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T. S. Aleroev and H. T. Aleroeva

**SOME APPLICATIONS OF THE PERTURBATION
THEORY TO FRACTIONAL CALCULUS**

Abstract. Spectral analysis of a class of integral operators associated with fractional order differential equations arising in mechanics is carried out. The connection between the eigenvalues of these operators and zeros of Mittag–Leffler type functions is established. Sufficient conditions for complete non-self-adjointness and completeness of the systems of eigenfunctions are given.

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რეზიუმე. ნაშრომში ჩატარებულია წილადი რიგის დიფერენციალურ განტოლებებთან ასოცირებული ინტეგრალური ოპერატორების ერთი კლასის სპექტრალური ანალიზი. დადგენილია კავშირი ამ ოპერატორების საკუთრივ რიცხვებსა და მიტაგ–ლეფლერის ტიპის ფუნქციების ნულებს შორის. მოყვანილია საკუთრივ ფუნქციათა სისტემის არათვითმუქლუბელობის და სისრულის საკმარისი პირობები.

The spectral analysis of operators of the kind

$$A_{\gamma}^{[\alpha, \beta]} u(x) = c_{\alpha} \int_0^x (x-t)^{\frac{1}{\alpha}-1} u(t) dt + c_{\beta, \gamma} \int_0^1 x^{\frac{1}{\beta}-1} (1-t)^{\frac{1}{\gamma}-1} u(t) dt$$

has been carried out in [1]. Here $\alpha, \beta, \gamma, c_{\alpha}, c_{\beta, \gamma}$ are real numbers, while α, β, γ are positive ones (similar operators were considered by G. M. Gubreev in [2]). These operators arise in studying boundary value problems for differential equations of fractional order (see [3] and references therein).

In particular, it is shown in [1] that the operator

$$\begin{aligned} A^{\rho} u(x) &= A_{\rho}^{[\rho, \rho]} u(x) = \\ &= \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{\frac{1}{\rho}-1} u(x) dt - \frac{1}{\Gamma(\rho^{-1})} \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t) dt \end{aligned}$$

is almost non-self-conjugate ([4] if $\rho > 1$), and if $0 < \rho < 1$, the kernel of the operator A^{ρ} coincides with the set of roots of the entire Mittag-Leffler type function

$$E_{\rho}(\lambda; \rho^{-1}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\rho^{-1} + k\rho^{-1})}.$$

Thus it follows from the above that all eigenvalues of the operator A^{ρ} are complex if $\rho > 1$, while if $0 < \rho < 1$ the operator A^{ρ} has real eigenvalues (in fact, if $\frac{1}{2} < \rho < 1$, then the set of real eigenvalues is finite). All zeros of the function $E_{\rho}(\lambda; \rho^{-1})$ are complex if $\rho > 1$, and if $0 < \rho < 1$ the function $E_{\rho}(\lambda; \rho^{-1})$ has real zeros. This confirms the hypothesis about the existence of real zeros of the function $E_{\rho}(\lambda; \rho^{-1})$ if $\frac{1}{2} < \rho < 1$, as is stated in the monograph ([5], p. 248).

The given paper is also devoted to investigation of boundary value problems for differential equations of fractional order and associated integral operators of the kind $A_{\gamma}^{[\alpha, \beta]}$.

To state the corresponding problems, we use some concepts from the fractional calculus.

Let $f(x) \in L_1(0, 1)$. Then the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \in L_1(0, 1)$$

is called the fractional integral of order $\alpha > 0$ with the origin at the point $x = 0$ [6] while the function

$$\frac{d^{-\alpha}}{d(1-x)^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1} f(t) dt \in L_1(0, 1)$$

is called the fractional integral of order $\alpha > 0$ with the end at the point $x = 1$ [6]. Here $\Gamma(\alpha)$ is Euler's gamma-function. In case $\alpha = 0$, it is natural

to put both fractional integrals equal to the function $f(x)$. As is known [6], the function $g(x) \in L_1(0, 1)$ is called a fractional derivative of the function $f(x) \in L_1(0, 1)$ of order $\alpha > 0$ with the origin at the point $x = 0$, if

$$f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} g(x).$$

Denote

$$g(x) = \frac{d^\alpha}{dx^\alpha} f(x).$$

In what follows, under the symbol $\frac{d^\alpha}{dx^\alpha}$, we will mean the fractional integral for $\alpha < 0$ and the fractional derivative for $\alpha > 0$. The fractional derivative $\frac{d^\alpha}{d(1-x)^\alpha}$ of the function $f(x) \in L_1(0, 1)$ of order $\alpha > 0$, with the end at the point $x = 1$ is defined similarly.

Let $\{\gamma_k\}_0^n$ be an arbitrary set of real numbers satisfying the condition $0 < \gamma_j \leq 1$ ($j = 0, 1, \dots, n$). Denote

$$\sigma_k = \sum_{j=0}^k \mu_j = \sigma_k + 1 = \sum_{j=0}^k \gamma_j \quad (k = 0, 1, \dots, n)$$

and assume that

$$\frac{1}{\rho} = \sum_{j=0}^n \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following [6], we introduce the differential operators

$$D^{(\sigma_0)} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x)$$

of, generally speaking, fractional order:

$$D^{(\sigma_1)} f(x) \equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),$$

$$D^{(\sigma_n)} f(x) \equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \cdots \frac{d^{\gamma_0}}{dx^{\gamma_0}}.$$

Note that if $\gamma_0 = \gamma_1 = \dots = \gamma_n = 1$, then it is obvious that

$$D^{(\sigma_k)} f(x) = f^{(k)}(x) \quad (k = 0, 1, 2, \dots, n).$$

Now,

a) a two-point Dirichlet boundary value problem

$$u(0) = 0, \quad u(1) = 0, \tag{1}$$

for the fractional oscillatory equation

$$u'' + \lambda D_{0x}^\alpha u = 0, \tag{2}$$

(D_{0x}^α is the operator of fractional differentiation of order $0 < \alpha < 1$ [2]) investigated by many authors is equivalent to the equation [3], [4]

$$u(x) + \frac{\lambda}{\Gamma(\rho-1)} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1} u(t) dt - \int_0^1 x(1-t)^{\frac{1}{\rho}-1} u(t) dt = 0, \quad (*) \right.$$

$$\left. \frac{1}{\rho} = 2 - \alpha, \quad \alpha = 2 - \frac{1}{\rho}. \right.$$

b) the operator, inverse to the operator generated by the differential expression

$$lu = \frac{1}{\Gamma(1-\gamma)} \frac{d^{n-1}}{dx^{n-1}} \int_0^x \frac{u'(t)}{(x-t)^\gamma} dt,$$

$0 < \gamma < 1$, and the natural boundary conditions

$$u(0) = 0, \quad u(1) = 0, \quad D^{(\sigma_1)}u \Big|_{t=0} = 0, \dots, D^{(\sigma_{n-2})}u \Big|_{t=0} = 0$$

can be represented in the form

$$A_\rho^{[\rho, \rho]} u(x) = \frac{1}{\Gamma(\rho-1)} \int_0^x (x-t)^{\frac{1}{\rho}-1} u(t) dt - \frac{1}{\Gamma(\rho-1)} \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t) dt$$

$$\left(\frac{1}{\rho} = n - 1 + \gamma, \quad n = 1, 2, 3, \dots \right).$$

A remarkable work of M. K. Gubreev in which similar operators are studied is worth mentioning. In [4], the operator $A_\rho^{[\rho, \rho]}$ (*) is investigated by methods of the perturbation theory. Assume that the operator

$$Au = \int_0^x (x-t)u(t) dt - \int_0^1 x(1-t)u(t) dt$$

is non-perturbed, and the operator

$$\tilde{A}u = \frac{1}{\Gamma(\rho-1)} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1} u(t) dt - \int_0^1 x(1-t)^{\frac{1}{\rho}-1} u(t) dt \right]$$

is perturbed. Towards this end, the use was made of the concept of opening between the closed operators. In [4], no explicit expression for the perturbation of the operator A was written. In the present work we obtain the corresponding expression and it seems to the authors that the present work will become a factor allowing one to insert the theory of differential equations of fractional order in a general scheme of the theory of perturbations [6], [7].

Thus using the methods of the theory of perturbations in $L_1(0, 1)$, we study the operator

$$A_\varepsilon u = \int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x(1-t)^{1+\varepsilon} u(t) dt.$$

Theorem 1. *The representation*

$$A_\varepsilon u = A + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^n A_n + \cdots \quad (\varepsilon > 0)$$

holds, where

$$Au = \int_0^x (x-t)u(t) dt - \int_0^1 x(1-t)u(t) dt,$$

$$A_n u = \frac{1}{n!} \left[\int_0^x (x-t)[\ln(x-t)]^n dt - \int_0^1 x(1-t)[\ln(x-t)]^n dt \right]$$

are operators with power-logarithmic kernels.

Proof. We rewrite the operator A_ε as follows:

$$A_\varepsilon u = M_\varepsilon u + N_\varepsilon u,$$

where

$$M_\varepsilon u = \int_0^x K_\varepsilon(x, t)u(t) dt,$$

$$K_\varepsilon(x, t) = \begin{cases} (x-t)^{1+\varepsilon}, & t < x \\ 0, & t \geq x \end{cases}$$

and

$$N_\varepsilon = \int_0^1 \tilde{K}_\varepsilon(x; t)u(t) dt,$$

$$\tilde{K}_\varepsilon(x, t) = \begin{cases} x(1-t)^{1+\varepsilon}, & t \neq 1 \\ 0, & t = 1 \end{cases}.$$

Find the operator A_ε as follows:

$$(A - A_\varepsilon)u = (M - M_\varepsilon)u - (N - N_\varepsilon)u.$$

First, we find $(M - M_\varepsilon)u$. Obviously,

$$(M - M_\varepsilon)u = \int_0^1 [K(x, t) - K_\varepsilon(x, t)]u(t) dt.$$

Since

$$[K(x, t) - K_\varepsilon(x, t)] = \begin{cases} (x - t) - (x - t)^{1+\varepsilon}, & t < x \\ 0, & t \geq x \end{cases},$$

we have

$$K(x, t) - K_\varepsilon(x, t) = \begin{cases} (x - t)[1 - (x - t)^\varepsilon], & t < x \\ 0, & t \geq x \end{cases}.$$

As far as

$$a^x = 1 + \frac{x}{1!} + \frac{(x)^2}{2!} + \dots + \frac{(x)^n}{n!} + \dots \quad (a > 0, \quad x < \infty),$$

we find

$$\begin{aligned} & K(x, t) - K_\varepsilon(x, t) = \\ & = \begin{cases} (x-t) \left[\varepsilon \frac{\ln(x-t)}{1!} + \varepsilon^2 \frac{(\ln(x-t))^2}{2!} + \dots + \varepsilon^n \frac{(\ln(x-t))^n}{n!} + \dots \right], & t < x \\ 0, & t \geq x \end{cases}. \end{aligned}$$

From the last expression we find that

$$(M - M_\varepsilon)u = \varepsilon \int_0^1 K_1(x, t)u(t) dt + \dots + \varepsilon^n \int_0^1 K_n(x, t)u(t) dt + \dots,$$

where

$$K_n(x, t) = \begin{cases} \frac{(x-t)[\ln(x-t)]^n}{n!}, & t < x \\ 0, & t \geq x \end{cases}.$$

It follows from (4) that

$$M_\varepsilon = \int_0^1 K(x, t)u(t) dt - \varepsilon \int_0^1 K_1(x, t)u(t) dt - \dots - \varepsilon^n \int_0^1 K_n(x, t)u(t) dt.$$

In the same way we can obtain the representation

$$N_\varepsilon u = \int_0^1 \tilde{K}(x, t)u(t) dt - \varepsilon \int_0^1 \tilde{K}_1(x, t)u(t) dt - \dots - \varepsilon^n \int_0^1 \tilde{K}_n(x, t)u(t) dt + \dots,$$

where

$$\tilde{K}_n(x, t) = \begin{cases} \frac{x(1-t)[\ln(x-t)]^n}{n!}, & t < 1 \\ 0, & t = 1 \end{cases}$$

which was to be proved. □

Theorem 2. All eigenvalues $\lambda_n(\varepsilon)$ of the operator $A(\varepsilon)$ are real.

Proof. We have

$$\lambda_n(\varepsilon) = \pi n^2 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots, \quad (3)$$

$$\varphi_n(\varepsilon) = \sin nx + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \cdots, \quad (4)$$

where

$$\lambda_n = \sum_{k=1}^n (A_k \varphi_{n-k}, \sin nx), \quad (5)$$

$$\varphi_n = R \sum_{k=1}^n (\lambda_k - A_k) \varphi_{n-k}. \quad (6)$$

Here, R is the reduced resolvent of the operator A corresponding to the eigenvalue πn^2 . This resolvent is an integral operator with the kernel

$$S(x, y) = \left[-\frac{y}{n} \cos ny \sin nx + \frac{1-x}{n} \sin ny \cos nx + \frac{1}{2n^2} \sin ny \sin nx \right], \quad y \leq x$$

(if $y > x$, in the middle part of this expression y and x should be interchanged).

Clearly, R is a one-to-one transformation of H_0 into itself which annuls $\sin \pi nx$ (H_0 is the orthogonal complement of the function $\sin \pi nx$).

It follows from [8] that

$$\lambda_1 = (A_1 \sin nx, \sin x),$$

because the kernel of the operator A_1 , and $\sin \lambda_1 = 0$ is real-valued.

Next, from (6) it follows

$$\varphi_1 = R(nk^2 - A_1) \sin nx,$$

and the kernels of the operators R and A_1 are real-valued. Hence $Im \varphi_1 = 0$. Consequently, we can successively prove that all λ_i are real, and since ε is real, therefore $\lambda u(\varepsilon)$ is real too. \square

Theorem 3. For the eigenvalues $\lambda_n(x)$ and eigenfunctions $\varphi_n(\varepsilon)$ of the operator $A(\varepsilon)$ the estimates

$$\begin{aligned} |\lambda_n(\varepsilon) - \pi n^2| &< \frac{\pi(2n-1)}{2}, \\ |\varphi_n(\varepsilon) - \sin nx| &< \frac{1}{2} \end{aligned}$$

hold.

Proof. From (5) and (6), under the assumption that

$$\|A_n u\| \leq p^{n-1} \{u\|u\| + b\|A_0 u\|\}, \quad m = \|A_0\|, \quad \frac{1}{d} = \|R\|, \quad |\varepsilon| < \frac{1}{c},$$

where

$$c = \max \left\{ \frac{8(a+mb)}{d}, 8p + 4\frac{a+mb}{d} \right\},$$

we obtain the following simple formulas [8]:

$$|\lambda(\varepsilon) - \lambda_0 - \varepsilon\lambda_1 - \varepsilon^n\lambda_n| \leq \frac{d}{2} (|\varepsilon|c)^{n+1}, \quad (7)$$

$$|\varphi(\varepsilon) - \varphi_0 - \varepsilon\varphi_1 - \varepsilon^n\varphi_n| \leq \frac{1}{2} (|\varepsilon|c)^{n+1}. \quad (8)$$

Let us calculate the values of the parameters a, b, c, d, m . First, we find m .

$m = \|A\| = \sup A$, where $\sup A$ is the spectral radius of the operator A . Since

$$\sup A = \pi^{-1},$$

we have $m = \pi^{-1}$. Further, we find d .

$$d = \text{dist}(\pi n^2; \Sigma'')$$

(d is an isolating distance). Here Σ'' is the spectrum of the operator A^{-1} with the unique excluded point πn^2 . Clearly, $d = \pi(2n - 1)$. To find the remaining parameters a, b, c , we have to estimate the norm of the operator A_n ,

$$\|A_n\varphi\|_{L(0,1)} \leq \int_0^1 \int_0^1 |K_n(x,t)| |\varphi(t)| dt dx.$$

Here, $K_n(x, t)$ is the kernel of the operator A_n ,

$$\begin{aligned} \int_0^1 \int_0^1 |K_n(x,t)| |\varphi(t)| dt dx &= \int_0^1 \int_0^x \left| \frac{(x-t)[\ln(x-t)]^n}{n!} \right| |\varphi(t)| dt dx = \\ &= \int_0^1 |\varphi(t)| \int_0^{1-t} \frac{z|\ln z|^n}{n!} dz dt \leq \frac{1}{n!} \int_0^1 z|\ln z|^n dz \|\varphi\|_{L_1(0,1)}. \end{aligned}$$

We now calculate the integral $\int z \ln z^n dz$. It is equal to

$$\begin{aligned} &\int z \ln z^n dz = \\ &= \frac{z^2(\ln z)^n}{2} - \frac{nx^2(\ln x)^{n-1}}{2^2} + \dots + \frac{(-1)^{n-1}n(n-1)(n-2)\dots 2}{2^{n-1}} \left[\frac{x^2}{2} - \frac{1}{2^2} \right], \end{aligned}$$

whence $\|A_n\| \leq \frac{1}{2^{n+1}}$. It is now obvious that we can take $\alpha = \frac{1}{4}$, $p = \frac{1}{2}$, and $b = 0$. Since

$$c = \max \left\{ 8 \frac{a+mb}{d} + 8p + 4 \frac{a+mb}{d} \right\},$$

we have

$$c = \max \left\{ \frac{2}{d}; 4 + \frac{1}{d} \right\} = 4 + \frac{1}{d} = 5.$$

Thus

$$|\lambda(\varepsilon) - \lambda| \leq \frac{1}{2} \frac{\pi(2n-1)}{2}.$$

In a similar way we find

$$|\varphi_\varepsilon - \varphi_0| \leq \frac{1}{2},$$

which proves Theorem 3. □

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Authors' addresses:

T. S. Aleroev
 Moscow Institute of a Municipal Services and Construction
 Faculty of Higher Mathematics
 Russia
 The Academy of National Economy
 under the Government of the Russian Federation
 Moscow 119571, Russia
 E-mail: aleroev@mail.ru
 Moscow Institute of Municipal Economy and Construction

H. T. Aleroeva
 Moscow Technical University of Communications and Informatics
 Russia
 E-mail: BinSabanur@gmail.com