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ON SOLVABILITY AND WELL-POSEDNESS OF TWO-POINT WEIGHTED SINGULAR BOUNDARY VALUE PROBLEMS

Abstract. For second order nonlinear ordinary differential equations with strong singularities, unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of two-point weighted boundary value problems are established.

რეზიუმე. მეორე რიგის არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებებისათვის ძლიერი სინგულარობებით დადგენილია ორწერტილოვანი წონიანი სასაზღვრო ამოცანების ამოხსნადობისა და კორექტულობის არაგაუმჯობესებადი საკმარისი პირობები.

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In an open interval $]a,b[$, we consider the second order nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with two-point weighted boundary conditions of one of the following two types:

$$\limsup_{t\to a}\frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \limsup_{t\to b}\frac{|u(t)|}{(b-t)^\beta} < +\infty \tag{2}$$

and

$$\limsup_{t\to a}\frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \lim_{t\to b} u'(t) = 0. \tag{3}$$

Here $f:]a,b[\times R \to R$ is a continuous function, $\alpha \in]0,1[$, and $\beta \in]0,1[$.

Eq. (1) is said to be regular if

$$\int\limits_a^b f^*(t,x)dt < +\infty \quad \text{for } x > 0,$$

where

$$f^*(t,x) = \max\big\{|f(t,y)|:\ 0 \le y \le x\big\} \quad \text{for } a < t < b, \ \ x \ge 0. \tag{4}$$

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And if

$$\int_a^{t_0} f^*(t,x)dt = +\infty \quad \left(\int_{t_0}^{b} f^*(t,x)dt = +\infty\right) \quad \text{for } a < t_0 < b, \ \ x > 0,$$

then it is said that Eq. (1) with respect to the time variable has a singularity at the point a (at the point b). In that case Eq. (1) is called singular, and boundary value problems for such equations are called singular boundary value problems.

Following R. P. Agarwal and I. Kiguradze [2, 8] we say that Eq. (1) with respect to the time variable has a strong singularity at the point a (at the point b) if for any $t_0 \in \,]a, b[$ and $x > 0$ the condition

$$\int_a^{t_0} (t-a)\big[|f(t,x)| - f(t,x)\operatorname{sgn} x\big]dt = +\infty$$

$$\left(\int_{t_0}^{b} (b-t)\big[|f(t,x)| - f(t,x)\operatorname{sgn} x\big]dt = +\infty\right)$$

is satisfied.

The boundary conditions (2) and (3), respectively, yield the conditions

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0, \tag{2_0}$$

and

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0. \tag{3_0}$$

On the other hand, if $\alpha = \beta = \frac{1}{2}$, then the conditions

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty, \tag{$2'$}$$

and

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty \tag{$3'$}$$

imply the conditions (2) and (3), respectively.

In the case, where Eq. (1) is regular, the problems (1),(2); (1),(2_0), and (1),($2'$) (the problems (1),(3); (1),(3_0), and (1),($3'$)) are equivalent to each other. However, if Eq. (1) is singular, then the above-mentioned problems are not equivalent. Precisely, if Eq. (1) with respect to the time variable has singularities at the points a and b (has a singularity at the point a), then from the solvability of the problem (1),(2_0) (of the problem (1),(3_0)), generally speaking, it does not follow the solvability of the problem (1),(2) or the problem (1),($2'$) (of the problem (1),(3) or the problem (1),($3'$)). On the other hand, in the above-mentioned cases the unique solvability of the problem (1),(2) or the problem (1),(2′) (of the problem (1),(3) or the problem (1),(3′)) does not imply the unique solvability of the problem (1),(2_0) (of the problem (1),(3_0)).

The investigation of two-point boundary value problems for second order singular ordinary differential equations was initiated by I. Kiguradze [4,5]. Nowadays the singular problems (1),(2_0) and (1),(3_0) are studied in full detail (see, e.g., [1,3–7,10–17,19,20], and the references therein).

The problems (1),(2′) and (1),(3′) and the analogous problems for higher order differential equations with strong singularities are studied in [2, 8, 9, 18].

As for the singular problems (1),(2) and (1),(3), they remain still unstudied. In the present paper, an attempt is made to fill this gap. Theorems 1 and 2 (Theorems 3 and 4) below contain unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of the problem (1),(2) (of the problem (1),(3)), at that these theorems, unlike the results from the above-mentioned works [1, 2–7, 10–17, 19, 20], cover the case, where Eq. (1) with respect to the time variable has strong singularities at the points a and b (has a strong singularity at the point a).

Before passing to the formulation of the main results, we introduce some definitions and notation.

By G_0 and G_1 we denote the Green functions of the problems

$$u''=0; \quad u(a)=u(b)=0$$

and

$$u''=0; \quad u(a)=u'(b)=0,$$

respectively, i.e.,

$$G_0(t,s)=\begin{cases}\dfrac{(s-a)(t-b)}{b-a} & \text{for } a\le s\le t\le b,\\ \dfrac{(t-a)(s-b)}{b-a} & \text{for } a\le t<s\le b,\end{cases}$$

and

$$G_1(t,s)=\begin{cases}a-s & \text{for } a\le s\le t\le b,\\ a-t & \text{for } a\le t<s\le b.\end{cases}$$

For any continuous function $h:]a,b[\to R$, we assume

$$\nu_{\alpha,\beta}(h)=\sup\Big\{(t-a)^{-\alpha}(b-t)^{-\beta}\int_a^b |G_0(t,s)h(s)|ds \ : a<t<b\Big\},$$

$$\nu_{\alpha}(h)=\sup\Big\{(t-a)^{-\alpha}\int_a^b |G_1(t,s)h(s)|ds \ : a<t<b\Big\}.$$

Definition 1. A function $u:]a,b[\to R$ is said to be **a solution of Eq. (1)** if it is twice continuously differentiable and satisfies that equation at each point of the interval $]a,b[$. A solution of Eq. (1), satisfying the boundary conditions (2) (the boundary conditions (3)), is said to be **a solution of the problem (1),(2)** (**of the problem (1),(3)**).

Definition 2. The problem (1),(2) (the problem (1),(3)) is said to be **well-posed** if for any continuous function $h :]a,b[\to R$, satisfying the condition

$$\nu_{\alpha,\beta}(h) < +\infty \quad \big(\nu_\alpha(h) < +\infty\big), \tag{5}$$

the perturbed differential equation

$$v'' = f(t,v) + h(t) \tag{6}$$

has a unique solution, satisfying the boundary conditions (2) (the boundary conditions (3)), and there exists a positive constant r, independent of the function h, such that in the interval $]a,b[$ the inequality

$$|u(t) - v(t)| \le r\nu_{\alpha,\beta}(h)(t-a)^\alpha(b-t)^\beta \quad \Big(|u(t) - v(t)| \le r\nu_\alpha(h)(t-a)^\alpha\Big)$$

is satisfied, where u and v are the solutions of the problems (1),(2) and (6),(2) (of the problems (1),(3) and (6),(3)), respectively.

It is clear that

$$\nu_{\alpha,\beta}(h) \le (b-a)^{-1}\int_a^b (s-a)^{1-\alpha}(b-s)^{1-\beta}|h(s)|ds,$$

$$\nu_\alpha(h) \le \int_a^b (s-a)^{1-\alpha}|h(s)|ds.$$

Thus for the condition (5) to be fulfilled it is sufficient that

$$\int_a^b (s-a)^{1-\alpha}(b-s)^{1-\beta}|h(s)|ds < +\infty \quad \left(\int_a^b (s-a)^{1-\alpha}|h(s)|ds < +\infty\right).$$

Now we formulate the main results. First we consider the problem (1),(2).

Theorem 1. *Let there exist continuous functions p and $q :]a,b[\to [0,+\infty[$ such that*

$$f(t,x)\operatorname{sgn} x \ge -(t-a)^{-\alpha}(b-t)^{-\beta}p(t)|x| - q(t) \quad \text{for } a<t<b, \ \ x\in R,$$

$$\nu_{\alpha,\beta}(p) < 1, \quad \nu_{\alpha,\beta}(q) < +\infty. \tag{7}$$

Then the problem (1),(2) has at least one solution.

Corollary 1. *Let there exist a constant $\ell \in [0,1[$ and a continuous function $q :]a,b[\to R$ such that*

$$f(t,x)\operatorname{sgn} x \ge -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)|x| - q(t)$$

$$\text{for } a<t<b, \ \ x\in R,$$

and $\nu_{\alpha,\beta}(q) < +\infty$. *Then the problem* (1),(2) *has at least one solution.*

Theorem 2. *Let there exist a continuous function* $p :]a,b[\to [0,+\infty[$ *such that*

$$f(t,x) - f(t,y) \geq -(t-a)^{-\alpha}(b-t)^{-\beta}p(t)(x-y) \quad for\ \ a<t<b, \ \ x>y.$$

If, moreover, the condition (7) *holds, where* $q(t) \equiv f(t,0)$, *then the problem* (1),(2) *is well-posed.*

Corollary 2. *Let there exist a constant* $\ell \in [0,1[$ *such that*

$$f(t,x) - f(t,y) \geq -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)(x-y)$$
$$for\ \ a<t<b, \ \ x>y,$$

and $\nu_{\alpha,\beta}(f(\cdot,0)) < +\infty$. *Then the problem* (1),(2) *is well-posed.*

A particular case of (1) is the differential equation

$$u'' = f_1(t)u + f_2(t)|u|^{\mu}\operatorname{sgn} u + f_0(t), \tag{8}$$

where $f_i :]a,b[\to R$ $(i=0,1,2)$ are continuous functions, and $\mu > 0$.
Corollary 2 yields

Corollary 3. *Let there exist a constant* $\ell \in [0,1[$ *such that*

$$f_1(t) \geq -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right) \quad for\ \ a<t<b.$$

If, moreover, $f_2(t) \geq 0$ *for* $a<t<b$, *and* $\nu_{\alpha,\beta}(f_0) < +\infty$, *then the problem* (8),(2) *is well-posed.*

Example 1. Let us consider the differential equation

$$u'' = -\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)\Big(\ell|u| + (s-a)^{\alpha}(b-s)^{\beta}\Big), \tag{9}$$

where ℓ is a nonnegative constant. If $\ell < 1$, then by virtue of Corollary 2 the problem (9),(2) is well-posed. Let us show that if $\ell \geq 1$, then that problem has no solution. Assume the contrary that the problem (9),(2) has a solution u. If we suppose

$$\delta = \inf\left\{\frac{|u(t)|}{(t-a)^{\alpha}(b-t)^{\beta}} \ : a<t<b\right\},$$

then from the representation

$$u(t) =$$
$$= \int_a^b |G_0(t,s)|\left(\frac{\alpha(1-\alpha)}{(s-a)^2} + \frac{2\alpha\beta}{(s-a)(b-s)} + \frac{\beta(1-\beta)}{(b-s)^2}\right)\Big(\ell|u(s)| + (s-a)^{\alpha}(b-s)^{\beta}\Big)ds$$

we get

$$u(t) \ge (1+\delta)\times$$

$$\times \int_a^b |G_0(t,s)| \left(\frac{\alpha(1-\alpha)}{(s-a)^2} + \frac{2\alpha\beta}{(s-a)(b-s)} + \frac{\beta(1-\beta)}{(b-s)^2} \right) (s-a)^\alpha (b-s)^\beta ds =$$

$$= (1+\delta)(t-a)^\alpha (b-t)^\beta \quad \text{for } \ a<t<b.$$

Hence we obtain the contradiction $\delta \ge 1+\delta$. Thus we have proved that the problem (9),(2) has no solution.

The above-constructed example shows that the condition $\nu_{\alpha,\beta}(p) < 1$ in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition $\nu_{\alpha,\beta}(p) \le 1$. Moreover, the strict inequality $\ell < 1$ in Corollaries 1–3 cannot be replaced by the non-strict one $\ell \le 1$.

Now we consider the problem (1),(3).

Theorem 3. *Let*

$$\int_t^b f^*(s,x)ds < +\infty \quad \textit{for } \ a<t<b, \ \ x>0, \tag{10}$$

and let the condition

$$f(t,x)\operatorname{sgn} x \ge -(t-a)^{-\alpha} p(t)|x| - q(t) \quad \textit{for } \ a<t<b, \ \ x \in R$$

be fulfilled, where f^* is a function, given by the equality (4)*, and $p, q :]a,b[\to [0,+\infty[$ are continuous functions such that*

$$\nu_\alpha(p) < 1, \quad \nu_\alpha(q) < +\infty. \tag{11}$$

Then the problem (1),(3) *has at least one solution.*

Corollary 4. *Let there exist a constant $\ell < \alpha(1-\alpha)$ and a continuous function $q :]a,b[\to [0,+\infty[$ such that*

$$f(t,x)\operatorname{sgn} x \ge -\frac{\ell}{(t-a)^2}|x| - q(t) \quad \textit{for } \ a<t<b, \ \ x \in R$$

and $\nu_\alpha(q) < +\infty$. If, moreover, the condition (10) *holds, then the problem* (1),(3) *has at least one solution.*

Theorem 4. *Let there exist a continuous function $p :]a,b[\to [0,+\infty[$ such that*

$$f(t,x) - f(t,y) \ge -(t-a)^{-\alpha} p(t)(x-y) \quad \textit{for } \ a<t<b, \ \ x>y,$$

and the conditions (11) *are satisfied, where $q(t) \equiv f(t,0)$. If, moreover, the condition* (10) *holds, then the problem* (1),(3) *is well-posed.*

Corollary 5. *Let there exist a constant $\ell < \alpha(1-\alpha)$ such that*

$$f(t,x) - f(t,y) \ge -\frac{\ell}{(t-a)^2}(x-y) \quad \textit{for } \ a<t<b, \ \ x>y.$$

If, moreover, $\nu_\alpha(f(\cdot,0)) < +\infty$ *and the condition* (10) *holds, then the problem* (1),(3) *is well-posed.*

For the Eq. (8), Corollary 5 yields

Corollary 6. *Let there exist a constant* $\ell < \alpha(1-\alpha)$ *such that*

$$f_1(t) \geq -\frac{\ell}{(t-a)^2} \quad \textit{for } a < t < b.$$

If, moreover, $f_2(t) \geq 0$ *for* $a < t < b$, *and* $\nu_\alpha(f_0) < +\infty$, *then the problem* (8),(3) *is well-posed.*

Example 2. Let us consider the differential equation

$$u'' = -\frac{\ell}{(t-a)^2}|u| - (t-a)^{\alpha-2}, \tag{12}$$

where $\alpha \in \,]0,1[$ and ℓ is a nonnegative constant. If $\ell < \alpha(1-\alpha)$, then according to Corollary 5 the problem (12),(3) is well-posed. On the other hand, it is easy to show that if $\ell \geq \alpha/(1-\alpha)$, then the problem (12),(3) has no solution.

The above-constructed example shows that the condition $\nu_\alpha(p) < 1$ in Theorems 3 and 4 is unimprovable and it cannot be replaced by the condition $\nu_\alpha(p) = 1 + \varepsilon$ no matter how small $\varepsilon > 0$ would be. Analogously, the condition $\ell < \alpha(1-\alpha)$ in Corollaries 4–6 cannot be replaced by the condition $\ell = \alpha(1-\alpha)(1+\varepsilon)$.

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