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OPTIMAL SOLVABILITY CONDITIONS OF THE CAUCHY–NICOLETTI PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. For the systems of singular functional differential equations the unimprovable sufficient conditions of solvability of the Cauchy–Nicoletti problem are established.

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Let $-\infty < a < b < +\infty$,

$$I = [a, b], \;\; t_i \in I, \quad I_i = I \setminus \{t_i\} \;\; (i = 1, \dots, n).$$

In the interval I we consider a system of functional differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1, \dots, x_n)(t) \;\; (i = 1, \dots, n) \tag{1}$$

with the boundary conditions

$$x_i(t_i) = 0 \;\; (i = 1, \dots, n). \tag{2}$$

Here every f_i is the operator acting from the space of continuous on I n-dimensional vector functions to the space of functions, Lebesgue integrable on every closed interval contained in I_i. We are, in the main, interested in a singular case, in which there exist $i \in \{1, \dots, n\}$ and continuous functions $x_k : I \to R$ $(k = 1, \dots, n)$, such that

$$\int_a^b \big| f_i(x_1, \dots, x_n)(t) \big| \, dt = +\infty.$$

(2) are called the boundary conditions of Cauchy–Nicoletti. In the case, where $t_1 = \dots = t_n$, these conditions represent the initial, i.e. the Cauchy conditions.

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I. Kiguradze [4]–[8] has developed technique for a priori estimates of solutions of one-sided differential inequalities allowing one to investigate the Cauchy and Cauchy–Nicoletti problems for a singular differential system

$$\frac{dx_i(t)}{dt} = f_{0i}\big(t, x_1(t), \dots, x_n(t)\big) \ \ (i = 1, \dots, n) \tag{3}$$

which is a particular case of system (1). The singular problem (3), (2) is investigated also in [19].

I. Kiguradze and Z. Sokhadze [12], [13], [21] have found the sufficient conditions of local and global solvability of the Cauchy problem for evolution singular functional differential systems of type (1) and proved Kneser type theorem on the structure of a set of solutions of the above-mentioned problem [14].

Optimal sufficient conditions of solvability of two-point problems of Cauchy–Nicoletti type for singular differential equations of second and higher orders and for linear singular differential systems can be found in [1]–[3], [9], [11], [15]–[18], [20].

In the case, where f_i $(i = 1, \dots, n)$ are not evolution operators, for the singular functional differential system (1) not only the Cauchy–Nicoletti problem, but also the Cauchy problem remain little studied. Just that very case our work is devored to.

Throughout the paper, we adopt the following notation:

$\mathbb{R} = \,]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.

$\mathbb{R}^n$ is n-dimensional real Euclidean space.

$x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are the n-dimensional column vector and $n \times n$-matrix with elements x_i and $x_{ik} \in \mathbb{R}$ $(i = 1, \dots, n)$.

$r(X)$ is the spectral radius of the matrix X.

$C(I; \mathbb{R}^n)$ is the Banach space of the n-dimensional continuous vector functions $x = (x_i)_{i=1}^n : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{ \sum_{i=1}^n |x_i(t)| : \ t \in I \Big\}.$$

$L(I; \mathbb{R})$ is the Banach space of the Lebesgue integrable functions $y : I \to \mathbb{R}$ with the norm

$$\|y\|_L = \int_a^b |y(s)|\, ds.$$

$L_{loc}(I_i; \mathbb{R})$ is the space of functions $y : I_i \to \mathbb{R}$, Lebesgue integrable on every closed interval contained in I_i.

$\mathcal{K}_{loc}(I \times \mathbb{R}^m; \mathbb{R})$ is the set of functions $g : I \times \mathbb{R}^m \to \mathbb{R}$ satisfying the local Carathéodory conditions, i.e., such that $g(\cdot, x_1, \dots, x_n) : I \to \mathbb{R}$ is measurable for any $(x_k)_{k=1}^m \in \mathbb{R}^m$, $g(t, \cdot, \dots, \cdot) : \mathbb{R}^m \to \mathbb{R}$, continuous almost for all $t \in I$ and

$$g^*_\rho \in L(I; \mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where

$$g^*_\rho(t) = \max\Big\{\big|g(t,x_1,\dots,x_n)\big| : \ \sum_{k=1}^m |x_k| \le \rho\Big\}. \tag{4}$$

$\mathcal{K}_{loc}(I_i \times \mathbb{R}^m;\mathbb{R})$ is the set of functions $g : I_i \times \mathbb{R}^m \to \mathbb{R}$, such that $g(\cdot,x_1,\dots,x_m) : I \to \mathbb{R}$ is measurable for any $(x_k)_{k=1}^m \in \mathbb{R}^m$, $g(t,\cdot,\dots,\cdot) : \mathbb{R}^m \to \mathbb{R}$, continuous for almost all $t \in I$ and

$$g^*_\rho \in L_{loc}(I_i;\mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where g^*_ρ is the function defined by the equality (4).

$\mathcal{K}_{loc}(C(I;\mathbb{R}^n);L(I;\mathbb{R}))$ is the set of continuous operators $f : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$, such that

$$g^*_\rho \in L(I;\mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where

$$g^*_\rho(t) = \sup\Big\{\big|f(x_1,\dots,x_n)(t)\big| : \ \sum_{k=1}^n \big\|x_k\big\|_C \le \rho\Big\}.$$

$\mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(I_i;\mathbb{R}))$ is the set of operators $f : C(I;\mathbb{R}^n) \to L_{loc}(I_i;\mathbb{R})$, such that

$$f \in \mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(J;\mathbb{R}))$$

for an arbitrary closed interval J contained in I_i.

We investigate the problem (1), (2) in the case, where

$$f_i \in \mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(I_i;\mathbb{R})) \ \ (i=1,\dots,n). \tag{5}$$

A vector function $(x_k)_{k=1}^n : I \to \mathbb{R}^n$ with absolutely continuous components $x_k : I \to \mathbb{R}$ $(k=1,\dots,n)$ is said to be a solution of the system (1) if it satisfies this system almost everywhere on I. The solution of the system (1), satisfying the boundary conditions (2), is said to be a solution of the problem (1), (2).

For an arbitrary $\delta > 0$, we put

$$\chi_i(t,\delta) = \begin{cases} 0 & \text{for } \ t \in [t_i-\delta, t_i+\delta] \\ 1 & \text{for } \ t \notin [t_i-\delta, t_i+\delta] \end{cases}$$

and along with (1) consider the functional differential system

$$\frac{dx_i(t)}{dt} = \lambda \chi_i(t,\delta) f_i(x_1,\dots,x_n)(t) \ \ (i=1,\dots,n) \tag{6}$$

depending on the parameters $\lambda \in]0,1]$ and $\delta > 0$.

The following propositions hold.

Theorem 1 (Principle of a Priori Boundedness)**.** *Let the condition* (5) *be fulfilled and there exist a positive number* δ_0 *and continuous functions* $\rho_i : I \to \mathbb{R}_+$ $(i=1,\dots,n)$*, such that*

$$\rho_i(t_i) = 0 \ \ (i=1,\dots,n)$$

and for arbitrary $\delta \in]0, \delta_0[$ *and* $\lambda \in]0, 1]$ *every solution* $(x_i)_{i=1}^n$ *of the problem* $(6), (2)$ *admits the estimates*

$$|x_i(t)| \le \rho_i(t) \ \text{ for } \ a \le t \le b \ \ (i = 1, \dots, n).$$

Then the problem $(1), (2)$ *has at least one solution.*

Theorem 2. *Let the condition* (5) *be fulfilled and there exist nonnegative operators*

$$p_i \in \mathcal{K}_{loc}(C(I; \mathbb{R}^n); L_{loc}(I; \mathbb{R})) \ \ (i = 1, \dots, n),$$

nonnegative numbers h_{ik}, h_i $(i, k = 1, \dots, n)$ *and nonnegative functions* $q_{ik} \in L(I; \mathbb{R})$, $q_i \in L(I; \mathbb{R})$ $(i, k = 1, \dots, n)$, *such that for any* $(x_k)_{k=1}^n \in C(I; \mathbb{R}^n)$, *almost everywhere on* I *the inequalities*

$$\begin{aligned} f_i(x_1, \dots, x_n)(t) \operatorname{sgn}\big((t - t_i)x_i(t)\big) \le \\ \le p_i(x_1, \dots, x_n)(t)\Big(- |x_i(t)| + \sum_{k=1}^n h_{ik} \|x_k\|_C + h_i\Big) + \\ + \sum_{k=1}^n q_{ik}(t) \|x_k\|_C + q_i(t) \ \ (i = 1, \dots, n) \end{aligned}$$

hold. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^n$ *satisfies the condition*

$$r(H) < 1, \tag{7}$$

then the problem $(1), (2)$ *has at least one solution.*

For regular systems (1) and (3), the results analogous to Theorem 2 are contained in [10] and [22].

An important particular case (1) is the differential system with deviating arguments

$$\frac{dx_i(t)}{dt} = g_i\Big(t, x_1(\tau_1(t)), \dots, x_n(\tau_n(t)), x_i(t)\Big) \ \ (i = 1, \dots, n), \tag{8}$$

where

$$g_i \in \mathcal{K}_{loc}(I_i \times \mathbb{R}^{n+1}; \mathbb{R}) \ \ (i = 1, \dots, n), \tag{9}$$

and $\tau_i : I \to I$ $(i = 1, \dots, n)$ are measurable functions.

If

$$f_i(x_1, \dots, x_n)(t) \equiv g_i\Big(t, x_1(\tau_1(t)), \dots, x_n(\tau_n(t)), x_i(t)\Big) \ \ (i = 1, \dots, n),$$

then the condition (9) ensures the fulfilment of the condition (5). Thus from Theorem 2 we arrive at the following proposition.

Corollary 1. *Let the condition* (9) *be fulfilled and there exist nonnegative numbers* h_{ik}, h_i $(i, k = 1, \dots, n)$ *and nonnegative functions* $q_{ik} \in L(I; \mathbb{R})$,

$q_i \in L(I;\mathbb{R})$, $g_{0i} \in \mathcal{K}_{loc}(I \times \mathbb{R}^{n+1};\mathbb{R})$ $(i,k = 1,\dots,n)$, such that for every $i \in \{1,\dots,n\}$ the inequality

$$g_i(t,y_1,\dots,y_{n+1})\operatorname{sgn}\big((t-t_i)y_{n+1}\big) \leq$$
$$\leq g_{0i}(t,y_1,\dots,y_{n+1})\Big(-|y_{n+1}|+\sum_{k=1}^{n}h_{ik}|y_k|+h_i\Big)+\sum_{k=1}^{n}q_{ik}(t)|y_k|+q_i(t) \quad (10)$$

holds on the set $I_i \times \mathbb{R}^n$. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^n$ satisfies the condition (7), then the problem (8), (2) has at least one solution.

Example 1. Let

$$\mu_i = \max\big\{t_i - a, b - t_i\big\}, \quad \tau_i = \begin{cases} a & \text{for } \mu_i = t_i - a \\ b & \text{for } \mu_i = t_i - b \end{cases}, \quad (11)$$
$$h_{ik} = 0 \ \text{ for } \ k < i, \quad h_{ik} > 0 \ \text{ for } \ k \geq i. \quad (12)$$

Consider the differential system

$$\frac{dx_i(t)}{dt} = \frac{1+|x_i(\tau_i)|}{\mu_i}\times$$
$$\times\bigg(-\frac{\mu_i x_i(t)}{|t-t_i|} + \sum_{k=1}^{n} h_{ik}|x_k(\tau_i)| + 2\bigg)\operatorname{sgn}(t-t_i) \ \ (i = 1,\dots,n). \quad (13)$$

Clearly, for every $i \in \{1,\dots,n\}$ the function

$$g_i(t,y_1,\dots,y_{n+1}) \equiv \frac{1+|y_i|}{\mu_i}\bigg(-\frac{\mu_i y_{n+1}}{|t-t_i|} + \sum_{k=1}^{n} h_{ik}|y_k| + 2\bigg)\operatorname{sgn}(t-t_i)$$

on $I_i \times \mathbb{R}^{n+1}$ satisfies the inequality (10), where

$$g_{0i}(t,y_1,\dots,y_{n+1}) \equiv \frac{1+|y_i|}{\mu_i}, \quad h_i = 2,$$
$$q_{ik}(t) \equiv 0, \quad q_i(t) \equiv 0 \ \ (i,k = 1,\dots,n).$$

Moreover, taking into account (12), we have

$$H = \big(h_{ik} + \|q_{ik}\|_L\big)_{i,k=1}^n = (h_{ik})_{i,k=1}^n, \quad r(H) = \max\{h_{11},\dots,h_{nn}\}. \quad (14)$$

If the inequality (7) is fulfilled, then according to Corollary 1, the problem (13), (2) has at least one solution. Consider now the case, where inequality (7) is violated. Then in view of (14), there exists $i \in \{1,\dots,n\}$, such that

$$r(H) = h_{ii} \geq 1. \quad (15)$$

Assume that the problem (13), (2) has in this case a solution $(x_k)_{k=1}^n$, as well. Then

$$x_i(t) = \frac{1+|x_i(\tau_i)|}{\mu_i(2+|x_i(\tau_i)|)}\Big(2 + \sum_{k=1}^{n} h_{ik}|x_k(\tau_i)|\Big)|t-t_i| \ \text{ for } \ a \leq t \leq b.$$

Taking into account (11), (12) and (15), the above equality results in

$$x_i(\tau_i) = \frac{1 + |x_i(\tau_i)|}{2 + |x_i(\tau_i)|} \Big(2 + \sum_{k=1}^{n} h_{ik} |x_k(\tau_i)| \Big) \geq 1 + |x_i(\tau_i)|.$$

The obtained contradiction proves that the problem (13), (2) is solvable iff the inequality (7) is fulfilled. Consequently, the condition (7) in Theorem 2 and in Corollary 1 is optimal and it cannot be replaced by the condition

$$r(H) \leq 1.$$

Example 2. Consider the differential system

$$\begin{aligned} \frac{dx_i(t)}{dt} = &- \bigg[\exp \bigg(|t - t_i|^{-1} + \sum_{k=1}^{n} |x_k(\tau_k(t))| + |x_i(t)| \bigg) \operatorname{sgn}(t - t_i) \bigg] x_i(t) + \\ &+ g_{1i} \Big(t, x_1(\tau_1(t)), \dots, x_n(\tau_n(t)), x_i(t) \Big), \end{aligned} \tag{16}$$

where $\tau_k : I \to I$ $(k = 1, \dots, n)$ are measurable functions, and $g_{1i} \in \mathcal{K}_{loc}(I \times \mathbb{R}^n; \mathbb{R})$ $(i = 1, \dots, n)$ are the functions satisfying the inequality

$$\sum_{i=1}^{n} \big| g_{1i}(t, y_1, \dots, y_{n+1}) \big| \leq \ell \exp \Big(\sum_{k=1}^{n+1} |y_k| \Big),$$

where $l = \text{const} > 0$. Then for any $i \in \{1, \dots, n\}$, the function

$$\begin{aligned} &g_i(t, y_1, \dots, y_{n+1}) \equiv \\ &\equiv - \bigg[\exp \Big(|t - t_i|^{-1} + \sum_{k=1}^{n+1} |y_k| \Big) \operatorname{sgn}(t - t_i) \bigg] y_{n+1} + g_{1i}(t, y_1, \dots, y_{n+1}) \end{aligned}$$

on the set $I_i \times \mathbb{R}^n$ admits the estimate (10), where

$$g_{0i}(t, y_1, \dots, y_{n+1}) \equiv \exp \Big(\sum_{k=1}^{n+1} |y_k| \Big), \quad h_{ik} = 0, \quad h_i = \ell,$$
$$q_{ik}(t) \equiv 0, \quad q_i(t) \equiv 0 \quad (i = 1, \dots, n).$$

Moreover, $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^n$ is a zero matrix, and hence $r(H) = 0$. Thus according to Corollary 1, it follows that the problem (16), (2) is solvable. On the other hand, it is evident that the system (16) in the case under consideration is superlinear, and the order of singularity for every function $g_i(\cdot, y_1, \dots, y_n) : I \to \mathbb{R}$ at the point t_i is equal to infinity, or more exactly, for an arbitrary natural m we have

$$\int_a^b |t - t_i|^m \big| g_i(t, y_1, \dots, y_{n+1}) \big| \, dt = +\infty$$

if only $y_{n+1} \neq 0$.

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