwhile mentioning that, due to Proposition 1.3, our results apply to a very wide class of equations (see Examples 2.1, 2.2).

Example 2.1. Consider differential equation (A) with

$$\phi(x) \sim x^{\gamma}\log(x+1) \ \text{ and } \ q(t) \sim \frac{3r(t)t^{\frac{\gamma-5}{2}}(\log t)^{\frac{1-\gamma}{2}}}{4\log(t^{-1/2}(\log t)^{1/2}+1)}, \quad t \to \infty, \tag{2.38}$$

where $0 < \gamma < 1$ and $r(t)$ is a continuous function on $[e, \infty)$ such that $\lim\limits_{t\to\infty} r(t) = 1$.

The function $q(t)$ is a regularly varying function of index $\sigma = \frac{\gamma-5}{2}$, which satisfies $\sigma < -2$, while $\phi(x) \in \mathcal{RV}(\gamma)$ fulfills the condition (2.3). Then $\rho = -1/2$ and it is easy to check that

$$\frac{t^2 q(t)}{\rho(\rho-1)} \sim \frac{t^{\frac{\gamma-1}{2}}(\log t)^{\frac{1-\gamma}{2}}}{\log(t^{-1/2}(\log t)^{1/2}+1)}, \quad t \to \infty.$$

Therefore, it follows from Theorem 2.1 that the equation possesses decreasing regularly varying solutions $x(t)$ of index $\rho = -1/2$, satisfying $x(t) \sim Y_0(t)$, $t \to \infty$ i.e.

$$\frac{x(t)^{1-\gamma}}{\log(x(t)+1)} = \frac{x(t)}{\phi(x(t))} \sim \frac{Y_0(t)}{\phi(Y_0(t))}, \quad t \to \infty.$$

In view of (2.8), we have

$$\frac{Y_0(t)}{\phi(Y_0(t))} \sim \left(\frac{\log t}{t}\right)^{\frac{1-\gamma}{2}}\left[\log\left(\left(\frac{\log t}{t}\right)^{\frac{1}{2}}+1\right)\right]^{-1}, \quad t \to \infty,$$

implying that

$$x(t) \sim \sqrt{\frac{\log t}{t}}, \quad t \to \infty.$$

Observe that if in (2.38) instead of "$\sim$" one has "$=$" and

$$r(t) = 1 - \frac{4}{3\log t} - \frac{1}{3(\log t)^2},$$

then, $x(t) = (\frac{\log t}{t})^{\frac{1}{2}} \in \mathrm{RV}(-1/2)$ is an exact solution.

Example 2.2. Consider equation (A) with

$$\phi(x) \sim x^{\gamma} \log(x^{\delta} + 1) \text{ and}$$
$$q(t) \sim \frac{f(t)}{2t^2 (\log t)^{\frac{3-\gamma}{2}} \log((\log t)^{-\delta/2} + 1)}, \quad t \to \infty, \tag{2.39}$$

where $\gamma \in (0,1)$, $\delta > 0$ and $f(t)$ is a continuous function on $[e, \infty)$ such that $\lim_{t\to\infty} f(t) = 1$. Clearly, $q(t)$ is a regularly varying function of index $\sigma = -2$ and satisfies

$$Q(t) = \int_t^{\infty} sq(s)\, ds \sim \frac{1}{\delta(1-\gamma)(\log t)^{\frac{1-\gamma}{2}} \log(\log t)^{-1/2}} \to 0, \quad t \to \infty. \tag{2.40}$$

Also, $\phi(x) \in \mathcal{RV}(\gamma)$ and

$$\Phi(x) = \int_0^x \frac{dv}{\phi(v)} \sim \frac{1}{\delta(1-\gamma)x^{\gamma-1}\log x}, \quad x \to 0. \tag{2.41}$$

By Theorems 2.2 and 2.3, equation (A) has, along with a trivial slowly varying solution, a nontrivial SV-solution $x(t)$ whose asymptotic behavior is given by (2.31) or equivalently

$$\Phi(x(t)) \sim Q(t) = \int_t^{\infty} sq(s)\, ds, \quad t \to \infty. \tag{2.42}$$

Using (2.40) and (2.41), (2.42) is reduced to

$$\delta(1-\gamma)x(t)^{\gamma-1} \log x(t) \sim \delta(1-\gamma)\big((\log t)^{-1/2}\big)^{\gamma-1} \log(\log t)^{-1/2}, \quad t \to \infty,$$

implying that $x(t) \sim (\log t)^{-1/2}$ as $t \to \infty$. If in (2.39) instead of "$\sim$" one has "$=$" and, in particular, $f(t) = 1 + 3/2 \log t$, then (A) possesses an exact nontrivial SV-solution $x(t) = (\log t)$.

Example 2.3. Consider equation (A) with

$$\phi(x) = x^{\gamma}\log(x+1), \quad q(t) = \big(t^{\gamma+1}(\log t)^{\gamma}\log(t\log t + 1)\big)^{-1}$$

with $\gamma \in (0,1)$. Note that ϕ fulfills the condition (2.4) with $\lambda = 1$. Also, $q(t) \in \mathrm{RV}(-\gamma - 1)$ and satisfies

$$q(t)\phi(t) \sim t(\log t)^{\gamma}, \quad t \to \infty$$

which for $t \to \infty$ gives

$$\int_{t_0}^{t} q(s)\phi(s)\, ds \sim \frac{(\log t)^{1-\gamma}}{1-\gamma} \to \infty.$$

Thus, by Theorem 2.4, the above-considered equation possesses nontrivial RV(1) solutions all of which have the same asymptotic behavior $x(t) \sim t \log t$, $t \to \infty$. In fact, an exact solution is $x(t) = t \log t$.

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