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PARTIAL DIFFERENTIAL EQUATIONS ON HYPERSURFACES

Dedicated to Mikheil Basheleishvili
on the occasion of his 80-th birthday anniversary

Abstract. We propose an approach which allows global representation of basic differential operators (such as Laplace–Beltrami, Hodge–Laplacian, Lamé, Navier–Stokes, etc.) and of corresponding boundary value problems on a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, in terms of the standard spatial coordinates in $\mathbb{R}^n$. The tools we develop also provide, in some important cases, useful simplifications as well as new interpretations of classical operators and equations.

The obtained results are applied to the Dirichlet and Neumann boundary value problems for the Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ and to the system of anisotropic elasticity on an open smooth hypersurface $\mathscr{C} \subset \mathscr{S}$ with the smooth boundary $\Gamma := \partial\mathscr{C}$. We prove the solvability theorems for the Dirichlet and Neumann BVPs on open hypersurfaces in the Bessel potential spaces.

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1. Introduction

The purpose of this work, which is based on the joint paper with D. Mitrea & M. Mitrea [16], is to provide a (relatively) simple calculus of Boundary value problems (BVP's) for partial differential equations (PDE's) on hypersurfaces in $\mathbb{R}^n$. Such BVPs arise in a variety of situations and have many practical applications. See, for example, [21, § 72] for the heat conduction by surfaces, [4, § 10] for the equations of surface flow, [8], [3] for the vacuum Einstein equations describing gravitational fields, [38] for the Navier-Stokes equations on spherical domains, as well as the references therein.

A hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ has the natural structure of a $(n-1)$-dimensional Riemannian manifold and the aforementioned PDE's are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathscr{S}$ such as curvature. Inherently, these PDE's are originally written in local coordinates, intrinsic to the manifold structure of $\mathscr{S}$.

The main aim of this paper is to demonstrate the approach which allows representation of the most basic partial differential operators (PDO's), as well as their associated boundary value problems, on a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, in global form, in terms of the standard spatial coordinates in $\mathbb{R}^n$. It turns out that a convenient way to carry out this program is by employing the the so-called Günter derivatives-the column of surface gradient

$$\mathscr{D} := (\mathscr{D}_1, \mathscr{D}_2, \dots, \mathscr{D}_n)^\top \tag{1.1}$$

(cf. [20], [23], [13]). Here, for each $1 \leq j \leq n$, the first-order differential operator $\mathscr{D}_j$ is the directional derivative along $\pi\, e_j$, where $\pi : \mathbb{R}^n \to T\mathscr{S}$ is the orthogonal projection onto the tangent plane to $\mathscr{S}$ and, as usual, $e_j = (\delta_{jk})_{1\leq k\leq n} \in \mathbb{R}^n$, with δ_{jk} denoting the Kronecker symbol.

The operator $\mathscr{D}$ is globally defined on (as well as tangential to) $\mathscr{S}$, and has a relatively simple structure. In terms of (1.1), the Laplace–Beltrami operator on $\mathscr{S}$ simply becomes (see [26, pp. 2ff and p. 8])

$$\Delta_{\mathscr{S}} = \mathscr{D}^* \mathscr{D} \quad \text{on} \quad \mathscr{S}. \tag{1.2}$$

Alternatively, this is the natural operator associated with the Euler–Lagrange equations for the variational integral

$$\mathscr{E}[u] = -\frac{1}{2} \int\limits_{\mathscr{S}} \|\mathscr{D}u\|^2 \, dS. \tag{1.3}$$

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy, leads to the derivation of the equation for the elastic hypersurface (cf. [16, 15] for the isotropic case).

These results are useful in numerical and engineering applications (cf. [2], [5], [7], [10], [12], [6], [34]) and we plan to treat a number of special surfaces in greater detail in a subsequent publication.

The layout of the paper is as follows. In § 2–§ 3 we review some basic differential-geometric concepts which are relevant for the work at hand (e.g., hypersurfaces and different methods of their identification). In § 4–§ 5 we identify the most important partial differential operators on hypersurfaces, such as gradient, divergence, Laplace–Beltrami operator. In § 5, starting from first principles, we identify the natural operator of anisotropic elasticity on a general (elastic, linear) hypersurface $\mathscr{S}$ (see [16] for the isotropic Lamé operator). Our approach is based on variational methods.

In § 7, § 8 we study the Dirichlet and Neumann boundary value problems (BVPs) on an open hypersurface. We apply two approaches-the functional-analytic based on the Lax–Milgram Lemma, which requires less smoothness of the underlying hypersurface, and the potential method, which appliues the fundamental solution and imposes the condition of infinite smoothness on the hypersurface, also allows investigation of the equivalent boundary pseudodifferential equations in the scale of Bessel potential spaces $\mathbb{H}^s_p(\Gamma)$, where $|s| \leq \ell$ and $1 < p < \infty$, provided the boundary $\Gamma := \partial\mathscr{S}$ is ℓ-smooth.

The same project is carried out in § 9-§ 12 for the equations of anisotropic elasticity and we study the Dirichlet and Neumann BVPs for them on an open hypersurface.

2. Brief Review of the Classical Theory of Hypersurfaces

The next definition of a hypersurface is basic in the present chapter and we give two further definitions later. The alternative definitions are very useful treating various problems and later, in Lemma 2.5, we prove equivalence of all three definitions.

The next definition is most universal and can be used for manifolds.

Definition 2.1. A Subset $\mathscr{S} \subset \mathbb{R}^n$ of the Euclidean space is called a **hypersurface** if it has a covering $\mathscr{S} = \bigcup_{j=1}^{M} \mathscr{S}_j$ and coordinate mappings

$$\Theta_j :\ \omega_j \to \mathscr{S}_j := \Theta_j(\omega_j) \subset \mathbb{R}^n, \quad \omega_j \subset \mathbb{R}^{n-1}, \ \ j = 1, \dots, M, \tag{2.1}$$

such that the corresponding differentials

$$D\Theta_j(p) := \operatorname{matr}\left[\partial_1\Theta_j(p), \dots, \partial_{n-1}\Theta_j(p)\right], \tag{2.2}$$

have the full rank

$$\operatorname{rank} D\Theta_j(p) = n - 1, \ \ \forall\, p \in Y_j, \ \ k = 1, \dots, n, \ \ j = 1, \dots, M,$$

i.e., all points of ω_j are regular for Θ_j for all $j = 1, \dots, M$.

Such mapping is called an **immersion** as well.

The hypersurface is called **smooth** if the corresponding coordinate diffeomorphisms Θ_j in (2.1) are smooth (C^∞-smooth). Similarly is defined a **μ-smooth** hypersurface.

Next we expose yet another definition of a hypersurface. Definition 2.1 is a particular (canonical) case of a hypograph surface represented by a single coordinate function $M = 1$, while Definition 2.2 deals with a general hypersurface.

Definition 2.2. An open subset

$$\Omega_\Phi = \Big\{p = (p', p_n) \in \mathbb{R}^n : \ p' \in \mathbb{R}^{n-1}, \ p_n \in \mathbb{R}, \ p_n < \Phi(p')\Big\}. \tag{2.3}$$

in the Euclidean space $\mathbb{R}^n$, generated by a real-valued function $\Phi : \ \mathbb{R}^{n-1} \to \mathbb{R}$, is called a **hypograph domain**.

The boundary

$$\mathscr{S}_\Phi = \Big\{z \in \mathbb{R}^n : \ z = (p', \Phi(p')), \ p' \in \omega \subset \mathbb{R}^{n-1}\Big\} \tag{2.4}$$

of a hypograph domain Ω_Φ is called a **hypograph surface**. If Φ is μ-smooth, $\mathscr{S}$ is referred to a μ-smooth hypersurface.

If Φ is a Lipschitz continuous

$$\big|\Phi(p') - \Phi(q')\big| \leq L|p' - q'|, \ \ p', q' \in \mathbb{R}^{n-1}. \tag{2.5}$$

$\mathscr{S}$ is referred to as a Lipschitz hypersurface.

Definition 2.3. An open subset $\Omega \subset \mathbb{R}^n$ (compact or with outlets at infinity) is called a **domain with smooth boundary** (with a μ-smooth or with the Lipschitz boundary) if there exists a finite family of open sets $\big\{\Omega_j\big\}_{j=1}^N$ such that:

i. each Ω_j, $j = 1, \ldots, N$ can be transformed into a hypograph domain by an **affine transformation**, i.e., by a rotation and a translation;

ii. $\Omega = \bigcap\limits_{j=1}^N \Omega_j$ and $\partial\Omega \subset \bigcap\limits_{j=1}^N \partial\Omega_j$.

The C^k-smooth (the Lipschitz) boundary $\mathscr{S} := \partial\Omega$ of a hypograph domain $\Omega \subset \mathbb{R}^n$ is called a **hypograph surface**.

The third definition of a hypersurface is **implicit**.

Definition 2.4. Let $k \geq 1$ an $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth (an implicit Lipschitz) hypersurface in $\mathbb{R}^n$ is defined as the set

$$\mathscr{S} = \big\{\mathscr{X} \in \omega : \ \Psi_{\mathscr{S}}(\mathscr{X}) = 0\big\}, \tag{2.6}$$

where $\Psi_{\mathscr{S}} : \ \omega \to \mathbb{R}$ is a C^k-mapping (or is a Lipschitz mapping) which is regular $\nabla \Psi(\mathscr{X}) \neq 0$.

Note, that by taking a single function $\Psi_{\mathscr{S}}$ for the implicit definition of a hypersurface $\mathscr{S}$ we does not restrict the generality: if

$$\mathscr{S} = \bigcup_{j=1}^M \mathscr{S}_j, \ \text{ and } \ \mathscr{S}_j = \big\{\mathscr{X} \in \omega_j \subset \mathbb{R}^n : \ \Psi_j(\mathscr{X}) = 0\big\},$$

we pick up a partition of unity $\{\psi_j\}_{j=1}^M$ subordinated to the covering $\{\omega_j\}_{j=1}^M$. The surface $\mathscr{S}$ is then represented by formula (2.6) and a single implicit function

$$\Psi_{\mathscr{S}} := \sum_{j=1}^M \psi_j \Psi_j. \tag{2.7}$$

Lemma 2.5. *Definition* 2.1*, Definition* 2.3 *and Definition* 2.4 *of a hypersurface* $\mathscr{S}$ *are all equivalent.*

Proof. Let us fix an arbitrary point $p \in \mathscr{S} = \partial\Omega$ at the boundary. According to Definition 2.3 locally, after an affine transformation, which brings p to the origin $p = 0$ and the tangential surface at p to the hyperplane $p_n = 0$, a neighborhood $\mathscr{S}_j \subset \mathscr{S}$ of the point p is given by the surface equation $\mathscr{S}_j = \{p_n = \Phi_j(p') : \ p' \in \Omega_j \subset \mathbb{R}^{n-1}\}$. Thus, modulo an affine transformation, $\mathscr{S}_j = \big\{(x', \Phi(x')) : \ x' \in \Omega_j \subset \mathbb{R}^{n-1}\big\}$ represents the image of the mapping $\Theta_j(\cdot) = (\cdot, \Phi(\cdot)) : \ \Omega_j \mapsto \mathscr{S}_j \subset \mathscr{S}$ and, for some integer $M \in \mathbb{N}$, $\mathscr{S} = \bigcup\limits_{j=1}^{M} \mathscr{S}_j$ is a hypersurface according to Definition 2.1.

Vice versa, let a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ be given by the definition 2.1. Fixing arbitrary point $p \in \mathscr{S}$ we recall that the Jacoby matrix $D\Theta_j = \nabla\Theta_j$ of the coordinate diffeomorphism has rank $n-1$. We choose a non-degenerate $(n-1)\times(n-1)$ minor among n minors of $D\Theta_j(p_1, \dots, p_n)$ and let g^k_j be the distinguished component of the vector-function $\Theta_j = (g^1_j, \dots, g^n_j)^\top$ not present in this minor. Due to the implicit function theorem (cf., e.g., [37, V. I]) there exists a small neighborhood ω_j of $p = 0$ and the implicit function $\Phi_j(p')$ such that $g^m_j(\Phi_j(p')) = p_m$, $m = 1, \dots, k-1, k+1, \dots, n$ for $(p', p_n) \in \mathscr{S}_j$.

Next we shift the point p to the origin $p = 0$ and apply the rotation which interchanges the distinguished variable p_k with p_n. Then, modulo an affine transformation of the variable p, the part $\mathscr{S}_j$ of the surface $\mathscr{S}$ is represented as the graph $(p', g^k_j(\Phi_j(p')))^\top$, i.e. as $p_n = \Psi_j(p) := g^k_j(\Phi_j(p'))$ and $\mathscr{S}$ is a hypersurface according the Definition 2.3.

The implication Definition 2.3 $\Longrightarrow$ Definition 2.4 is trivial: a piece $\mathscr{S}^j_\Phi$ of a hypograph surface $\mathscr{S}_\Phi$ defined by a function $\Phi_j \in C^k(V)$, $V \subset \mathbb{R}^{n-1}$, is an implicitly defined hypersurface and the corresponding function is

$$\begin{aligned} \Psi^j_{\mathscr{S}}(\Theta) := x_n - \Phi_j(x'), \quad x = (x', x_n) \in \omega_j := V_j \times [-\varepsilon, \varepsilon], \\ \varepsilon > 0, \quad j = 1, \dots, M. \end{aligned} \tag{2.8}$$

How to convert a local implicit representation into a global one is shown in (2.7).

To complete the proof we only need check the implication: Definition 2.4 $\Longrightarrow$ Definition 2.3.

Let $\mathscr{S}_j$ be a part of a hypersurface $\mathscr{S}$ given implicitly by a single function $\Psi_j \in C^k(\omega_j)$, $\omega_j \subset \mathbb{R}^n$ and $\partial_{k_j}\Psi_j(x) \neq 0$. Due to the implicit function theorem there exists the implicit functions $\Phi_j \in C^k(\Omega_j)$, $\Omega_j \subset \mathbb{R}^{n-1}$ such that

$$\begin{aligned} \Psi\Big(x_1, \dots, x_{k_j-1}, \Phi_j(x_1, \dots, x_{k_j-1}, x_{k_j-1}, \dots, x_n), x_{k_j-1}, \dots, x_n\Big) \equiv 0 \\ \forall\, x \in U_j, \quad j = 1, \dots, n. \end{aligned}$$

Then, modulo the affine transformation

$$(x_1, \dots, x_{k_j-1}, x_{k_j-1}, \dots, x_n) \mapsto (p_1, \dots, p_{n-1}), \quad p_n = x_{k_j},$$

the part $\mathscr{S}_j := U_j \cap \mathscr{S}$ of the surface is represented as the graph $p_n = \Phi_j(p')$ and $\mathscr{S}$ is a hypersurface according the Definition 2.3. $\square$

Remark 2.6. Redefinition of a C^k-smooth hypograph hypersurface as an implicit hypersurface in (2.8) is not unique: we can also take

$$\Psi_{\mathscr{S}}(\Theta) := x_n - \Phi(x') + G(x), \quad x = (x', x_n) \in \omega := V \times \mathbb{R}, \tag{2.9}$$

where $G(\mathscr{X}) = 0$ for $\forall\, \mathscr{X} \in \mathscr{S}$. Moreover, $G(x)$ might be non-properly smooth $G \in C^m(\omega)$ with $m < k$.

Definition 2.4 is a powerful source of hypersurfaces.

Example 2.7. For a fixed pair $R > 0$ and $p \in \mathbb{R}^n$ the set

$$\mathbb{S}^{n-1}_R(p) := \Big\{ x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n : \ \Psi_{R,p}(x) = |x - p|^2 - R^2 = 0 \Big\}, \tag{2.10}$$

defines the sphere of radius R centered at p.

Similarly, for a pair of vectors $p \in \mathbb{R}^n$ and of $r = (r_1, \dots, r_n)^\top$ with positive components $r_1 > 0, \dots, r_n > 0$ the set

$$\mathscr{E}^{n-1}_{r,p} := \left\{ x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n : \ \Psi_{r,p}(x) = \sum_{j=1}^n \Big(\frac{x_j - p_j}{r_j} \Big)^2 - 1 = 0 \right\} \tag{2.11}$$

defines the ellipsoid.

Both, $\mathbb{S}^{n-1}_R(p)$ and $\mathscr{E}^{n-1}_{r,p}$ are hypersurfaces in $\mathbb{R}^n$.

In some applications it is necessary to extend the outer unit vector field to a hypersurface in a neighborhood of $\mathscr{S}$, preserving some important features. For example, such extension is needed to define correctly the normal derivative (the derivative along normal vector fields, outer or inner). We consider here a natural extension based on implicit representation of a surface $\mathscr{S}$ and note that another possible extension is based on the hypograph representation (2.4).

Lemma 2.8. *Let $\mathscr{S} \subset \mathbb{R}^n$ be a k-smooth hypersurface, $k = 1, 2, \dots$, given implicitly $\Psi_{\mathscr{S}}(\mathscr{X}) = 0$ by the function $\Psi_{\mathscr{S}} \in C^k(\Omega_{\mathscr{S}})$ defined in a neighborhood $\Omega_{\mathscr{S}}$ of the surface $\mathscr{S} \subset \Omega_{\mathscr{S}} \subset \mathbb{R}^n$.*

i. *The unit vector field*

$$\mathscr{N} := \frac{\nabla \Psi_{\mathscr{S}}}{|\nabla \Psi_{\mathscr{S}}|} = \{\mathscr{N}_1, \dots, \mathscr{N}_n\}^\top, \quad \mathscr{N}_j = \frac{\partial_j \Psi_{\mathscr{S}}}{|\nabla \Psi_{\mathscr{S}}|}, \ \ j = 1, \dots, n \tag{2.12}$$

is C^{k-1}-smooth and, for any (fixed) point $x \in \Omega_{\mathscr{S}}$ it is normal vector to the level surface

$$\mathscr{S}_C := \big\{ y \in \mathbb{R}^n : \ \Psi_{\mathscr{S}}(y) = C := \Psi_{\mathscr{S}}(x) \big\}. \tag{2.13}$$

In particular, on the initial surface $\mathscr{S}$ it coincides with the unit normal vector field

$$\mathscr{N}(x) = \boldsymbol{\nu}(x) \ \textit{ for all } \ x \in \mathscr{S}.$$

ii. *If $k \geq 2$ the following equality holds:*

$$\begin{aligned} \mathscr{N}(x) &= \nabla \frac{\Psi_{\mathscr{S}}(x) - C}{|\nabla \Psi_{\mathscr{S}}(x)|} \ \textit{ or, componentwise,} \\ \mathscr{N}_j(x) &= \partial_j \frac{\Psi_{\mathscr{S}}(x) - C}{|\nabla \Psi_{\mathscr{S}}(x)|}\,, \ \ \forall\, x \in \mathscr{S}_C, \ \ j = 1, \dots, n. \end{aligned} \tag{2.14}$$

iii. *The following equalities*

$$\partial_j \mathscr{N}_k(x) = \partial_k \partial \mathscr{N}_j \quad \textit{hold for all} \quad x \in \mathscr{S}_C, \ \ j, k = 1, \dots, n. \tag{2.15}$$

Proof. Let $\{\mathscr{S}_j, \Theta_j\}_{j=1}^M$ be the atlas which defines $\mathscr{S}$ (cf. Definition 2.1). The pull-back functions $\Psi^*_j(x) = (\Theta_{j,*}\Psi_{\mathscr{S}})(x) = \Psi_j(\Theta_j(x))$, $x \in \omega_j \subset \mathbb{R}^{n-1}$, are immersions: the corresponding gradient has maximal rank

$$\begin{gathered} \nabla \Psi^*_j(x) := \operatorname{matr}\big[\partial_1 \Psi^*_j(x), \dots, \partial_{n-1}\Psi^*_j(x)\big], \\ \operatorname{rank} \nabla \Psi^*_j(x) = n - 1 \ \ \forall\, x \in \omega_j, \ \ j = 1, \dots, M. \end{gathered}$$

Since $\Psi^*_j(x) \equiv 0$ for $x \in \omega_j$, the chain rule provides

$$\partial_k \Psi^*_j(x) = \sum_{m=1}^{n-1} (\partial_m \Psi_{\mathscr{S}})(\Theta_j(x))(\partial_k \Theta_j)_m(x) = 0, \ \ k = 1, \dots, n-1$$

and justifies that the gradient of the hypograph function is orthogonal to all tangential vectors

$$\big\langle \partial_k \Theta_j(x), (\nabla \Psi_{\mathscr{S}})(\Theta_j(x)) \big\rangle \equiv 0 \ \ \forall\, x \in \omega_j, \ k = 1, \dots, n, \ j = 1, \dots, M. \tag{2.16}$$

Therefore, the normed gradient

$$\boldsymbol{\nu}(\mathscr{X}) = \frac{(\nabla \Psi_{\mathscr{S}})(\mathscr{X})}{|(\nabla \Psi_{\mathscr{S}})(\mathscr{X})|}\,, \ \ \mathscr{X} \in \mathscr{S} \tag{2.17}$$

coincides with the outer normal vector on the surface (cf. Fig. 1).

The same holds for the level surfaces $\mathscr{S}_C$, since this surface is defined by the implicit function $\Psi_{\mathscr{S}} - C$.

The equality (2.14) follows taking into account that $\Psi_{\mathscr{S}}(x) - C \equiv 0$ for all $x \in \mathscr{S}_C$:

$$\begin{aligned} \partial_j \frac{\Psi_{\mathscr{S}}(x) - C}{|\nabla \Psi_{\mathscr{S}}(x)|} = \frac{(\partial_j \Psi_{\mathscr{S}})(x)}{|\nabla \Psi_{\mathscr{S}}(x)|} - (\Psi_{\mathscr{S}}(x) - C)\, \frac{\partial_j |\nabla \Psi_{\mathscr{S}}(x)|}{|\nabla \Psi_{\mathscr{S}}(x)|^2} = \\ = \frac{(\partial_j \Psi_{\mathscr{S}})(x)}{|\nabla \Psi_{\mathscr{S}}(x)|} = \mathscr{N}_j(x) \ \text{ for all } \ x \in \mathscr{S}_C. \end{aligned}$$

Equalities (2.15) are simple consequences of (2.14). □

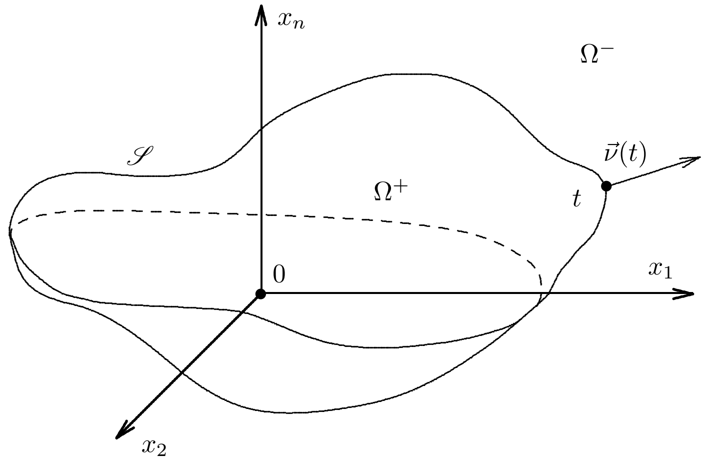

Fig. 1

Definition 2.9. Let $\mathscr{S}$ be a surface in $\mathbb{R}^n$ with the unit normal $\boldsymbol{\nu}$. A vector filed $\mathscr{N} \in C^1(\Omega_{\mathscr{S}})$ in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, will be referred to as a **proper extension** if $\mathscr{N}\Big|_{\mathscr{S}} = \boldsymbol{\nu}$, it is unitary $|\mathscr{N}| = 1$ in $\Omega_{\mathscr{S}}$ and if $\mathscr{N}$ satisfies the condition

$$\partial_j \mathscr{N}_k(x) = \partial_k \mathscr{N}_j(x) \qquad \text{for all} \quad x \in \Omega_{\mathscr{S}}, \quad j, k = 1, \ldots, n. \tag{2.18}$$

not only on the surface $\mathscr{S}$ but in the neighborhood (cf. (2.15)).

The proper extension of the unit normal vector filed $\boldsymbol{\nu}$ is organized as follows: $\mathscr{N}(x) = \boldsymbol{\nu}(\mathscr{x})$ for all $x = \mathscr{x} + t\nu(\mathscr{x}) \subset \Omega_{\mathscr{S}}$, where $\mathscr{x} \in \mathscr{S}$ and $-\varepsilon < t < \varepsilon$, i.e., we extend the unit normal vector field in the direction of the normal vectors (positive and negative) as constant vectors. Obviously, $\partial_{\mathscr{N}} \mathscr{N}(x) \equiv 0$ in $\Omega_c S$ and the extension is proper.

In the sequel we will dwell on a proper extension and apply the above properties of $\mathscr{N}$.

Corollary 2.10. *For any proper extension $\mathscr{N}(x)$, $x \in \Omega_{\mathscr{S}} \subset \mathbb{R}^n$ of the unit normal vector field $\boldsymbol{\nu}$ to the surface $\mathscr{S} \subset \Omega_{\mathscr{S}}$ the equality*

$$\partial_{\mathscr{N}} \mathscr{N}(x) = 0 \;\; \textit{holds for all} \;\; x \in \Omega_{\mathscr{S}}. \tag{2.19}$$

In particular, for the derivatives

$$\mathscr{D}_k = \partial_k - \mathscr{N}_k \partial_{\mathscr{N}}, \;\; k = 1, \ldots, n, \tag{2.20}$$

which are extension into the domain $\Omega_{\mathscr{S}}$ of Günter's derivatives $\mathscr{D}_k = \partial_k - \nu_k \partial_{\boldsymbol{\nu}}$ on the surface $\mathscr{S}$, we have the equality:

$$\mathscr{D}_k \mathscr{N}_j = \partial_k \mathscr{N}_j - \mathscr{N}_k \partial_{\mathscr{N}} = \partial_k \mathscr{N}_j \;\; j, k = 1, \ldots, n. \tag{2.21}$$

Proof. We apply (2.18) and proceed as follows:

$$\partial_{\mathscr{N}}\mathscr{N}_j=\sum_{k=1}^{n}\mathscr{N}_k\partial_k\mathscr{N}_j=\sum_{k=1}^{n}\mathscr{N}_k\partial_j\mathscr{N}_k=\frac{1}{2}\sum_{k=1}^{n}\partial_j\mathscr{N}_k^2=\partial_j 1=0$$

for all $j=1,\dots,n$. □

Remark 2.11. Lemma 2.8 was proved partly in [16, § 3] for a particular implicit function representing the given hypersurface $\mathscr{S}$, namely for the signed distance

$$\Psi_{\mathscr{S}}(x):=\pm\operatorname{dist}(x,\mathscr{S}),\quad x\in\Omega_{\mathscr{S}},\tag{2.22}$$

where the signs "+" and "−" are chosen for x "above" (in the direction of the unit normal vector) and "below" $\mathscr{S}$, respectively.

Lemma 2.12. *For an arbitrary unitary extension $\mathscr{N}(x)\in C^1(\Omega_{\mathscr{S}})$, $|\mathscr{N}(x)|\equiv 1$, of $\boldsymbol{\nu}(\mathscr{X})$, in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, the following conditions are equivalent:*

i. $\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}}=0$, *i.e.,* $\partial_{\mathscr{N}}\mathscr{N}_j(x)\to 0$ *for* $x\to\mathscr{X}\in\mathscr{S}$ *and* $j=1,2,\dots,n$;
ii. $[\partial_k\mathscr{N}_j-\partial_j\mathscr{N}_k]\big|_{\mathscr{S}}=\mathscr{D}_k\nu_j-\mathscr{D}_j\nu_k=0$ *for* $k,j=1,2,\dots,n$.

Proof. The implication $(ii)\Rightarrow(i)$ follows readily by writing

$$\begin{aligned}\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}}&=\Big\{\sum_{j=1}^{n}\mathscr{N}_j\partial_j\mathscr{N}_k\Big\}_{k=1}^{n}\Big|_{\mathscr{S}}=\Big\{\sum_{j=1}^{n}\mathscr{N}_j\partial_k\mathscr{N}_j\Big\}_{k=1}^{n}\Big|_{\mathscr{S}}=\\&=\frac{1}{2}\nabla_x|\mathscr{N}|^2\Big|_{\mathscr{S}}=\frac{1}{2}\nabla_x 1=0.\end{aligned}\tag{2.23}$$

As for the inverse implication, we first observe that, in general,

$$\partial_{\boldsymbol{V}}\mathscr{N}\big|_{\mathscr{S}}=0\ \ \&\ \ \mathscr{N}\big|_{\mathscr{S}}=\boldsymbol{\nu}\ \text{ imply }\ \partial_{\boldsymbol{V}}\mathscr{N}\big|_{\mathscr{S}}\ \text{ depends only on }\ \boldsymbol{\nu}\tag{2.24}$$

and does not depend on a particular extension $\mathscr{N}$ for arbitrary vector field $\boldsymbol{V}$.

Let

$$\begin{gathered}\pi_{\mathscr{S}}:\mathbb{R}^n\to\mathscr{V}(\mathscr{S}),\\ \pi_{\mathscr{S}}(t)=I-\boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t)=\big[\delta_{jk}-\nu_j(t)\nu_k(t)\big]_{n\times n},\quad t\in\mathscr{S}\end{gathered}\tag{2.25}$$

denote the canonical orthogonal projection $\pi_{\mathscr{S}}^2=\pi_{\mathscr{S}}$ onto the space of tangential vector fields to $\mathscr{S}$ at the point $t\in\mathscr{S}$:

$$(\boldsymbol{\nu},\pi_{\mathscr{S}}\boldsymbol{V})=\sum_j\nu_jV_j-\sum_{j,k}\nu_j^2\nu_kV_k=0\ \text{ for all }\ \boldsymbol{V}=(V_1,\dots,V_n)^\top\in\mathbb{R}^n.$$

In the sequel we shall tacitly assume that the projection $\pi_{\mathscr{S}}$ is extended to the neighborhood $\Omega_{\mathscr{S}}$

$$\widetilde{\pi}_{\mathscr{S}}(x)=\big[\delta_{jk}-\mathscr{N}_j(x)\mathscr{N}_k(x)\big]_{n\times n},\quad\widetilde{\pi}_{\mathscr{S}}^2=\widetilde{\pi}_{\mathscr{S}},\quad x\in\Omega_{\mathscr{S}}.\tag{2.26}$$

Note that $\boldsymbol{U} = \widetilde{\pi}_{\mathscr{S}}\boldsymbol{U} + \langle \boldsymbol{U}, \mathscr{N}\rangle\mathscr{N}$ for arbitrary field $\boldsymbol{U}$ in the neighborhood $\Omega_{\mathscr{S}}$. Then

$$\partial_{\boldsymbol{U}}\mathscr{N}\big|_{\mathscr{S}} = \partial_{\widetilde{\pi}_{\mathscr{S}}\boldsymbol{U}}\mathscr{N}\big|_{\mathscr{S}} + (\boldsymbol{U}, \mathscr{N})\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}} = \partial_{\widetilde{\pi}_{\mathscr{S}}\boldsymbol{U}}\mathscr{N}\big|_{\mathscr{S}} = \partial_{\pi_{\mathscr{S}}\boldsymbol{U}}\boldsymbol{\nu},$$

because $\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}} = 0$ and $\pi_{\mathscr{S}}\boldsymbol{U}$ is a tangential field to $\mathscr{S}$. Thus, we can dwell on the particular extension (2.14) and observe

$$\partial_k\mathscr{N}_j\big|_{\mathscr{S}} = \partial_k\partial_j\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\bigg|_{\mathscr{S}} = \partial_j\partial_k\frac{\Psi_{\mathscr{S}}}{|\nabla\Psi_{\mathscr{S}}|}\bigg|_{\mathscr{S}} = \partial_j\mathscr{N}_k\big|_{\mathscr{S}},$$

which proves the implication $(i) \Rightarrow (ii)$. □

Remark 2.13. It is clear that a normal vector field and it's (non-unique) extension exists for arbitrary Lipschitz surface, but almost everywhere on $\mathscr{S}$.

Moreover to enjoy the properties listed in Lemma 2.8, we have to consider smoother than Lipschitz surfaces and assume C^2-smoothness of $\mathscr{S}$.

3. Gauss and Stoke's Formulae for Domains in $\mathbb{R}^n$

In the present section we consider a hypersurface $\mathscr{S}$, which is a boundary of some domain $\Omega \subset \mathbb{R}^n$. We dwell on Definition 2.1 and 2.2 of a (hypograph) hypersurface $\mathscr{S}$, which are most convenient for the present purposes.

The Gauß formula (3.1) is a basic result in calculus on surfaces. We refer to [27] for the simplest proof of the following proposition.

Proposition 3.1 (Gauß formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with the Lipschitz boundary $\mathscr{S} := \partial\Omega$, $\boldsymbol{\nu}(t) = (\nu_1(t), \dots, \nu_n(t))^\top$ be the outer unit normal vector to $\mathscr{S}$ and $f \in \mathbb{W}^1_1(\Omega)$. Then*

$$\int\limits_{\Omega}\partial_j f(y)\,dy = \oint\limits_{\mathscr{S}}\nu_j(\tau)f(\tau)\,dS \tag{3.1}$$

in the following sense: the integral in the left hand side exists (since, by the condition, $\partial_j f \in \mathbb{L}_1(\Omega)$) and the integral in the right-hand side is defined by the above equality.

Remark 3.2. The last statement of the foregoing Proposition 3.1 explains the traces $\gamma_{\mathscr{S}}\partial_j f(\mathscr{x})$ of $f \in \mathbb{W}^1_1(\Omega)$ despite, a well known theorem that the trace $\gamma_{\mathscr{S}}\partial_j f(\mathscr{x}) = \partial_j f(\mathscr{x})\big|_{\mathscr{S}}$ of a function $f \in \mathbb{W}^1_1(\Omega)$ on the boundary surface $\mathscr{S} = \partial\Omega$ does not exist for sure. The assertion does not contradicts the trace theorem, because states existence of the trace in combination with components of the normal vector $\nu_j(x)f(x)$.

Next we are going to derive some important consequences of the Gauß formula.

Corollary 3.3. *Let Ω, $\mathscr{S} = \partial\Omega$ and $\boldsymbol{\nu}(\tau) = (\nu_1(\tau), \dots, \nu_n(\tau))^\top$ be as in Lemma* 3.1.

i. *The divergence formula*

$$\int_\Omega \operatorname{div} F(y)\,dy = \oint_{\mathscr{S}} \langle \boldsymbol{\nu}(\tau), F(\tau)\rangle\, dS \tag{3.2}$$

holds for the divergence

$$\operatorname{div} F(x) := \partial_1 f_1(x) + \cdots + \partial_n f_n(x) \tag{3.3}$$

of a vector field $F = (f_1, \ldots, f_n)^\top \in \mathbb{W}^1(\Omega)$.

ii. *The integration by parts*

$$\int_\Omega \partial_j f(y) g(y)\,dy = \oint_{\mathscr{S}} \nu_j(\tau) f(\tau) g(\tau)\,dS - \int_\Omega f(y)\partial_j g(y)\,dy \tag{3.4}$$

holds for arbitrary $f, g \in \mathbb{W}^1(\mathscr{S})$.

Proof. Formula (3.2) is a direct consequences of the Gauß formula (3.1):

$$\int_\Omega \operatorname{div} F(y)\,dy = \sum_j \int_\Omega \partial_j f_j(y)\,dy =$$
$$= \sum_j \oint_{\mathscr{S}} \nu_j(\tau), f_j(\tau)\,dS = \oint_{\mathscr{S}} \langle \boldsymbol{\nu}(\tau), F(\tau)\rangle\, dS.$$

Since $f, g \in \mathbb{W}^2_2(\mathscr{S})$ implies $fg \in \mathbb{W}^2_1(\mathscr{S})$, we can apply the Gauß formula (3.1) to the Leibnitz equality $\partial_j[\psi(y)\varphi(y)] = \varphi(y)\partial_j\psi(y) + \psi(y)\partial_j\varphi(y)$ and get (3.4) readily. □

Let us consider the *normal derivative*

$$\partial_{\boldsymbol{\nu}}\varphi := \boldsymbol{\nu}\cdot\nabla\varphi = \sum_{j=1}^n \nu_j\partial_j\varphi, \quad \varphi \in C^1(\Omega). \tag{3.5}$$

Corollary 3.4 (Green's formula)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with Lipschitz boundary.*

For the Laplace operator

$$\Delta := \partial_1^2 + \cdots + \partial_n^2 \tag{3.6}$$

and functions $\varphi, \psi \in \mathbb{W}^1_2(\Omega)$ *the following* I *and* II *Green formulae are valid:*

$$\int_\Omega (\Delta\psi)(y)\varphi(y)\,dy = \oint_{\partial\Omega} (\partial_{\boldsymbol{\nu}}\psi)(\tau)\varphi(\tau)\,dS - \sum_{j=1}^n \int_\Omega (\partial_j\psi)(y)(\partial_j\varphi)(y)\,dy, \tag{3.7}$$

$$\int_\Omega (\Delta\psi)(y)\varphi(y)\,dy =$$
$$= \int_\Omega \psi(y)(\Delta\varphi)(y)\,dy + \oint_{\partial\Omega} \left[(\partial_{\boldsymbol{\nu}}\psi)(\tau)\varphi(\tau) + \psi(\tau)(\partial_{\boldsymbol{\nu}}\varphi)(\tau)\right] dS. \tag{3.8}$$

Proof. Let, for time being, $\varphi, \psi \in C^2(\Omega)$. By applying (3.4) we prove I Green formulae in (3.7).

By writing a similar formula

$$\int\limits_{\Omega} (\psi)(y)\Delta\varphi(y)\,dy =$$

$$= \oint\limits_{\partial\Omega} (\partial_{\boldsymbol{\nu}}\psi)(\tau)\varphi(\tau)\,dS_{\mathscr{S}} - \sum_{j=1}^{n} \int\limits_{\Omega} (\partial_j\psi)(y)(\partial_j\varphi)(y)dy \tag{3.9}$$

and taking the difference with (3.7), we prove II Green formulae in (3.8).

For arbitrary $\varphi, \psi \in \mathbb{W}^1_2(\Omega)$ the Green formulae (3.7) and (3.7) follow by approximation $\varphi_j \to \varphi$, $\psi_j \to \psi$, $\varphi_j, \psi_j \in C^2(\Omega)$. □

Stoke's derivatives are concrete examples of weakly tangential operators

$$\mathscr{M}_{\mathscr{S}} := [\mathscr{M}_{jk}]_{n\times n}\,, \quad \mathscr{M}_{jk} := \nu_j\partial_k - \nu_k\partial_j = \partial_{\mathfrak{m}_{j,k}}. \tag{3.10}$$

These derivatives are directional with respect to a tangential vector fields to $\mathscr{S}$ (cf. (4.8) and (4.10)). Indeed, the directing vector $\mathfrak{m}_{jk}(\mathscr{X}) = \nu_j(\mathscr{X})\boldsymbol{e}^k - \nu_k(\mathscr{X})\boldsymbol{e}^j$ of $\mathscr{M}_{jk}$, where $\{\boldsymbol{e}^j\}_{j=1}^n$ is the Cartesian frame in $\mathbb{R}^n$, is tangential to $\mathscr{S}$:

$$\boldsymbol{\nu}(\mathscr{X})\cdot\mathfrak{m}_{jk}(\mathscr{X}) = \nu_j(\mathscr{X})\nu_k(\mathscr{X}) - \nu_k(\mathscr{X})\nu_j(\mathscr{X}) \equiv 0, \quad \mathscr{X}\in\mathscr{S}. \tag{3.11}$$

Therefore the Stoke's derivative $\mathscr{M}_{jk}$ can be applied to functions defined on the surface $\mathscr{S}$ only.

Corollary 3.5. *Let Ω, $\mathscr{S} = \partial\Omega$ and $\boldsymbol{\nu}(\tau) = (\nu_1(\tau),\dots,\nu_n(\tau))^\top$ be as in Lemma* 3.1.

The following Stoke's formulae

$$\oint\limits_{\mathscr{S}} (\mathscr{M}_{jk}f)(\tau)\,dS = 0 \tag{3.12}$$

holds for $j,k = 1,\dots,n$ and for all $f \in \mathbb{W}^1_1(\mathscr{S})$.

The Stokes derivatives $\mathscr{M}_{j,k}$ are skew-symmetric:

$$\oint\limits_{\mathscr{S}} (\mathscr{M}_{jk}\psi)(\tau)\varphi(\tau)\,dS = -\oint\limits_{\mathscr{S}} \psi(\tau)(\mathscr{M}_{jk}\varphi)(\tau)\,dS \tag{3.13}$$

for $j,k = 1,\dots,n$ and for arbitrary pair $\varphi,\psi \in \mathbb{W}^2_2(\mathscr{S})$.

Proof. We assume temporarily that $f \in C^1(\mathscr{S})$ and extend this function into the domain $F \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ with the trace on the boundary $F\big|_{\mathscr{S}} = f$. Such extension is possible since the boundary is a Lipschitz hypersurface. It is possible to construct a direct extension by means of function theory (cf. E. Stein [35]). But we consider here the following indirect construction: consider the Dirichlet problem for the Laplace operator $\Delta F = 0$ in Ω with a boundary condition $F\big|_{\mathscr{S}} = f$. It is well known that the solution exists and, moreover, $F \in C^\infty(\Omega)$ (cf., e.g., [24]). Herewith we have found the extension.

Now apply the Gauß formula (3.1) to a function $\partial_j\partial_k f = \partial_k\partial_j f$ twice:

$$\int_\Omega (\partial_j\partial_k F)(y)\,dy = \oint_{\mathscr{S}} \nu_j(\tau)(\partial_k f)(\tau)\,dS,$$
$$\int_\Omega (\partial_k\partial_j F)(y)\,dy = \oint_{\mathscr{S}} \nu_k(\tau)(\partial_j f)(\tau)\,dS.$$

By taking the difference we get (3.12) immediately.

Note that formula (3.12) is valid for arbitrary $f \in C^1(\mathscr{S})$ without knowing an extension $F(x)$ of $f(\mathscr{X})$ into the domain Ω, because the Stoke's derivative $\mathscr{M}_{jk}$ can be applied to a function defined only on the surface.

For a function $\psi \in \mathbb{W}^1_2(\mathscr{S})$ formula (3.12) is proved by approximation (cf. the concluding part of the proof of Lemma 3.1).

Formula (3.13) follows from (3.12) Since $\mathscr{M}_{jk}$ is a linear differential operator

$$\mathscr{M}_{jk}[\varphi\psi] = (\mathscr{M}_{jk}\varphi)\psi + \varphi(\mathscr{M}_{jk}\psi)$$

and by applying (3.12) we get

$$0 = \oint_{\mathscr{S}} \big(\mathscr{M}_{jk}[\psi\varphi]\big)(\tau)\,dS = \oint_{\mathscr{S}} (\mathscr{M}_{jk}\varphi)(\tau)\psi(\tau)\,dS + \oint_{\mathscr{S}} \varphi(\tau)(\mathscr{M}_{jk}\psi)(\tau)\,dS.$$

The obtained equality completes the proof of (3.13). □

4. Calculus of Tangential Differential Operators

The content of the present section partly follows [16, § 4].

Throughout the present section we keep the following convention: $\mathscr{S}$ is a hypersurface in $\mathbb{R}^n$, given by an immersion

$$\Theta:\ \omega \to \mathscr{S},\ \ \omega \subset \mathbb{R}^{n-1} \tag{4.1}$$

with a boundary $\Gamma = \partial\mathscr{S}$, given by another immersion

$$\Theta_\Gamma:\ \omega \to \Gamma := \partial\mathscr{S},\ \ \omega \subset \mathbb{R}^{n-2}, \tag{4.2}$$

$\boldsymbol{\nu}(\mathscr{X})$ is the outer unit normal vector field to $\mathscr{S}$ an $\mathscr{N}(x)$ denotes an extended unit field in a neighborhood $\omega_{\mathscr{S}}$ of $\mathscr{S}$ (cf. Definition 2.9). $\boldsymbol{\nu}_\Gamma(t)$ is the outer normal vector field to the boundary Γ, which is tangential to $\mathscr{S}$.

A **curve** on a smooth surface $\mathscr{S}$ is a mapping

$$\gamma:\ \mathscr{I} \mapsto \mathscr{S},\ \ \mathscr{I} := (a,b] \subset \mathbb{R}, \tag{4.3}$$

of a line interval $\mathscr{I}$ to $\mathscr{S}$.

A **vector field** on a domain Ω in $\mathbb{R}^n$ is a mapping

$$\boldsymbol{U}:\ \Omega \to \mathbb{R}^n, \quad \boldsymbol{U}(x) = \sum_{j=1}^n U_j(x)\boldsymbol{e}^j, \tag{4.4}$$

where $U^j \in C_0^\infty(\Omega)$ and $\boldsymbol{e}^j$ is the element of the natural Cartesian basis in $\mathbb{R}^n$

$$e^1 := (1, 0, \dots, 0), \ \dots, \ e^n := (0, \dots, 0, 1), \tag{4.5}$$

in the Euclidean space $\mathbb{R}^n$. $\{\boldsymbol{e}_j\}_{j=1}^n$ is also called the **natural frame** or the **Cartesian frame**.

By $\mathscr{V}(\Omega)$ we denote the set of all smooth vector fields on Ω.

Let $\boldsymbol{U} \in \mathscr{V}(\Omega)$ and consider the corresponding ordinary differential equations (ODE):

$$y' = \boldsymbol{U}(y), \ \ y(0) = x, \ \ x \in \Omega. \tag{4.6}$$

A solution $y(t)$ of (4.6) is called an **integral curve** (or **orbit**) of the vector field $\boldsymbol{U}$. The mapping

$$y = y(t, x) = \mathscr{F}_{\boldsymbol{U}}^t(x) : \ \Omega \to \Omega \tag{4.7}$$

is called the **flow** generated by the vector field $\boldsymbol{U}$ at the point x.

A vector field $\boldsymbol{U} \in \mathscr{V}(\Omega)$ defines the **first order differential operator**

$$\boldsymbol{U} f(x) = \partial_{\boldsymbol{U}} f(x) := \lim_{h\to 0} \frac{f(\mathscr{F}_{\boldsymbol{U}}^h(x)) - f(x)}{h} = \frac{d}{dt} f(\mathscr{F}_{\boldsymbol{U}}^t(x))\big|_{t=0}. \tag{4.8}$$

By applying the chain rule to (4.8) we get

$$\partial_{\boldsymbol{U}} f(x) = \big\langle \boldsymbol{U}(x), \nabla f(x)\big\rangle = \sum_{j=1}^n U_j(x) \frac{\partial f}{\partial x_j} \, . \tag{4.9}$$

By $\mathscr{V}(\mathscr{S})$ we denote the set of all smooth vector fields, tangential to the hypersurface $\mathscr{S}$. Note that if the vector $\boldsymbol{U}$ is tangential, i.e., $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$, its orbit can be chosen as a curve on the surface $\mathscr{S}$,

$$\mathscr{F}_{\boldsymbol{U}}^t(x) : \ \mathscr{I} \to \mathscr{S}, \ \ \mathscr{I} \subset \omega \subset \mathbb{R}^{n-1}. \tag{4.10}$$

Then the derivative $\partial_{\boldsymbol{U}}$ defined by (4.8) is applicable to a function $f \in C^1(\mathscr{S})$ which is defined on the surface $\mathscr{S}$ only.

Note, that if a function f is defined not only on the surface $\mathscr{S}$, but also in a neighborhood of $\mathscr{S} \subset \mathbb{R}^n$, formula (4.9) gives the rule for the differentiation of f along a vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$.

Definition 4.1. A derivative $\partial_{\boldsymbol{U}}^{\mathscr{S}} : \ C^1(\mathscr{S}) \to C^1(\mathscr{S})$, $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is called **covariant** if it is a linear automorphism of the space of tangential vector fields:

$$\partial_{\boldsymbol{U}}^{\mathscr{S}} : \ \mathscr{V}(\mathscr{S}) \longrightarrow \mathscr{V}(\mathscr{S}). \tag{4.11}$$

If $\mathscr{S}$ is embedded in $\mathbb{R}^n$, a directional derivative $\partial_{\boldsymbol{U}}$ along a tangential vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ maps the space of tangential vector fields to the space of possibly non-tangential vector fields

$$\partial_{\boldsymbol{U}} : \ \mathscr{V}(\mathscr{S}) \not\longrightarrow \mathscr{V}(\mathscr{S}).$$

If composed with the projection

$$\partial_{\boldsymbol{U}}^{\mathscr{S}} \boldsymbol{V} := \pi_{\mathscr{S}} \partial_{\boldsymbol{U}} \boldsymbol{V} = \partial_{\boldsymbol{U}} \boldsymbol{V} - \langle \boldsymbol{\nu}, \partial_{\boldsymbol{U}} \boldsymbol{V} \rangle \boldsymbol{\nu} \tag{4.12}$$

(cf. (2.25)), it becomes a covariant derivative, i.e., becomes an automorphism of the space of tangential vector fields (cf. (4.11)).

The Günter's derivatives $\{\mathscr{D}_j\}_{j=1}^n$ are tangent and represent a full system (cf. (4.37)-(4.39)). But the derivative $\mathscr{D}_j\boldsymbol{V}$ is not covariant and maps the tangential vectors to non-tangential ones $\mathscr{D}_j : \mathscr{V}(\mathscr{S}) \not\to \mathscr{V}(\mathscr{S})$. To improve this we just eliminate the normal component of the vector by applying the canonical orthogonal projection $\pi_{\mathscr{S}}$ onto $\mathscr{V}(\mathscr{S})$ (cf. (2.25))

$$\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V} := \pi_{\mathscr{S}}\mathscr{D}_j\boldsymbol{V} = \mathscr{D}_j\boldsymbol{V} - \langle\boldsymbol{\nu}, \mathscr{D}_j\boldsymbol{V}\rangle\boldsymbol{\nu} = \mathscr{D}_j\boldsymbol{V} + (\partial_{\boldsymbol{V}}\nu_j)\boldsymbol{\nu}, \quad (4.13)$$
$$\text{where } \partial_{\boldsymbol{V}}\varphi := \sum_{k=1}^n V_k^0\partial_k\varphi = \sum_{k=1}^n V_k^0\mathscr{D}_k\varphi$$

and obtain an automorphisms of the space of tangential vector fields

$$\mathscr{D}_j^{\mathscr{S}} : \mathscr{V}(\mathscr{S}) \to \mathscr{V}(\mathscr{S}). \quad (4.14)$$

To check the equalities in (4.13) we recall $\langle\boldsymbol{\nu}, \boldsymbol{V}\rangle = \sum\limits_{j=1}^n \nu_j V_j^0 = 0$ and proceed as follows

$$\begin{aligned}
\partial_{\boldsymbol{V}}\varphi &= \sum_{k=1}^n V_k^0\partial_k\varphi = \sum_{k=1}^n V_k^0\mathscr{D}_k\varphi + \sum_{k=1}^n V_k^0\nu_k\partial_{\boldsymbol{\nu}}\varphi = \sum_{k=1}^n V_k^0\mathscr{D}_k\varphi,\\
\langle\boldsymbol{\nu}, \mathscr{D}_jV\rangle &= \sum_{m=1}^n \nu_m\mathscr{D}_jV_m^0 = \sum_{m=1}^n \left[\mathscr{D}_j(\nu_mV_m^0) - V_m^0\mathscr{D}_j\nu_m\right] =\\
&= -\sum_{m=1}^n V_m^0\mathscr{D}_j\nu_m = -\sum_{m=1}^n V_m^0\mathscr{D}_m\nu_j = -\partial_{\boldsymbol{V}}\nu_j. \qquad (4.15)
\end{aligned}$$

Note that if $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is tangent then

$$\boldsymbol{U} = \sum_{j=1}^n U_j^0\boldsymbol{e}^j = \sum_{j=1}^n U_j^0\boldsymbol{d}^j \text{ since } \sum_{j=1}^n \nu_jU_j^0 = \langle\boldsymbol{\nu}, \boldsymbol{U}\rangle \equiv 0, \quad (4.16)$$

i.e. the system $\{\boldsymbol{d}^j\}_{j=1}^n$ is full in $\mathscr{V}(\mathscr{S})$. Although this system is linearly dependent, the representation of a tangential vector by $\{\boldsymbol{d}^j\}_{j=1}^n$ is unique.

Definition 4.2. A tangential vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is called **Killing's field**, if it generates a flow consisting of isometries and preserves the metric on the surface $\mathscr{S}$ (cf. [37, V. I, Ch. 2, § 3]).

In other words the metric $g(\boldsymbol{V}, \boldsymbol{W})$ is invariant under the flow $\mathscr{F}_U^t$ generated by the vector field $\boldsymbol{U}$ and can be recorded in terms of the **Lie derivative $\mathfrak{L}_{\boldsymbol{U}}$** (cf. [37, V. I, Ch. 2], [16]) as follows:

$$\mathfrak{L}_Ug(\boldsymbol{V}, \boldsymbol{W}) \equiv 0 \text{ for all } \boldsymbol{V}, \boldsymbol{W} \in \mathscr{V}(\mathscr{S}). \quad (4.17)$$

The representation matrix $\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}$ of the bilinear form

$$2(\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V}, \boldsymbol{W}) := \mathfrak{L}_Ug(\boldsymbol{V}, \boldsymbol{W}), \ \forall\,\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W} \in \mathscr{V}(\mathscr{S}) \quad (4.18)$$

is called the **deformation tensor** (cf., e.g., [37, V. I, Ch. 5, § 12]).

Note that the deformation tensor is the symmetrized covariant derivative (cf., e.g., [37, V. I, Ch. 5, § 12]).

$$\begin{aligned}(\mathrm{Def}_{\mathscr{S}}\boldsymbol{U})(\boldsymbol{V},\boldsymbol{W}) &= \frac{1}{2}\Big\{\langle\partial_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle+\langle\partial_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\} = \\ &= \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle+\langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\},\ \ \forall\,\boldsymbol{V},\boldsymbol{W}\in\mathscr{V}(\mathscr{S}). \quad (4.19)\end{aligned}$$

Let

$$\boldsymbol{d}_j := \pi_{\mathscr{S}}\boldsymbol{e}_j \in \mathscr{V},\ \ j=1,\dots,n, \quad (4.20)$$

be the projection of the Cartesian frame onto the tangent space $\mathscr{V}(\mathscr{S})$ to the hypersurface $\mathscr{S}$. Obviously, the frame $\{\boldsymbol{d}_j\}_{j=1}^n$ is linearly dependent

$$\langle\boldsymbol{\nu},\boldsymbol{d}_j\rangle = \sum_{j=1}^n \nu_j\boldsymbol{d}_j = 0,\ \ j=1,\dots,n.$$

Then any tangential vector field $\boldsymbol{U}\in\mathscr{V}(\mathscr{S})$ has the following representation

$$\boldsymbol{U} = \sum_{j=1}^n U_j^0\boldsymbol{e}^j = \sum_{j=1}^n U_j^0\boldsymbol{d}^j \in \mathscr{V}(\mathscr{S}) \quad (4.21)$$

in the canonical Cartesian frame and its projection.

Lemma 4.3. *In Cartesian coordinates the deformation tensor* $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})=\big[\mathfrak{D}^0_{jk}(\boldsymbol{U})\big]_{n\times n}$ *has order n and of type* $(0,2)$ *and*

$$\begin{aligned}\mathfrak{D}^0_{jk}(\boldsymbol{U}) = (\mathrm{Def}_{\mathscr{S}}\,(\boldsymbol{U}))_{jk} &= \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j+(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k\big] = \\ &= \frac{1}{2}\big[\mathscr{D}_jU^0_k+\mathscr{D}_kU^0_j+\partial_{\boldsymbol{U}}(\nu_j\nu_k)\big],\ \ \forall\,j,k=1,\dots,n. \quad (4.22)\end{aligned}$$

where $(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k$ *denotes the* $k-th$ *component of the covariant derivative* $\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}$.

Proof. For the proof we refer to [16]. □

Remark 4.4. Let us introduce the linearly dependent but full system of vectors

$$\big\{\boldsymbol{d}^{jk} := \boldsymbol{d}^j\otimes\boldsymbol{d}^k\big\}_{j=1}^n,\quad \boldsymbol{d}^j = \boldsymbol{e}^j-\nu_j\boldsymbol{\nu},\ \ j,k=1,\dots,n \quad (4.23)$$

in contrast to the system

$$\big\{\boldsymbol{e}^{jk} := \boldsymbol{e}^j\otimes\boldsymbol{e}^k\big\}_{j=1}^n. \quad (4.24)$$

which is linearly independent. Then the deformation tensor can be written as follows

$$
\begin{aligned}
\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U}) &= \big[\mathfrak{D}^0_{jk}(\boldsymbol{U})\big]_{n\times n} = \sum_{j,k=1}^{n} \mathfrak{D}^0_{jk}(\boldsymbol{U})\boldsymbol{d}^{jk} = \\
&= \sum_{j,k=1}^{n} V^0_k W^0_j \big[(\mathscr{D}^{\mathscr{S}}_k \boldsymbol{U})_j + (\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U})_k\big]\boldsymbol{d}^{jk} = \\
&= \sum_{j,k=1}^{n} V^0_k W^0_j \big[\mathscr{D}_k U^0_j + \mathscr{D}_j U^0_k + \partial_{\boldsymbol{U}}\nu_j\nu_k\big]\boldsymbol{d}^{jk} = \\
&= \sum_{j,k=1}^{n} V^0_k W^0_j \big[\mathscr{D}_k U^0_j + \mathscr{D}_j U^0_k\big]\boldsymbol{d}^{jk},
\end{aligned} \tag{4.25}
$$

since, due to (4.38)

$$
\sum_{j,k=1}^{n} \partial_{\boldsymbol{U}}(\nu_j\nu_k)\boldsymbol{d}^{jk} = \sum_{j,k,m=1}^{n} \big[\nu_j U^0_m \mathscr{D}_m \nu_k + \nu_k U^0_m \mathscr{D}_m \nu_j\big]\boldsymbol{d}^{jk} = 0.
$$

The obtained formulae prompts the following representation for the entries of the deformation tensor $\mathfrak{D}_{jk}(\boldsymbol{U}) = \dfrac{1}{2}\big[(\mathscr{D}_j\boldsymbol{U})_k + (\mathscr{D}_k\boldsymbol{U})_j\big]$, which is **false** since all rows of the deformation tensor $\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U})$ (and all columns-since the tensor is a tensor $\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U})$ is symmetric) should be tangent for $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$. This is the case if $\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U})$ is written in the form (4.22).

Definition 4.5. Let $\mathscr{S}$ be a Lipschitz hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open subsurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

We say that a class of functions $\mathscr{U}(\Omega)$ has the *strong unique continuation property from the boundary* if a vector-function $\boldsymbol{U} \in \mathscr{U}(\Omega)$ which vanishes $\boldsymbol{U}(\mathfrak{s}) = 0$, $\forall\,\mathfrak{s} \in \gamma$ on an open subset of the boundary $\gamma \subset \Gamma$, vanishes on the entire $\mathscr{C}$.

Let $\mathscr{R}(\mathscr{S})$ denote the linear space of all deformation-free tangential vector fields (or Killing's vector fields; see Lemma 4.3).

For the proof of the next Proposition 4.6 we refer to [15].

Proposition 4.6. *The set of Killing's vector fields $\mathscr{R}(\mathscr{S})$ coincides with the set of all solutions to the following system of partial differential equations*

$$
\mathfrak{D}^0_{jk}(\boldsymbol{U}) = (\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U})_k + (\mathscr{D}^{\mathscr{S}}_k \boldsymbol{U})_j = \mathscr{D}_j U^0_k + \mathscr{D}_k U^0_j + \partial_{\boldsymbol{U}}(\nu_j\nu_k) \equiv 0 \quad \text{for } 1 \le j \le k \le n \tag{4.26}
$$

provided that $\langle \boldsymbol{\nu}, \boldsymbol{U}\rangle = \sum\limits_{j=1}^{n} \nu_j U^0_j = 0$ *and is finite dimensional, i.e.,* $\dim \mathscr{R}(\mathscr{S}) < \infty$ (*cf.* (4.21)) *and* $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$ *is the surface* $\mathscr{S}$ *is infinitely smooth.*

If $\mathscr{S}$ is a C^2-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ is an open C^2-smooth subsurface, the set $\mathscr{R}(\mathscr{S})$ has the strong unique continuation property from the boundary.

Let us find a formally adjoint operator to $\mathscr{D}_j$.

With (4.50) and with (2.18) we get

$$\begin{aligned}\mathscr{D}_j^*\varphi &= -\partial_j\varphi + \sum_{k=1}^n \partial_k\big(\nu_j\nu_k\varphi\big) = \\ &= -\partial_j\varphi + \sum_{k=1}^n \big[\nu_j\nu_k\partial_k\varphi + \big(\nu_k\partial_k\nu_j\big)\varphi + \nu_j\big(\partial_k\nu_k\big)\varphi\big] = \\ &= -\mathscr{D}_j\varphi - \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi + (\partial_\nu\nu_j)\varphi, \quad \varphi \in C^1(\Omega_{\mathscr{S}}),\end{aligned} \tag{4.27}$$

since, like (2.23),

$$\partial_{\boldsymbol{\nu}}\nu_j = \sum_{k=1}^n \nu_k\partial_k\nu_j = \sum_{k=1}^n \nu_k\partial_j\nu_k = \frac{1}{2}\sum_{k=1}^n \partial_j\nu_k^2 = \frac{1}{2}\partial_j 1 = 0 \tag{4.28}$$

(cf. Lemma 2.12.ii). Here

$$\mathscr{H}^0_{\mathscr{S}}(\mathscr{X}) = -\sum_{k=1}^n \mathscr{D}_k\nu_k(\mathscr{X}) \tag{4.29}$$

and $(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{X}) = \mathscr{H}_{\mathscr{S}}(\mathscr{X})$ is actually the mean curvature of the surface at $\mathscr{X} \in \mathscr{S}$.

It is obvious that the formal adjoint to the derivation $\partial_{\boldsymbol{U}}$ with respect to the vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ in Cartesian coordinates $\boldsymbol{U} = \sum\limits_{j=1}^n U_j^0\boldsymbol{d}^j$, can be written as follows

$$\begin{aligned}\partial^*_{\boldsymbol{U}} f = \left[\sum_{j=1}^n U_j^0\mathscr{D}_j\right]^* f &= -\sum_{j=1}^n \mathscr{D}_j^*(U_j^0 f) = -\sum_{j=1}^n(\mathscr{D}_j + \mathscr{H}_{\mathscr{S}}\nu_j)(U_j^0 f) = \\ &= -\sum_{j=1}^n \mathscr{D}_j(U_j^0 f) = -\partial_{\boldsymbol{U}} f - \big(\operatorname{div}_{\mathscr{S}}\boldsymbol{U}\big)f,\end{aligned} \tag{4.30}$$

since $\mathscr{D}_j^* = -\mathscr{D}_j - \nu_j\mathscr{H}^0_{\mathscr{S}}$ (cf. (4.52)) and $\sum\limits_{j=1}^n \mathscr{H}_{\mathscr{S}}\nu_j U_j^0 f = \mathscr{H}_{\mathscr{S}} f\langle\boldsymbol{\nu}, \boldsymbol{U}\rangle = 0$. This further entails that

$$(\partial^{\mathscr{S}}_{\boldsymbol{U}})^* = \pi_{\mathscr{S}}(\partial_{\boldsymbol{U}})^* = -\partial^{\mathscr{S}}_{\boldsymbol{U}} - \operatorname{div}_{\mathscr{S}}\boldsymbol{U}, \quad \forall\, \boldsymbol{U} \in \mathscr{V}(\mathscr{S}). \tag{4.31}$$

In particular, for $\boldsymbol{U} = \boldsymbol{d}^j$,

$$\big(\partial^{\mathscr{S}}_{\boldsymbol{d}^j}\big)^* = \big(\mathscr{D}^{\mathscr{S}}_j\big)^* = -\mathscr{D}^{\mathscr{S}}_j - \operatorname{div}_{\mathscr{S}}\boldsymbol{d}^j = -\mathscr{D}^{\mathscr{S}}_j - \nu_j\mathscr{H}^0_{\mathscr{S}}, \quad j = 1, \dots, n, \tag{4.32}$$

since

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{d}^j = \sum_{k=1}^n \mathscr{D}_k(\nu_j\nu_k) =$$

$$= \sum_{k=1}^n \big[\nu_j\mathscr{D}_k\nu_k + \nu_k\mathscr{D}_k\nu_j\big) = \nu_j \operatorname{div}_{\mathscr{S}}\boldsymbol{\nu} + \partial_{\boldsymbol{\nu}}\nu_j = \nu_j\mathscr{H}^0_{\mathscr{S}}.$$

The adjoint $\mathrm{Def}^*_{\mathscr{S}}$ to $\mathrm{Def}_{\mathscr{S}}$ is defined in Cartesian coordinates by

$$(\mathrm{Def}^*_{\mathscr{S}} Z)^k = \frac{1}{2}\,(\mathscr{D}^{\mathscr{S}}_j)^*\big[Z^{jk} + Z^{kj}\big] \tag{4.33}$$

for each tensor field $Z = \big[Z^{jk}\big]$ of type $(0,2)$. Indeed, by assuming $\mathscr{S}$ a closed surface, we get

$$\int\limits_{\mathscr{S}} \langle \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}, Z\rangle\, dS =$$

$$= \int\limits_{\mathscr{S}} \mathrm{Tr}\big[(\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U})Z^\top\big]\,dS = \frac{1}{2}\sum_{j,k}\int\limits_{\mathscr{S}}\big[\mathscr{D}_kU_j + \mathscr{D}_jU_k\big]Z^{jk}\,dS$$

$$= \frac{1}{2}\sum_{j,k}\int\limits_{\mathscr{S}} U_k\big[\big(\mathscr{D}^*_j Z^{jk} + \big(\mathscr{D}^*_k Z^{kj}\big]\,dS = \int\limits_{\mathscr{S}}\langle \boldsymbol{U}, \mathrm{Def}^*_{\mathscr{S}} Z\rangle\,dS$$

which holds for all tangential vectors $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ and all tensor fields $Z = \big[Z^{jk}\big]$ of type $(0,2)$ and $\mathrm{Def}^*_{\mathscr{S}}$ defined in (4.33).

Let

$$P(D)u = \sum_{j=1}^n a_j\partial_j u + bu, \quad a_j, b \in C^1(\mathbb{R}^{m\times m}) \tag{4.34}$$

be a first-order differential operator with real valued (variable) matrix coefficients, acting on vector-valued functions $u = (u_\beta)_\beta$ in $\mathbb{R}^n$ and its **principal symbol** is given by the matrix-valued function

$$\sigma(P;\xi) := \sum_{j=1}^n a_j\xi_j, \quad \xi = \{\xi_j\}_{j=1}^n \in \mathbb{R}^n. \tag{4.35}$$

Definition 4.7. We say that P is a *weakly tangential operator* to the hypersurface $\mathscr{S}$, with unit normal $\boldsymbol{\nu}$, provided that

$$\sigma(P;\boldsymbol{\nu}) = 0 \ \text{ on the hypersurface } \ \mathscr{S}. \tag{4.36}$$

The most important weakly tangential differential operators to the hypersurface for us are the following:

A. The weakly tangential Günter's derivatives

$$\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}} = \partial_j - \nu_j\sum_{k=1}^n \nu_k\partial_k, \quad j = 1,\dots,n,$$

introduced in (2.20);

B. The weakly tangential Stoke's derivatives $\mathscr{M}_{jk} = \nu_j\partial_k - \nu_k\partial_j$, introduced in § 3.

The Günter's and Stoke's derivatives are tangent since their directing vector fields are tangent

$$\begin{aligned}&\mathscr{D}_j := \partial_{\boldsymbol{d}^j} = \boldsymbol{d}^j \cdot \nabla, \quad \mathscr{M}_{jk} := \partial_{\mathfrak{m}_{jk}} = \mathfrak{m}_{jk} \cdot \nabla,\\ &\boldsymbol{d}^j := \pi_{\mathscr{S}}\boldsymbol{e}^j = \boldsymbol{e}^j - \nu_j\boldsymbol{\nu} = \boldsymbol{\nu} \wedge \left(\boldsymbol{\nu} \wedge \boldsymbol{e}^j\right) = \sum_{k=1}^{n}(\delta_{jk} - \nu_j\nu_k)\boldsymbol{e}^k, \qquad (4.37)\\ &\mathfrak{m}_{jk} := \nu_j e_k - \nu_k e_j, \quad \langle \boldsymbol{d}^j, \boldsymbol{\nu}\rangle = 0, \quad \langle \mathfrak{m}_{jk}, \boldsymbol{\nu}\rangle = 0, \quad j,k = 1,\dots,n.\end{aligned}$$

Here $\pi_{\mathscr{S}}$ is the projection on the tangential space to the surface. Therefore $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ can be applied to functions which are defined on the surface $\mathscr{S}$ only.

The generating vector fields $\{\boldsymbol{d}^j\}_{j=1}^n$ $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ are not frame since they are linearly dependent

$$\sum_{j=1}^{n}\nu_j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X}) \equiv 0, \quad \mathfrak{m}_{jj} = 0, \qquad (4.38)$$

but both systems $\{\boldsymbol{d}^j\}_{j=1}^n$ and $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ are complete in the space of all tangential vector fields: any vector field $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is represented as follows

$$\boldsymbol{U}(\mathscr{X}) = \sum_{j=1}^{n}U^j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X}) = \sum_{0\le j<k\le 1}^{n} c_{jk}(\mathscr{X})\mathfrak{m}_{jk}(\mathscr{X}). \qquad (4.39)$$

Let $\mathscr{N}$ be a proper extension of the unit normal vector field $\boldsymbol{\nu}$ to $\mathscr{S}$ (cf. Definition 2.9). Then each operator $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ extends accordingly by setting (cf. (2.20))

$$\mathscr{D}_j = \partial_j - \mathscr{N}_j\partial_{\mathscr{N}}, \quad \mathscr{M}_{jk} := \mathscr{N}_j\partial_k - \mathscr{N}_k\partial_j, \quad 1 \le j,k \le n \qquad (4.40)$$

In the sequel, we shall make no distinction between the operator $\mathscr{D}_j$ or $\mathscr{M}_{jk}$ on $\mathscr{S}$ and the extended one in $\mathbb{R}^n$ given by (4.40).

Note that in a weakly tangential operator P (cf. (4.34)) the coordinate derivatives ∂_j can be replaced by the Günter's derivatives $\mathscr{D}_j$:

$$P(D)u = \sum_{j=1}^{n}a_j\partial_j u + bu = \sum_{j=1}^{n}a_j\mathscr{D}_j u + \sigma(P;\boldsymbol{\nu})u = P(\mathscr{D})u. \qquad (4.41)$$

Therefore, any weakly tangential operator P in (4.34) is *strongly tangential* to $\mathscr{S}$, which means the following: there exists an extended unit field $\mathscr{N}$ such that

$$\sigma(P;\mathscr{N}) = 0 \ \text{ in an open neighborhood of } \mathscr{S} \text{ in } \mathbb{R}^n. \qquad (4.42)$$

In particular, the extended operators $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ are strongly tangential.

For further reference, below we collect some of the most basic properties of this system of differential operators.

Lemma 4.8. *Let $\mathscr{N}$ be a proper extension of the unit vector field of normal vectors $\boldsymbol{\nu}$ to $\mathscr{S}$. The following relations are valid for $j,k=1,\dots,n$:*

i. $\mathscr{M}_{jj}=0$, $\mathscr{M}_{jk}=-\mathscr{M}_{kj}$;

ii. $\partial_k=\sum\limits_{j=1}^n \mathscr{N}_j\mathscr{M}_{jk}+\mathscr{N}_k\partial_{\mathscr{N}}=-\sum\limits_{k=1}^n \mathscr{N}_k\mathscr{M}_{jk}+\mathscr{N}_j\partial_{\mathscr{N}}$;

iii. $\sum\limits_{k=1}^n \mathscr{M}_{jk}\mathscr{N}_k=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}$, *where* $\mathscr{H}^0_{\mathscr{S}}(\mathscr{X})=-\operatorname{div}_{\mathscr{S}}\boldsymbol{\nu}(\mathscr{X})$ *and* $\mathscr{H}_{\mathscr{S}}(\mathscr{X}):=(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{X})$ *is the mean curvature at* $\mathscr{X}\in\mathscr{S}$;

iv. $\mathscr{D}_j=\sum\limits_{k=1}^n \mathscr{N}_k\mathscr{M}_{kj}$;

v. $\mathscr{M}_{jk}=\mathscr{N}_j\mathscr{D}_k-\mathscr{N}_k\mathscr{D}_j$;

vi. $\sum\limits_{j=1}^n \mathscr{N}_j\mathscr{D}_j=0$;

vii. $\sum\limits_{r,j,k=m-1}^{m+1}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=2\sum\limits_{\{r,j,k\}\subset\{(m-1),m,(m+1)\}}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=0$ *for* $m=2,\dots,n-1$, *where* $\sigma(r,j,k)$ *is the permutation sign;*

viii. $[\mathscr{D}_j,\mathscr{D}_k]=\sum\limits_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r+\big[\mathscr{N}_j\partial_{\mathscr{N}}\mathscr{N}_k-\mathscr{N}_k\partial_{\mathscr{N}}\mathscr{N}_j\big]\partial_{\mathscr{N}}$;

ix. $[\mathscr{D}_j,\mathscr{D}_k]=\sum\limits_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r=\mathscr{N}_k[\mathscr{D}_{\mathscr{N}},\partial_j]-\mathscr{N}_j[\mathscr{D}_{\mathscr{N}},\partial_k]$;

x. $\partial_j\mathscr{N}_k=\mathscr{D}_j\mathscr{N}_k=\mathscr{D}_k\mathscr{N}_j$.

Proof. The identities (i)–(ii) and (iv)–(vii) are simple consequences of the definitions. For the equality (iii) we have

$$\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^n(\mathscr{N}_j\partial_k-\mathscr{N}_k\partial_j)\mathscr{N}_k=$$
$$=\mathscr{N}_j\operatorname{div}\mathscr{N}-\frac{1}{2}\partial_j(\|\mathscr{N}\|^2)=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}},$$

as claimed.

To prove (viii) we calculate

$$\mathscr{D}_j\mathscr{D}_k=(\partial_j-\mathscr{N}_j\partial_{\mathscr{N}})(\partial_k-\mathscr{N}_k\partial_{\mathscr{N}})=\partial_j\partial_k-(\partial_j\mathscr{N}_k)\partial_{\mathscr{N}}-$$
$$-\sum_{r=1}^n\big[\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r+\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j+\mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\big]+\mathscr{N}_j\big(\partial_{\mathscr{N}}\mathscr{N}_k\big)\partial_{\mathscr{N}}+\mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}=$$
$$=-\sum_{r=1}^n\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r+\mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}}+B_{jk}=$$
$$=-\sum_{r=1}^n\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{D}_r+\mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}}+B_{jk},\tag{4.43}$$

since

$$\sum_{r=1}^n\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{N}_r\partial_{\mathscr{N}}=\frac{1}{2}\sum_{r=1}^n\mathscr{N}_k(\partial_j\mathscr{N}_r^2)\partial_{\mathscr{N}}=\frac{1}{2}\mathscr{N}_k(\partial_j1)\partial_{\mathscr{N}}=0.$$

The operator

$$B_{jk} = \partial_j\partial_k - (\partial_j\mathscr{N}_k)\partial_{\mathscr{N}} - \sum_{r=1}^{n}\big[\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j + \mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\big] + \mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}$$

is symmetric $B_{jk} = B_{kj}$ and the desired commutator identity in (viii) follows from (4.43).

The first commutator identity in (ix) utilizes the facts that $\partial_{\mathscr{N}}\mathscr{N}_k = 0$ (cf. Lemma 2.12) and follows from the identity in (viii). The second commutator identity in (ix) applies the same identity $\partial_{\mathscr{N}}\mathscr{N}_k = 0$, the identity $\partial_j\mathscr{N}_k = \partial_k\mathscr{N}_j$ (cf. (2.19)), and follows by a routine calculations.

The identities in (x) are already proved in (2.18) and (2.21). □

The next proposition generalizes Stoke's formulae (3.12) and (3.13). Since the proof applies some properties of differential forms on hypersurfaces, we drop the proof and refer [37, § 2.2, Theorem 2.1], where the case a compact Riemannian manifolds is considered.

Proposition 4.9. *Let $\boldsymbol{\nu}_\Gamma(\xi) = \big(\nu^1_\Gamma(\xi), \dots, \nu^n_\Gamma(\xi)\big)^\top$ be the unit tangential vector to $\mathscr{S}$ at the boundary point $\xi \in \Gamma := \partial\mathscr{S}$ and outward (unit) normal vector to the boundary $\Gamma = \partial\mathscr{S}$. Then*

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi\, dS = \oint_{\Gamma} \big[\nu_j\nu^k_\Gamma - \nu_k\nu^j_\Gamma\big]\varphi^+\, d\mathfrak{s}, \tag{4.44}$$

$$\int_{\mathscr{S}} \mathscr{D}_j\varphi\, dS = \oint_{\Gamma} \nu^j_\Gamma\varphi^+\, d\mathfrak{s} \tag{4.45}$$

for any real-valued function $\varphi \in C^1(\mathscr{S})$, its trace φ^+ on the boundary Γ, and any $j \neq k$, $j,k = 1,\dots,n$.

The formal adjoint in $\mathbb{R}^n$ to P in (4.34) is defined by

$$P^*u = -\sum_j \partial_j a_j^\top u + b^\top u. \tag{4.46}$$

Moreover, if $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, and if P is a first-order operator, weakly tangential to $\partial\Omega$, then, applying (3.4), P can be integrated by parts over Ω without boundary terms, i.e.

$$(Pu, v)_\Omega := \int_\Omega \langle Pu, v\rangle\, dx = \int_\Omega \langle u, P^*v\rangle\, dx =: (u, P^*v)_\Omega \tag{4.47}$$

for all vector-valued sections of vector fields $u, v \in C^1(\bar\Omega)$.

For a weakly tangential differential operator P on a closed hypersurface $\mathscr{S}$ let $Q^*_{\mathscr{S}}$ denote the "surface" adjoint:

$$(Q_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} := \oint_{\mathscr{S}} \langle Q_{\mathscr{S}}\varphi, \psi\rangle\, dS = \oint_{\mathscr{S}} \langle \varphi, Q^*_{\mathscr{S}}\psi\rangle\, dS = (\varphi, Q^*_{\mathscr{S}}\psi)_{\mathscr{S}}, \tag{4.48}$$

$$\forall\, \varphi, \psi \in C^1(\bar\Omega).$$

Throughout the paper we use the following notation

$$(u,v)_{\mathscr{S}} := \oint\limits_{\mathscr{S}} u^\top(t)\overline{v(t)}\,dS, \quad (\varphi,\ v)_\Gamma := \oint\limits_\Gamma \varphi^\top(s)\,\overline{v(s)}\,d\mathfrak{s}. \tag{4.49}$$

Corollary 4.10. *For a weakly tangential differential operator P in* (4.34) *the surface-adjoint and the formally adjoint operators coincide, i.e.,*

$$P^*_{\mathscr{S}}\varphi = P^*\varphi = -\sum_{j=1}^n \partial_j a_j^\top \varphi + b^\top\varphi. \tag{4.50}$$

In particular, the Stoke's derivatives are skew-symmetric

$$(\mathscr{M}^*_{jk})_{\mathscr{S}} = \mathscr{M}^*_{jk} = -\mathscr{M}_{jk} = \mathscr{M}_{kj}, \quad \forall\, j,k = 1,\dots,n, \tag{4.51}$$

while the adjoint operator to the operator $\mathscr{D}_j$ is given by formula

$$(\mathscr{D}_j)^*_{\mathscr{S}}\varphi = \mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi - \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi, \quad \varphi\in C^1(\mathscr{S}). \tag{4.52}$$

For any real-valued function $\varphi\in C^1(\mathscr{S})$, any $1\le j<k\le n$ and for $\boldsymbol{\nu}_\Gamma = (\nu^1_\Gamma,\dots,\nu^n_\Gamma)^\top$ being the the same as in Theorem 4.9 *the following integration by parts formula is valid:*

$$\int\limits_{\mathscr{S}} \big[(\mathscr{D}_j\varphi)\psi - \varphi\mathscr{D}^*_j\psi\big]\,dS = \oint\limits_\Gamma \nu^j_\Gamma\varphi\psi\,d\mathfrak{s}. \tag{4.53}$$

Proof. We start by proving (4.51): applying the the Stoke's formulae (3.12) from § A.5, we get

$$\oint\limits_{\mathscr{S}}(\mathscr{M}_{jk}\varphi)\psi\,dS = \oint\limits_{\mathscr{S}}(\mathscr{M}_{jk}\varphi\psi)\,dS - \oint\limits_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,dS = -\oint\limits_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,dS$$

and the equality

$$(\mathscr{M}^*_{jk})_{\mathscr{S}} = -\mathscr{M}_{jk} = \mathscr{M}_{kj} \tag{4.54}$$

follows. Moreover, note that the formal adjoint to $\mathscr{M}_{jk} = \mathscr{N}_j\mathscr{D}_k - \mathscr{N}_k\mathscr{D}_j$ is

$$\begin{aligned}\mathscr{M}^*_{jk}\varphi &= \big(\mathscr{N}_j\partial_k - \mathscr{N}_k\partial_j\big)^*\varphi = -\partial_j(\mathscr{N}_k\varphi) + \partial_k(\mathscr{N}_j\varphi) =\\ &= \mathscr{N}_k\partial_j\varphi - \mathscr{N}_j\partial_k\varphi + (\partial_j\mathscr{N}_k)\varphi - (\partial_k\mathscr{N}_j)\varphi = -\mathscr{M}_{jk}\varphi\end{aligned}$$

(cf. (2.18)), where $\varphi\in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$. (4.51) is proved.

To prove (4.50) we note that, on $\mathscr{S}$,

$$\begin{aligned}P\varphi &= \sum_{j=1}^n a_j\partial_j\varphi + b\varphi = \sum_j a_j\big[\mathscr{D}_j + \nu_j\partial_{\boldsymbol{\nu}}\big]\varphi =\\ &= \sum_{j=1}^n a_j\mathscr{D}_j\varphi + b\varphi + \sigma(P;\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\varphi = \sum_{j=1}^n a_j\mathscr{D}_j\varphi \qquad (4.55)\\ &= \sum_{j,k=1}^n a_j\nu_k\mathscr{M}_{kj}\varphi \qquad (4.56)\end{aligned}$$

due to Lemma 4.8.iv and the weak tangentiality of P. The property postulated in (4.50) follows from (4.56) and (4.51):

$$P^*_{\mathscr{S}}\varphi = \sum_{j,k=1}^{n} (\mathscr{M}_{kj})^*_{\mathscr{S}} a_j^\top \nu_k \varphi + b^\top \varphi = \sum_{j,k=1}^{n} (\mathscr{M}_{kj})^* a_j^\top \nu_k \varphi + b^\top \varphi = P^*\varphi.$$

(4.52) follows as in (4.27), since (cf. (2.19)) $\partial_{\mathscr{N}} \mathscr{N}_j = 0$.

To prove (4.29) we apply (2.23) and proceed as follows

$$\sum_{k=1}^{n} \mathscr{D}_k \nu_k = \sum_{k=1}^{n} \Big(\partial_k \nu_k - \nu_k \sum_{j=1}^{n} \nu_j \partial_j \nu_k\Big) = -\mathscr{H}^0_{\mathscr{S}} - \sum_{j=1}^{n} \frac{\nu_j}{2} \partial_j 1 = -\mathscr{H}^0_{\mathscr{S}}.$$

For the proof of the last formula (4.53) we apply Lemma 4.8.iv, (4.51), the equalities $\sum\limits_{k=1}^{n} \nu_k^2 = 1$, $\sum\limits_{k=1}^{n} \nu_k \nu_\Gamma^k = 0$ and proceed as follows:

$$\oint_{\mathscr{S}} (\mathscr{D}_j \varphi)\psi\, dS = \sum_{k=1}^{n} \oint_{\mathscr{S}} \nu_k (\mathscr{M}_{jk}\varphi)\psi\, dS - \sum_{k=1}^{n} \oint_{\mathscr{S}} \psi(\mathscr{M}_{jk}\nu_k\psi)\, dS +$$
$$+ \sum_{k=1}^{n} \oint_{\Gamma} (\nu_k^2 \nu_\Gamma^j - \nu_k \nu_j \nu_\Gamma^k)\varphi\psi\, d\mathfrak{s} = \oint_{\mathscr{S}} \psi(\mathscr{D}_j^*\psi)\, dS + \oint_{\Gamma} \nu_\Gamma^j \varphi\psi\, d\mathfrak{s}. \quad \square$$

Lemma 4.11. *Let P be, as in (4.34), a first-order differential operator with C^1-smooth coefficients. P is weakly/strongly tangential if and only if the adjoint P^* operator is so.*

If P is weakly tangential to $\mathscr{S}$ and P is defined in a neighborhood of $\mathscr{S}$, then

$$(P\varphi)\big|_{\mathscr{S}} = P(\varphi|_{\mathscr{S}}) \tag{4.57}$$

for every C^1 function φ defined in a neighborhood of $\mathscr{S}$. In particular,

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}} = \mathscr{D}_j(\varphi|_{\mathscr{S}}),\ \ \mathscr{M}_{jk}\varphi\big|_{\mathscr{S}} = \mathscr{M}_{jk}(\varphi|_{\mathscr{S}}),\ \ j,k = 1,\dots,n. \tag{4.58}$$

Furthermore, (4.57) is true for the adjoint P^*, and

$$\int_{\mathscr{S}} \langle P\varphi, \psi\rangle\, dS = \int_{\mathscr{S}} \langle \varphi, P^*\psi\rangle\, dS + \oint_{\Gamma} \langle \sigma(P; \boldsymbol{\nu}_\Gamma)\varphi, \psi\rangle\, d\mathfrak{s} \tag{4.59}$$

for any vector-valued functions $\varphi, \psi \in \mathscr{S}$.

Proof. The first assertion follows since $\sigma(P^*;\xi) = -\sigma(P;\xi)^\top$, for each $\xi \in \mathbb{R}^n$.

Due to the representation (4.55) it suffices to prove (4.57) for only the operator $\mathscr{D}_j = \boldsymbol{d}^j \cdot \nabla$, where $\boldsymbol{d}^j = \pi_{\mathscr{S}} \boldsymbol{e}^j = \mathscr{N} \wedge (\mathscr{N} \wedge \boldsymbol{e}^j)$ is at least C^1-smooth vector field in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, tangent to the surface $\mathscr{S}$ at surface points (cf. (4.37)). Thus, we have to justify the following equality:

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}} = \big(\boldsymbol{d}^j \cdot \nabla\big)\varphi\big|_{\mathscr{S}} = \boldsymbol{d}^j \cdot \nabla(\varphi|_{\mathscr{S}}) = \mathscr{D}_j(\varphi|_{\mathscr{S}}). \tag{4.60}$$

The vector field $\boldsymbol{d}^j(x) = \boldsymbol{d}^j(\theta, \mathscr{X})$ depends on the signed distance $\theta = \mathrm{dist}(x, \mathscr{S}) = \pm|x - \mathscr{X}|$ continuously ($\theta > 0$ for the outer domain and $\theta > 0$ for the inner one). Let $\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)}$ be the integral curve of the vector field $\boldsymbol{d}^j$ and

$$\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)} : \ \Omega_{\mathscr{S}} \to \Omega_{\mathscr{S}}, \ \ \mathscr{F}^t_{\boldsymbol{d}^j(0,\cdot)} = \mathscr{F}^t_{\boldsymbol{d}^j}(\cdot) : \ \mathscr{S} \to \mathscr{S} \tag{4.61}$$

be the flow generated by this vector field ℓ_θ in the neighborhood $\Omega_{\mathscr{S}}$ (cf. (4.7)). Since the flow depends continuously on the parameter θ, we get

$$\big(\boldsymbol{d}^j(\theta, \mathscr{X}) \cdot \nabla\big)\varphi\Big|_{\mathscr{S}} = \lim_{\theta\to 0} \frac{d}{dt}\, \varphi\big(\mathscr{F}^t_{\boldsymbol{d}^j(\theta,\mathscr{X})}\big)\Big|_{t=0} = \frac{d}{dt}\, \varphi(\mathscr{F}^t_{\boldsymbol{d}^j})\big|_{t=0} =$$
$$= \boldsymbol{d}^j \cdot \nabla(\varphi|_{\mathscr{S}}) = \mathscr{D}_j(\varphi|_{\mathscr{S}})$$

and (4.60) is proved.

Next, using (4.55), (4.53) and integrating by parts we get

$$\int\limits_{\mathscr{S}} \langle P\varphi, \psi\rangle\, dS = \sum_{j=1}^n \int\limits_{\mathscr{S}} \langle a_j \mathscr{D}_j\varphi, \psi\rangle\, dS + \int\limits_{\mathscr{S}} \langle b\varphi, \psi\rangle\, dS =$$
$$= \sum_{j=1}^n \int\limits_{\mathscr{S}} \langle \varphi, \mathscr{D}^*_j a^\top_j \psi\rangle\, dS + \int\limits_{\mathscr{S}} \langle \varphi, b^\top\psi\rangle\, dS + \sum_{j=1}^n \oint\limits_{\Gamma} \langle \varphi, \nu^j_\Gamma a^\top_j \psi\rangle\, dS =$$
$$= \int\limits_{\mathscr{S}} \langle \varphi, P^*\psi\rangle\, dS + \oint\limits_{\Gamma} \langle \sigma(P; \boldsymbol{\nu}_\Gamma)\varphi, \psi\rangle\, d\mathfrak{s}$$

and this completes the proof. □

Based on the above formulae it is easy to write adjoint to a high order partial differential operator

$$\mathbf{G}(\mathscr{D}) = \sum_{|\alpha|\le k} \boldsymbol{g}_\alpha(\mathscr{X})\mathscr{D}^\alpha = \sum_{|\beta|\le k} f_\beta(\mathscr{X})\mathscr{M}^\beta, \ \ \mathscr{X} \in \mathscr{S},$$
$$\nabla^\alpha_{\mathscr{S}} := \mathscr{D}^{\alpha_1}_1 \dots \mathscr{D}^{\alpha_n}_n, \ \ \alpha \in \mathbb{N}^n_0,$$
$$\mathscr{M}^\beta_{\mathscr{S}} := \mathscr{M}^{\beta_1}_1 \dots \mathscr{M}^{\beta_m}_m, \ \ \beta \in \mathbb{N}^m_0, \ \ m = \frac{n(n-1)}{2}$$

on a hypersurface $\mathscr{S}$ and find ample examples of self adjoint operators among them. Below we will consider concrete examples of such self adjoint operators which encounter in applications.

5. Differential Operators on Hypersurfaces in $\mathbb{R}^n$

Let us start by the definition of the surface divergence $\mathrm{div}_{\mathscr{S}}$, the surface gradient $\nabla_{\mathscr{S}}$ and the surface Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$.

Consider the following **differential 1-form**

$$\omega_f(\boldsymbol{V}) := \partial_V f = \sum_{k=1}^{n-1} V^k \partial_k f \ \text{ for } \ f \in C^1(\mathscr{S}), \ \ \boldsymbol{V} = \sum_{k=1}^{n-1} V^k \boldsymbol{g}_k \in \mathscr{V}(\mathscr{S}), \tag{5.1}$$

where $\mathscr{V}(\mathscr{S})$ denotes the linear space of tangential vector fields to a surface $\mathscr{S}$. The form is well defined because the differential operator $\partial_{\boldsymbol{V}}$ is tangential and can be applied to a function f defined on the surface $\mathscr{S}$ only.

Due to the Riesz theorem for a given f there exists a vector field $\nabla_{\mathscr{S}} f \in \mathscr{V}(\mathscr{S})$ such that

$$\omega_f(\boldsymbol{V}) := \langle \nabla_{\mathscr{S}} f, \boldsymbol{V}\rangle \ \text{ for all } \ \boldsymbol{V} \in \mathscr{V}(\mathscr{S}), \tag{5.2}$$

which is, according the classical differential geometry, the **surface Gradient** of a function $f \in C^1(\mathscr{S})$ and maps

$$\nabla_{\mathscr{S}} : \ C^\infty(\mathscr{S}) \to \mathscr{V}(\mathscr{S}). \tag{5.3}$$

The **surface divergence**

$$\operatorname{div}_{\mathscr{S}} : \ \mathscr{V}(\mathscr{S}) \to C^\infty(\mathscr{S}) \tag{5.4}$$

of a smooth tangential vector field $\boldsymbol{V}$ in (5.1) is, by the definition,

$$\operatorname{div}_{\mathscr{S}}\boldsymbol{V} := \sum_{k=1}^{n-1} V^j_{;j}, \ \ V^j_{;k} := \partial_k V^j + \sum_{m=1}^{n-1} \Gamma^j_{km} V^m \tag{5.5}$$

where Γ^j_{km} denotes the **Christoffel symbols**:

$$\Gamma^j_{km} := \frac{1}{2}\sum_{k=1}^{n-1} g^{j\ell}\big[\partial_m g_{k\ell} + \partial_k g_{m\ell} - \partial_\ell g_{km}\big] = \Gamma^j_{mk}. \tag{5.6}$$

$\operatorname{div}_{\mathscr{S}}$ is the negative dual to the surface gradient:

$$\langle \operatorname{div}_{\mathscr{S}}\boldsymbol{V}, f\rangle := -\langle \boldsymbol{V}, \nabla_{\mathscr{S}} f\rangle, \ \ \forall\, \boldsymbol{V} \in \mathscr{V}(\mathscr{S}), \ \ \forall\, f \in C^1(\mathscr{S}). \tag{5.7}$$

The Laplace–Beltrami operator $\Delta_{\mathscr{S}}$ on $\mathscr{S}$ is defined as the composition

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \psi = \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}}\, \psi = -\nabla^*_{\mathscr{S}}\Big(\nabla_{\mathscr{S}}\psi\Big). \tag{5.8}$$

Expressions of the surface divergence and gradient in intrinsic parameters of the surface $\mathscr{S}$ (tangential vector fields, Metric tensor etc.) are rather complicated (cf. e.g., [36]). We suggests an alternative, much simpler interpretation.

Theorem 5.1. *For any function $\varphi \in C^1(\mathscr{S})$ we have*

$$\nabla_{\mathscr{S}}\varphi = \big\{\mathscr{D}_1\varphi, \mathscr{D}_2\varphi, \ldots, \mathscr{D}_n\varphi\big\}^\top. \tag{5.9}$$

Also, for a 1-smooth tangential vector field $\boldsymbol{V} = \sum\limits_{j=1}^{n} V^j e_j \in \mathscr{V}(\mathscr{S})$,

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{V} = -\nabla^*_{\mathscr{S}}\boldsymbol{V} := \sum_{j=1}^{n} \mathscr{D}_j V^j. \tag{5.10}$$

The Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ on $\mathscr{S}$ takes the form

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \psi = \sum_{j=1}^{n} \mathscr{D}_j^2\psi = \sum_{j<k} \mathscr{M}^2_{jk}\psi = \frac{1}{2}\sum_{j,k=1}^{n} \mathscr{M}^2_{jk}\psi, \ \ \forall\, \psi \in C^2(\mathscr{S}). \tag{5.11}$$

Proof. Any function $\varphi \in C^1(\mathscr{S})$ is approximated, $\|\varphi - \varphi_k | C^1(\mathscr{S})\| \to 0$ as $k \to \infty$, by a functions $\varphi_k \in C^1(U_{\mathscr{S}})$, $k = 1, 2, \dots$, defined in a neighborhood $U_{\mathscr{S}} \subset \mathbb{R}^n$ of $\mathscr{S}$. Then, from the definition of the surface gradient (5.2), follows

$$\langle \nabla_{\mathscr{S}} \varphi, \boldsymbol{V} \rangle := \omega_\varphi(\boldsymbol{V}) := \partial_{\boldsymbol{V}} \varphi = \lim_{k\to\infty} \partial_{\boldsymbol{V}} \varphi_k = \lim_{k\to\infty} \sum_{j=1}^n V^j \partial_j \varphi_k =$$
$$= \lim_{k\to\infty} \langle \nabla \varphi_k, \boldsymbol{V} \rangle = \lim_{k\to\infty} \langle \pi_{\mathscr{S}} \nabla \varphi_k, \boldsymbol{V} \rangle \ \text{ for } \ \varphi \in C^1(\mathscr{S}),$$
$$\boldsymbol{V} = \sum_{k=1}^{n-1} \widetilde{\boldsymbol{V}}^k \boldsymbol{g}_k = \sum_{k=1}^n V^k e_k \in \mathscr{V}(\mathscr{S}),$$

where $\pi_{\mathscr{S}}$ denotes the orthogonal projection onto the tangential vector fields $\mathscr{V}(\mathscr{S})$ (cf. (2.25)); we get finally

$$\nabla_{\mathscr{S}} \varphi = \lim_{k\to\infty} \pi_{\mathscr{S}} \nabla \varphi_k = \lim_{k\to\infty} \left\{ \partial_j \varphi_k - \nu_j \sum_{m=1}^n \nu_m \partial_m \varphi_k \right\}_{j=1}^n =$$
$$= \lim_{k\to\infty} \big(\mathscr{D}_1 \varphi_k, \dots, \mathscr{D}_n \varphi_k \big)^\top = \big(\mathscr{D}_1 \varphi, \dots, \mathscr{D}_n \varphi \big)^\top .$$

Now we consider the divergence operator $\operatorname{div}_{\mathscr{S}} = \nabla^*_{\mathscr{S}}$ (cf. (5.4), (5.7)). Let a scalar function φ and a tangential vector field $\boldsymbol{V} \in \mathscr{V}(\mathscr{S})$ be both smooth, $\mathscr{S}$ be non-closed with the boundary $\partial\mathscr{S} \neq \varnothing$, and the supports have no intersections with the boundary $\operatorname{supp} \varphi \cap \partial\mathscr{S} = \varnothing$, $\operatorname{supp} \boldsymbol{V} \cap \partial\mathscr{S} = \varnothing$. By applying the duality, the proved formulae (5.9) and (4.52) for the dual $(\mathscr{D}_j)^*_{\mathscr{S}}$, we get:

$$(\operatorname{div}_{\mathscr{S}} \boldsymbol{V}, \varphi)_{\mathscr{S}} = -(\boldsymbol{V}, \nabla_{\mathscr{S}} \psi)_{\mathscr{S}} =$$
$$= \oint_{\mathscr{S}} \sum_{j=1}^n V^j(\mathscr{x}) \mathscr{D}_j \varphi(\mathscr{x})\, dS = -\oint_{\mathscr{S}} \sum_{j=1}^n (\mathscr{D}_j)^*_{\mathscr{S}} V^j(\mathscr{x}) \varphi(\mathscr{x})\, dS =$$
$$= \oint_{\mathscr{S}} \sum_{j=1}^n \mathscr{D}_j V^j(\mathscr{x}) \varphi(\mathscr{x})\, dS + \mathscr{H}^0_{\mathscr{S}} \oint_{\mathscr{S}} \sum_{j=1}^n \nu_j(\mathscr{x}) V^j(\mathscr{x}) \varphi(\mathscr{x})\, dS =$$
$$= \sum_{j=1}^n (\mathscr{D}_j V^j, \varphi)_{\mathscr{S}}.$$

We applied above that $\boldsymbol{V}$ is tangent $\langle \boldsymbol{\nu}(\mathscr{x}), \boldsymbol{V}(\mathscr{x}) \rangle = \sum\limits_{j=1}^n \nu_j(\mathscr{x}) V^j(\mathscr{x}) \equiv 0$. Since the function φ is arbitrary, (5.10) follows.

To prove (5.8) we apply (5.9), (4.52) and proceed as follows

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \psi = \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}}\, \psi =$$
$$= -\sum_{j=1}^n (\mathscr{D}_j)^* \mathscr{D}_j \psi = \sum_{j=1}^n \mathscr{D}_j^2 \psi + \mathscr{H}^0_{\mathscr{S}} \sum_{j=1}^n \nu_j \mathscr{D}_j \psi = \sum_{j=1}^n \mathscr{D}_j^2 \psi,$$

since $\langle \boldsymbol{\nu}, \mathscr{D}\rangle = \sum\limits_{j=1}^{n} \nu_j \mathscr{D}_j = 0$ (cf. Lemma 4.8.v)).

To prove the last equality (5.11) we note that (cf. (4.28))

$$\sum_{j=1}^{n} \nu_j \mathscr{D}_k(\nu_j \psi) = \boldsymbol{\nu}^2 \mathscr{D}_k \psi + \sum_{j=1}^{n} \nu_j (\mathscr{D}_k \nu_j)\psi = \mathscr{D}_k \psi \tag{5.12}$$

and $\sum\limits_{j=1}^{n} \nu_j \mathscr{D}_j = 0$, $\mathscr{M}_{jk} = \nu_j \mathscr{D}_k - \nu_k \mathscr{D}_j$ for $j, k = 1, \dots, n$; (cf. Lemma 4.8.vi, 4.8.v). Then

$$\frac{1}{2} \sum_{j,k=1}^{n} \mathscr{M}_{jk}^2 \psi = \frac{1}{2} \sum_{j,k=1}^{n} \big[\nu_j \mathscr{D}_k - \nu_k \mathscr{D}_j\big]^2 \psi =$$

$$= \frac{1}{2} \sum_{j,k=1}^{n} \Big[\nu_j \mathscr{D}_k \nu_j \mathscr{D}_k \psi - \nu_j \mathscr{D}_k \nu_k \mathscr{D}_j \psi + \nu_k \mathscr{D}_j \nu_k \mathscr{D}_j \psi - \nu_k \mathscr{D}_j \nu_j \mathscr{D}_k \psi\Big] =$$

$$= \sum_{j,k=1}^{n} \Big[\nu_j \mathscr{D}_k \nu_j \mathscr{D}_k \psi - \nu_j \mathscr{D}_k \nu_k \mathscr{D}_j \psi\Big] =$$

$$= \sum_{k=1}^{n} \mathscr{D}_k^2 \psi - \sum_{j,k=1}^{n} \Big[\nu_j \nu_k \mathscr{D}_k \mathscr{D}_j \psi + (\mathscr{D}_k \nu_k) \nu_j \mathscr{D}_j \psi\Big] = \sum_{k=1}^{n} \mathscr{D}_k^2 \psi = \Delta_{\mathscr{S}} \psi. \quad \square$$

Lemma 5.2. *Let $\mathscr{S}$ be μ–smooth and $\ell \in \mathbb{N}_0$, $\ell \leq \mu$. The Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ is elliptic on the hypersurface $\mathscr{S}$ and self adjoint, i.e.,*

$$\boldsymbol{\Delta}_{\mathscr{S}}(t, \xi) \equiv |\xi|^2, \ \ \forall\, (t, \xi) \in \mathscr{T}^*(\mathscr{S}), \ \ (\boldsymbol{\Delta}_{\mathscr{S}})^*_{\mathscr{S}} = \boldsymbol{\Delta}_{\mathscr{S}}. \tag{5.13}$$

For arbitrary $\ell = 0, \pm 1, \dots$ the operator

$$-\boldsymbol{\Delta}_{\mathscr{S}} : \ \mathbb{W}^1_2(\mathscr{S}) \to \mathbb{W}^{-1}_2(\mathscr{S}) \tag{5.14}$$

is positive definite (coercive) on non–constant functions

$$(-\boldsymbol{\Delta}_{\mathscr{S}} \varphi, \varphi)_{L_2(\mathscr{S})} = \sum_{k=1}^{n} (\mathscr{D}_k \varphi, \mathscr{D}_k \varphi)_{L_2(\mathscr{S})} = \big\| \nabla_{\mathscr{S}} \varphi | L_2(\mathscr{S}) \big\| > 0 \tag{5.15}$$

$$\text{for } \ \forall \varphi \in \mathbb{W}^1_2(\mathscr{S}), \ \ \varphi \neq \text{const}.$$

Proof. Let us prove that $\boldsymbol{\Delta}_{\mathscr{S}}$ is elliptic. We proceed straightforwardly:

$$\begin{aligned} \boldsymbol{\Delta}_{\mathscr{S}}(t, \xi) &= \sum_{k=1}^{n} \mathscr{D}_k^2(t, \xi) = \sum_{k=1}^{n} \big[\xi_k - \nu_k(t)(\vec{\nu}(t), \xi)\big]^2 \\ &= |\xi|^2 - 2(\vec{\nu}(t), \xi)^2 + |\vec{\nu}(t)|^2 (\vec{\nu}(t), \xi)^2 = \\ &= |\xi|^2 - (\vec{\nu}(t), \xi)^2 = |\xi|^2 \ \text{ for } \ (t, \xi) \in \mathscr{T}^*(\mathscr{S}). \end{aligned} \tag{5.16}$$

From the definition (5.8) and the property (5.7) it follows easily that $\boldsymbol{\Delta}_{\mathscr{S}}$ is self adjoint and non-negative:

$$(\boldsymbol{\Delta}_{\mathscr{S}} \varphi, \varphi)_{\mathbb{W}^\ell_2(\mathscr{S})} = (\nabla_{\mathscr{S}} \varphi, \nabla_{\mathscr{S}} \varphi)_{\mathbb{W}^\ell_2(\mathscr{S})} = (\varphi, \boldsymbol{\Delta}_{\mathscr{S}} \varphi)_{\mathbb{W}^\ell_2(\mathscr{S})},$$

$$(\boldsymbol{\Delta}_{\mathscr{S}}\varphi,\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}=-(\nabla_{\mathscr{S}}\varphi,\nabla_{\mathscr{S}}\varphi)_{\mathbb{W}^\ell_2(\mathscr{S})}=\big\|\nabla_{\mathscr{S}}\varphi|\mathbb{W}^\ell_2(\mathscr{S})\big\|>0$$
$$\text{provided }\ \varphi\in\mathbb{W}^{\ell+2}_2(\mathscr{S}),\quad \varphi\neq\text{const}.$$

The last assertion (5.15) also follows from the definition (5.8) and the property (5.7):

$$(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi,\varphi)_{L_2(\mathscr{S})}=(\nabla_{\mathscr{S}}\varphi,\nabla_{\mathscr{S}}\varphi)_{\mathbb{L}_2(\mathscr{S})}=\big\|\nabla_{\mathscr{S}}\varphi|\mathbb{W}^\ell_2(\mathscr{S})\big\|>0$$
$$\text{for }\ \varphi\in\mathbb{W}^1_2(\mathscr{S}),\quad \varphi\neq\text{const}. \qquad \square$$

We remind that the surface gradient $\nabla_{\mathscr{S}}$ maps scalar functions to the tangential vector fields

$$\nabla_{\mathscr{S}}:\ C^\infty(\mathscr{S})\to\mathscr{V}(\mathscr{S}):=\mathbb{C}(\mathscr{S},\mathscr{V}(\mathscr{S})) \tag{5.17}$$

and the scalar product with the normal vector vanishes everywhere on the surface $\mathscr{S}$:

$$\langle\boldsymbol{\nu}(\mathscr{x}),\nabla_{\mathscr{S}}\varphi(\mathscr{x})\rangle\equiv 0\ \text{ for all }\ \varphi\in C^1(\mathscr{S}). \tag{5.18}$$

Tangential derivatives can be applied to the definition of Sobolev spaces $\mathbb{W}^\ell_p(\mathscr{S})=\mathbb{H}^\ell(\mathscr{S})$, $\ell\in\mathbb{N}^0$, $1\leq p<\infty$ on an ℓ-smooth surface $\mathscr{S}$

$$\mathbb{H}^\ell(\mathscr{S})=\mathbb{W}^\ell_p(\mathscr{S}):=$$
$$:=\Big\{\varphi\in D'(\mathscr{S}):\ \nabla^\alpha_{\mathscr{S}}\varphi\in\mathbb{L}_p(\mathscr{S}),\ \forall\,\alpha\in\mathbb{N}^n_0,\ |\alpha|\leq\ell\Big\}. \tag{5.19}$$

Equivalently, $\mathbb{W}^\ell_p(\mathscr{S})$ is the closure of the space $C^\ell(\mathscr{S}$ with respect to the norm

$$\big\|\varphi|\mathbb{W}^\ell_p(\mathscr{S})\big\|:=\Big[\sum_{|\alpha|\leq\ell}\big\|\mathscr{D}_\alpha\varphi|\mathbb{L}_p(\mathscr{S})\big\|_p\Big]^{1/p}.$$

The space $\mathbb{W}^\ell_p(\mathscr{S})$ can also be understood in distributional sense: derivative $\mathscr{D}_j\varphi\in\mathbb{L}_2(\mathscr{S})$ means that there exists a function in $\mathbb{L}_2(\mathscr{S})$ denoted by $\mathscr{D}_j\varphi$ such that

$$(\mathscr{D}_j\varphi,\psi):=(\varphi,\mathscr{D}^*_j\psi):=\int\limits_{\mathscr{S}}\varphi(\mathscr{x})\overline{\mathscr{D}^*_j\psi(\mathscr{x})}\,dS,\quad\forall\,\psi\in\mathbb{L}_2(\mathscr{S})$$

(cf. (4.52) for the formal dual $\mathscr{D}^*_m$).

Moreover, $\mathbb{W}^\ell_2(\mathscr{S})$ is a Hilbert space with the scalar product

$$(\varphi,v)^{(\ell)}_{\mathscr{S}}:=\sum_{|\alpha|\leq\ell}\oint\limits_{\mathscr{S}}\mathscr{D}^\alpha_x\varphi)(\mathscr{x})\overline{\mathscr{D}^\alpha_x\ v(\mathscr{x})}\,dS. \tag{5.20}$$

Under the space $\mathbb{W}^{-\ell}_2(\mathscr{S})$ with a negative order $-\ell$, $\ell\in\mathbb{N}$, is understood, as usual, the dual space of distributions to the Sobolev space $\mathbb{W}^\ell_2(\mathscr{S})$.

The following Proposition 5.3 accomplishes the definition of the Banach spaces $\mathbb{H}^m_p(\mathscr{S})$ (cf. [15] for a simple proof).

Proposition 5.3. *For $\varphi\in C^1(\mathscr{S})$ the surface gradient vanishes $\nabla_{\mathscr{S}}\varphi\equiv 0$ if and only if $\varphi(\mathscr{x})\equiv$ const.*

Remark 5.4. For any smooth scalar function f, defined in a neighborhood of $\mathscr{S}$, there holds (see [16])

$$(\boldsymbol{\Delta}_{\mathbb{R}^n} f)\big|_{\mathscr{S}} = \boldsymbol{\Delta}_{\mathscr{S}}(f|_{\mathscr{S}}) + \mathscr{H}^0_{\mathscr{S}}(\partial_{\boldsymbol{\nu}} f)\big|_{\mathscr{S}} + (\partial^2_{\boldsymbol{\nu}} f)\big|_{\mathscr{S}}. \tag{5.21}$$

In particular, for the case the unit sphere in $\mathbb{R}^n$, i.e., $\mathscr{S} = S^{n-1}$ one can choose $\boldsymbol{\nu}(x) := x/\|x\|$, $x \in \mathbb{R}^n \setminus 0$, so that $\mathscr{H}^0_{\mathscr{S}} := \operatorname{div} \boldsymbol{\nu} = (n-1)/\|x\|$, and $\partial_{\boldsymbol{\nu}} = \sum (x_j/\|x\|)\partial_j = \partial/\partial r$, the radial derivative in $\mathbb{R}^n$. Then (5.21) becomes, after a rescaling, the classical formula

$$\boldsymbol{\Delta}_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_{S^{n-1}}.$$

A number of related identities, at least for $n = 3$ and special extensions of the unit normal, can be found in [12], [7], [10], [23], [26], [31] [33] and the references therein.

6. The Equation of Anisotropic Elastic Hypersurface

One way of understanding the genesis of the Laplace-Beltrami operator (5.8) is to consider the energy functional

$$\mathscr{E}[u] := \int\limits_{\mathscr{S}} \|\nabla u\|^2 \, dS, \quad u \in C^\infty(\mathscr{S}). \tag{6.1}$$

Then any minimizer u of the functional (6.1) should satisfy

$$0 = \frac{d}{dt}\,\mathscr{E}[u + tv]\big|_{t=0} = \int\limits_{\mathscr{S}} \Big[\langle \nabla u, \nabla v\rangle + \langle \nabla v, \nabla u\rangle\Big]\, dS =$$

$$= 2\mathrm{Re}\int\limits_{\mathscr{S}} \langle \nabla u, \nabla v\rangle \, dS, \quad u \in C^\infty(\mathscr{S}), \quad \forall\, v \in C^\infty_0(\mathscr{S}), \tag{6.2}$$

which implies

$$\Delta u = 0 \quad \text{on} \quad \mathscr{S}. \tag{6.3}$$

In other words, (6.3) is the Euler-Lagrange equation associated with the integral functional (6.1).

We assume that the closed hypersurface $\mathscr{S}$ is ℓ-smooth and $\ell \geq 1$.

Our aim is to adopt a similar point of view in the case of anisotropic (Lamé) system of elasticity on $\mathscr{S}$. The starting point is to consider the total free (elastic) energy

$$\mathscr{E}[\boldsymbol{U}] \coloneqq \int\limits_{\mathscr{S}} E(y, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y))\, dS, \quad \mathscr{D}^{\mathscr{S}}\boldsymbol{U} \coloneqq \big[(\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U})^0_k\big]_{n\times n}, \quad \boldsymbol{U} \in \mathscr{V}(\mathscr{S}), \tag{6.4}$$

ignoring at the moment the displacement boundary conditions (Koiter's model). As before, equilibria states correspond to minimizers of the above variational integral (see [32, § 5.2]). First we should identify the correct form of the stored energy density $E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, $E = (\mathfrak{S}_{\mathscr{S}}, \mathrm{Def}_{\mathscr{S}})$ depends bilinearly on the stress tensor $\mathfrak{S}_{\mathscr{S}} = [\mathfrak{S}^{jk}]_{n\times n}$ and the deformation (strain) tensor

$$\mathrm{Def}_{\mathscr{S}} = [\mathfrak{D}_{jk}]_{n\times n},$$

$$\mathfrak{D}_{jk}\boldsymbol{U} := \frac{1}{2}\left[(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j + (\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k\right] = \frac{1}{2}\left[\mathscr{D}_j U^0_k + \mathscr{D}_k U^0_j + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\right] =$$
$$= \frac{1}{2}\left[\mathscr{D}_j U^0_k + \mathscr{D}_k U^0_j + \sum_{q=1}^{n} U_q\mathscr{D}_q(\nu_j\nu_k)\right], \ \ \forall\, j,k = 1,\dots,n \qquad (6.5)$$

(cf. [16]) which, according to Hooke's law, satisfy $\mathfrak{S}_{\mathscr{S}} = \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}$, for some linear, fourth-order tensor $\mathbb{T}$. If the medium is also homogeneous (i.e. the density and elastic parameters are position-independent), it follows that E *depends quadratically on the covariant derivative* $\mathscr{D}^{\mathscr{S}}\boldsymbol{U}$, i.e.

$$E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) = \left\langle \mathbb{T}\,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\right\rangle \qquad (6.6)$$

for a linear operator

$$\mathbb{T}: \ \mathbb{M}_{n,n}(\mathbb{R}) \to \mathbb{M}_{n,n}(\mathbb{R}), \qquad (6.7)$$

where $\mathbb{M}_{n,n}(\mathbb{R})$ stands for the vector space of all $n\times n$ matrices with real entries. Hereafter, we organize $\mathbb{M}_{n,n}(\mathbb{R})$ as a real Hilbert space with respect to the inner product

$$\langle A, B\rangle := \mathrm{Tr}(AB^\top) = \sum_{i,j} a_{ij}b_{ij}, \ \ \forall\, A = [a_{ij}]_{i,j}, \ \ B = [b_{ij}]_{i,j} \in \mathbb{M}_{n,n}(\mathbb{R}), \ (6.8)$$

where $B^\top$ denotes transposed matrix, and Tr is the usual trace operator for square matrices.

A linear operator (6.7) is a tensor of order 4, i.e., $\mathbb{T} = [c_{ijk\ell}]_{ijk\ell}$, and

$$\mathbb{T}\,A = \Big[\sum_{k,\ell} c_{ijk\ell}a_{k\ell}\Big]_{ij}, \ \ \text{for } \ A = [a_{k\ell}]_{k\ell} \in \mathbb{M}_{n,n}(\mathbb{R}). \qquad (6.9)$$

$\mathbb{T}$ will be referred to in the sequel as the *elasticity tensor*. It is customary to assume that the elasticity tensor (6.7) is self-adjoint

$$\langle \mathbb{T}\,A, B\rangle = \langle A, \mathbb{T}\,B\rangle, \ \ A, B \in \mathbb{M}_{n,n}(\mathbb{R}). \qquad (6.10)$$

The condition rescaling (6.10), written in coordinate notation, is equivalent to the following equality

$$c_{ijk\ell} = c_{k\ell ij}, \ \ \forall\, i,j,k,\ell. \qquad (6.11)$$

Indeed, the equality

$$\mathrm{Tr}((\mathbb{T}\,A)B^\top) = \sum_{i,j,k,\ell} c_{ijk\ell}a_{k\ell}b_{ij} = \sum_{i,j,k,\ell} c_{k\ell ij}a_{k\ell}b_{ij} = \mathrm{Tr}(A(\mathbb{T}\,B)^\top)$$

holds, for arbitrary $A = [a_{k\ell}]_{k\ell}$ and $B = [b_{k\ell}]_{k\ell}$, if and only if (6.11) holds: by inserting the delta functions $a_{k\ell} = \delta_{k\ell}$, $b_{ij} = \delta_{ij}$ we get the equality (6.11).

It is also customary to impose a symmetry condition, presented with two natural options:

$$\mathbb{T}\,(A^\top) = \mathbb{T}\,A \quad\text{and}\quad (\mathbb{T}\,A)^\top = \mathbb{T}\,A, \quad \forall\, A \in \mathbb{M}_{n,n}(\mathbb{R}). \tag{6.12}$$

Then (6.12) amounts to the following symmetry in the indices of the elastic tensor:

$$c_{ijk\ell} = c_{ij\ell k} \quad\text{and}\quad c_{ijk\ell} = c_{jik\ell}, \quad \forall\, i, j, k, \ell, \tag{6.13}$$

where the second (the first) equality follows already from (6.11) and the first (the second) equality in (6.13).

Remark 6.1. The conditions (6.10) and the first equality in (6.12) imply the second equality in (6.12) as well as the conditions (6.10) and the second equality in (6.12) imply the first equality in (6.12). This is evident if we apply an equivalent formulation for corresponding tensors and matrices: (6.11) and (6.13).

A linear operator $\mathbb{T}$ in the energy functional of anisotropic elasticity (6.6) satisfies the symmetry conditions (6.10), and (6.12). Equivalently, the corresponding elasticity tensor $\mathbb{T} = [c_{ijk\ell}]_{ijk\ell}$ has the symmetries (6.11), (6.13) and, therefore, might have $n + n^2(n-1)^2/2$ different entries only.

By inserting the value (6.5) of deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}$ and applying the symmetry properties (6.13), we obtain

$$\begin{aligned} 4\big\langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x)\big\rangle &= \\ &= \big\langle \mathbb{T}\,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\big\rangle = E\big(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\big) \end{aligned} \tag{6.14}$$

(cf. (6.6)) which means that *the density of the elastic energy functional depends quadratically also on the deformation tensor.*

The density of the potential energy of an elastic medium should be strictly positive for the non-vanishing deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U} \neq 0$ (the energy conservation law!). This leads to the following.

Lemma 6.2. *There exists a constant $C_0 > 0$ such that*

$$\langle \mathbb{T}\,\zeta, \zeta\rangle := \sum_{i,j,k,\ell} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geq C_0 \sum_{i,j} |\zeta_{i,j}|^2 := C_0|\zeta|^2 \tag{6.15}$$

for all symmetric and complex valued $\zeta_{ij} = \zeta_{ji} \in \mathbb{C}$ tensors $\zeta := [\zeta_{ij}]_{n\times n}$.

Proof. The sum in the left hand side of (6.15) is real $\langle \mathbb{T}\,\zeta, \zeta\rangle = \overline{\langle \mathbb{T}\,\zeta, \zeta\rangle}$ (easy to check applying the symmetry properties (6.13) of the real valued coefficients). Dividing equality in (6.15) by $|\zeta|^2 = \sum\limits_{lm} |\zeta_{lm}|^2$ we find that it suffices to prove

$$\inf_{|\zeta|=1} \sum_{i,j,k,\ell} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geq C_0 > 0. \tag{6.16}$$

If otherwise $C_0 = 0$, we select a sequence $\zeta^{(q)}_{jk} = \zeta^{(q)}_{kj} \in \mathbb{C}$, $q = 1, 2, \ldots$ such that

$$\lim_{m\to\infty} \sum_{i,j,k,\ell} c_{ijk\ell}\zeta^{(q)}_{ij}\overline{\zeta^{(q)}_{k\ell}} = 0, \ \ |\zeta^{(q)}| = 1.$$

Since the space of tensors $[\zeta^{(q)}_{jk}]_{n\times n}$ is finite dimensional, there exists a convergent subsequence $\zeta^{(q_r)}_{k\ell} \to \zeta^{(0)}_{k\ell}$ as $r \to \infty$. Then we get an obvious contradiction

$$\sum_{i,j,k,\ell} c_{ijk\ell}\zeta^{(0)}_{ij}\overline{\zeta^{(0)}_{k\ell}} = 0, \ \ |\zeta^{(0)}| = 1.$$

which proves that $C_0 > 0$. $\square$

Theorem 6.3. *The total free* (*elastic*) *energy functional* (*cf.* (6.4)) *acquires the form*

$$\mathscr{E}[\boldsymbol{U}] := \int_{\mathscr{S}} \big\langle \mathbb{T}\,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y)\big\rangle\, dS = \\ = 4\int_{\mathscr{S}} \big\langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\big\rangle\, dS, \ \ \boldsymbol{U} \in \mathscr{V}(\mathscr{S}) \quad (6.17)$$

and the Euler–Lagrange equation associated with the energy functional (6.17) *for a linear anisotropic elastic medium, reads*

$$\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U} = \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U} = \Bigg\{ \sum_{j,k,m=1}^{n} \Big[- c_{jklm}\mathscr{D}_m - \mathscr{H}^0_{\mathscr{S}} c_{jklm}\nu_j \\ + \nu_m \sum_{q=1}^{n} c_{jkqm}\mathscr{D}_l\nu_q \Big] \big[\mathscr{D}_k U_j + \nu_k\langle \mathscr{D}_j\boldsymbol{\nu}, \boldsymbol{U}\rangle\big] \Bigg\}_{l=1}^{n} \quad (6.18)$$

for $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$. *Here again* $\mathbb{T} = [c_{ijk\ell}]_{ijk\ell}$ *is the elasticity tensor which is positive definite* (*cf.* (6.15)) *and has the symmetry properties* (6.11), (6.13).

Proof. The representation (6.17) follows from (6.4) and (6.14).

The Euler–Lagrange equation (6.18) is derived from (6.17) as a similar equation e3.3 is derived from (6.1):

$$\mathscr{E}[\boldsymbol{U}] = 4\int_{\mathscr{S}} \big\langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\big\rangle\, dS = \\ = 4\int_{\mathscr{S}} \big\langle \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \boldsymbol{U}(y)\big\rangle\, dS = 0$$

if and only if $\boldsymbol{U} \in \mathscr{V}(\mathscr{S})$ is a solution of equation (6.18) due to the positive definiteness of the elasticity tensor $\mathbb{T}$ (cf. (6.15)).

The vector-function $\boldsymbol{U}(t) = (U_1(t), \dots, U_n(t))^\top$ denotes the tangential field of elastic displacement. The strain (the deformation) tensor has the following mapping properties

$$\mathrm{Def}_{\mathscr{S}} : \ \mathbb{H}^\theta_p(\mathscr{S}) := (\mathbb{H}^\theta_p)^n(\mathscr{S}) \to (\mathbb{H}^{\theta-1}_p)^{n\times n}(\mathscr{S}) \tag{6.19}$$

for arbitrary $\theta \in \mathbb{R}$, $1 \le p \le \infty$ and maps displacement vector field to the tensors of order 2. The dual operator

$$\mathrm{Def}^*_{\mathscr{S}} w = \{\mathfrak{D}^*_k w\}^n_{k=1}, \tag{6.20}$$

$$\mathfrak{D}^*_k w = \frac{1}{2}\Big[\sum^n_{j=1} \mathscr{D}^*_k(w_{jk} + w_{kj}) + \sum^n_{j,m=1} w_{jm}\mathscr{D}_k(\nu_j\nu_m)\Big] \ \text{ for } \ w = \|w_{jk}\|_{n\times n}$$

(cf. (4.33)) maps tensor functions to vector functions and has the following mapping properties

$$\mathrm{Def}^*_{\mathscr{S}} : \ (\mathbb{H}^\theta_p)^{n\times n}(\mathscr{S}) \to (\mathbb{H}^{\theta-1})^n_p(\mathscr{S}) \tag{6.21}$$

for arbitrary $\theta \in \mathbb{R}$, $1 \le p \le \infty$. Moreover,

$$\mathfrak{D}^*_k w = \sum^n_{j=1} \mathscr{D}^*_k w_{jk} + \sum^n_{j,m=1} \nu_m(\partial_j\nu_k)w_{jm} \ \text{ for symmetric } \ w_{jk} = w_{kj} \tag{6.22}$$

due to the curl-free condition $\partial_k\nu_j = \partial_j\nu_k$ (see Lemma 2.12.ii). Then, by applying the equality

$$\begin{aligned}\sum^n_{j,k=1} c_{jklm}\mathfrak{D}_{j,k}U &= \frac{1}{2}\sum^n_{j,k=1} c_{jklm}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \big\langle \boldsymbol{U}, \nabla_{\mathscr{C}}(\nu_j\nu_k)\big\rangle\Big] = \\ &= \sum^n_{j,k=1} c_{jklm}\Big[\mathscr{D}_k U_j + \nu_k\sum^n_{q=1}(\mathscr{D}_j\nu_q)U_q\Big] = \\ &= \sum^n_{j,k=1} c_{jklm}\big[\mathscr{D}_k U_j + \nu_k\langle\mathscr{D}_j\boldsymbol{\nu}, \boldsymbol{U}\rangle\big],\end{aligned} \tag{6.23}$$

which exploits the symmetry of coefficients (6.13) and the symmetry properties of the deformation tensor (6.22), we finally prove (6.18)

$$\begin{aligned}\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U} &= \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U} = \mathrm{Def}^*_{\mathscr{S}}\Big\|\sum^n_{j,k=1} c_{jklm}\mathfrak{D}_{j,k}\boldsymbol{U}\Big\|_{n\times n} = \\ &= \mathrm{Def}^*_{\mathscr{S}}\Big\|\sum^n_{j,k=1} c_{jklm}\big[\mathscr{D}_k U_j + \nu_k\langle\mathscr{D}_j\boldsymbol{\nu}, \boldsymbol{U}\rangle\big]\Big\|_{n\times n} = \\ &= \Big\{\sum^n_{j,k,m=1}\Big[c_{jklm}\mathscr{D}^*_m + \nu_m\sum^n_{q=1} c_{jkqm}\mathscr{D}_l\nu_q\Big]\big[\mathscr{D}_k U_j + \nu_k\langle\mathscr{D}_j\boldsymbol{\nu}, \boldsymbol{U}\rangle\big]\Big\}^n_{l=1} = \\ &= \Big\{\sum^n_{j,k,m=1}\Big[- c_{jklm}\mathscr{D}_m - \mathscr{H}^0_{\mathscr{S}} c_{jklm}\nu_j + \nu_m\sum^n_{q=1} c_{jkqm}\mathscr{D}_l\nu_q\Big]\times\end{aligned}$$

$$\times\big[\mathscr{D}_k U_j+\nu_k\langle\mathscr{D}_j\boldsymbol{\nu},\boldsymbol{U}\rangle\big]\Bigg\}_{l=1}^{n}.$$

since $\mathscr{D}_j^*=-\mathscr{D}_j-\nu_j\mathscr{H}^0_{\mathscr{S}}$ (cf. (4.52)). $\square$

If the surface $\mathscr{S}$ is isotropic, i.e., has the corresponding energy functional is invariant with respect to any rotation, the elasticity tensor $\mathbb{T}$ has the properties

$$\begin{gathered}\mathbb{T}\,(BAB^{-1})=B(\mathbb{T}\,A)B^{-1},\ \ \forall\,A,B\in\mathbb{M}_{n,n}(\mathbb{R})\\ \text{and unitary}\ \ B^{\top}=B^{-1}.\end{gathered}\tag{6.24}$$

Moreover, then the tensor $\mathbb{T}$ has the form

$$\mathbb{T}\,A=\lambda\,(\operatorname{Tr}A)I+\mu\,(A+A^{\top}),\ \ A\in\mathbb{M}_{n,n}(\mathbb{R}),\tag{6.25}$$

where $\lambda,\mu\in\mathbb{R}$ are some constants. The corresponding *lamé operator* $\boldsymbol{A}(D)=\mathscr{L}_{\mathscr{S}}(t,\mathscr{D})$ (cf. (6.18)) on the hypersurface acquires the form

$$\begin{aligned}&\mathscr{L}_{\mathscr{S}}(t,\mathscr{D})=\mu\,\pi_{\mathscr{S}}\nabla^*_{\mathscr{S}}\,\nabla_{\mathscr{S}}+(\lambda+\mu)\,\nabla_{\mathscr{S}}\,\nabla^*_{\mathscr{S}}-\mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}=\\ &=-\mu\,\boldsymbol{\Delta}_{\mathscr{S}}-(\lambda+\mu)\,\nabla_{\mathscr{S}}\operatorname{div}_{\mathscr{S}}-\mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}},\ \ \mathscr{W}_{\mathscr{S}}=-\big[\mathscr{D}_j\nu_k\big]_{n\times n}.\end{aligned}\tag{6.26}$$

For details of the formulated assertions we refer to [16].

The next Proposition 6.4 is proved in [15, Theorem 3.5] for a isotropic case. For the anisotropic case the proof is similar.

Proposition 6.4. *Let $\mathscr{S}$ be an ℓ-smooth closed hypersurface in $\mathbb{R}^n$. The operator $\boldsymbol{A}_{\mathscr{S}}(D)$ for anisotropic/isotropic media* (*cf.* (6.18) *and* (6.26)) *is elliptic. Therefore the mapping*

$$\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D}):\ \mathbb{H}^{s+1}_p(\mathscr{S})\to\mathbb{H}^{s-1}_p(\mathscr{S})\tag{6.27}$$

is Fredholm and has the trivial index $\operatorname{Ind}\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})=0$ *for all* $1<p<\infty$ *and all* $s\in\mathbb{R}$, *provided* $|s|\le\ell$.

The kernel of the operator $\operatorname{Ker}\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\subset\mathbb{H}^s_p(\mathscr{S})$ *is independent of the parameters p and s, is finite dimensional* $\dim\mathscr{R}(\mathscr{S})=\dim\operatorname{Ker}\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})<\infty$ *and coincides with the space of Killing's vector fields*

$$\operatorname{Ker}\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})=\big\{\boldsymbol{U}\in\mathscr{V}(\mathscr{S}):\ \boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U}=0\big\}=\mathscr{R}(\mathscr{S}).\tag{6.28}$$

$\mathscr{L}_{\mathscr{S}}$ *is non-negative on the space* $\mathbb{H}^1(\mathscr{S})$ *and positive definite on the orthogonal complement* $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ *to the kernel*

$$(\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}\ge 0\ \ \text{for all}\ \ \boldsymbol{U}\in\mathbb{H}^1(\mathscr{S}),\tag{6.29}$$

$$(\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}\ge C\big\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\big\|^2\ \ \text{for all}\ \ \boldsymbol{U}\in\mathbb{H}^1_{\mathscr{R}}(\mathscr{S}),\ C>0,\tag{6.30}$$

where $\mathbb{H}^1(\mathscr{S})=\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})\oplus\mathscr{R}(\mathscr{S})$.

Moreover, the following Gårding's inequality

$$(\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}}\ge C_1\big\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\big\|^2-C_0\big\|\boldsymbol{U}|\mathbb{H}^{-r}(\mathscr{S})\big\|^2\tag{6.31}$$

holds for all $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{S})$, *with arbitrary* $0<r\le\ell$ *and positive constants* $C_0>0$, $C_1>0$.

7. Boundary Integral Equations for the Laplace–Beltrami Operator

To apply the *potential method* to the investigation of BVPs (8.1) and (8.2) for the Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{C}}$ on an open hypersurface $\mathscr{C}$ in the next section, we need a fundamental solution for $\boldsymbol{\Delta}_{\mathscr{S}}$ when $\mathscr{S}$ is a closed hypersurface, which coincides with the Schwartz kernel of the inverse operator (see [14]). Such fundamental solution might fail to exist and we consider an alternative.

Theorem 7.1. *Let $\mathscr{S}$ be μ-smooth and $\ell \in \mathbb{N}_0$, $\ell \leq \mu$. Assume $\mathscr{H} \in C^\ell(\mathbb{R}^n)$ is real valued and non-negative $\mathscr{H} \geq 0$ with non-trivial support $0 \neq \operatorname{mes}\operatorname{supp}\mathscr{H}$.*

The perturbed Laplace–Beltrami operator

$$\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I \ : \ \mathbb{H}^{s+1}_2(\mathscr{S}) \to \mathbb{H}^{s-1}_2(\mathscr{S}) \tag{7.1}$$

is invertible for arbitrary $s \in \mathbb{R}$, i.e. $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$ has the fundamental solution.

Proof. As an elliptic operator on the closed hypersurface $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$ in (7.1) is Fredholm for $s = 0, 1, \ldots.$ On the other hand,

$$\begin{aligned}\big(-(\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H})\varphi, \varphi\big)_{L_2(\mathscr{S})} &= \\ = \big\|\nabla_{\mathscr{S}}\varphi | L_2(\mathscr{S})\big\| + \mathscr{H}\|\varphi | L_2(\mathscr{S})\|, &\ \ \forall\, \varphi \in \mathbb{W}^1_2(\mathscr{S}).\end{aligned} \tag{7.2}$$

and, therefore, $\operatorname{Ker}(\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I) = \varnothing$.

The same is true for the dual operator, which is the same and, therefore, $\operatorname{Coker}(\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I) = \varnothing$, which yields the invertibility.

The dual operator, which is again $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$, but between spaces $\mathbb{W}^1_2(\mathscr{S}) \to \mathbb{W}^{-1}_2(\mathscr{S})$, is also invertible. Then for non-integer $s \in \mathbb{R}$ the invertibility of the operator (7.1) follows by the interpolation (see [39]). $\square$

Remark 7.2. $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$ is invertible as an operator between more general Sobolev–Slobodetski spaces $\mathbb{W}^{s+1}_p(\mathscr{S}) \to \mathbb{W}^{-1}_p(\mathscr{S})$ and the Bessel potential spaces $\mathbb{H}^{s+1}_p(\mathscr{S}) \to \mathbb{H}^{s-1}_p(\mathscr{S})$ for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$.

In fact, for $p = 2$ this follows from Theorem 7.1. For arbitrary $1 < p < \infty$ the assertion follows since the operator $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$ has the same kernel and cokernel in all these spaces (see [17]).

Remark 7.3. The function

$$g_\mu(x, y) := \frac{1}{\cos(\pi\mu)}\, P_{\mu-1/2}(-x \cdot y), \ \ \mu \in \mathbb{R}, \ \ x, y \in \mathbb{S}^2, \tag{7.3}$$

where $P_\gamma(t)$, $-1 \leq t \leq 1$, is the Legendre special function of the first kind of order γ, represents the fundamental solution to the Laplace–Beltrami equation

$$\Big(\boldsymbol{\Delta}^2_{\mathbb{S}} + \mu^2 - \frac{1}{4}\Big) g_\mu(x, y) = \delta(x - y) \tag{7.4}$$

on the unit sphere $\mathbb{S}^2 := \big\{u \in \mathbb{R}^3 : \ || = 1\big\}$ (cf. [6], [34]).

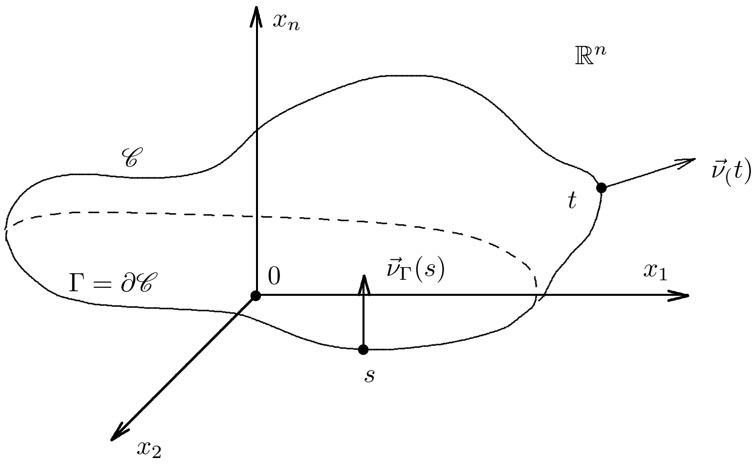

Fig. 2

Thus, $g_\mu(x, y)$ is the fundamental solution to the perturbed Laplace–Beltrami operator $\mathbf{\Delta}^2_{\mathbb{S}} + \mu^2 - 1/4$.

Now let $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ and $\gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary $\partial\mathscr{C} = \Gamma$ (see Fig. 2).

Following [39], by $\widetilde{\mathbb{W}}{}^s_p(\mathscr{C})$ (and by $\widetilde{\mathbb{H}}{}^s_p(\mathscr{C})$) we denote the subspace of $\mathbb{W}^s_p(\mathscr{S})$ (of $\mathbb{H}^s_p(\mathscr{S})$, respectively) obtained by closure of the subset $C_0^\infty(\mathscr{C})$. If $s > 0$, by an equivalent definition,

$$\widetilde{\mathbb{W}}{}^s_p(\mathscr{C}) := \Big\{ u: \ u \in \widetilde{\mathbb{W}}{}^p_s(\mathscr{S}), \ (\partial^k_\nu u)^+(t) = 0 \text{ for } k = 0, \ldots, m, \ t \notin \mathscr{S} \Big\}, \tag{7.5}$$

where $m = [s]$ is the integer part of s. Similar definition holds can be given for $\widetilde{\mathbb{H}}{}^s_p(\mathscr{C})$, $s > 0$.

$\mathbb{W}^1_2(\mathscr{C})$ and $\mathbb{H}^s_p(\mathscr{C})$ denote the quotient spaces

$$\begin{aligned} \mathbb{W}^s_p(\mathscr{C}) &= \mathbb{W}^s_p(\mathscr{S}) / \widetilde{\mathbb{W}}{}^s_p(\mathscr{S} \setminus \mathscr{C}), \\ \mathbb{H}^s_p(\mathscr{C}) &= \mathbb{H}^s_p(\mathscr{S}) / \widetilde{\mathbb{H}}{}^s_p(\mathscr{S} \setminus \mathscr{C}). \end{aligned} \tag{7.6}$$

The next Corollary 7.4 is a standard consequence of the Stoke's formulae (4.45).

Corollary 7.4. *For the Laplace–Beltrami operator $\mathbf{\Delta}_{\mathscr{C}}$ on the open hypersurface $\mathscr{C}$ with the boundary $\partial\mathscr{C} := \Gamma$ the following Green formulae are valid*

$$(\mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})\varphi, \psi)_{\mathscr{C}} + (\nabla_{\mathscr{C}}\varphi, \nabla_{\mathscr{C}}\psi)_{\mathscr{C}} = -(\mathscr{D}_{\vec{\nu}_\Gamma}\varphi^+, \psi^+)_\Gamma, \tag{7.7}$$

$$(\mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})\varphi, \psi)_{\mathscr{C}} - (\mathscr{D}_{\vec{\nu}_\Gamma}\varphi^+, \psi^+)_\Gamma = (\varphi, \mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})\psi)_{\mathscr{C}} - (\varphi^+, \mathscr{D}_{\vec{\nu}_\Gamma}\psi^+)_\Gamma \tag{7.8}$$

for arbitrary $\varphi, \psi \in C^\infty(\mathscr{C})$, where $(\varphi, \psi)_{\mathscr{C}}$ and $(\varphi, \psi)_\Gamma$ denote the appropriate scalar products (*cf.* (4.49)).

By continuity the Green formulae (7.7) *and* (7.8) *are extended to arbitrary functions* $\varphi \in \mathbb{W}^1_p(\mathscr{C})$, $\psi \in \mathbb{W}^1_{p'}(\mathscr{C})$, $1 < p < \infty$, $p' := \dfrac{p}{p-1}$.

Let us consider the following volume (Newton), the double and the single layer potentials, respectively

$$\begin{aligned} (\mathbf{N}_{\mathscr{C}} f)(t)\varphi(t) &:= \oint_{\mathscr{C}} \mathscr{K}_{\boldsymbol{\Delta}}(t, t-\tau) f(\tau)\, dS, \\ (\mathbf{W}_\Gamma \psi)(t) &:= \oint_\Gamma \left[(\mathscr{D}_{\vec{\nu}_\Gamma(s)} \mathscr{K}_{\boldsymbol{\Delta}})(t, s-t)\right]^\top \psi^+(s)\, d\mathfrak{s}, \\ (\boldsymbol{V}_\Gamma \psi)(t) &:= \oint_\Gamma \mathscr{K}_{\boldsymbol{\Delta}}(t, t-s)\psi^+(s)\, d\mathfrak{s}, \quad t \in \mathscr{C}, \end{aligned} \tag{7.9}$$

where $\mathscr{K}_{\boldsymbol{\Delta}})(t,\tau)$ is a fundamental solution to the Laplace–Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}} - \mathscr{H} I$ with some function $\mathscr{H} \in C^\infty(\mathbb{R}^n)$.

Theorem 7.5. *Let* $1 < p < \infty$, $r \in \mathbb{R}$. *Then the direct values of the double and the single layer potential operators are bounded between the spaces:*

$$\begin{aligned} \mathbf{N}_{\mathscr{C}} :\ & \mathbb{H}^s_p(\mathscr{C}) \to \mathbb{H}^{s+2}_p(\mathscr{C}), \\ :\ & \mathbb{W}^s_p(\mathscr{C}) \to \mathbb{W}^{s+2}_p(\mathscr{C}) \cap \mathbb{H}^{s+2}_p(\mathscr{C}), \\ \boldsymbol{V}_\Gamma :\ & \mathbb{H}^s_p(\Gamma) \to \mathbb{H}^{s+1+\frac{1}{p}}_p(\mathscr{C}), \\ :\ & \mathbb{W}^s_p(\Gamma) \to \mathbb{W}^{s+1+\frac{1}{p}}_p(\mathscr{C}) \cap \mathbb{H}^{s+1+\frac{1}{p}}_p(\mathscr{C}), \\ \mathbf{W}_\Gamma :\ & \mathbb{H}^s_p(\Gamma) \to \mathbb{H}^{s+\frac{1}{p}}_p(\mathscr{C}), \\ :\ & \mathbb{W}^s_p(\Gamma) \to \mathbb{W}^{s+\frac{1}{p}}_p(\mathscr{C}) \cap \mathbb{H}^{s+\frac{1}{p}}_p(\mathscr{C}). \end{aligned} \tag{7.10}$$

The following Plemelj formulae for the layer potentials hold:

$$\begin{aligned} (\mathbf{W}_\Gamma \varphi)^\pm(s) &= \pm\frac{1}{2}\,\varphi(s) + \mathbf{W}_0(s, \mathscr{D}_s)\varphi(s), \\ (\mathscr{D}_{\vec{\nu}_\Gamma} \boldsymbol{V}_\Gamma \varphi)^\pm(s) &= \mp\frac{1}{2}\,\varphi(s) + \mathbf{W}^*_0(s, \mathscr{D}_s)\varphi(s), \\ (\boldsymbol{V}_\Gamma \varphi)^-(s) &= (\boldsymbol{V}_\Gamma \varphi)^+(s) = \boldsymbol{V}_{-1}(s, \mathscr{D}_s)\varphi(s), \\ (\mathscr{D}_{\vec{\nu}_\Gamma} \mathbf{W}_\Gamma \varphi)^-(s) &= (\mathscr{D}_{\vec{\nu}_\Gamma} \mathbf{W}_\Gamma \varphi)^+(s) = \boldsymbol{V}_{+1}(s, \mathscr{D}_s)\varphi(s). \end{aligned} \tag{7.11}$$

Here $\Phi^-(s)$ *denotes the trace of* $\Phi(t)$ *on* Γ *from the hypersurface* $\mathscr{C}^c$*, complemented to* $\mathscr{C}$ (*outer with respect of* Γ*, which is the common boundary* $\Gamma = \partial\mathscr{C} = \partial\mathscr{C}^c$). *The operators* $\mathbf{W}_0(s, \mathscr{D}_s)$ *and* $\boldsymbol{V}_{-1}(s, \mathscr{D}_s)$ *are the direct values of the corresponding double and the single layer potentials on the boundary* Γ *and represent PsDOs of order* -1. $\mathbf{W}^*_0(s, \mathscr{D}_s)$ *is the dual* (*adjoint*) *PsDO to* $\mathbf{W}_0(s, \mathscr{D}_s)$. $\boldsymbol{V}_{+1}(s, \mathscr{D}_s)$ *is the direct values of the operator* $\mathscr{D}_{\vec{\nu}_\Gamma} \mathbf{W}_\Gamma$ *on the boundary* Γ *and represent a PsDO of order* $+1$.

Proof. The proof is verbatim to the case of domains in $\mathbb{R}^n$ and we quote for details [14], [19], [23] etc. □

By a standard approach it is proved that the operator

$$\begin{aligned} \boldsymbol{V}_{-1}:\ & \mathbb{H}^s_p(\Gamma) \to \mathbb{H}^{s+1}_p(\Gamma),\\ :\ & \mathbb{W}^s_p(\Gamma) \to \mathbb{W}^{s+1}_p(\Gamma) \end{aligned} \tag{7.12}$$

is invertible for all $s \in \mathbb{R}$, $1 < p < \infty$ (is positive definite for $p = 2$, $s = -\dfrac{1}{2}$) while the operator

$$\begin{aligned} \boldsymbol{V}_{+1}:\ & \mathbb{H}^s_p(\Gamma) \to \mathbb{H}^{s-1}_p(\Gamma),\\ :\ & \mathbb{W}^s_p(\Gamma) \to \mathbb{W}^{s-1}_p(\Gamma) \end{aligned} \tag{7.13}$$

has one dimensional kernel and cokernel for all $s \in \mathbb{R}$, $1 < p < \infty$, is non-negative for $p = 2$, $s = \dfrac{1}{2}$ (cf. [11], [17], [18], [29], [28] for a similar assertions). Theorem 8.2 follows from these results by standard arguments (see [11], [17], [18], [29], [28]).

Remark 7.6. The "indirect potential method" is also applicable: if we look for a solution of the Dirichlet BVP (8.1) as the double layer potential and for a solution of the Neumann BVP (8.2) as the single layer potential with unknown densities, from boundary conditions we derive appropriate boundary integral equations, which are Fredholm integral equations. These equations can be investigated by a standard procedure (see, e.g., [23]). Later we apply these results to prove Theorem 8.2.

In conclusion of the present section we formulate the following auxiliary assertions.

Lemma 7.7 (Lax–Milgram)**.** *Let $\mathfrak{B}$ be a Banach space, $A(\varphi,\psi)$ be a continuous, bilinear, symmetric form*

$$A(\cdot,\cdot):\ \mathfrak{B}\times\mathfrak{B} \to \mathbb{R} \tag{7.14}$$

and positive definite

$$A(\varphi,\varphi) \geq C\|\varphi|\mathfrak{B}\|^2, \ \ \forall\,\varphi \in \mathfrak{B}, \ \ C > 0. \tag{7.15}$$

Let $L(\cdot):\ \mathfrak{B} \to \mathbb{R}$ be a continuous linear form (a functional).

A linear equation

$$A(\varphi,\psi) = L(\psi) \tag{7.16}$$

has a unique solution $\varphi \in \mathfrak{B}$ for arbitrary $\psi \in \mathfrak{B}$. Moreover, the same φ minimizes the functional

$$F(\psi) := \frac{1}{2}A(\psi,\psi) - L(\psi), \tag{7.17}$$

i.e., represents a unique solution to the following problem

$$\min_{\psi\in\mathfrak{B}}\Big[\frac{1}{2}\,A(\psi,\psi) - L(\psi)\Big] = \frac{1}{2}\,A(\varphi,\varphi) - L(\varphi). \tag{7.18}$$

Proof. For the proof we refer to [9, § 6.3]. □

8. Boundary Value Problems for the Laplace–Beltrami Operator

Let again $\mathscr{C} \subset \mathscr{S}$ be a smooth subsurface of a closed hypersurface $\mathscr{S}$ and $\gamma = \partial\mathscr{C} \neq \varnothing$ be its smooth boundary $\partial\mathscr{C} = \Gamma$ (see Fig. 2). Let $\mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})$ be the Laplace–Beltrami operator restricted to the hypersurface $\mathscr{C}$. Consider the Dirichlet

$$\begin{cases} (\mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})\varphi)(t) = f(t), & t \in \mathscr{C}, \\ \varphi^+(s) = g(s), & s \in \Gamma = \partial\mathscr{C} \end{cases} \tag{8.1}$$

and the Neumann

$$\begin{cases} (\mathbf{\Delta}_{\mathscr{C}}(t, \mathscr{D})\varphi)(t) = f(t), & t \in \mathscr{C}, \\ (\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi)^+(s) = h(s), & s \in \Gamma = \partial\mathscr{C} \end{cases} \tag{8.2}$$

boundary value problems for the Laplace–Beltrami operator $\mathbf{\Delta}_{\mathscr{C}}$ (see (5.11)) on the open hypersurface $\mathscr{C}$ with the boundary Γ. The derivative $\mathscr{D}_{\vec{\nu}_\Gamma(s)}$ is defined as follows

$$\mathscr{D}_{\vec{\nu}_\Gamma(s)} := \sum_{k=1}^{n} \nu_{\Gamma,k}(s)\mathscr{D}_k, \quad \vec{\nu}_\Gamma(s) := (\nu_{\Gamma,1}(s), \ldots, \nu_{\Gamma,n}(s))^\top, \quad s \in \Gamma, \tag{8.3}$$

where $\mathscr{D}_{\vec{\nu}_\Gamma(s)}$ is a tangent derivative on the hypersurface $\mathscr{C}$ and the normal derivative with respect to the boundary Γ.

Note, that BVPs (8.1) and (8.2) describe the stationary heat transfer process in a thin conductor having the shape of the hypersurface * $\mathscr{S}$ (see [21, § 72]).

Corollary 8.1. *For arbitrary solution $\varphi \in \mathbb{W}^1_p(\mathscr{C})$ of the equation $\mathbf{\Delta}_{\mathscr{S}}\varphi = f$, $f \in \mathbb{W}^{-1}_p(\mathscr{C})$ the trace $(\mathscr{D}_{\vec{\nu}_\Gamma}\varphi)^+$ exists and belongs to $\mathbb{W}^{-\frac{1}{p}}_p(\Gamma)$.*

Proof. Let $\varphi \in \mathbb{W}^1_p(\mathscr{C})$ be a solution of the equation $\mathbf{\Delta}_{\mathscr{S}}\varphi = f$, $f \in \mathbb{W}^{-1}_p(\mathscr{C})$ and $\psi \in \mathbb{W}^1_{p'}(\mathscr{C})$ be arbitrary. Then (7.7) gives

$$(\mathscr{D}_{\vec{\nu}_\Gamma}\varphi^+, \psi^+)_\Gamma = -(f, \psi)_{\mathscr{C}} - (\nabla_{\mathscr{C}}\varphi, \nabla_{\mathscr{C}}\psi)_{\mathscr{C}}. \tag{8.4}$$

Since the right-hand side in (7.8) is correctly defined and $\psi^+ \in \mathbb{W}^{1-\frac{1}{p'}}_{p'}(\Gamma) = \mathbb{W}^{\frac{1}{p}}_{p'}(\Gamma)$, the functional in the left-hand side is defined correctly and the inclusion for the trace $\mathscr{D}_{\vec{\nu}_\Gamma}\varphi^+ \in \mathbb{W}^{-\frac{1}{p}}_p(\mathscr{S})$ holds by duality. $\square$

We impose the following constraints on the participating functions in BVPs (8.1) and (8.2):

$$\begin{gathered} f \in \mathbb{W}^{s-2}_p(\mathscr{C}), \quad \varphi \in \mathbb{W}^s_p(\mathscr{C}), \\ g \in \mathbb{W}^{s-\frac{1}{p}}_p(\Gamma), \quad h \in \mathbb{W}^{s-1-\frac{1}{p}}_p(\Gamma), \quad 1 < p < \infty, \quad s \geq 1. \end{gathered} \tag{8.5}$$

*We consider the stationary heat conduction only for simplicity. For the time dependent process, which is represented by a Hypoelliptic operator, similar results can be obtained.

Note that due to Corollary 7.4 the traces of solutions to the equation $\mathbf{\Delta}_{\mathscr{C}}\varphi = f$ in BVPs (8.1) and (8.2) under constraints (8.5) are defined correctly.

In the perturbed Laplace–Beltrami operator $\mathbf{\Delta}_{\mathscr{S}} - \mathscr{H}I$ (see (7.1)) we choose the function $\mathscr{H} \in C^\infty(\mathbb{R}^n)$ which is supported in the complemented domain $supp\,\mathscr{H} \subset \mathscr{C}^c := \mathscr{S} \setminus \overline{\mathscr{C}}$. Then any solution of the Dirichlet (8.1), (8.5) and the Neumann (8.2), (8.5) boundary value problems is represented as follows

$$\varphi(t) = (\mathbf{N}_{\mathscr{C}} f)(t) + (\mathbf{W}_\Gamma \varphi^+)(t) - \big(\boldsymbol{V}_\Gamma(\mathscr{D}_{\vec{\nu}_\Gamma}\varphi)^+\big)(t), \quad t \in \mathscr{C}, \tag{8.6}$$

where the potential operators are defined in (7.9).

The proof of (8.6) is standard: by inserting the solution φ of $\mathbf{\Delta}_{\mathscr{C}}\varphi = f$ and the fundamental solution $\psi = \mathscr{K}_{\mathbf{\Delta}}(t, t-\tau)$,

$$\begin{gathered}\mathbf{\Delta}_{\mathscr{C}}\mathscr{K}_{\mathbf{\Delta}}(t, t-\tau) = \\ = \chi_{\mathscr{C}}(\mathbf{\Delta}_{\mathscr{S}} - \mathscr{H}I)\mathscr{K}_{\mathbf{\Delta}}(t, t-\tau) = \chi_{\mathscr{C}}\delta(t-\tau) = \delta(t-\tau), \quad t, \tau \in \mathscr{C}\end{gathered}$$

truncated properly around the diagonal $t = \tau$ on the distance $\varepsilon > 0$, into the Green formula (7.8), written for $\mathbf{\Delta}_{\mathscr{C}} - \mathscr{H}I$, we derive the representation formula (8.6) by sending $\varepsilon \to 0$.

Following the "direct potential method" we apply the representation formulae (8.6) and note that one of the densities either φ^+ or $(\mathscr{D}_{\vec{\nu}_\Gamma(s)}\varphi)^+$ is already known and given by the boundary conditions in (8.1) or in (8.2), respectively. Applying also the appropriate Plemelj formulae from (7.11) we get the following equivalent boundary pseudodifferential equations:

A. For the Dirichlet BVP (8.1)

$$\boldsymbol{V}_{-1}(s, \mathscr{D}_s)\psi(s) = \big(\mathbf{N}_{\mathscr{C}}(s, \mathscr{D}_s)f\big)(s) - \frac{1}{2}\,g + \big(\mathbf{W}_0(s, \mathscr{D}_s)g\big)(s), \quad s \in \Gamma, \tag{8.7}$$

where $\psi(s) := (\mathscr{D}_{\vec{\nu}_\Gamma}\varphi)^+(s)$ is the unknown function and the right-hand side is known.

B. For the Neumann BVP (8.2)

$$\boldsymbol{V}_{+1}(s, \mathscr{D}_s)\omega(s) = -\big(\mathbf{N}_{\mathscr{C}}(s, \mathscr{D}_s)f\big)(s) + \frac{1}{2}\,h + \big(\mathbf{W}_0^*(s, \mathscr{D}_s)h\big)(s), \quad s \in \Gamma, \tag{8.8}$$

where $\omega(s) := \varphi^+(s)$ is the unknown function and the right-hand side is known again.

Theorem 8.2. *Let* $1 < p < \infty$, $s \geq 11$. *The Dirichlet problem* (8.1), (8.5) *has a unique solution* $\varphi \in \mathbb{W}^s_p(\mathscr{C})$ *for arbitrary right-hand side* $g \in \mathbb{W}^{s-\frac{1}{p}}_p(\Gamma)$.

The Neumann problem (8.2), (8.5) *has a solution* $\varphi \in \mathbb{W}^s_p(\mathscr{C})$ *only for those right-hand sides* $h \in \mathbb{W}^{s-1-\frac{1}{p}}_p(\Gamma)$ *which satisfy the condition*

$$\oint_\Gamma h(s)\, d\mathfrak{s} = 0. \tag{8.9}$$

If the condition (8.9) *holds, the Neumann problem has a solution* $\varphi_0 \in \mathbb{W}^s_p(\mathscr{C})$ *and a general solution reads* $\varphi = \varphi_0 + \mathrm{const}$.

Proof. For the proof of existence in the restricted space settings (8.5) we recall that the equivalent boundary pseudodifferential equations (8.7) and (8.8) to BVPs (8.1) and (8.2), respectively, are Fredholm and have indices zero. Moreover, the operator in (8.7) is even invertible, while the kernel and cokernel of the equation in (8.8) coincide with constants (cf. (7.12) and (7.13)). Therefore, the Dirichlet BVP (8.1) is solvable uniquely, while for the solvability of the Neumann problem there must hold the orthogonality condition (8.9) for the data with the solution $v(t) \equiv \text{const}$ of the homogeneous equation. □

9. BVPs for an Elastic Hypersurface and Green's Formulae

Throughout the present section $\mathscr{S}$ is an open C^2-smooth hypersurface (or: the derivative of the corresponding diffeomorphisms are Lipscitz continuous) with the Lipschitz boundary $\partial\mathscr{C} = \Gamma \neq \varnothing$, a subsurface of a closed C^2-smooth hypersurface $\mathscr{S}$. $r_{\mathscr{C}}$ denotes the restriction to the surface $\mathscr{C}$ from $\mathscr{S}$ and

$$\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}) := r_{\mathscr{C}}\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}), \;\; \mathscr{L}_{\mathscr{C}}(t,\mathscr{D}) := r_{\mathscr{C}}\mathscr{L}_{\mathscr{C}}(t,\mathscr{D}).$$

Note that the imposed constraint on the surface $\mathscr{C}$ can not be relaxed, because in the definition of the equation

$$\boldsymbol{A}_{\mathscr{C}}(D)\boldsymbol{U} = \boldsymbol{F}, \;\; \boldsymbol{U} \in \mathbb{H}^1(\mathscr{C}), \;\; \boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \tag{9.1}$$

is participating the gradient $\nabla_{\mathscr{S}}\boldsymbol{\nu} = [\mathscr{D}_j\nu_k]_{n\times n}$ of the unit normal vector field $\boldsymbol{\nu}$ (cf. (6.18) and (6.26)). $\boldsymbol{\nu}(t)$ is defined almost everywhere on $\mathscr{C}$ is just C^1-smooth (or is Lipschitz continuous, respectively).

Equation (9.1) is actually understood in a weak sense:

$$\big(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{V}\big)_{\mathscr{C}} := \big(\mathbb{T}\,\mathrm{Def}_{\mathscr{C}}\boldsymbol{U},\mathrm{Def}_{\mathscr{C}}\boldsymbol{V}\big)_{\mathscr{C}} = \big(\boldsymbol{F},\boldsymbol{V}\big)_{\mathscr{C}}, \tag{9.2}$$
$$\forall\,\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C}), \boldsymbol{V} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$$

(cf. (6.18)). In particular, for the Lamé operator in isotropic medium we have

$$\begin{aligned}\big(\mathscr{L}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{V}\big)_{\mathscr{C}} := \lambda\big(\nabla_{\mathscr{C}}\boldsymbol{U},\nabla_{\mathscr{C}}\boldsymbol{V}\big)_{\mathscr{C}} + \\ + (\lambda+\mu)\big(\mathrm{div}_{\mathscr{C}}\boldsymbol{U},\mathrm{div}_{\mathscr{C}}\boldsymbol{V}\big)_{\mathscr{C}} = \big(\boldsymbol{F},\boldsymbol{V}\big)_{\mathscr{C}}, \;\; \forall\,\boldsymbol{V} \in \widetilde{\mathbb{H}}^1_2(\mathscr{S})\end{aligned} \tag{9.3}$$

(cf. (6.26)).

Let $\boldsymbol{\nu}_\Gamma = (\nu_\Gamma^1,\dots,\nu_\Gamma^n)^\top$ be the tangential to $\mathscr{C}$ and outer unit normal vector field to Γ.

If a tangential vector field $\boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{C}) \cap \mathscr{V}(\mathscr{C})$ denotes the displacement, the natural boundary value problems for $\mathscr{L}_{\mathscr{C}}$ are the following:

I. The *Dirichlet problem* when the displacement is prescribed on the boundary

$$\begin{cases}(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U})(t) = \boldsymbol{F}(t), & t \in \mathscr{C},\\ \boldsymbol{U}^+(\tau) = \boldsymbol{G}(\tau), & \tau \in \Gamma,\end{cases} \tag{9.4}$$

$$\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \quad \boldsymbol{G} \in \mathbb{H}^{1/2}(\Gamma), \quad \boldsymbol{U} \in \mathbb{H}^{1}(\mathscr{C});$$

the first (basic) equation in the domain is understood in a weak sense (see (9.2), (9.3)) and

$$\gamma_D^+ \boldsymbol{U} := \boldsymbol{U}^+ \tag{9.5}$$

is the Dirichlet trace operator on the boundary.

II. The *Neumann problem* when the traction is prescribed on the boundary:

$$\begin{cases} \big(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U}\big)(t) = \boldsymbol{F}(t), & t \in \mathscr{C}, \\ \big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\, \boldsymbol{U}\big)^+(\tau) = \boldsymbol{H}(\tau), & \tau \in \Gamma, \end{cases} \tag{9.6}$$

$$\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \quad \boldsymbol{H} \in \mathbb{H}^{-1/2}(\Gamma), \quad \boldsymbol{U} \in \mathbb{H}^{1}(\mathscr{C});$$

here

$$\gamma_N^+ \boldsymbol{U} := \big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U}\big)^+ \tag{9.7}$$

and

$$\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U} := \Big[\sum_{k,m=1}^{n} c_{jklm}\nu_\Gamma^j \big[\mathscr{D}_k U_j + \nu_k\langle \mathscr{D}_j \boldsymbol{\nu}, \boldsymbol{U}\rangle\big]\Big]_{n\times n}. \tag{9.8}$$

In particular, for an isotropic case,

$$\begin{aligned} \mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U} &:= -\lambda(\operatorname{div}_{\mathscr{C}}\boldsymbol{U})\boldsymbol{\nu}_\Gamma - 2\mu\sum_{j=1}^{n}\big\{(\nu_\Gamma^j + \mathscr{H}^0_{\mathscr{C}}\nu_j)\mathfrak{D}_{jk}(\boldsymbol{U})\big\}_{k=1}^{n} = \\ &= -\mu\mathscr{D}_{\boldsymbol{\nu}_\Gamma}\boldsymbol{U} - (\lambda+\mu)(\operatorname{div}_{\mathscr{C}}\boldsymbol{U})\boldsymbol{\nu}_\Gamma \end{aligned} \tag{9.9}$$

is the Neumann trace operator on the boundary (the traction) with

$$\mathscr{D}_{\boldsymbol{\nu}_\Gamma}\varphi := \sum_{j=1}^{n}\nu_\Gamma^j \mathscr{D}_j\varphi, \quad \varphi \in \mathbb{H}^1(\mathscr{C}). \tag{9.10}$$

The trace $\gamma_N^+\boldsymbol{U}$ exists provided that $\boldsymbol{U}$ is a solution to the basic (first) equation in (9.6) (see Corollary 9.2 below).

Later we will relax constraints on the data and the solution and replace them by constraints in $\mathbb{H}^s_p$-setting to gain some a priori smoothness of solution. On the other hand we should raise constraints on the underlying hypersurface $\mathscr{C}$ and require the infinite smoothness to apply the potential method.

A crucial role in the investigation of BVPs (9.4)–(9.6) belongs to the Green formula.

Lemma 9.1. *For the operator $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ on the open hypersurface $\mathscr{C}$ the following Green formulae are valid:*

$$(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) + ((\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U})^+,\boldsymbol{V}^+)_\Gamma, \tag{9.11}$$

$$\begin{aligned} &(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} - ((\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U})^+,\boldsymbol{V}^+)_\Gamma = \\ &= (\boldsymbol{U},\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{V})_{\mathscr{C}} - (\boldsymbol{U}^+,(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{V})^+)_\Gamma \end{aligned} \tag{9.12}$$

for arbitrary $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{H}^1_p(\mathscr{S})$ *and the traction operator is defined in* (9.8) (*see* (9.9) *for an isotropic case*). *The energy bilinear form* $\mathscr{E}(\boldsymbol{U}, \boldsymbol{V})$ *is defined by the formulae*

$$\mathscr{E}(\boldsymbol{U}, \boldsymbol{V}) := \int\limits_{\mathscr{S}} \big\langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\big\rangle\, dS, \quad \boldsymbol{U} \in \mathscr{V}(\mathscr{S}) \tag{9.13}$$

(*cf.* (6.17)) *and, in particular,*

$$\mathscr{E}(\boldsymbol{U}, \boldsymbol{V}) := \int\limits_{\mathscr{C}} \Big[\mu\big\langle \nabla_{\mathscr{S}}\boldsymbol{U}, \nabla_{\mathscr{S}}\boldsymbol{V}\big\rangle + (\lambda+\mu)\big\langle \mathrm{div}_{\mathscr{C}}\,\boldsymbol{U}, \mathrm{div}_{\mathscr{C}}\,\boldsymbol{V}\big\rangle\Big]\, dS \tag{9.14}$$

for an isotropic case.

Proof. Using the first representation of $\boldsymbol{A}_{\mathscr{C}}(t, \mathscr{D})$ in (6.18) (for an isotropic case – in (6.26)) and the integration by parts on surfaces (Stoke's formulae) (4.45) we get the following

$$\oint\limits_{\mathscr{C}} \big[\boldsymbol{A}_{\mathscr{C}}(t, \mathscr{D})\boldsymbol{U}(t)\big]^{\top}\overline{\boldsymbol{V}(t)}\, dS = \big((\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U})^+, \boldsymbol{V}^+\big)_\Gamma + \mathscr{E}(\boldsymbol{U}, \boldsymbol{V}), \tag{9.15}$$

where is defined in (9.13) (in (9.14) for an isotropic case).

To find the expression for the traction operator $\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D}$ we apply the second representation of $\boldsymbol{A}_{\mathscr{C}}(t, \mathscr{D})$ in (6.18), the integration by parts on surfaces (Stoke's formulae) (4.45) and get the following:

$$\oint\limits_{\mathscr{C}} \big[\boldsymbol{A}_{\mathscr{C}}(t, \mathscr{D})\boldsymbol{U}(t)\big]^{\top}\overline{\boldsymbol{V}(t)}\, dS =$$

$$\oint\limits_{\mathscr{C}} \sum_{j,k,m,l=1}^{n} \Big[-c_{jklm}\mathscr{D}_j - \mathscr{H}^0_{\mathscr{S}}c_{jklm}\nu_j + \nu_m(t)\sum_{q=1}^{n} c_{jkqm}\mathscr{D}_l\nu_q(t)\Big] \times$$
$$\times\Big[\mathscr{D}_k U_j(t) + \nu_k\big\langle\mathscr{D}_j\boldsymbol{\nu}(t), \boldsymbol{U}(t)\big\rangle\Big]\overline{V_l(t)}\, dS =$$
$$= \oint\limits_{\Gamma} \sum_{j,k,m,l=1}^{n} c_{jklm}\nu^j_\Gamma(s)\Big[(\mathscr{D}_k U_j)^+(s) + \nu_k\big\langle\mathscr{D}_j\boldsymbol{\nu}(s), \boldsymbol{U}(s)\big\rangle\Big]\overline{V^+_l(s)}\, d\mathfrak{s} +$$
$$+\mathscr{E}(\boldsymbol{U}, \boldsymbol{V}) = \big((\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U})^+, \boldsymbol{V}^+\big)_\Gamma + \mathscr{E}(\boldsymbol{U}, \boldsymbol{V}).$$

For an isotropic case we apply the representation of $\mathscr{L}_{\mathscr{S}}(t, D)$ in (6.26) and proceed similarly. □

Corollary 9.2. *For arbitrary solution* $\boldsymbol{U} \in \mathbb{H}^1_p(\mathscr{S})$ *to the equation* $\boldsymbol{A}_{\mathscr{S}}(t, \mathscr{D})\boldsymbol{U} = \boldsymbol{F}$, $\boldsymbol{F} \in \mathbb{H}^{-1}_p(\mathscr{C})$, *the trace* $(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U})^+$ *exists and belongs to* $\mathbb{W}^{-\frac{1}{p}}_p(\mathscr{S})$.

Proof. The proof is based on (9.11) and is verbatim to the proof of Corollary 8.1. □

Theorem 9.3. *Let $\mathscr{S}$ be μ-smooth and $\ell \in \mathbb{N}_0$, $\ell \leq \mu$. Assume $\mathscr{H} \in C^\ell(\mathbb{R}^n)$ is real valued and non-negative $\mathscr{H} \geq 0$ with non-trivial support $0 \neq \operatorname{mes}\operatorname{supp}\mathscr{H}$.*

The perturbed operator of anisotropic elasticity

$$\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D}) - \mathscr{H}I \;:\; \mathbb{H}_2^{s+1}(\mathscr{S}) \to \mathbb{H}_2^{s-1}(\mathscr{S}) \tag{9.16}$$

is invertible for arbitrary $s \in \mathbb{R}$, i.e. $\boldsymbol{A}_{\mathscr{S}}(t,\mathscr{D}) - \mathscr{H}I$ has the fundamental solution.

Proof. The proof is based on Proposition 6.4 and follows the proof of Theorem 7.1. □

10. The Dirichlet BVP for the Equation of Anisotropic Elasticity

Throughout this section $\mathscr{C}$ is a C^2-smooth hypersurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

Theorem 10.1. *The Dirichlet problem* (9.4) *has a unique solution $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$ for arbitrary data $\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ and $\boldsymbol{G} \in \mathbb{H}^{1/2}(\Gamma)$.*

The proof will be exposed at the end of the section after we prove some auxiliary results.

Lemma 10.2 (Gårding's inequality "with boundary condition")**.** *The operator*

$$\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}) :\; \widetilde{\mathbb{H}}^1(\mathscr{C}) \to \mathbb{H}^{-1}(\mathscr{C}) \tag{10.1}$$

is positive definite: there exists some constant $C > 0$ such that

$$(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} \geq C\big\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{C})\big\|^2, \;\; \forall\, \boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\mathscr{C}). \tag{10.2}$$

Proof. Due to (6.30) inequality (10.1) holds for all $\boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$, i.e., for $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$ and $\boldsymbol{U} \notin \mathscr{R}(\mathscr{S})$. Since $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$ due to the strong unique continuation from the boundary (cf. Proposition 4.6), all Killing's vector fields $\boldsymbol{K} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$ are identically 0. Therefore, (6.30) holds for all $\boldsymbol{U} \in \widetilde{\mathbb{H}}^1(\mathscr{C})$. □

Corollary 10.3. *The operator $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ in* (10.1) *is invertible.*

Proof. From the inequality (10.2) follows that $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ is normally solvable (has the closed range) and the trivial kernel $\operatorname{Ker}\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}) = \{0\}$. Since $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ is self adjoint, the co-kernel (the kernel of the adjoint operator) is trivial as well $\operatorname{Ker}\boldsymbol{A}^*_{\mathscr{C}}(t,\mathscr{D}) = \operatorname{Ker}\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}) = \{0\}$. Therefore $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ is invertible. □

Definition 10.4 (see [25, Ch. 2, § 1.4])**.** A partial differential operator

$$\mathbf{B}(x,\mathscr{D}) := \sum_{|\alpha|\leq m} a_\alpha(x)\nabla^\alpha_{\mathscr{C}}, \;\; \nabla^\alpha_{\mathscr{C}} = \mathscr{D}_1^{\alpha_1}\cdots\mathscr{D}_n^{\alpha_n}, \;\; a_\alpha \in C(\mathscr{C}, C^{N\times N}) \tag{10.3}$$

is called normal on Γ if

$$\inf\left|\det\mathscr{B}_0(t,\boldsymbol{\nu}(t))\right|\neq 0,\quad t\in\Gamma,\quad |\xi|=1, \tag{10.4}$$

where $\mathscr{B}_0(x,\xi)$ is the homogeneous principal symbol of $\mathbf{A}$

$$\mathscr{B}_0(x,\xi):=\sum_{|\alpha|=m}a_\alpha(x)(-i\xi)^\alpha,\quad x\in\overline{\mathscr{C}},\quad \xi\in\mathbb{R}^n. \tag{10.5}$$

Definition 10.5. A system $\{\mathbf{D}_j(t,D_t)\}_{j=0}^{k-1}$ of differential operators with matrix $N\times N$ coefficients is called a Dirichlet system of order k if all participating operators are normal on Γ (see Definition 10.4) and $\operatorname{ord}\mathbf{D}_j=j$, $j=0,1,\dots,k-1$.

Let us assume $\mathscr{C}$ is k-smooth and $m\leq k$ $(m,k=1,2,\dots)$ and define the trace operator (cf. (9.10)):

$$\mathscr{R}_m\boldsymbol{U}:=\left\{\gamma_\Gamma\mathbf{D}_1\boldsymbol{U},\dots,\gamma_\Gamma\mathbf{D}_m\boldsymbol{U}\right\}^\top,\quad \boldsymbol{U}\in\mathbb{C}^k_0(\overline{\mathscr{C}}). \tag{10.6}$$

Proposition 10.6. *Let $\mathscr{C}$ be k-smooth, $1\leq p\leq\infty$, $m=1,2,\dots$, $m\leq k$ and $m<s-1/p\notin\mathbb{N}_0$. The trace operator*

$$\mathscr{R}_m:\ \mathbb{H}^s_p(\overline{\mathscr{C}})\to\bigotimes_{j=0}^{m}\mathbb{W}^{s-1/p-j}_p(\Gamma), \tag{10.7}$$

where $\mathbb{W}^r_p(\overline{\mathscr{C}})=\mathbb{B}^r_{p,p}(\overline{\mathscr{C}})$ is the Sobolev–Slobodecki–Besov space (cf. [39] for details) is a retraction, i.e., is continuous and has a continuous right inverse, called a coretraction

$$\begin{aligned}&(\mathscr{R}_m)^{-1}:\ \bigotimes_{j=0}^{m}\mathbb{W}^{s-1/p-j}_p(\mathscr{S})\to\mathbb{H}^s_p(\overline{\Omega}),\\&\mathscr{R}_m(\mathscr{R}_m)^{-1}\Phi=\Phi,\quad\forall\Phi\in\bigotimes_{j=0}^{m}\mathbb{W}^{s-1/p-j}_p(\mathscr{S}).\end{aligned} \tag{10.8}$$

Proof. The result was proved in [39, Theorem 2.7.2, Theorem 3.3.3] for a domain $\Omega\subset\mathbb{R}^{n-1}$ and the classical Dirichlet trace operator $\mathscr{R}_mu:=\{\gamma_\Gamma\partial_{\boldsymbol{\nu}}u,\dots,\gamma_\Gamma\partial^m_{\boldsymbol{\nu}}u\}^\top$. In [15] the theorem was proved for a domain $\Omega\subset\mathbb{R}^{n-1}$ and for arbitrary trace operator $\mathscr{R}_mu$.

A surface $\mathscr{C}=\cup_{j=1}^N\mathscr{C}_j$ is covered by a finite number of local coordinate charts $\varkappa_j:\ \Omega_j\to\mathscr{C}_j$, $\Omega_j\subset\mathbb{R}^{n-1}$. After transformation, the Dirichlet trace operator $\mathscr{R}_mu$ on a portion $\mathscr{C}_j$ of the surface transform into another Dirichlet trace operator on the coordinate domains Ω_j. Therefore, we prove the assertion locally on each coordinate chart $\mathscr{C}_j\subset\mathscr{C}$ and, by applying a partition of unity, extend it to the entire surface $\mathscr{C}$. □

Proof of Theorem 10.1. Let $\widetilde{\boldsymbol{G}}=(\mathscr{R}_0)^{-1}\boldsymbol{G}\in\mathbb{H}^1(\mathscr{C})$ be the continuation of the Dirichlet boundary data $\boldsymbol{G}\in\mathbb{H}^{1/2}(\Gamma)$ from BVP (9.4) into the surface $\mathscr{C}$ from the boundary Γ, found with the help of a coretraction from Proposition 10.6. Then the Dirichlet BVP

$$\begin{cases}(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\widetilde{\boldsymbol{U}})(t)=\boldsymbol{F}_0(t), & t\in\mathscr{C},\\ \widetilde{\boldsymbol{U}}^+(\tau)=0, & \tau\in\Gamma,\end{cases} \tag{10.9}$$

$$\boldsymbol{F}_0 := \boldsymbol{F} - \boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\widetilde{\boldsymbol{G}} \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}),$$

is an equivalent reformulation of BVP (9.4) and the solutions are related by the equality $\widetilde{\boldsymbol{U}} := \boldsymbol{U} - \widetilde{\boldsymbol{G}}$. On the other hand, since

$$\widetilde{\mathbb{H}}^{-1}(\mathscr{C}) := \big\{\boldsymbol{U} \in \mathbb{H}^{-1}(\mathscr{C}) :\ \boldsymbol{U}^+ = 0\big\},$$

the solvability of BVP (10.9) is equivalent to the invertibility of the operator $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})$ in (10.1). Now the unique solvability of BVP (10.9) (and of the equivalent BVP (9.4)) follows from Corollary 10.3. □

11. The Neumann BVP for the Equation of Anisotropic Elasticity

Throughout this section $\mathscr{C}$ is a C^2-smooth hypersurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.

Theorem 11.1. *The Neumann problem* (9.6) *has a solution* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$ *only for those right-hand sides* $\boldsymbol{F} \in \widetilde{\mathbb{H}}^{-1}(\Gamma)$ *and* $\boldsymbol{H} \in \mathbb{H}^{-1/2}(\Gamma)$ *which satisfy the equality*

$$\int\limits_{\mathscr{C}} \boldsymbol{F}(t)\boldsymbol{K}(t)\,dS = \oint\limits_{\Gamma} \boldsymbol{H}(\tau)\gamma_D^+\boldsymbol{K}(\tau)\,d\mathfrak{s}, \quad \forall \boldsymbol{K} \in \mathscr{R}(\mathscr{C}). \tag{11.1}$$

If the condition (11.1) *holds, the Neumann problem has a general solution* $\boldsymbol{U} = \boldsymbol{U}^0 + \boldsymbol{K} \in \mathbb{H}^1(\mathscr{C})$*, where* $\boldsymbol{U}^0 \in \mathbb{H}^1(\mathscr{C})$ *is a particular solution and* $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ *is a Killing's vector field.*

The proof will be exposed at the end of the section after we prove some auxiliary results.

Lemma 11.2. *The condition* (11.1) *is necessary for the Neumann problem* (9.6) *to have a solution* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{C})$.

Proof. First note that for a Killing's vector field $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$,

$$\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{K} = 0 \ \text{ and } \ \gamma_N^+\boldsymbol{K} = \big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{K}\big)^+ = 0. \tag{11.2}$$

Indeed, if $\boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ is naturally extended to $\widetilde{\boldsymbol{K}} \in \mathscr{R}(\mathscr{S})$, then $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{K}(t) = \boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\widetilde{\boldsymbol{K}}(t) = 0$ for $t \in \mathscr{C}$ (cf. (6.28)) and the first equality follows.

The second equality in (11.2) follows from (9.9) if we recall that $\mathrm{Def}_{\mathscr{C}}(\boldsymbol{K}) = 0$ (see 4.26) and this implies

$$\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{K} = \mathrm{Def}^*_{\mathscr{S}}(\boldsymbol{\nu}_\Gamma)\mathbb{T}\,\mathrm{Def}_{\mathscr{S}}(D)\boldsymbol{K} = 0, \ \ \boldsymbol{U} \in \mathscr{V}(\mathscr{S}). \tag{11.3}$$

The latter formula can easily be seen analyzing (9.15).

From (9.13) and $\mathrm{Def}_{\mathscr{C}}(\boldsymbol{K}) = 0$ it follows

$$\mathscr{E}(\boldsymbol{K},\boldsymbol{U}) := \int\limits_{\mathscr{S}} \langle \mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{K}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\,dS = 0 \tag{11.4}$$

$$\text{for all } \ \boldsymbol{U} \in \mathbb{H}^1(\mathscr{C}) \ \text{ and all } \ \boldsymbol{K} \in \mathscr{R}(\mathscr{C}).$$

Introducing into the Green formula (9.11) $\boldsymbol{F} = \boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U}$, $\boldsymbol{V} = \boldsymbol{K} \in \mathscr{R}(\mathscr{C})$ and the obtained equality, we get the claimed orthogonality condition (11.1). □

Lemma 11.3. *The bilinear form*

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V}) := (\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} - (\gamma_N^+\boldsymbol{U},\gamma_D^+\boldsymbol{V})_\Gamma = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) \tag{11.5}$$

is well defined, symmetric $\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V}) = \mathbb{A}_N(\boldsymbol{V},\boldsymbol{U})$ *for all* $\boldsymbol{U}$, $\boldsymbol{V} \in \mathbb{H}^1(\mathscr{C})$ *and non-negative* $\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) \geq 0$ *for* $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$ *(cf.* (9.13)*). Moreover, the form is positive definite*

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{U}) \geq M_3\big\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\big\|^2, \quad \forall \boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \tag{11.6}$$

on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ *to the finite dimensional subspace of Killing's vector fields* $\mathscr{R}(\mathscr{C})$ *in the Hilbert–Sobolev space* $\mathbb{H}^1(\mathscr{C})$*.*

Proof. The proof is a direct consequence of the equality

$$\mathbb{A}_N(\boldsymbol{U},\boldsymbol{V}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) := (\mathbb{T}\,\mathrm{Def}_{\mathscr{S}}\boldsymbol{U},\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}), \quad \forall \boldsymbol{U} \in \mathbb{H}^1(\mathscr{S}) \tag{11.7}$$

(cf. (9.13)) if we recall that the tensor $\mathbb{T}$ is positive definite (cf. Lemma 6.2). □

Proof of Theorem 11.1. The space of Killing's vector fields $\mathscr{R}(\mathscr{S})$ is finite dimensional and consists of continuous vector-fields with bounded second derivatives (these fields are actually as smooth as the surface $\mathscr{C}$, i.e., are infinitely smooth if $\mathscr{S}$ is infinitely smooth; see Proposition 4.6). Let $\boldsymbol{K}_1,\dots,\boldsymbol{K}_m$ be a finite dimensional orthonormal basis in $\mathscr{R}(\mathscr{C})$, $(\boldsymbol{K}_j,\boldsymbol{K}_r)_{\mathscr{C}} = \delta_{jr}$, $j,r = 1,\dots,m$. Consider the finite rank smoothing operator

$$\boldsymbol{T}\boldsymbol{U}(\mathscr{X}) := \sum_{j=1}^m (\boldsymbol{K}_j,\boldsymbol{U})_{\mathscr{S}}\ \boldsymbol{K}_j(\mathscr{X}), \quad \mathscr{X} \in \mathscr{S}. \tag{11.8}$$

The operator $\boldsymbol{T}$ is symmetric and non-negative:

$$(\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} = (\boldsymbol{T}\boldsymbol{V},\boldsymbol{U})_{\mathscr{C}}, \quad (\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} = \sum_{j=1}^m (\boldsymbol{U},\boldsymbol{K}_j)^2_{\mathscr{C}} \geq 0, \tag{11.9}$$
$$\forall \boldsymbol{U},\boldsymbol{V} \in \mathbb{H}^1(\mathscr{C}).$$

Consider the modified bilinear form

$$\begin{aligned}\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{V}) &:= ((\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D}) + \boldsymbol{T})\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} - (\gamma_N^+\boldsymbol{U},\gamma_D^+\boldsymbol{V})_\Gamma = \\ &= \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}}\ \ \boldsymbol{U},\boldsymbol{V} \in \mathbb{H}^1(\mathscr{C}).\end{aligned} \tag{11.10}$$

The form is symmetric because both summands are

$$\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{V}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{V}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{V})_{\mathscr{C}} = \mathscr{E}(\boldsymbol{V},\boldsymbol{U}) + (\boldsymbol{T}\boldsymbol{V},\boldsymbol{U})_{\mathscr{C}} = \mathbb{A}_N^{\#}(\boldsymbol{V},\boldsymbol{U})$$

(cf. Lemma 11.3 and the first equality in (11.9)).

Moreover, the corresponding quadratic form is strongly positive

$$\mathbb{A}_N^{\#}(\boldsymbol{U},\boldsymbol{U}) = \mathscr{E}(\boldsymbol{U},\boldsymbol{U}) + (\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}} \geq C\big\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{C})\big|\| \tag{11.11}$$

for some $C>0$. Indeed, due to the positivity of the summands the equality $\mathbb{A}^{\#}_N(\boldsymbol{U},\boldsymbol{U})$ implies $\mathscr{E}(\boldsymbol{U},\boldsymbol{U})=0$, and further $\boldsymbol{U}\in\mathscr{R}(\mathscr{C})$ (cf. Lemma 11.3). Also $(\boldsymbol{T}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}}=0$ and further $(\boldsymbol{U},\boldsymbol{K}_j)=0$ for all $j=1,\dots,m$. Then $\boldsymbol{U}=\sum\limits_{j=1}^{m}(\boldsymbol{U},\boldsymbol{K}_j)\boldsymbol{K}_j=0$. A non-negative symmetric form with the property $\mathbb{A}^{\#}_N(\boldsymbol{U},\boldsymbol{U})=0$ if and only if $\boldsymbol{U}=0$ is positive definite.

According to Lax–Milgram's Lemma 7.7 the equation

$$\mathbb{A}^{\#}_N(\boldsymbol{U},\boldsymbol{V})=(\boldsymbol{F},\boldsymbol{V})_{\mathscr{C}}-(\boldsymbol{H},\boldsymbol{V}^+)_{\Gamma} \tag{11.12}$$

has a unique solution $\boldsymbol{U}\in\mathbb{H}^1(\mathscr{C})$ for all $\boldsymbol{V}\in\mathbb{H}^1(\mathscr{C})$. This solves the problem

$$\begin{cases}(\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U})(t)+\boldsymbol{T}\boldsymbol{U}(t)=\boldsymbol{F}(t), & t\in\mathscr{C},\\ (\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{U})^+(\tau)=\boldsymbol{H}(\tau), & \tau\in\Gamma,\end{cases} \tag{11.13}$$

which is a modified Neumann's problem (9.6).

Now assume that the vector-functions $\boldsymbol{F}\in\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ and $\boldsymbol{H}\in\mathbb{H}^{-1/2}(\Gamma)$ satisfy the orthogonality condition (11.1) from Theorem 11.1 and $\boldsymbol{U}^0\in\mathbb{H}^1(\mathscr{C})$ be a solution of (11.13). Since

$$\begin{gathered}(\boldsymbol{T}\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}}=(\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}},\\ \mathbb{A}_N(\boldsymbol{U}^0,\boldsymbol{K}_k)=\mathscr{E}(\boldsymbol{U}^0,\boldsymbol{K}_k)=0,\quad k=1,2,\dots,m\end{gathered}$$

(cf. (11.3)) from (11.12) we get

$$\begin{aligned}0=(\boldsymbol{F},\boldsymbol{K}_k)_{\mathscr{C}}-(\boldsymbol{H},\boldsymbol{K}_k)_{\Gamma}&=\mathbb{A}^{\#}_N(\boldsymbol{U}^0,\boldsymbol{K}_k)=\\ &=\mathbb{A}_N(\boldsymbol{U}^0,\boldsymbol{K}_k)+(\boldsymbol{T}\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}}=(\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}},\quad k=1,2,\dots,m.\end{aligned}$$

Therefore, $\boldsymbol{T}\boldsymbol{U}^0=\sum\limits_{k=1}^{m}(\boldsymbol{U}^0,\boldsymbol{K}_k)_{\mathscr{C}}\boldsymbol{K}_k=0$ and BVP (11.13), which is uniquely solvable, coincides with BVP (9.6) provided that the right hand sides satisfy the orthogonality condition (11.1). Since the kernel of BVP (9.6) coincides with the space of Killing's vector fields $\mathscr{R}(\mathscr{C})$, a general solution of BVP (9.6) has the form $\boldsymbol{U}=\boldsymbol{U}^0+\boldsymbol{K}$ with arbitrary $\boldsymbol{K}\in\mathscr{R}(\mathscr{C})$.

12. Potential Method and Boundary Integral Equations

In the present section we relax the constraints on the data for the BVPs in (9.4) and (9.6):

$$\begin{gathered}\boldsymbol{F}\in\mathbb{W}^{s-2}_p(\mathscr{C}),\quad \boldsymbol{U}\in\mathbb{W}^s_p(\mathscr{C}),\\ \boldsymbol{G}\in\mathbb{W}^{s-\frac1p}_p(\Gamma),\quad \boldsymbol{H}\in\mathbb{W}^{s-1-\frac1p}_p(\Gamma),\quad 1<p<\infty,\quad s\geq 1.\end{gathered} \tag{12.1}$$

Note that due to Corollary 9.2 the traces of solutions to the equation $\boldsymbol{A}_{\mathscr{C}}(t,\mathscr{D})\boldsymbol{U}=f$ in BVPs (9.4) and (9.6) under constraints (12.1) are defined correctly.

To apply the potential method and relax constraints on the data of BVPs we have to restrict ourselves with smooth hypersurfaces to ensure the existence of a fundamental solution to the basic equation. Thus, throughout this section a hypersurface $\mathscr{S}$ will be infinitely smooth and $\mathscr{C}$ will be a subsurface with the ℓ-smooth boundary $\Gamma = \partial\mathscr{C}$. A function $\mathscr{B} \in C^\infty(\mathscr{C})$ is supported in the complemented domain $\operatorname{supp}\mathscr{B} \subset \mathscr{C}^c := \mathscr{S} \setminus \overline{\mathscr{C}}$ and let $\mathscr{K}_{\boldsymbol{A}}(t, t-\tau)$ be the fundamental solution to the perturbed elasticity operator $\boldsymbol{A}_\mathscr{S}(t, \mathscr{D}) + \mathscr{B}I$, which exists due to Theorem 9.3. Then any solution to the BVPs (9.4) and (9.6) is represented by the formulae

$$\boldsymbol{U}(t) = (\mathbf{N}_\mathscr{C}\boldsymbol{F})(t) + (\mathbf{W}_\Gamma \boldsymbol{U}^+)(t) - \big(\boldsymbol{V}_\Gamma(\mathfrak{T}_\mathscr{C}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{U})^+\big)(t), \quad t \in \mathscr{C}, \tag{12.2}$$

where the corresponding potential operators are defined as follows

$$\begin{aligned}
(\mathbf{N}_\mathscr{C}(t,\mathscr{D})\varphi)(t) &:= \oint_\mathscr{C} \mathscr{K}_{\boldsymbol{A}}(t, t-\tau)\varphi(\tau)\, dS,\\
(\mathbf{W}_\Gamma(t,\mathscr{D})\varphi)(t) &:= \oint_\Gamma \Big[\big(\mathfrak{T}_\mathscr{C}(\boldsymbol{\nu}_\Gamma(\tau), \mathscr{D}_\tau)\mathscr{K}_{\boldsymbol{A}}\big)(t, \tau - t)\Big]^\top \varphi(\tau)\, d\mathfrak{s},\\
(\boldsymbol{V}_\Gamma(t,\mathscr{D})\varphi)(t) &:= \oint_\Gamma \mathscr{K}_{\boldsymbol{A}}(t, t-\tau)\varphi(\tau)\, d\mathfrak{s}, \quad t \in \mathscr{C}.
\end{aligned} \tag{12.3}$$

The proof of (12.2) is standard: by inserting the solution $\boldsymbol{U}$ to $\boldsymbol{A}_\mathscr{C}(t,\mathscr{D})\boldsymbol{U} = \boldsymbol{F}$ and the fundamental solution $\boldsymbol{V} = \mathscr{K}_{\boldsymbol{A}}(t, t-\tau)$,

$$\begin{aligned}
\boldsymbol{A}_\mathscr{C}\mathscr{K}_{\boldsymbol{A}}(t, t-\tau) &= \chi_\mathscr{C}(\boldsymbol{A}_\mathscr{S} - \mathscr{H}I)\mathscr{K}_{\boldsymbol{A}}(t, t-\tau) =\\
&= \chi_\mathscr{C}\delta(t-\tau) = \delta(t-\tau), \quad t, \tau \in \mathscr{C}
\end{aligned}$$

truncated properly around the diagonal $t = \tau$ on the distance $\varepsilon > 0$, into the Green formula (9.12) we get the representation formula (12.2) by sending $\varepsilon \to 0$.

Let us consider the following pseudodifferential operators on the boundary Γ, which are direct values of potential operators and their compositions with the boundary operator $\mathfrak{T}_{\Omega^\varepsilon}(\boldsymbol{\nu}_\Gamma, \mathscr{D})$ (cf. (9.9)):

$$\begin{aligned}
\boldsymbol{V}_{-1}(t,\mathscr{D})\boldsymbol{U} &:= \boldsymbol{V}_\Gamma(x,\mathscr{D})\boldsymbol{U}\big|_\mathscr{S},\\
\mathbf{W}_0^*(t,\mathscr{D})\boldsymbol{U} &:= \mathfrak{T}_{\Omega^\varepsilon}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\boldsymbol{V}_\Gamma(x,\mathscr{D})\boldsymbol{U}\big|_\mathscr{S},\\
\mathbf{W}_0(t,\mathscr{D}) &:= \mathbf{W}_\Gamma(x,\mathscr{D})\boldsymbol{U}\big|_\mathscr{S},\\
\boldsymbol{V}_{+1}(\tau,\mathscr{D}) &:= \mathfrak{T}_{\Omega^\varepsilon}(\boldsymbol{\nu}_\Gamma, \mathscr{D})\mathbf{W}_\Gamma(x,\mathscr{D})\boldsymbol{U}\big|_\mathscr{S}.
\end{aligned} \tag{12.4}$$

For these operators we have the standard Plemelji formulae, proved in [15]:

$$\begin{gathered}
(\mathbf{W}_\Gamma\varphi)^\pm(\tau)=\pm\frac{1}{2}\,\varphi(\tau)+\mathbf{W}_0(\tau,\mathscr{D})\varphi(\tau),\\
\big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\boldsymbol{V}_\Gamma\varphi\big)^\pm(\tau)=\mp\frac{1}{2}\,\varphi(\tau)+\mathbf{W}_0^*(\tau,\mathscr{D})\varphi(\tau),\\
(\boldsymbol{V}_\Gamma\varphi)^-(\tau)=(\boldsymbol{V}_\Gamma\varphi)^+(\tau)=\mathbf{V}_{-1}(\tau,\mathscr{D})\varphi(\tau),\\
\big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\mathbf{W}_\Gamma\varphi\big)^-(\tau)=\\
=\big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_\Gamma,\mathscr{D})\mathbf{W}_\Gamma\varphi\big)^+(\tau)=\boldsymbol{V}_{+1}(\tau,\mathscr{D})\varphi(\tau),\quad \tau\in\Gamma.
\end{gathered}\tag{12.5}$$

Moreover, if Γ is ℓ-smooth and $|s|\le\ell$, $1<p<\infty$, the pseudodifferential operators

$$\boldsymbol{V}_{-1}=\boldsymbol{V}_{-1}(\tau,\mathscr{D}):\ \mathbb{H}^s_p(\Gamma)\to\mathbb{H}^{s+1}_p(\Gamma),\tag{12.6a}$$

$$\boldsymbol{V}_{+1}=\boldsymbol{V}_{+1}(\tau,\mathscr{D}):\ \mathbb{H}^s_p(\Gamma)\to\mathbb{H}^{s-1}_p(\Gamma),\tag{12.6b}$$

$$\mathbf{W}_0=\mathbf{W}_0(\tau,\mathscr{D}):\ \mathbb{H}^s_p(\Gamma)\to\mathbb{H}^s_p(\Gamma).\tag{12.6c}$$

are bounded (cf. similar assertions in [13], [17], [18]).

Lemma 12.1. *The pseudodifferential operators $\boldsymbol{V}_{-1}$ is elliptic, positive definite* (*and, therefore, self adjoint*)

$$(\boldsymbol{V}_{-1}\boldsymbol{U},\boldsymbol{U})_{\mathscr{C}}\ge C\big\|\boldsymbol{U}|\mathbb{H}^{-1/2}(\Gamma)\big\|^2\tag{12.7}$$

for some $C>0$.

The pseudodifferential operators

$$\boldsymbol{V}_{+1}=\boldsymbol{V}_{+1}(\tau,\mathscr{D}):\ \mathbb{H}^{1/2}(\Gamma)\to\mathbb{H}^{-1/2}(\Gamma),\tag{12.8}$$

is elliptic, non-positive

$$-(\boldsymbol{V}_{-1}\boldsymbol{Z},\boldsymbol{Z})_\Gamma\ge0,\quad\forall\,\boldsymbol{Z}\in\mathbb{H}^{1/2}(\Gamma)\tag{12.9}$$

and has the trivial index $\operatorname{Ind}\boldsymbol{V}_{+1}=0$.

Proof. For the proof of (12.7) we refer to [11], [17], [18], [29], [28] where similar assertions are proved. □

Corollary 12.2. *Let Γ is ℓ-smooth and $|s|\le\ell$, $1<p<\infty$.*

The pseudodifferential operators $\boldsymbol{V}_{-1}$ in (12.6a) *is invertible.*

The pseudodifferential operators $\boldsymbol{V}_{+1}$ in (12.6a) *is Fredholm, has the trivial index, i.e.,* $\operatorname{Ind}\boldsymbol{V}_{+1}=0$ *and Killing's vector fields all belong to the kernel $\mathscr{R}(\mathscr{S})\subset\operatorname{Ker}\boldsymbol{V}_{+1}$.*

Proof. For $p=2$ the first two assertions are direct consequences of the inequalities (12.7), (12.14) and of ellipticity of the corresponding ΨDOs. Concerning the last assertion about the kernel-the proof is standard and we refer to [11], [17], [18], [29], [28] for such proofs.

For arbitrary $1<p<\infty$ the we quote [17] (also see [1], [14], [22]) where is proved that an elliptic pseudodifferential operator on closed manifold have the same kernel and cokernel in the spaces $\mathbb{H}^s_p(\mathscr{S})$ for all $|s| \leq \ell$ and all $1 < p < \infty$. $\square$

As a byproduct we prove in the next Theorem 12.3 that the kernel $\operatorname{Ker} \boldsymbol{V}_{+1}$ consists of only Killing's vector fields $\operatorname{Ker} \boldsymbol{V}_{+1} = \mathscr{R}(\mathscr{S})$ (cf. Corollary 12.4.

Theorem 12.3. *Let $1 < p < \infty$ and $s \geq 1$.*

The Dirichlet problem (9.4), (12.1) has a unique solution $\boldsymbol{U} \in \mathbb{H}^s_p(\mathscr{C})$ for arbitrary data $\boldsymbol{G} \in \mathbb{H}^{s-1/p}_p(\Gamma)$. This solution is written in the form

$$\boldsymbol{U}(\mathscr{X}) = (\mathbf{N}_{\mathscr{C}}\boldsymbol{F})(\mathscr{X}) + (\mathbf{W}_{\Gamma}\boldsymbol{G})(\mathscr{X}) - (\boldsymbol{V}_{\Gamma}\boldsymbol{Z})(\mathscr{X}), \quad \mathscr{X} \in \mathscr{C}, \tag{12.10}$$

where $\boldsymbol{Z} \in \mathbb{H}^{s-1/p-1}_p(\Gamma)$ is a unique solution to the boundary pseudodifferential equation

$$(\boldsymbol{V}_{-1}\boldsymbol{Z})(t) = (\mathbf{N}_{\mathscr{C}}\boldsymbol{F})(t) - \frac{1}{2}\boldsymbol{G} + (\mathbf{W}_0\boldsymbol{G})(t), \quad t \in \Gamma. \tag{12.11}$$

The Neumann problem (9.6), (12.1) has a solution $\boldsymbol{U} \in \mathbb{H}^s_p(\mathscr{C})$ for those data $\boldsymbol{H} \in \mathbb{H}^{s-1/p-1}_p(\Gamma)$ which satisfy the condition (11.1). If this is the case, a solution is written in the form

$$\boldsymbol{U}(\mathscr{X}) = (\mathbf{N}_{\mathscr{C}}\boldsymbol{F})(\mathscr{X}) + (\mathbf{W}_{\Gamma}\boldsymbol{Z})(\mathscr{X}) - (\boldsymbol{V}_{\Gamma}\boldsymbol{H})(\mathscr{X}) + \boldsymbol{V}(\mathscr{X}), \quad \mathscr{X} \in \mathscr{C}, \tag{12.12}$$

where $\boldsymbol{V} \in \mathscr{R}(\Gamma)$ is arbitrary Killing's vector field and $\boldsymbol{Z} \in \mathbb{H}^{s-1/p-1}_p(\Gamma)$ is a solution to the boundary pseudodifferential equation

$$(\boldsymbol{V}_{+1}\boldsymbol{Z})(t) = -\big(\mathfrak{T}_{\mathscr{C}}(\boldsymbol{\nu}_{\Gamma}, \mathscr{D})\mathbf{N}_{\mathscr{C}}\boldsymbol{F}\big)(t) + \frac{1}{2}\boldsymbol{H}(t) + (\mathbf{W}^*_0\boldsymbol{H})(t), \quad t \in \Gamma. \tag{12.13}$$

Proof. By introducing the representation of a solution (12.10) into the boundary condition in (9.4), invoking Plemelji formulae (12.5), we obtain an equivalent boundary pseudodifferential equation (12.11). Since this boundary ΨDE is uniquely solvable (see Corollary 12.2), the initial BVP has a unique solution, the first part of the theorem is proved.

Similarly, by introducing the representation of a solution (12.12) into the boundary condition in (9.6), invoking Plemelji formulae (12.5), we obtain an equivalent boundary integral (pseudodifferential) equation (12.13). Due to the equivalence, the homogeneous equation $\boldsymbol{V}_{+1}\boldsymbol{Z} = 0$ has as solutions Killing's vector fields $\boldsymbol{Z} \in \mathscr{R}(\mathscr{C})$ only. The solvability condition (11.1) is a consequence of the definition of a Fredholm operator. $\square$

Corollary 12.4. *The pseudodifferential operators $\boldsymbol{V}_{+1}$ satisfies the Gårding's inequality*

$$-(\boldsymbol{V}_{-1}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{C}} \geq C_1\big\|\boldsymbol{U}|\mathbb{H}^{1/2}(\Gamma)\big\|^2 - C_2\big\|\boldsymbol{U}|\mathbb{H}^{-r}(\Gamma)\big\|^2 \tag{12.14}$$

for some $C_1 > 0$, $C_2 > 0$ and arbitrary $0 < r \leq \ell$.

Proof. Let $\{\boldsymbol{K}_j\}_{j=1}^m$ be a biorthogonal basis $(\boldsymbol{K}_j, \boldsymbol{K}_k)_{\Gamma} = \delta_{jk}$ in the finite dimensional space of traces of Killing's vector fields $\mathscr{R}(\Gamma)$ on the boundary Γ. Let us consider the smoothing (infinitely smoothing if $\ell = \infty$) finite rank operator operator $\boldsymbol{T}: \mathbb{H}^{-r}(\Gamma) \to \mathbb{H}^{r}(\Gamma)$ defined in (11.8). We remind that $\{\boldsymbol{K}_j\}_{j=1}^m \subset C^\ell(\Gamma)$ is the orthonormal system of Killing's vector fields. Then, the operator

$$-\boldsymbol{V}_{+1} + \boldsymbol{T}: \mathbb{H}^{1/2}(\Gamma) \to \mathbb{H}^{-1/2}(\Gamma)$$

is invertible and non-negative

$$((-\boldsymbol{V}_{+1} + \boldsymbol{T})\boldsymbol{U}, \boldsymbol{U})_\Gamma = -(\boldsymbol{V}_{+1}\boldsymbol{U}, \boldsymbol{U})_\Gamma + \sum_{j=1}^m (\boldsymbol{K}_j, \boldsymbol{U})_\Gamma^2 \geq 0$$

(cf. (12.9)). This implies that $-\boldsymbol{V}_{+1} + \boldsymbol{T}$ is positive definite

$$((-\boldsymbol{V}_{+1} + \boldsymbol{T})\boldsymbol{U}, \boldsymbol{U})_\Gamma \geq C_1 \big\| \boldsymbol{U} | \mathbb{H}^{1/2}(\Gamma) \big\|^2$$

and we write

$$\begin{gathered} -(\boldsymbol{V}_{+1}\boldsymbol{U}, \boldsymbol{U})_\Gamma := (-\boldsymbol{V}_{+1} + \boldsymbol{T}\boldsymbol{U}, \boldsymbol{U})_\Gamma - (\boldsymbol{T}\boldsymbol{U}, \boldsymbol{U})_\Gamma \geq \\ \geq C_1 \big\| \boldsymbol{U} | \mathbb{H}^{1/2}(\Gamma) \big\|^2 - (\boldsymbol{T}\boldsymbol{U}, \boldsymbol{U})_\Gamma \geq C_1 \big\| \boldsymbol{U} | \mathbb{H}^{1/2}(\Gamma) \big\|^2 - C_2 \big\| \boldsymbol{U} | \mathbb{H}^{-r}(\Gamma) \big\|^2, \end{gathered}$$

which proves (12.14). $\square$

Remark 12.5. Not only the pseudodifferential operator $\boldsymbol{V}_{-1}$ in (12.6a) is invertible for closed surface Γ of codimension 2, but also for an open part of it $\Gamma_D \subset \Gamma$:

$$r_D \boldsymbol{V}_{-1}: \widetilde{\mathbb{H}}_p^s(\Gamma_D) \to \mathbb{H}_p^{s+1}(\Gamma_D), \tag{12.15}$$

provided that

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}, \quad 1 < p < \infty. \tag{12.16}$$

Here r_D is the restriction of functions from Γ to the subsets Γ_D.

The proof is standard and can be retrieved from [17], [18], [30] and other sources.

This assertion can be used for the investigation of the mixed type BVPs, associated with (9.4) and (9.6).

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