Mem. Differential Equations Math. Phys. 54 (2011), 117–125

Short Communications

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ON THE FREDHOLM PROPERTY FOR GENERAL LINEAR BOUNDARY VALUE PROBLEMS FOR IMPULSIVE SYSTEMS WITH SINGULARITIES

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. A general linear singular boundary value problem

$$\frac{dx_i}{dt} = P_i(t) \cdot x_{3-i} + q_i(t) \ \ (i = 1, 2),$$
$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1, 2; \ k = 1, 2, \dots);$$
$$l_i(x_1, x_2) = c_i \ \ (i = 1, 2)$$

is considered, where $P_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i \times n_{3-i}})$, $q_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i})$, $G_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i \times n_{3-i}}$, $h_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, and l_i is a linear bounded operator $(i = 1, 2)$.

The singularity is understood in the sense that $P_i \notin L([a, b], \mathbb{R}^{n_i \times n_{3-i}})$, $q_j \notin L([a, b], \mathbb{R}^{n_j})$ or $\sum\limits_{k=1}^{\infty} \big(\|G_i(k)\| + \|h_j(k)\|\big) = +\infty$ for some $i, j \in \{1, 2\}$.

The conditions are established under which this problem is uniquely solvable if and only if the corresponding homogeneous boundary value problem has only the trivial solution.

Analogous problems for similar impulsive systems with small parameters are also considered.

რეზიუმე. განხილულია ზოგადი სახის წრფივი იმპულსური სასაზღვრო ამოცანა

$$\frac{dx_i}{dt} = P_i(t) \cdot x_{3-i} + q_i(t) \ \ (i = 1, 2),$$
$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i=1, 2; \ k = 1, 2, \dots);$$
$$l_i(x_1, x_2) = c_i \ \ (i = 1, 2),$$

სადაც $P_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i \times n_{3-i}})$, $q_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i})$, $G_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i \times n_{3-i}}$, $h_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, ხოლო l_i $(i = 1, 2)$ წრფივი შემოსაზღვრული ოპერატორია.

Reported on the Tbilisi Seminar on Qualitative Theory of Differential Equations on December 13, 2010.

სინგულარობა გაიგება იმ აზრით, რომ $P_i \not\in L([a,b],\mathbb{R}^{n_i\times n_{3-i}})$, $q_j \not\in L([a,b],\mathbb{R}^{n_j})$ ან $\sum\limits_{k=1}^{\infty}\big(\|G_i(k)\|+\|h_j(k)\|\big)=+\infty$ გარკვეული $i,j\in\{1,2\}$-თვის.

მიღებულია პირობები, რომელთა შესრულების შემთხვევაში აღნიშნული ამოცანა ცალსახად ამოხსნადია მაშინ და მხოლოდ მაშინ, როდესაც ერთგვაროვან სასაზღვრო ამოცანას გააჩნია მხოლოდ ტრივიალური ამონახსნი.

აგრეთვე განხილულია ანალოგიური ამოცანები პარამეტრიანი წრფივი იმპულსური სისტემებისათვის.

2010 Mathematics Subject Classification. 34B37.
Key words and phrases. Linear impulsive systems, singularities, systems with small parameters, general linear boundary value problems, Fredholm property.

1. Statement of the Problem and Basic Notation

Let n_1 and n_2 be natural numbers; $-\infty<a<b<+\infty$, $a<\tau_1<\tau_2<\dots<b$ and $\lim\limits_{k\to\infty}\tau_k=b$.

On the interval $]a,b[$ we consider the linear system of impulsive systems with singularities

$$\frac{dx_i}{dt}=P_i(t)\cdot x_{3-i}+q_i(t)\ \ (i=1,2), \tag{1}$$

$$x_i(\tau_k+)-x_i(\tau_k-)=G_i(k)\cdot x_{3-i}(\tau_k)+h_i(k)\ \ (i=1,2;\ k=1,2,\dots) \tag{2}$$

under the following two-point boundary value conditions:

$$l_i(x_1,x_2)=c_i\quad (i=1,2), \tag{3}$$

where $P_i\in L_{loc}(]a,b[\,;\mathbb{R}^{n_i\times n_{3-i}})$, $q_i\in L_{loc}(]a,b[\,;\mathbb{R}^{n_i})$, $G_i:\{1,2,\dots\}\to\mathbb{R}^{n_i\times n_{3-i}}$, $h_i:\{1,2,\dots\}\to\mathbb{R}^{n_i}$, $c_i\in\mathbb{R}^{n_i}$ $(i=1,2)$, $l_i:\mathrm{BV}([a_1,b_1],\mathbb{R}^{n_1})\times\mathrm{BV}([a_2,b_2],\mathbb{R}^{n_2})\to\mathbb{R}^{n_i}$ $(i=1,2)$ are linear bounded operators and $[a_i,b_i]$ $(i=1,2)$ are some closed intervals from $[a,b]$.

In the case, where P_i $(i=1,2)$ and q_i $(i=1,2)$ are the integrable on $[a,b]$ matrix- and vector-functions and $\sum\limits_{k=1}^{\infty}\big(\|G_i(k)\|+\|h_i(k)\|\big)<\infty$ $(i=1,2)$, in [1, 5, 11, 12], the conditions are established for as wether the problem $(1),(2);(3)$ is Fredholm, i.e., the conditions under which the problem $(1),(2);(3)$ is uniquely solvable if and only if the corresponding homogeneous system

$$\frac{dx_i}{dt}=P_i(t)\cdot x_{3-i}\ \ (i=1,2), \tag{1_0}$$

$$x_i(\tau_k+)-x_i(\tau_k-)=G_i(k)\cdot x_{3-i}(\tau_k)\ \ (i=1,2;\ \ k=1,2,\dots) \tag{2_0}$$

under the conditions

$$l_i(x_1,x_2)=0\quad (i=1,2) \tag{3_0}$$

has only trivial solutions. In the case, where the system (1), (2) has singularities at the points a and b, i.e.,

$$\int_a^b \|P_i(t)\|\,dt + \sum_{k=1}^{\infty} \|G_i(k)\| = +\infty,$$

$$\int_a^b \|q_j(t)\|\,dt + \sum_{k=1}^{\infty} \|h_j(k)\| = +\infty$$

for some $i, j \in \{1, 2\}$, the question as to wether the problem (1), (2); (3) is Fredholm remains open. The present paper fills in this gap.

The results obtained in the paper are improved for the case, where the boundary condition (3) has the form

$$\sum_{k=1}^{m} \big[B_{1ik}x_1(t_{1ik}) + B_{2ik}x_2(t_{2ik})\big] = c_i \quad (i = 1, 2), \tag{4}$$

where $B_{jik} \in \mathbb{R}^{n_i \times n_j}$ $t_{jik} \in \mathbb{R}$ $(i, j = 1, 2;\ k = 1, \dots, m)$.

The impulsive system (1), (2) is a particular case of the so-called generalized ordinary differential system (see, e.g., [1–5, 10, 11] and the references therein). The analogous questions and some singular boundary value problems are investigated in [2], [3] for the generalized ordinary differential systems, and in [6, 8, 9] for ordinary differential systems.

In the present paper, on the basis of the results presented in [2, 3], we obtain tests for the Fredholm property for the above impulsive problem. Similar tests are obtained for every of the two linear singular impulsive systems with a small parameter $\varepsilon > 0$,

$$\frac{dx_i(t)}{dt} = \varepsilon^{i-1} P_i(t) \cdot x_{3-i}(t) + q_i(t) \ \ (i = 1, 2), \tag{5_ε}$$

$$x_i(\tau_k+) - x_i(\tau_k-) = \\ = \varepsilon^{i-1} G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1, 2;\ \ k = 1, 2, \dots) \tag{6_ε}$$

and

$$\frac{dx_i(t)}{dt} = \varepsilon^{2-i} P_i(t) \cdot x_{3-i}(t) + q_i(t) \ \ (i = 1, 2), \tag{7_ε}$$

$$x_i(\tau_k+) - x_i(\tau_k-) = \\ = \varepsilon^{2-i} G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1, 2;\ \ k = 1, 2, \dots) \tag{8_ε}$$

under the condition (3).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, the closed and open intervals.

$\mathbb{I}$ is an arbitrary closed or open interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|.$$

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the inverse to X matrix, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a)=0$, $V(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\leq b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-)=X(a),\ X(b+)=X(b))$.

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<\infty$).

$\mathrm{BV}_{loc}(\mathbb{I};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_a^b(X)<+\infty$ for $a,\ b\in\mathbb{I}$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, measurable and integrable in the Lebesgue sense on the closed interval $[a,b]$.

$L_{loc}(\mathbb{I};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}$ belong to $L([a,b];\mathbb{R}^{n\times m})$.

$\widetilde{C}([a,b],\mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$.

$\widetilde{C}_{loc}(\mathbb{I},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}$ belong to $\widetilde{C}([a,b],\mathbb{R}^{n\times m})$.

$\widetilde{C}_{loc}(\mathbb{I}\setminus\{\tau_k\}_{k=1}^{\infty},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}\setminus\{\tau_k\}_{k=1}^{\infty}$ belong to $\widetilde{C}([a,b],\mathbb{R}^{n\times m})$.

If $X\in L_{loc}(]a,b[;\mathbb{R}^{l\times n})$, $G:\mathbb{N}\to\mathbb{R}^{l\times n}$, $Y\in L_{loc}(]a,b[;\mathbb{R}^{n\times m})$ and $Q:\mathbb{N}\to\mathbb{R}^{n\times m}$, then

$$\mathcal{F}_1(X,G;Y,Q)(s,t)=\int\limits_s^t dA(X,G)(\tau)\cdot\big(A(Y,Q)(t)-A(Y,Q)(\tau)\big)$$

and

$$\mathcal{F}_2(X,G;Y,Q)(s,t)=\mathcal{F}_1(X,G;Y,Q)(t,s) \ \text{ for } s,t\in\,]a,b[\,,$$

where

$$A(Y,Q)(t)=\begin{cases}\int\limits_c^t Y(\tau)\,d\tau+\sum\limits_{c\le\tau_k<t}Q(k) & \text{for } \ c\le t<b,\\ \int\limits_c^t Y(\tau)\,d\tau-\sum\limits_{t<\tau_k\le c}Q(k) & \text{for } \ a<t<c,\\ O_{n\times m} & \text{for } \ t=c,\end{cases} \tag{9}$$

and $c=(a+\tau_1)/2$.

Using the formulae of integration-by-parts, formula I.4.33 and Lemma I.4.23 from [10], it is not difficult to verify that

$$\mathcal{F}_1(X,G;Y,Q)(s,t)=\int\limits_s^t\int\limits_s^\tau X(r)dr\cdot Y(\tau)\,d\tau+$$

$$+\sum_{s\le\tau_k<t}\bigg(G(k)\int\limits_{\tau_k}^t Y(\tau)\,d\tau+\int\limits_s^{\tau_k}X(\tau)\,d\tau\cdot Q(k)+\sum_{l=1}^k G(l)\cdot Q(k)\bigg) \tag{10}$$

$$\text{for } \ a<s<t<b.$$

Moreover, we introduce the operator

$$\mathcal{F}_0(X,G;Y,Q)(s,t)=$$

$$=\int\limits_s^t\bigg(\int\limits_s^\tau X(r)\,dr+\sum_{s<\tau_k<\tau}G(k)\bigg)\cdot\bigg(\int\limits_\tau^t X(r)\,dr+\sum_{\tau<\tau_k<t}G(k)\bigg)Y(\tau)\,d\tau+$$

$$+\sum_{s<\tau_k<t}\bigg(\int\limits_s^{\tau_k}X(r)\,dr+\sum_{s<\tau_l<\tau_k}G(l)\bigg)\cdot\bigg(\int\limits_{\tau_k}^t X(r)\,dr+\sum_{\tau_k<\tau_l<t}G(l)\bigg)\cdot Q(k) \tag{11}$$

$$\text{for } \ a<s<t<b.$$

Under a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $(x_i)_{i=1}^2$, $x_i\in\widetilde{C}_{loc}(]a,b[\setminus\{\tau_k\}_{k=1}^\infty,\mathbb{R}^{n_i})\cap \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n_i})$ $(i=1,2)$, satisfying both the system

$$\frac{dx_i(t)}{dt}=P_i(t)x_{3-i}(t)+q_i(t) \ \text{ for a.e. } \ t\in]a,b[\setminus\{\tau_k\}_{k=1}^\infty \tag{12}$$

and the relation (2) for every $k\in\{1,2,\dots\}$. If the component x_i has a right (respectively, left) limit at the point a (respectively, at the point b), then this limit is assumed to be equal to $x_i(a)$ (respectively, to $x_i(b)$). Thus x_i is assumed to be continues at this point.

A solution of the impulsive system (1), (2) is said to be a solution of the problem (1), (2); (3) if there exist one-sided limits $x_i(a_i+)$ and $x_i(b_i-)$

$(i = 1, 2)$ and the function $x = (x_i)_{i=1}^2$ defined at the endpoints of the closed intervals $[a_i, b_i]$ $(i = 1, 2)$ by the continuity, satisfy the relation (3).

Consider now the general linear impulsive system

$$\frac{dz_i(t)}{dt} = P_{i1}(t) \cdot z_1(t) + P_{i2}(t) \cdot z_2(t) + \widetilde{q}_i(t) \ \ (i = 1, 2), \tag{13}$$

$$z_i(\tau_k+) - x_i(\tau_k-) =$$
$$= G_{i1}(k) \cdot z_1(\tau_k) + G_{i2}(k) \cdot z_2(\tau_k) + \widetilde{h}_i(k) \ \ (i = 1, 2; \ \ k = 1, 2, \ldots), \tag{14}$$

for the boundary value problem

$$\widetilde{l}_i(z_1, z_2) = c_i \ \ (i = 1, 2), \tag{15}$$

where $P_{ij} \in L_{loc}(]a, b[\,; \mathbb{R}^{n_i \times n_j})$, $\widetilde{q}_i \in L_{loc}(]a, b[\,; \mathbb{R}^{n_i})$, $G_{i,j} : \mathbb{N} \to \mathbb{R}^{n_i \times n_j}$, $\widetilde{h}_i : \mathbb{N} \to \mathbb{R}^{n_i}$, and $\widetilde{l}_i$ is the linear bounded operator $(i = 1, 2)$.

For the general system (13), (14), we assume that

$$\det(I_n + G_{ii}(\tau_k)) \neq 0 \ \ (i = 1, 2; \ \ k = 1, 2, \ldots).$$

Under this condition, there exists the fundamental matrix Y_i of the homogeneous system

$$\frac{dy_i(t)}{dt} = P_{i1}(t) \cdot y_1(t) + P_{i2}(t) \cdot y_2(t) \ \ (i = 1, 2),$$
$$y_i(\tau_k+) - y_i(\tau_k-) =$$
$$= G_{i1}(k) \cdot y_1(\tau_k) + G_{i2}(k) \cdot y_2(\tau_k) \ \ (i = 1, 2; \ \ k = 1, 2, \ldots),$$

satisfying the condition $Y_i(c) = I_{n_i}$ for every $i \in \{1, 2\}$ (see, for example, [10]).

Then it is not difficult to verify that the substitution $z_i(t) = Y_i(t)x_i(t)$ $(i = 1, 2)$ reduces the problem (13), (14); (15) to the problem (1), (2); (3), where

$$P_i(t) \equiv Y_i^{-1}(t)P_{i3-i}(t)Y_{3-i}(t), \ q_i(t) \equiv Y_i^{-1}(t)\widetilde{q}_i(t) \ \ (i = 1, 2);$$
$$G_i(k) \equiv Y_i^{-1}(\tau_k)(I_{n_i} + G_{ii}(k))^{-1}G_{i3-i}(k)Y_{3-i}(\tau_k) \ \ (i = 1, 2),$$
$$h_i(k) \equiv Y_i^{-1}(\tau_k)(I_{n_i} + G_{ii}(k))^{-1}\widetilde{h}_i(k) \ \ (i = 1, 2)$$

and

$$l_i(x_1, x_2) \equiv \widetilde{l}_i(Y_1 x_1, Y_2 x_2) \ \ (i = 1, 2).$$

2. Statement of the Main Results

Theorem 1. *Let $a_0 \in]a, b[$ and $b_0 \in]a_0, b[$, and let*

$$l_i : \mathrm{BV}([a, b], \mathbb{R}^{n_1}) \times \mathrm{BV}([a_0, b_0], \mathbb{R}^{n_2}) \to \mathbb{R}^{n_i} \ \ (i = 1, 2)$$

be linear bounded operators. (16)

In addition, suppose that

$$\int\limits_a^b \big(\|P_1(t)\| + \|q_1(t)\|\big)\,dt + \sum_{k=1}^{\infty} \big(\|G_1(k)\| + \|h_1(k)\|\big) < +\infty, \tag{17}$$

$$\|\mathcal{F}_0(|P_1|, |G_1|; |P_2|, |G_2|)(a+, b-)\| < +\infty, \tag{18}$$

$$\|\mathcal{F}_0(|P_1|, |G_1|; |q_2|, |h_2|)(a+, b-)\| < +\infty. \tag{19}$$

Then the problem (1), (2); (3) *is the Fredholm one, i.e., it is uniquely solvable if and only if the corresponding homogeneous problem* (1_0), (2_0); (3_0) *has only a trivial solution.*

Theorem 2. *Let* $b_0 \in]a, b[$ *and* $a_0 \in]a, b_0[$ *and let*

$$l_i : \mathrm{BV}([a, b_0], \mathbb{R}^{n_1}) \times \mathrm{BV}([a_0, b], \mathbb{R}^{n_2}) \to \mathbb{R}^{n_i} \quad (i = 1, 2)$$

be linear bounded operators. (20)

In addition, suppose that

$$\begin{aligned} &\int\limits_a^{a_0} \big(\|P_1(t)\| + \|q_1(t)\|\big)\,dt + \sum_{a<\tau_k<a_0} \big(\|G_1(k)\| + \|h_1(k)\|\big) < +\infty, \\ &\int\limits_{a_0}^{b} \big(\|P_2(t)\| + \|q_2(t)\|\big)\,dt + \sum_{a_0<\tau_k<b} \big(\|G_2(k)\| + \|h_2(k)\|\big) < +\infty; \end{aligned} \tag{21}$$

$$\Big\|\mathcal{F}_1(|P_1|, |G_1|; |P_2|, |G_2|)(a+, a_0)\| + \|\mathcal{F}_1(|P_1|, |G_1|; |q_2|, |h_2|)(a+, a_0)\Big\| < \infty, \tag{22}$$

$$\Big\|\mathcal{F}_2(|P_2|, |G_2|; |P_1|, |G_1|)(a_0, b-)\| + \|\mathcal{F}_2(|P_2|, |G_2|; |q_1|, |h_1|)(a_0, b-)\Big\| < \infty. \tag{23}$$

Then the assertion of Theorem 1 *is valid.*

Corollary 1. *Let either* $t_{1ik} \in [a, b]$, $t_{2ik} \in]a, b[$ $(i = 1, 2;\ k = 1, \dots, m)$ *and the conditions* (17)–(19) *be satisfied, or* $t_{1ik} \in [a, b[$, $t_{2ik} \in]a, b]$ $(i = 1, 2;\ k = 1, \dots, m)$ *and the conditions* (21)–(23) *be satisfied for some* $a_0 \in]a, b[$. *Then for the unique solvability of the problem* (1), (2); (4) *it is necessary and sufficient that the system* (1), (2) *under the homogeneous boundary condition*

$$\sum_{k=1}^{m} \big[B_{1ik} x_1(t_{1ik}) + B_{2ik} x_2(t_{2ik})\big] = 0 \quad (i = 1, 2) \tag{4_0}$$

has only a trivial solution.

Corollary 2. *Let* $P_i \in L([a,b];\mathbb{R}^{n_i\times n_{3-i}})$*,* $q_i \in L([a,b];\mathbb{R}^{n_i})$ *and*

$$\sum_{k=1}^{\infty}\big(\|G_i(k)\| + \|h_i(k)\|\big) < +\infty \ \ (i=1,2).$$

Let, moreover, either the condition (17) *or the condition* (21) *be fulfilled for, respectively, some* $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$ *or for some* $b_0 \in]a,b[$ *and* $a_0 \in]a,b_0[$*. Then the problem* (1), (2)*;* (3) *is the Fredholm one.*

Theorem 3. *Let the conditions* (16)–(19) *hold for some* $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$*. Let, moreover,* $\Delta \neq 0$*, where* Δ *is the determinant of the system* $l_i(c_1 + A(P_1,G_1)c_2, c_2) = 0$ $(i=1,2)$*, and the matrix-function is defined by* (1.9)*. Then there exists a positive number* ε_0*, independent of* P_i*,* G_i*,* q_i*,* h_i *and* c_i $(i=1,2)$*, such that the problem* (5_ε), (6_ε)*;* (3) *has one and only one solution for each* $\varepsilon \in [0,\varepsilon_0]$*.*

Theorem 4. *Let the conditions* (20)–(23) *hold for some* $b_0 \in]a,b[$ *and* $a_0 \in]a,b_0[$*. Let, moreover,* $\Delta_0 \neq 0$*, where* Δ_0 *is the determinant of the system* $l_i(c_1,c_2) = 0$ $(i=1,2)$*. Then there exists a positive number* ε_0 *independent of* P_i*,* G_i*,* q_i*,* h_i *and* c_i $(i=1,2)$ *such that the problem* (7_ε), (8_ε)*;* (3) *has one and only one solution for each* $\varepsilon \in [0,\varepsilon_0]$*.*

Finally, it should be noted that the vector-function $x = (x_i)_{i=1}^2$, with components $x_i \in \widetilde{C}_{loc}(]a,b[\setminus\{\tau_k\}_{k=1}^{\infty}, \mathbb{R}_i^n) \cap \mathrm{BV}([a,b];\mathbb{R}_i^n)$, is a solution of the impulsive system (1), (2) if and only if it is a solution of the generalized ordinary differential system

$$dx_i(t) = dA_i(t)\cdot x_{3-i}(t) + df_i(t) \ \ (i=1,2),$$

where $A_i(t) \equiv A(P_i,G_i)(t)$ and $f_i(t) \equiv A(q_i,h_i)(t)$ $(i=1,2)$, and the matrix- and vector-functions $A(P_i,G_i)$ $(i=1,2)$ and $A(q_i,h_i)$ $(i=1,2)$ are defined by (9).

Acknowledgement

This work is supported by the Georgian National Science Foundation (Project # GNSF/ST09_175_3-101).

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(Received 21.10.2010)

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