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G. Berikelashvili, O. Jokhadze, S. Kharibegashvili, and B. Midodashvili

FINITE-DIFFERENCE METHOD OF SOLVING THE DARBOUX PROBLEM FOR THE NONLINEAR KLEIN–GORDON EQUATION

Abstract. The first Darboux problem for cubic nonlinear Klein–Gordon equation is considered, with a nonhomogeneous condition on the characteristic line. Solvability and convergence of the proper difference scheme is investigated in Sobolev spaces.

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რეზიუმე. კუბური არაწრფივობის მქონე კლეინ–გორდონის განტოლებისათვის განხილულია დარბუს პირველი ამოცანა არაერთგვაროვანი პირობით მახასიათებელ წირზე. გამოკვლეულია შესაბამისი სხვაობიანი სქემის ამოხსნადობა და კრებადობა სობოლევის სივრცეებში.

1. Introduction

We consider the cubic nonlinear Klein–Gordon equation. This equation arises in the quantum theory of field [1], solid and high energy physics [2], radiation theory [3], investigation of thermal equilibrium properties of solitary wave solutions ("kinks") in the classical ϕ^4 field theory [4].

Many works are dedicated to the investigation of boundary value problems for these equations, among which we mention [5]–[9]. The difference schemes of some problems for nonlinear wave equations have been studied in [10]–[13].

For the mentioned equation we consider the first Darboux problem with a non-homogeneous condition on the characteristic line. Note that such problems arise in mathematical modeling of various physical processes, e.g., in the study of harmonic oscillations of a chock in a supersonic flow [14], as well as in investigation of oscillation of a string with a piston beaded on it, which is immersed in a cylinder filled with viscous liquid [15].

2. Statement of the Problem and Main Results

Consider the nonlinear Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u + \lambda u^3 = 0, \tag{2.1}$$

where $\lambda > 0,\ m \geq 0$ are constants.

Let $D_\tau := \big\{(x,t)|\ 0 < x < t,\ 0 < t < \tau\big\}$, $\Gamma_\tau := \big\{(x,\tau)|\ 0 < x < \tau\big\}$.

In the domain D_T for the equation (2.1) consider the first Darboux problem with the following boundary conditions

$$u(0,t) = 0, \quad u(t,t) = \varphi(t), \ \ 0 \leq t \leq T. \tag{2.2}$$

By $W_p^q(D)$ we denote the Sobolev space with the norm defined by

$$\|u\|_{W_p^q(D)} := \Big(\sum_{k=0}^{q} |u|^p_{W_p^k(D)}\Big)^{1/p},$$

$$|u|_{W_p^k(D)} := \Big(\sum_{\alpha+\beta=k} \Big\|\frac{\partial^{\alpha+\beta}u}{\partial x^\alpha \partial t^\beta}\Big\|^p_{L_p(D)}\Big)^{1/p}.$$

In particular, for $q = 0$ we have $W_p^0 = L_p$. Moreover, let

$$|u|^2_{W_2^1(\Gamma_\tau)} := \int\limits_0^\tau \bigg(\Big(\frac{\partial u(x,\tau)}{\partial x}\Big)^2 + \Big(\frac{\partial u(x,\tau)}{\partial t}\Big)^2\bigg)\, dx.$$

As in [16], one can prove that if $\varphi \in C^1$, then the problem (2.1), (2.2) has a classical solution. In the investigation of the difference scheme we require that the solution of the problem (2.1), (2.2) belongs to the space W_2^2.

Denote $h := T/n$. Using the straight lines $t \pm x = 2ih$, $i = 0, 1, 2, \ldots$, let us cover $\overline{D}_T$ by a mesh $\overline{Q}_n$. Denote by Q_n the set of internal nodes (including the nodes lying on the line $t = T$), and by γ_i the set of the internal nodes lying on the layers $t = ih$, $\quad i = 3, 4, \dots$:

$$\gamma_{2k} := \big\{(x_{2\alpha}, t_{2k})|\ \alpha = 1, 2, \dots, k-1\big\},$$

$$\gamma_{2k-1} := \big\{(x_{2\alpha-1}, t_{2k-1})|\ \alpha = 1, 2, \dots, k-1\big\},\ \ x_i = ih,\ \ t_j = jh.$$

Let

$$\gamma^+ := \big\{(jh, jh)|\ j = 0, 1, \dots, n\big\},\quad \gamma^- := \big\{(0, 2jh)|\ j = 0, 1, \dots, [n/2]\big\},$$

where $[\,\cdot\,]$ denotes the integer part of a number.

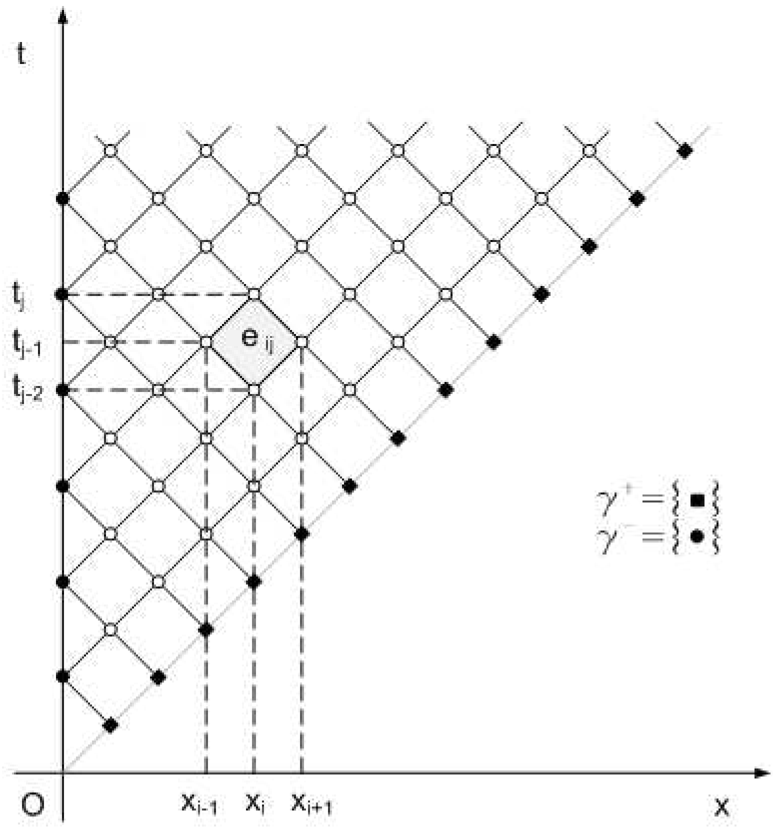

FIGURE 1. Meshes

For summing up on subsets of Q_n, we use the identities

$$\sum_{Q_{2k}} f = \sum_{j=1}^{k-1} \sum_{i=1}^{2k-2j} f_i^{i+2j}, \quad \sum_{Q_{2k-1}} f = \sum_{j=1}^{k-1} \sum_{i=1}^{2k-2j-1} f_i^{i+2j}.$$

For mesh functions we use the notation $U_i^j := U(x_i, t_j)$.

Let us approximate the problem (2.1), (2.2) by the difference scheme

$$\mathcal{L}U := \mathcal{L}_0 U + m^2 \mathcal{L}_m U + \lambda \mathcal{L}_\lambda U = 0, \quad (x,t) \in Q_n, \tag{2.3}$$

$$U\big|_{\gamma^+} = \Phi, \quad U\big|_{\gamma^-} = 0, \tag{2.4}$$

where

$$(\mathcal{L}_0 U)_i^j := (U_i^j + U_i^{j-2} - U_{i-1}^{j-1} - U_{i+1}^{j-1})/h^2, \quad (\mathcal{L}_m U)_i^j := 0.5(U_i^j + U_i^{j-2}),$$
$$(\mathcal{L}_\lambda U)_i^j := (U_i^j + U_i^{j-2})\big((U_{i-1}^{j-1})^2 + (U_{i+1}^{j-1})^2\big)/4, \quad \Phi_i := \varphi(ih).$$

Denote by ∂_1, ∂_2, ∂_t difference quotients along lines $x - t = 0$, $x + t = 0$ and t, respectively:

$$(\partial_1 U)_i^j := \frac{1}{\sqrt{2}h}\big(U_i^j - U_{i-1}^{j-1}\big), \quad (\partial_2 U)_i^j := \frac{1}{\sqrt{2}h}\big(U_i^j - U_{i+1}^{j-1}\big),$$
$$(\partial_t U)_i^j := \frac{1}{2h}\big(U_i^j - U_i^{j-2}\big).$$

Let

$$|U|^2_{W_2^1(\gamma_{2k})} := h\sum_{\alpha=1}^{k}(\partial_1 U_{2\alpha}^{2k})^2 + h\sum_{\alpha=0}^{k-1}(\partial_2 U_{2\alpha}^{2k})^2,$$
$$|U|^2_{W_2^1(\gamma_{2k-1})} := h\sum_{\alpha=1}^{k}(\partial_1 U_{2\alpha-1}^{2k-1})^2 + h\sum_{\alpha=1}^{k-1}(\partial_2 U_{2\alpha-1}^{2k-1})^2, \quad k \ge 2,$$
$$(U,V)_{Q_s} := h^2 \sum_{(x,t)\in Q_s} U(x,t)V(x,t),$$
$$\|U\|^2_{Q_s} := (U,U)_{Q_s}, \quad \|U\|_{C(Q)} := \max_Q |U|.$$

Let $Z := u - U$, where u is the exact solution of the problem (2.1), (2.2) and U is the solution of the finite difference scheme (2.3), (2.4). For the discretization error Z we obtain the following problem

$$\mathcal{L}_0 Z + m^2 \mathcal{L}_m Z = \lambda(\mathcal{L}_\lambda U - \mathcal{L}_\lambda u) + \mathcal{L}u. \tag{2.5}$$

Theorem 2.1. *For the error of the solution of the difference scheme* (2.3), (2.4) *the following estimate*

$$\big|U - u\big|_{W_2^1(\gamma_s)} \le c\|\mathcal{L}u\|_{Q_s} \tag{2.6}$$

is valid, where the constant $c > 0$ *does not depend on* h.

Theorem 2.2. *The solution of the difference scheme* (2.3), (2.4) *converges to the solution of the problem* (2.1), (2.2) *and the following estimates*

$$|U - u|_{W_2^1(\gamma_n)} \le ch^2\|u\|_{W_2^2(Q_T)}, \quad \|U - u\|_{C(Q_n)} \le ch^2\|u\|_{W_2^2(Q_T)} \tag{2.7}$$

are valid, where the constant $c > 0$ *does not depend on* h.

3. Auxiliary Results

Lemma 3.1. *For any mesh function V defined on $\overline{Q}_s$ and satisfying $V|_{\gamma^-} = 0$, the following identity*

$$(\mathcal{L}_0 V, \partial_t V)_{Q_s} = 0.5|V|^2_{W_2^1(\gamma_s)} - 0.5h \sum_{i=1}^{s} (\partial_1 V_i^i)^2$$

is valid.

Lemma 3.2. *For any mesh function V defined on $\overline{Q}_s$ and satisfying $V|_{\gamma^-} = 0$, the following inequality*

$$(\mathcal{L}_m V, \partial_t V)_{Q_s} \geq -\frac{h}{4} \sum_{i=1}^{s-2} (V_i^i)^2, \quad s \geq 3$$

is valid.

Lemma 3.3. *If $V|_{\gamma^-} = 0$, then*

$$(\mathcal{L}_\lambda V, \partial_t V)_{Q_s} \geq -\frac{h}{8} \sum_{i=2}^{s-1} (V_i^i V_{i-1}^{i-1})^2.$$

Lemma 3.4. *For the solutions of the difference scheme* (2.3), (2.4) *and the problem* (2.1), (2.2), *the following estimates*

$$\|U\|_{C(Q_n)} \leq \delta, \quad \|u\|_{C(\overline{D}_T)} \leq \delta$$

are valid, where $\delta := \big((1+m^2T^2+0,5\lambda T^3\varphi_*)T\varphi_*\big)^{1/2}$, $\varphi_* := \int_0^T \big(\varphi'(t)\big)^2\,dt$.

Lemma 3.5. *For any function V defined on $\overline{Q}_n$ and satisfying $V|_{\gamma^-} = V|_{\gamma^+} = 0$, the following inequality*

$$\|\partial_t V\|^2_{Q_s} \leq h \sum_{l=3}^{s} |V|^2_{W_2^1(\gamma_l)} - 0.5h|V|^2_{W_2^1(\gamma_s)}$$

is valid.

Lemma 3.6 (Discrete Gronwall's lemma)**.** *Let w_s, g_s be nonnegative sequences of numbers and g_s be nondecreasing. Then from the inequalities*

$$w_s \leq c \sum_{i=k}^{s-1} w_i + g_s, \quad s = k+1, k+2, \dots, n, \quad w_k \leq g_k, \quad c > 0,$$

it follows

$$w_s \leq g_s \exp(c(s-k)), \quad s = k, k+1, \dots, n.$$

Lemma 3.7. *Let u be the exact solution of the problem* (2.1), (2.2) *and U be the solution of the finite difference scheme* (2.3), (2.4). *Then the equality*

$$\|\mathcal{L}_\lambda U - \mathcal{L}_\lambda u\|^2_{Q_s} \leq$$

$$\le c_1 h \sum_{l=3}^{s-1} |U-u|^2_{W_2^1(\gamma_l)} + (2/\lambda^2)\|\mathcal{L}u\|^2_{Q_s}, \quad s \ge 4, \quad c_1 := 18\delta^4(sh)^2$$

is valid.

4. Proof of Main Results

Proof of Theorem 2.1. Multiplying the equation (2.5) by $\partial_t Z$ and summing up on Q_s, we obtain

$$(\mathcal{L}_0 Z + m^2\mathcal{L}_m Z, \partial_t Z)_{Q_s} = (\mathcal{L}u, \partial_t Z)_{Q_s} + \lambda(\mathcal{L}_\lambda U - \mathcal{L}_\lambda u, \partial_t Z)_{Q_s},$$

whence using Lemmas 3.1, 3.2 and the equality $Z_i^i = 0$, we receive

$$0.5|Z|^2_{W_2^1(\gamma_s)} \le \|\mathcal{L}u\|_{Q_s}\|\partial_t Z\|_{Q_s} + \lambda\|\mathcal{L}_\lambda U - \mathcal{L}_\lambda u\|_{Q_s}\|\partial_t Z\|_{Q_s} \le$$
$$\le \frac{\varepsilon_1}{2}\|\partial_t Z\|^2_{Q_s} + \frac{1}{2\varepsilon_1}\|\mathcal{L}u\|^2_{Q_s} + \frac{\varepsilon_2}{2}\|\partial_t Z\|^2_{Q_s} + \frac{\lambda^2}{2\varepsilon_2}\|\mathcal{L}_\lambda U - \mathcal{L}_\lambda u\|^2_{Q_s}.$$

Choose $\varepsilon_1 = 4/(3T)$, $\varepsilon_2 = 8/(3T)$ and use Lemma 3.7:

$$0.5|Z|^2_{W_2^1(\gamma_s)} \le \frac{3T}{4}\|\mathcal{L}u\|^2_{Q_s} + \frac{2}{T}\|\partial_t Z\|^2_{Q_s} + \frac{3T\lambda^2 c_1 h}{16}\sum_{l=3}^{s-1}|Z|^2_{W_2^1(\gamma_l)}.$$

Due to Lemma 3.5, we have

$$0.5|Z|^2_{W_2^1(\gamma_s)} \le$$
$$\le \Big(\frac{2h}{T} + \frac{3T\lambda^2 c_4 h}{16}\Big)\sum_{l=3}^{s-1}|Z|^2_{W_2^1(\gamma_l)} + \frac{h}{T}|Z|^2_{W_2^1(\gamma_s)} + \frac{3T}{4}\|\mathcal{L}u\|^2_{Q_s}, \quad s \ge 4.$$

Since $h/T = h/(nh) \le 1/4$ for $4 \le s \le n$, we find

$$|Z|^2_{W_2^1(\gamma_s)} \le$$
$$\le \Big(\frac{8}{T} + \frac{3T\lambda^2 c_1}{4}\Big)h\sum_{l=3}^{s-1}|Z|^2_{W_2^1(\gamma_l)} + 3T\|\mathcal{L}u\|^2_{Q_s}, \quad s = 4, 5, \ldots, n. \tag{4.1}$$

Now let us show that

$$|Z|^2_{W_2^1(\gamma_3)} \le 3T\|\mathcal{L}u\|^2_{Q_3}. \tag{4.2}$$

Indeed, first note that $|Z|^2_{W_2^1(\gamma_3)} = (1/h)(Z_1^3)^2$. The equation (2.5) on the grid Q_3 (consisting from one grid point only) can be rewritten as follows

$$\Big(\frac{1}{h^2} + \frac{m^2}{2} + \frac{\lambda}{4}(u_2^2)^2\Big)Z_1^3 = \mathcal{L}u_1^3,$$

whence $(Z_1^3)^2 \le h^4(\mathcal{L}u_1^3)^2$.

Therefore

$$|Z|^2_{W_2^1(\gamma_3)} \le h\|\mathcal{L}u\|^2_{Q_3} \le 3T\|\mathcal{L}u\|^2_{Q_3}.$$

Applying Lemma 3.6 to the inequalities (4.1), (4.2), we obtain the estimate (2.6) with $c = \sqrt{3T\exp(8 + 14\lambda^2T^4\delta^4)}$, where δ is defined in Lemma 3.4. □

Proof of Theorem 2.2. Let

$$l(u) := \frac{1}{2h^2} \int\limits_{e_{ij}} u(x,t)\, dx\, dt.$$

By introducing the notation

$$\begin{gathered} \psi_1(u) := l(u) - 0.5(u_i^j + u_i^{j-2}), \\ \psi_2(u) := l(u) - 0.5(u_{i-1}^{j-1} + u_{i+1}^{j-1}), \\ \psi_3(u) := l(u^2)\psi_1(u) + 0.5(u_i^j + u_i^{j-2})\psi_2(u^2), \end{gathered} \tag{4.3}$$

we write the truncation error in the form

$$\mathcal{L}u = -m^2\psi_1(u) - \lambda\big(l(u^3) - l(u)l(u^2) + \psi_3(u)\big). \tag{4.4}$$

The expressions for $\psi_\alpha(u)$, $\alpha = 1,2$, can be rewritten as follows:

$$\psi_\alpha(u) = \frac{1}{4h^2} \int\limits_{e_{ij}} \bigg((x - x_{i+\alpha-2})(x - x_{i-\alpha+2}) \frac{\partial^2 u}{\partial t^2} + \\ + (t - t_{j+\alpha-2})(t - t_{j-\alpha}) \frac{\partial^2 u}{\partial x^2} + 2(x - x_i)(t - t_{j-1}) \frac{\partial^2 u}{\partial x \partial t} \bigg)\, dx\, dt.$$

Hence it follows

$$|\psi_\alpha(u)| \le \frac{1}{4} \int\limits_{e_{ij}} \bigg(\Big|\frac{\partial^2 u}{\partial t^2}\Big| + \Big|\frac{\partial^2 u}{\partial x^2}\Big| + 2\Big|\frac{\partial^2 u}{\partial x \partial t}\Big| \bigg)\, dxdt, \quad \alpha = 1,2,$$

whence, using the Cauchy–Schwartz inequality and the algebraic inequality $(a + b + 2c)^2 \le 4(a^2 + b^2 + 2c^2)$, we have

$$|\psi_1(u)| \le \frac{h}{\sqrt{2}}\, |u|_{W_2^2(e_{ij})}, \quad |\psi_2(u^2)| \le \frac{h}{\sqrt{2}} |u^2|_{W_2^2(e_{ij})}. \tag{4.5}$$

Since

$$l(u^2) \le \|u\|^2_{C(\overline{D}_T)}, \quad 0.5|u_i^j + u_i^{j-2}| \le \|u\|_{C(\overline{D}_T)},$$

we have

$$|\psi_3(u)| \le \|u\|_{C(\overline{D}_T)}\big(|\psi_1(u)|\, \|u\|_{C(\overline{D}_T)} + \psi_2(u^2)\big).$$

Therefore from (4.3) it follows

$$|\psi_3(u)| \le \frac{h}{\sqrt{2}}\, \|u\|_{C(\overline{D}_T)} \Big(\|u\|_{C(\overline{D}_T)} |u|_{W_2^2(e_{ij})} + |u^2|_{W_2^2(e_{ij})} \Big). \tag{4.6}$$

It can be shown that

$$\big|l(u^3) - l(u)l(u^2)\big| \le 4h|u|^2_{W_4^1(e_{ij})} \|u\|_{C(\overline{D}_T)}. \tag{4.7}$$

According to the estimates (4.5), (4.6), (4.7) from (4.4), we obtain

$$|(\mathcal{L}u)_i^j| \le c_2 h\Big(|u|_{W_2^2(e_{ij})} + |u|^2_{W_4^1(e_{ij})} + |u^2|_{W_2^2(e_{ij})} \Big),$$

where $c_2 := m^2 + \lambda\delta^2 + 4\lambda\delta$ and δ is defined in Lemma 3.4.

Therefore

$$\|\mathcal{L}u\|^2_{Q_s} = h^2 \sum_{Q_s} (\mathcal{L}u)^2 \le 3c_2^2 h^4 \Big(|u|^4_{W_4^1(D_s)} + |u|^2_{W_2^2(D_s)} + |u^2|^2_{W_2^2(D_s)} \Big),$$

i.e., as we note that

$$|u^2|^2_{W_2^2(D_s)} \le 8\Big(\delta^2\, |u|^2_{W_2^2(D_s)} + |u|^4_{W_4^1(D_s)}\Big),$$

we have

$$\|\mathcal{L}u\|^2_{Q_s} \le 3c_2^2 h^4 \Big(9|u|^4_{W_4^1(D_s)} + (1+8\delta^2)|u|^2_{W_2^2(D_s)} \Big). \tag{4.8}$$

Since $W_2^2 \subset W_4^1$, then from (4.8) follows the validity of the first estimate in (2.7). The validity of the second estimate in (2.7) can be easily obtained using $\|Z\|^2_{C(\gamma_s)} \le 2sh|Z|^2_{W_2^1(\gamma_s)}$, $s = 3, 4, \ldots$. □

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Authors' addresses:

G. Berikelashvili, O. Jokhadze, S. Kharibegashvili
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: bergi@rmi.acnet.ge
ojokhadze@yahoo.com
kharib@rmi.acnet.ge

B. Midodashvili
Gori University
53, Chavchavadze St., Gori 1400
Georgia