Mem. Differential Equations Math. Phys. 5(1995), 117–118

M. Ashordia

ON THE STABILITY OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

(Reported on April 10 and 17, 1995)

Let $A=(a_{ij})_{i,j=1}^{n}\in BV_{n\times n}(a,b)$, $f=(f_i)_{i=1}^{n}\in K_n(a,b;A)$, and let $h: BV_n(a,b)\to R^n$ be a continuous operator such that the nonlinear boundary value problem

$$dx(t)=dA(t)\cdot f(t,x(t)),\quad h(x)=0 \tag{1}$$

has a solution x°.

Consider sequences $A_k\in BV_{n\times n}(a,b)$ $(k=1,2,\dots)$ and $f_k\in K_n(a,b;A_k)$ $(k=1,2\dots)$ and a sequence of continuous operators $h_k: BV_n(a,b)\to R^n$ $(k=1,2,\dots)$.

In this paper, sufficient conditions are given guaranteeing both the solvability of the problem

$$dx(t)=dA_k(t)\cdot f_k(t,x(t)),\quad h_k(x)=0 \tag{1_k}$$

for any sufficiently large k and the convergence of its solutions as $k\to\infty$ to the solution x° of the problem (1).

We use the following notation and definitions: $R=]-\infty,\infty[$, $R_+=[0,\infty[$; $R^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\{\sum\limits_{i=1}^{n}|x_{ij}|: j=1,\dots,m\}$; $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+=(|X|+X)/2$; I is the identity $n\times n$-matrix; $R^n=R^{n\times 1}$ is the space of all matrix-functions $X=(x_{ij})_{i,j=1}^{n,m}:[a,b]\to R^{n\times m}$ such that $\operatorname{var}_a^b x_{ij}<+\infty$ with the norm $\|X\|_s=\sup\{\|X(t)\|: t\in[a,b]\}$; $V(X)(t)\equiv(\operatorname{var}_a^t x_{ij})_{i,j=1}^{n,m}$, $d_1X(t)=X(t)-X(t-0)$, $d_2X(t)=X(t+0)-X(t)$. $U(y,r)=\{x\in BV_n(a,b):\|x-y\|_s<r\}$; $D(y,r)$ is the set of all $x\in R^n$ such that $\inf\{\|x-y(\tau)\|:\tau\in[a,b]\}<r$.

If $g\in BV_1(a,b)$, $x:[a,b]\to R$ and $a\le s<t\le b$, then $\int\limits_s^t x(\tau)dg(\tau)=\int\limits_{]s,t[} x(\tau)dg(\tau)+x(t)d_1g(t)+x(s)d_2g(s)$, where $\int\limits_{]s,t[} x(\tau)dg(\tau)$ is the Lebesque–Stieltjes integral over the open interval $]s,t[$. If $G(t)=(g_{ij}(t))_{i,j=1}^{l,n}$ and $X(t)=(x_{jk}(t))_{j,k=1}^{n,m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau)=\Big(\sum_{j=1}^{n}\int\limits_s^t x_{jk}(\tau)dg_{ij}(\tau)\Big)_{i,k=1}^{l,m}.$$

$L_{n\times m}(a,b;G)$ is the set of all matrix-functions $(x_{jk}(t))_{j,k=1}^{n,m}$ such that x_{jk} is integrable with respect to g_{ij} $(i=1,\dots,l)$. $K_{n\times m}(a,b;G)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{jk})_{j,k=1}^{n,m}:[a,b]\times R^n\to R^{n\times m}$ such that (a) $f_{ik}(\cdot,x)$ is measurable

---

1991 *Mathematics Subject Classification.* 34B15.

Key words and phrases. System of generalized ordinary differential equations, nonlinear boundary value problem, stability.

with respect to the measures $V(g_{ij})$ and $V(g_{ij})-g_{ij}$ for $x\in R^n$ $(i=1,\dots,l)$; (b) $F(t,\cdot)$ is continuous for $t\in[a,b]$ and $\sup\{|F(\cdot,x)|:x\in D\}\in L_{n\times m}(a,b;G)$ for every compact $D\subset R^n$.

Inequalities between the matrices are understood to be componentwise. If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ is called positively homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

A vector-function $x\in BV_n(a,b)$ is said to be a solution of the problem (1) if $h(x)=0$ and $x(t)=x(s)+\int\limits_s^t dA(\tau)\cdot x(\tau)$ for $a\le s\le t\le b$.

Let $\ell:BV_n(a,b)\to R^n$ be a linear continuous operator and let $\ell_0:BV_n(a,b)\to R^n_+$ be a positively homogeneous continuous operator. We say that a matrix-function $P\in K_{n\times n}(a,b;A)$ satisfies the Opial condition with respect to the triplet $(\ell,\ell_0;A)$ if (a) there exists a matrix-function $\Phi\in L_{n\times n}(a,b;A)$ such that $|P(t,x)|\le\Phi(t)$ on $[a,b]\times R^n$; b) $\det(I+(-1)^j d_jB(t))\ne 0$ for $t\in[a,b]$ $(j=1,2)$ and the problem $dx(t)=dB(t)\cdot x(t)$, $|\ell(x)|\le\ell_0(x)$ has only trivial solution for every $B\in BV_{n\times n}(a,b)$ for which there exists a sequence $y_k\in BV_n(a,b)$ $(k=1,2,\dots)$ such that $\lim\limits_{k\to\infty}\int\limits_a^t dA(\tau)\cdot P(\tau,y_k(\tau))=B(t)$ uniformly on $[a,b]$.

x° is said to be strongly isolated in the radius r if there exist $P\in K_{n\times n}(a,b;A)$, $q\in K_n(a,b;A)$, a linear continuous operator $\ell:BV_n(a,b)\to R^n$ and a positively homogeneous operator $\widetilde{\ell}:BV_n(a,b)\to R^n$ such that (a) $f(t,x)=P(t,x)x+q(t,x)$ for $t\in[a,b]$, $\|x-x^\circ(t)\|<r$ and $h(x)=\ell(x)+\widetilde{\ell}(x)$ is fulfilled on $U(x^\circ;r)$; (b) the vector-functions $\alpha(t,\rho)=\max\{|q(t,x)|:\|x\|\le\rho\}$ and $\beta(\rho)=\sup\{[|\widetilde{\ell}(x)|-\ell_0(x)]_+:\|x\|_s\le\rho\}$ satisfy $\lim\limits_{\rho\to\infty}\frac{1}{\rho}\int\limits_a^b dV(A)(t)\cdot\alpha(t,\rho)=0$, $\lim\limits_{\rho\to\infty}\beta(\rho)/\rho=0$; (c) the problem $dx(t)=dA(t)\cdot[P(t,x(t))x(t)+q(t,x(t))]$, $\ell(x)+\widetilde{\ell}(x)=0$ has no solution different from x°; (d) P satisfies the Opial condition with respect to the triplet $(\ell,\ell_0;A)$.

The notation $((A_k,f_k,h_k))_{k=1}^\infty\in W_r(A,f,h;x^\circ)$ means that (a) $\lim\limits_{k\to\infty}\int\limits_a^t dA_k(\tau)\times f_k(\tau,x)=\int\limits_a^t dA(\tau)\cdot f(\tau,x)$ uniformly on $[a,b]$ for every $x\in D(x^\circ;r)$; (b) $\lim\limits_{k\to\infty}h_k(x)=h(x)$ uniformly on $U(x^\circ;r)$; (c) $\lim\limits_{s\to 0+}\sup\{\|\int\limits_a^b dV(A_k)(t)\cdot\omega_k(t,s)\|:k=1,2,\dots\}=0$ on $[a,b]\times D(x^\circ;r)$, where $\omega_k(t,s)=\max\{|f_k(t,x)-f_k(t,y)|:\|x\|,\|y\|\le\|x^\circ\|_s+r;\ \|x-y\|\le s\}$.

The problem (1) is said to be $(x^\circ;r)$-correct if for every $\varepsilon\in]0,r[$ and $((A_k,f_k,h_k))_{k=1}^\infty\in W_r(A,f,h;x^\circ)$ there exists a natural number k_0 such that the problem (1_k) has at least one solution contained in $U(x^\circ;r)$, and any such solution belongs to $U(x^\circ;r)$ for any $k\ge k_0$.

Theorem. *If the problem* (1) *has a solution x° which is strongly isolated in the radius r, then it is $(x^\circ;r)$-correct.*

Similar results for ordinary differential equations can be found in [1].

References

1. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current Problems in Mathematics. Newest Results, vol.* 30 (Russian), 3–103, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vses. Inst. Nauchn. i Tekh. Inform., Moscow*, 1987.

Author's address:
A. Razmadze Mathematical Institute, Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093, Georgia