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REPRESENTATION FORMULAS OF GENERAL SOLUTIONS TO THE STATIC EQUATIONS OF THE HEMITROPIC ELASTICITY THEORY

Abstract. We consider the differential equations of statics of the theory of elasticity of hemitropic materials. We derive general representation formulas for solutions, i.e., for the displacement and microrotation vectors by means of three harmonic and three metaharmonic functions. These formulas are very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of these formulas to the Neumann type boundary value problem for a ball. We construct explicit solutions in the form of absolutely and uniformly convergent series.

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[illegible]

1. Introduction

Technological and industrial developments as well as great success in biological and medical sciences require to use more generalized and refined models for elastic bodies. In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model composites with a complex inner structure whose material particles have 6 degrees of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Experiments have shown that micropolar materials possess quite different properties in comparison with the classical elastic materials (see, e.g., [1]–[6] and the references therein). For example, in noncentrosymmetric micropolar materials (which are called also hemitropic or chiral materials) there propagate the left-handed and right-handed elastic waves. Moreover, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity.

Hemitropic solids are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

Materials may exhibit chirality on the atomic scale, as in quartz and biological molecules – DNA, as well as on a large scale, as in composites with helical or screw–shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For more details see the references [1], [2], [4], [7]–[15].

Mathematical models describing chiral properties of elastic hemitropic materials have been proposed by Aero and Kuvshinski [1], [2] (for historical notes see also [4], [12], [13] and the references therein).

In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and the moment stress tensor which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor via the constitutive equations. All these quantities are expressed in terms of the components of the displacement and microrotation vectors. In turn, the displacement and microrotation vectors satisfy a coupled complex system of second order partial differential equations of dynamics. When the mechanical characteristics (displacements, microrotations, body force and body couple vectors) do not depend on the time variable t, we have the differential equations of statics. These equations generate a 6×6 strongly elliptic, formally self-adjoint differential operator involving 9 material constants and have very complex form.

The Dirichlet, Neumann and mixed type boundary value problems (BVPs) corresponding to this model are well investigated for general domains of arbitrary shape and the uniqueness and existence theorems are proved. Regularity results for solutions are also established by potential as well as by variational methods (see [13], [16]–[19] and the references therein).

The main goal of this paper is to derive general representation formulas for the displacement and microrotation vectors by means of harmonic and metaharmonic functions. That is, we can represent solutions to the very complicated coupled system of simultaneous differential equations of hemitropic elasticity with the help of solutions of a simpler canonical equations (similar formulas in the classical elastostatics are well known as Papkovich–Neuber representation formulas).

Namely, we prove that the six components of the field vectors (three displacement and three microrotation components) can be expressed linearly by three harmonic and three metaharmonic scalar functions. Moreover, we show that this correspondence is one-to-one. The representation formulas obtained have proved to be very useful in the study of many problems for domains with concrete geometry.

In particular, here we apply these representation formulas to construct explicit solutions to the Neumann type boundary value problem for a ball. We represent the solution in the form of Fourier–Laplace series and show their absolute and uniform convergence along with their derivatives of the first order if the boundary data satisfy appropriate smoothness conditions.

The motivation for the choice of the transmission problems treated in the paper is that by the same approach one can construct explicit solutions to transmission problems for layered composites with finitely many spherical interfaces.

Moreover, the representations obtained can be applied to some generalizations of the classical Eshelby type inclusion problems for hemitropic materials (see [20], [21]). For a wider overview of the subject concerning different areas of application we refer to the references [5], [7], [9], [10], [15], and [22]–[24].

2. General Representation of Solutions

The basic equations of statics of the hemitropic elasticity read as follows [1], [2]

$$\begin{gathered}(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}u(x)+(\varkappa+\nu)\Delta\omega(x)+\\ +(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x)+2\alpha\operatorname{curl}\omega(x)=0,\\ (\varkappa+\nu)\Delta u(x)+(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}u(x)+2\alpha\operatorname{curl}u(x)+\\ +(\gamma+\varepsilon)\Delta\omega(x)+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x)+\\ +4\nu\operatorname{curl}\omega(x)-4\alpha\omega(x)=0,\end{gathered}\tag{2.1}$$

where $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator, $\partial_j=\partial/\partial x_j$, $u=(u_1,u_2,u_3)^\top$ and $\omega=(\omega_1,\omega_2,\omega_3)^\top$ are the *displacement vector* and the *micro-rotation vector*, respectively; α, β, γ, δ, λ, μ, ν, $\varkappa$ and ε are the material constants; here and in what follows the symbol $(\,\cdot)^\top$ denotes transposition.

The material constants satisfy the following inequalities

$$
\begin{gathered}
\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\
(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\\
(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\
d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\
d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0,\\
\mu[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\
\mu[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.
\end{gathered}
\tag{2.2}
$$

Now we formulate our basic assertion.

Theorem 2.1. *A vector* $u=(u,\omega)^\top$ *is a solution to the system* (2.1) *if and only if it is representable in the form*

$$
\begin{aligned}
u(x)&=\operatorname{grad}\Phi_1(x)-a\operatorname{grad}\big[r^2(r\partial_r+1)\Phi_2(x)\big]+\operatorname{rot}\operatorname{rot}[xr^2\Phi_2(x)]+\\
&\quad+\operatorname{rot}[x\Phi_3(x)]+(\delta+2\varkappa)\operatorname{grad}\Phi_4(x)+\\
&\quad+\sum_{j=1}^{2}\Big[\operatorname{rot}\operatorname{rot}[x\Psi_j(x)]+k_j\operatorname{rot}[x\Psi_j(x)]\Big],\\
\omega(x)&=\sigma\operatorname{grad}\big[(2r\partial_r+3)(r\partial_r+1)\Phi_2(x)\big]-\operatorname{rot}[x(2r\partial_r+3)\Phi_2(x)]+\\
&\quad+2^{-1}\operatorname{rot}\operatorname{rot}[x\Phi_3(x)]-(\lambda+2\mu)\operatorname{grad}\Phi_4(x)-\\
&\quad-\sum_{j=1}^{2}\eta_j\Big[\operatorname{rot}\operatorname{rot}[x\Psi_j(x)]+k_j\operatorname{rot}[x\Psi_j(x)]\Big],
\end{aligned}
\tag{2.3}
$$

where $x=(x_1,x_2,x_3)^\top$, $r=|x|$, $r\partial_r=x\cdot\operatorname{grad}$,

$$
\begin{gathered}
\Delta\Phi_j(x)=0,\ \ j=1,2,3,\quad (\Delta-\lambda_1^2)\Phi_4(x)=0,\\
(\Delta+k_j^2)\Psi_j(x)=0,\ \ j=1,2,\\
a=\mu/(\lambda+2\mu),\ \ \sigma=-\frac{1}{2\alpha}\big[a(\delta+2\varkappa)-\varkappa+\nu\big],\ \ \lambda_1^2=4\alpha(\lambda+2\mu)/d_2,\\
k_{1,2}=2d_1^{-1}\Big[\mu\nu-\alpha\varkappa\pm i\sqrt{(\mu+\alpha)[\mu(\alpha\varepsilon-\nu^2)+\alpha(\mu\gamma-\varkappa^2)]}\Big],\\
\sigma_1=\frac{d_1}{2\alpha\Delta_1}\Big[\alpha(\gamma+\varepsilon)-2\nu(\varkappa+\nu)\Big],\quad \eta_j=k_j\Big(\sigma_1k_j^2-\sigma_2k_j-\frac12\Big),\\
\sigma_2=\frac{1}{\alpha\Delta_1}\Big[\alpha(\varkappa+\nu)d_1+2(\mu\nu-\alpha\varkappa)(\alpha(\gamma+\varepsilon)-2\nu(\varkappa+\nu))\Big],\\
\Delta_1=4\alpha(\alpha\gamma+\varkappa^2+\alpha\varepsilon-\nu^2)>0,
\end{gathered}
$$

d_1 *and* d_2 *are defined in* (2.2).

Proof. Let u and ω solve the system (2.1). Let us show that then they admit the representation (2.3). Apply the divergence operation to both equations of the system (2.1)

$$(\lambda+2\mu)\Delta\operatorname{div}u+(\delta+2\varkappa)\Delta\operatorname{div}\omega=0,$$
$$(\delta+2\varkappa)\Delta\operatorname{div}u+\big[(\beta+2\gamma)\Delta-4\alpha\big]\operatorname{div}\omega=0.$$

From this system we get

$$(\Delta-\lambda_1^2)\operatorname{div}\omega=0,\quad \Delta(\Delta-\lambda_1^2)\operatorname{div}u=0, \tag{2.4}$$

where

$$\lambda_1^2=4\alpha(\lambda+2\mu)/d_2,\quad d_2=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2.$$

Now apply the curl operation to both equations of the system (2.1)

$$\begin{gathered}(\mu+\alpha)\Delta\operatorname{rot}u+\big[(\varkappa+\nu)\Delta+2\alpha\operatorname{rot}\big]\operatorname{rot}\omega=0,\\ \big[(\varkappa+\nu)\Delta+2\alpha\operatorname{rot}\big]\operatorname{rot}u+\big[(\gamma+\varepsilon)\Delta+4\nu\operatorname{rot}-4\alpha\big]\operatorname{rot}\omega=0.\end{gathered} \tag{2.5}$$

With the help of the identities $\operatorname{rot}\operatorname{rot}=\operatorname{grad}\operatorname{div}-\Delta$ and $\operatorname{div}\operatorname{rot}=0$ this system implies

$$\big[d_1\operatorname{rot}\operatorname{rot}+4(\alpha\varkappa-\mu\nu)\operatorname{rot}+4\alpha\mu\big](\Delta\operatorname{rot}u,\Delta\operatorname{rot}\omega)^\top=0. \tag{2.6}$$

Rewrite the first equation of the system (2.1) in the following form

$$\mu\Delta u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u=v(x)-(\delta+2\varkappa)\operatorname{grad}\operatorname{div}\omega, \tag{2.7}$$

where

$$v(x)=\alpha\operatorname{rot}(\operatorname{rot}u-2\omega)+(\varkappa+\nu)\operatorname{rot}\operatorname{rot}\omega. \tag{2.8}$$

From the second equation of the system (2.1) we get

$$\begin{gathered}2\alpha(\operatorname{rot}u-2\omega)=-(\varkappa+\nu)\Delta u-(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}u-(\gamma+\varepsilon)\Delta\omega-\\ -(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega-4\nu\operatorname{rot}\omega.\end{gathered}$$

In view of this equality, from (2.8) we obtain

$$v(x)=-\frac{\varkappa+\nu}{2}\,\Delta\operatorname{rot}u-\frac{\gamma+\varepsilon}{2}\,\Delta\operatorname{rot}\omega+(\varkappa-\nu)\operatorname{rot}\operatorname{rot}\omega. \tag{2.9}$$

The equation (2.5) yields

$$\operatorname{rot}\operatorname{rot}\omega=-\frac{\mu+\alpha}{2\alpha}\,\Delta\operatorname{rot}u-\frac{\varkappa+\nu}{2\alpha}\,\Delta\operatorname{rot}\omega.$$

We can rewrite the equation (2.9) as

$$v(x)=a_1\Delta\operatorname{rot}u+a_2\Delta\operatorname{rot}\omega, \tag{2.10}$$

where

$$a_1=-\frac{1}{2\alpha}\Big[\alpha(\varkappa+\nu)+(\mu+\alpha)(\varkappa-\nu)\Big],\quad a_2=-\frac{1}{2\alpha}\Big[\alpha(\gamma+\varepsilon)+\varkappa^2-\nu^2\Big].$$

From the equalities (2.10) and (2.6) we derive

$$\big[d_1\operatorname{rot}\operatorname{rot}+4(\alpha\varkappa-\mu\nu)\operatorname{rot}+4\alpha\mu\big]v(x)=0,$$

i.e.,

$$(\operatorname{rot}-k_1)(\operatorname{rot}-k_2)v(x)=0, \tag{2.11}$$

where k_1 and k_2 are the roots of the quadratic equation

$$d_1 z^2 + 4(\alpha\varkappa - \mu\nu)z + 4\alpha\mu = 0.$$

The discriminant of this equation is

$$D = -16(\mu+\alpha)\big[\mu(\alpha\varepsilon - \nu^2) + \alpha(\mu\gamma - \varkappa^2)\big] < 0.$$

Therefore

$$k_{1,2} = \frac{2}{d_1}\Big[\mu\nu - \alpha\varkappa \pm i\sqrt{(\mu+\alpha)(\mu(\alpha\varepsilon-\nu^2)+\alpha(\mu\gamma-\varkappa^2))}\,\Big].$$

A solution of the equation (2.11) can be represented as

$$v(x) = -\sum_{j=1}^{2} \mu k_j^2 v_j(x), \tag{2.12}$$

where

$$(\operatorname{rot} - k_j)v_j(x) = 0, \quad \operatorname{div} v_j(x) = 0. \tag{2.13}$$

Notice that $k_2 = \overline{k}_1$ and $v_2 = \overline{v}_1$ (the over-bar means complex conjugation). Moreover, remark that for a vector $v = (v_1, v_2, v_3)^\top$ to be a solution of the system

$$\operatorname{rot} v(x) \mp kv(x) = 0, \quad \operatorname{div} v(x) = 0,$$

necessary and sufficient conditions read as follows

$$v(x) = \operatorname{rot}\operatorname{rot}(x\Psi(x)) \pm k\operatorname{rot}(x\Psi(x)),$$

where Ψ is a scalar function satisfying the Helmholtz equation $(\Delta + k^2)\Psi = 0$. Due to this remark, we can represent a solution to the system (2.13) as

$$v_j(x) = \operatorname{rot}\operatorname{rot}(x\,\Psi_j(x)) + k_j\operatorname{rot}(x\,\Psi_j(x)), \quad j = 1, 2,$$

where $(\Delta + k_j^2)\Psi_j(x) = 0$. The functions Ψ_1 and Ψ_2 are mutually complex conjugate functions. Substituting the expressions of the vectors v_j into (2.12), we get

$$v(x) = -\sum_{j=1}^{2} \mu k_j^2 \Big[\operatorname{rot}\operatorname{rot}(x\Psi_j(x)) + k_j\operatorname{rot}(x\Psi_j(x))\Big]. \tag{2.14}$$

Introduce the function Φ_4 by the equation

$$\operatorname{div}\omega(x) = -\lambda_1^2(\lambda + 2\mu)\Phi_4(x). \tag{2.15}$$

From the equations (2.4) and (2.15) it follows that

$$(\Delta - \lambda_1^2)\Phi_4(x) = 0.$$

Taking into consideration the equations (2.14) and (2.15), we get from (2.7)

$$\mu\Delta u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u =$$

$$= \lambda_1^2(\lambda+2\mu)(\delta+2\varkappa)\operatorname{grad}\Phi_4(x) - \sum_{j=1}^{2}\mu k_j^2\Big[\operatorname{rot}\operatorname{rot}(x\Psi_j(x)) + k_j\operatorname{rot}(x\Psi_j(x))\Big].$$

The general solution of this equation is written as

$$u(x) = u_0(x) + (\delta + 2\varkappa)\,\mathrm{grad}\,\Phi_4(x) + \\ + \sum_{j=1}^{2} \Big[\mathrm{rot}\,\mathrm{rot}(x\Psi_j(x)) + k_j\,\mathrm{rot}(x\Psi_j(x)) \Big], \tag{2.16}$$

where u_0 is the general solution of the equation

$$\mu\Delta u_0(x) + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,u_0(x) = 0. \tag{2.17}$$

Now let us express the vector ω from (2.1) in terms of u and $\mathrm{div}\,\omega$

$$\omega = \sigma_1\Delta\,\mathrm{rot}\,u + \sigma_2\,\mathrm{rot}\,\mathrm{rot}\,u + \frac{1}{2}\,\mathrm{rot}\,u + \sigma_3\,\mathrm{grad}\,\mathrm{div}\,u + \sigma_4\,\mathrm{grad}\,\mathrm{div}\,\omega, \tag{2.18}$$

where

$$\begin{aligned}
\sigma_1 &= \frac{d_1}{2\alpha\Delta_1}\big[\alpha(\gamma+\varepsilon) - 2\nu(\varkappa+\nu)\big],\\
\sigma_2 &= \frac{1}{\alpha\Delta_1}\Big[\alpha(\varkappa+\nu)d_1 + 2(\mu\nu - \alpha\varkappa)\big(\alpha(\gamma+\varepsilon) - 2\nu(\varkappa+\nu)\big)\Big],\\
\sigma_3 &= \frac{\delta+2\varkappa}{4\alpha} - \frac{\lambda+2\mu}{\alpha\Delta_1}\big[\alpha(\gamma+\varepsilon)(\varkappa+3\nu) - 4\nu^2(\varkappa+\nu)\big],\\
\sigma_4 &= \frac{\beta+2\gamma}{4\alpha} - \frac{\delta+2\varkappa}{\alpha\Delta_1}\big[\alpha(\gamma+\varepsilon)(\varkappa+3\nu) - 4\nu^2(\varkappa+\nu)\big].
\end{aligned}$$

Further, substitute (2.16) into (2.18) and take into consideration the equations

$$\Delta\,\mathrm{rot}\,u_0 = 0, \quad \lambda_1^2\big[(\delta+2\varkappa)\sigma_3 - (\lambda+2\mu)\sigma_4\big] = -(\lambda+2\mu)$$

to obtain

$$\omega(x) = \sigma_2\,\mathrm{rot}\,\mathrm{rot}\,u_0(x) + \frac{1}{2}\,\mathrm{rot}\,u_0(x) + \sigma_3\,\mathrm{grad}\,\mathrm{div}\,u_0(x) - \\ - (\lambda+2\mu)\,\mathrm{grad}\,\Phi_4(x) - \sum_{j=1}^{2}\eta_j\Big[\mathrm{rot}\,\mathrm{rot}(x\Psi_j(x)) + k_j\,\mathrm{rot}(x\Psi_j(x))\Big], \tag{2.19}$$

where $\eta_j = k_j(\sigma_1 k_j^2 - \sigma_2 k_j - 1/2)$. It is known that a solution of the system (2.17) is representable in the form [28]

$$u_0(x) = \mathrm{grad}\,\Phi_1(x) - a\,\mathrm{grad}\,r^2\Big(r\partial r + 1\Big)\Phi_2(x) + \\ + \mathrm{rot}\,\mathrm{rot}(xr^2\Phi_2(x)) + \mathrm{rot}(x\Phi_3(x)), \tag{2.20}$$

where

$$x = (x_1,\ x_2,\ x_3)^\top, \quad r\partial r = x\cdot\mathrm{grad}, \quad r = |x|,$$
$$\Delta\Phi_j(x) = 0, \ \ j = 1,2,3, \quad a = \mu/(\lambda+2\mu).$$

Substitution of (2.20) into (2.16) and (2.19) completes the proof of the first part of the theorem. The sufficiency easily follows from the identities

$$\sigma_2 + a\sigma_3 = \frac{1}{4\alpha}\Big[a(\delta+2\varkappa) - \varkappa + \nu\Big] = -\frac{\sigma}{2}\,,$$
$$(\varkappa+\nu)\Big[\sigma_1 \frac{4(\mu\nu-\alpha\varkappa)}{d_1} - \sigma_2\Big] - 2\alpha\sigma_1 = -\frac{d_1}{4\alpha}\,,$$
$$(\varkappa+\nu)\Big(\frac{4\alpha\mu}{d_1}\sigma_1 + \frac{1}{2}\Big) - 2\alpha\sigma_2 = \frac{\alpha\varkappa-\mu\nu}{\alpha}\,,$$
$$(\gamma+\varepsilon)\Big(\frac{4\alpha\mu}{d_1}\sigma_1 + \frac{1}{2}\Big) - 4\nu\sigma_2 - 4\alpha\sigma_1 = \frac{\mu\nu-\alpha\varkappa}{\alpha\mu}\Big[4\alpha\sigma_3 - (\delta+2\varkappa)\Big]a,$$
$$(\gamma+\varepsilon)\Big[\sigma_1 \frac{4(\mu\nu-\alpha\varkappa)}{d_1} - \sigma_2\Big] - 4\nu\sigma_1 = \frac{d_1}{4\alpha\mu}\Big[4\alpha\sigma_3 - (\delta+2\varkappa)\Big]a,$$
$$4\alpha\sigma_2 + \varkappa - \nu = -a\Big[4\alpha\sigma_3 - (\delta+2\varkappa)\Big].$$

The proof is complete. □

3. Basic Boundary Value Problems

Let Ω_1 be a ball whose boundary is the sphere $\partial\Omega_1$ having radius R and centered at the origin:

$$\Omega_1 = \big\{x:\ x\in\mathbb{R}^3,\ |x|<R\big\},\quad \partial\Omega = \big\{x:\ x\in\mathbb{R}^3,\ |x|=R\big\}.$$

Denote $\Omega_2 = \mathbb{R}^3 \setminus \overline{\Omega_1}$. Assume that the regions Ω_1 and Ω_2 are filled by isotropic hemitropic materials.

Problem $(\mathrm{I})^{\pm}$. Find a regular vector $U=(u,\omega)^\top$ satisfying the differential equations (2.1) in Ω_1 (Ω_2) and the boundary condition

$$[U(z)]^{\pm} = F(z),\quad z\in\partial\Omega,$$

where

$$F(z) = \big(f^{(1)}(z), f^{(2)}(z)\big)^\top,\quad f^{(j)}(z) = \big(f_1^{(j)}(z), f_2^{(j)}(z), f_3^{(j)}(z)\big),\quad j=1,2.$$

In the case of the exterior domain Ω_2, the vector U has to satisfy the following decay conditions at infinity

$$U(x) = O(|x|^{-1}),\quad \frac{\partial}{\partial x_j}U(x) = O(|x|^{-2}),\quad j=1,2,3. \tag{3.1}$$

Problem $(\mathrm{II})^{\pm}$. Find a regular vector $U=(u,\omega)^\top$ satisfying the differential equations (2.1) in Ω_1 (Ω_2) and the boundary condition

$$\big[T(\partial z, n(z))U(z)\big]^{\pm} = F(z),\quad z\in\partial\Omega, \tag{3.2}$$

where $n(z)$ is the exterior normal vector to $\partial\Omega$, $T(\partial_z, n(z))U$ is the generalized stress vector [17]

$$T(\partial x, n)U(x) = \Big[H^{(1)}(\partial x, n)U(x), H^{(2)}(\partial x, n)U(x)\Big]^\top, \tag{3.3}$$

where

$$
\begin{gathered}
H^{(j)}(\partial x,n)U(x)=T^{(2j-1)}(\partial x,n)u(x)+T^{(2j)}(\partial x,n)\omega(x),\\
T^{(2j-1)}(\partial x,n)u(x)=\\
=\xi_j'\,\frac{\partial u(x)}{\partial n(x)}+\eta_j' n(x)\operatorname{div}u(x)+\zeta_j'[n(x)\times\operatorname{rot}u(x)],\\
T^{(2j)}(\partial x,n)\omega(x)=\xi_j''\,\frac{\partial\omega(x)}{\partial n(x)}+\\
+\eta_j'' n(x)\operatorname{div}\omega(x)+\zeta_j''[n(x)\times\operatorname{rot}\omega(x)]+\tau_j[n(x)\times\omega(x)],\quad j=1,2,
\end{gathered}
\tag{3.4}
$$

with

$$
\begin{gathered}
\xi_l'=\begin{cases}2\mu, & l=1,\\ 2\varkappa, & l=2;\end{cases}\quad
\eta_l'=\begin{cases}\lambda, & l=1,\\ \delta, & l=2;\end{cases}\\
\zeta_l'=\begin{cases}\mu-\alpha, & l=1,\\ \varkappa-\nu, & l=2;\end{cases}\quad
\xi_l''=\begin{cases}2\varkappa, & l=1,\\ 2\gamma, & l=2;\end{cases}\\
\eta_l''=\begin{cases}\delta, & l=1,\\ \beta, & l=2;\end{cases}\quad
\zeta_l''=\begin{cases}\varkappa-\nu, & l=1,\\ \gamma-\varepsilon, & l=2;\end{cases}\quad
\tau_l=\begin{cases}2\alpha, & l=1,\\ 2\nu, & l=2.\end{cases}
\end{gathered}
$$

Here and in what follows the symbol $a\times b$ denotes the cross product of two vectors a, $b\in\mathbb{R}^3$.

In the case of the exterior domain Ω_2, the vector U has to satisfy the decay conditions (3.1).

Transmission Problem (A). Assume that the domains Ω_j, $j=1,2$, are filled by isotropic hemitropic solids with material constants $\alpha_j,\beta_j,\dots,\varkappa_j$. Find a pair of regular vectors $U^{(j)}=(u^{(j)},\omega^{(j)})^\top$, $j=1,2$, satisfying the differential equations

$$
\begin{gathered}
(\mu_j+\alpha_j)\Delta u^{(j)}+(\lambda_j+\mu_j-\alpha_j)\operatorname{grad}\operatorname{div}u^{(j)}+(\varkappa_j+\nu_j)\Delta\omega^{(j)}+\\
+(\delta_j+\varkappa_j-\nu_j)\operatorname{grad}\operatorname{div}\omega^{(j)}2\alpha_j\operatorname{rot}\omega^{(j)}=0,\\
(\varkappa_j+\nu_j)\Delta u^{(j)}+(\delta_j+\varkappa_j-\nu_j)\operatorname{grad}\operatorname{div}u^{(j)}+(\gamma_j+\varepsilon_j)\Delta\omega^{(j)}+\\
+(\beta_j+\gamma_j-\varepsilon_j)\operatorname{grad}\operatorname{div}\omega^{(j)}+2\alpha_j\operatorname{rot}u^{(j)}+4\nu_j\operatorname{rot}\omega^{(j)}-4\alpha_j\omega^{(j)}=0,\quad j=1,2,
\end{gathered}
$$

in Ω_j and the following transmission conditions on $\partial\Omega$

$$
\begin{gathered}
\big[U^{(1)}(z)\big]^+-\big[U^{(2)}(z)\big]^-=F^{(1)}(z),\\
\big[T^1(\partial z,n)U^{(1)}(z)\big]^+-\big[T^2(\partial z,n)U^{(2)}(z)\big]^-=F^{(2)}(z).
\end{gathered}
$$

Again the vector $U^{(2)}$ has to satisfy the decay conditions (3.1). Here

$$
\begin{gathered}
F^{(1)}(z)=\big(f^{(1)}(z),f^{(2)}(z)\big)^\top,\quad F^{(2)}(z)=\big(f^{(3)}(z),f^{(4)}(z)\big)^\top,\\
f^{(j)}(z)=\big(f_1^{(j)}(z),f_2^{(j)}(z),f_3^{(j)}(z)\big)^\top,\quad j=1,2,3,4
\end{gathered}
$$

are given vector functions. The stress vector is defined by the formulas (3.3) with the appropriate material constants equipped with superscript (j) and with $U^{(j)}$ for U.

There hold the following uniqueness theorems [17].

Theorem 3.1. *Problems* $(\mathrm{I})^{\pm}$*,* $(\mathrm{II})^{-}$ *and* (A) *have at most one solution. Solutions to Problem* $(\mathrm{II})^{+}$ *are defined modulo the rigid displacement vectors of the type*

$$u(x)=[a\times x]+b,\quad \omega(x)=a,$$

where $x=(x_1,x_2,x_3)^{\top}$*, and* a *and* b *are arbitrary three dimensional constant vectors.*

4. Explicit Solutions of the Boundary Value Problems

Here we demonstrate our approach for Problem $(\mathrm{II})^{+}$, since other problems can be treated quite similarly.

We look for a solution to Problem $(\mathrm{II})^{+}$ in the form (2.3), where

$$\begin{aligned}
\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left(\frac{r}{R}\right)^{k}Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\quad j=1,2,3,\\
\Phi_4(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}g_k(\lambda_1 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(4)},\\
\Psi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(k_j r)Y_k^{(m)}(\vartheta,\varphi)B_{mk}^{(j)},\quad j=1,2.
\end{aligned}\tag{4.1}$$

Here $A_{mk}^{(j)}$, $j=1,2,3,4$, and $B_{mk}^{(j)}$, $j=1,2$, are unknown coefficients, while

$$g_k(\lambda_1 r)=\sqrt{\frac{R}{r}}\,\frac{I_{k+1/2}(\lambda_1 r)}{I_{k+1/2}(\lambda_1 R)}\,,\quad h_k(k_j r)=\sqrt{\frac{R}{r}}\,\frac{\mathcal{I}_{k+1/2}(k_j r)}{\mathcal{I}_{k+1/2}(k_j R)}\,,\quad j=1,2,$$

$$Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\,\frac{(k-m)!}{(k+m)!}}\,P_k^{(m)}(\cos\vartheta)e^{im\varphi}.$$

$P_k^{(m)}(\cos\vartheta)$ are the Legendre associated polynomials, $\mathcal{I}_{k+1/2}$ is the Bessel function of real argument and $I_{k+1/2}$is the Bessel function of complex (pure imaginary) argument [29].

We assume that the functions Φ_j, $j=1,3$, and Ψ_j, $j=1,2$, satisfy the conditions

$$\int\limits_{\partial\Omega'}\Phi_j(z)\,ds=0,\quad j=1,3,\qquad \int\limits_{\partial\Omega'}\Psi_j(z)\,ds=0,\quad j=1,2,\tag{4.2}$$

where

$$\partial\Omega'=\big\{x:\ x\in\mathbb{R}^3,\ |x|=R'<R\big\}.$$

Substituting Φ_j, $j=1,3$, and Ψ_j, $j=1,2$, into (4.2) and taking into account the equalities

$$\int\limits_{\partial\Omega} Y_k^{(m)}(\vartheta,\varphi)\,ds = \begin{cases} 2\sqrt{\pi}R^2 & \text{for } k=0, \ \ m=0,\\ 0 & \text{otherwise,}\end{cases}$$

we get that $A_{00}^{(j)}=0$, $j=1,3$, and $B_{00}^{(j)}=0$,, $j=1,2$.

Substitute Φ_j, $j=1,3$, and Ψ_j, $j=1,2$, into (2.3) and use the relations [28]

$$\begin{aligned}
\operatorname{grad}\big[a(r)Y_k^{(m)}(\vartheta,\varphi)\big] &= \frac{da(r)}{dr}\,X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big] &= \sqrt{k(k+1)}\,a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big] &= \frac{k(k+1)}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),
\end{aligned}$$

to get

$$\begin{aligned}
u(x) &= u_{00}(r)X_{00}(\vartheta,\varphi)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big[v_{mk}(r)Y_{mk}(\vartheta,\varphi)+\omega_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
\omega(x) &= \widetilde{u}_{00}(r)X_{00}(\vartheta,\varphi)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{\widetilde{u}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big[\widetilde{v}_{mk}(r)Y_{mk}(\vartheta,\varphi)+\widetilde{\omega}_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{aligned}\tag{4.3}$$

where [25], [26]

$$\begin{gathered}
X_{mk}(\vartheta,\varphi)=e_rY_k^{(m)}(\vartheta,\varphi),\ \ k\geq 0,\\
Y_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta\,\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\vartheta,\varphi),\ \ k\geq 1,\\
Z_{mk}(\vartheta,\varphi)=\\
=\frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}-e_\varphi\,\frac{\partial}{\partial\vartheta}\Big)Y_k^{(m)}(\vartheta,\varphi),\ \ k\geq 1,\ \ |m|\leq k,
\end{gathered}\tag{4.4}$$

e_r, e_ϑ and e_φ are the unit vectors

$$\begin{aligned}
e_r &= (\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)^\top,\\
e_\vartheta &= (\cos\varphi\cos\vartheta,\sin\varphi\cos\vartheta,-\sin\vartheta)^\top,\\
e_\varphi &= (-\sin\varphi,\cos\varphi,0)^\top.
\end{aligned}$$

The system of vectors $\{X_{mk}(\vartheta,\varphi), Y_{mk}(\vartheta,\varphi), Z_{mk}(\vartheta,\varphi)\}$, $|m|\le k$, $k=\overline{1,\infty}$, is orthogonal and complete in $L_2(\Sigma_1)$, where Σ_1 is the unit sphere,

$$
\begin{aligned}
u_{mk}(r) &= \frac{k}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(1)}_{mk} + (k+1)(bk-2a)R\Big(\frac{r}{R}\Big)^{k+1}A^{(2)}_{mk} + \\
&\quad + (\delta+2\varkappa)\,\frac{d}{dr}\,g_k(\lambda_1 r)A^{(4)}_{mk} + \frac{k(k+1)}{r}\sum_{j=1}^{2} h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 0,\\
v_{mk}(r) &= \frac{1}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(1)}_{mk} + (b(k+1)+2)R\Big(\frac{r}{R}\Big)^{k+1}A^{(2)}_{mk} + \\
&\quad + \frac{(\delta+2\varkappa)}{r}\,g_k(\lambda_1 r)A^{(4)}_{mk} + \Big(\frac{d}{dr}+\frac{1}{r}\Big)\sum_{j=1}^{2} h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 1,\\
\omega_{mk}(r) &= \Big(\frac{r}{R}\Big)^{k}A^{(3)}_{mk} + \sum_{j=1}^{2} k_j h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 1,\\
\widetilde{u}_{mk}(r) &= \frac{\sigma k(k+1)(2k+3)}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(2)}_{mk} + \frac{k(k+1)}{2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(3)}_{mk} - \\
&\quad - (\lambda+2\mu)\,\frac{d}{dr}g_k(\lambda_1 R)A^{(4)}_{mk} - \frac{k(k+1)}{r}\sum_{j=1}^{2}\eta_j h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 0,\\
\widetilde{v}_{mk}(r) &= \frac{\sigma(k+1)(2k+3)}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(2)}_{mk} + \frac{k+1}{2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(3)}_{mk} - \\
&\quad - \frac{\lambda+2\mu}{r}\,g_k(\lambda_1 r)A^{(4)}_{mk} - \Big(\frac{d}{dr}+\frac{1}{r}\Big)\sum_{j=1}^{2}\eta_j h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 1,\\
\widetilde{\omega}_{mk}(r) &= -(2k+3)\Big(\frac{r}{R}\Big)^{k}A^{(2)}_{mk} - \sum_{j=1}^{2}\eta_j k_j h_k(k_j r)B^{(j)}_{mk}, \quad k\ge 1,\ b=1-a.
\end{aligned}
\tag{4.5}
$$

Now substitute the vectors u and ω into (3.4) and employ the equalities

$$
\begin{gathered}
e_r\times X_{mk}(\vartheta,\varphi)=0, \quad e_r\times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi),\\
e_r\times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi),\\
\operatorname{div}\big[a(r)X_{mk}(\vartheta,\varphi)\big] = \Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y^{(m)}_k(\vartheta,\varphi),\\
\operatorname{div}\big[a(r)Y_{mk}(\vartheta,\varphi)\big] = -\sqrt{k(k+1)}\,\frac{a(r)}{r}\,Y^{(m)}_k(\vartheta,\varphi),\\
\operatorname{div}\big[a(r)Z_{mk}(\vartheta,\varphi)\big] = 0,\\
\operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big] = \sqrt{k(k+1)}\,\frac{a(r)}{r}\,Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big] = -\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big] = \sqrt{k(k+1)}\,\frac{a(r)}{r}\,X_{mk}(\vartheta,\varphi) + \Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi)
\end{gathered}
$$

to obtain

$$
\begin{aligned}
H^{(1)}(\partial x,n)U(x) &= a_{00}(r)X_{00}(\vartheta,\varphi)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{a_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\,\Big[b_{mk}(r)Y_{mk}(\vartheta,\varphi)+c_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
H^{(2)}(\partial x,n)U(x) &= \widetilde{a}_{00}(r)X_{00}(\vartheta,\varphi)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{\widetilde{a}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big[\widetilde{b}_{mk}(r)Y_{mk}(\vartheta,\varphi)+\widetilde{c}_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
a_{mk}(r) &= \Big[(\lambda+2\mu)\,\frac{d}{dr}+\frac{2\lambda}{r}\Big]u_{mk}(r)+\Big[(\delta+2\varkappa)\,\frac{d}{dr}+\frac{2\delta}{r}\Big]\widetilde{u}_{mk}(r)-\\
&\quad-\frac{k(k+1)}{r}\,\Big[\lambda v_{mk}(r)+\delta\widetilde{v}_{mk}(r)\Big],\quad k\geq 0,\\
b_{mk}(r) &= \frac{1}{r}\,\Big[(\mu-\alpha)(u_{mk}(r)-v_{mk}(r))+(\varkappa-\nu)(\widetilde{u}_{mk}(r)-\widetilde{v}_{mk}(r))\Big]+\\
&\quad+(\mu+\alpha)\,\frac{d}{dr}\,v_{mk}(r)+(\varkappa+\nu)\,\frac{d}{dr}\,\widetilde{v}_{mk}(r)+2\alpha\widetilde{\omega}_{mk}(r),\\
c_{mk}(r) &= \Big[(\mu+\alpha)\,\frac{d}{dr}-(\mu-\alpha)\,\frac{1}{r}\Big]\omega_{mk}(r)+\\
&\quad+\Big[(\varkappa+\nu)\,\frac{d}{dr}-(\varkappa-\nu)\,\frac{1}{r}\Big]\widetilde{\omega}_{mk}(r)-2\alpha\widetilde{v}_{mk}(r),\\
\widetilde{a}_{mk}(r) &= \Big[(\delta+2\varkappa)\,\frac{d}{dr}+\frac{2\delta}{r}\Big]u_{mk}(r)+\Big[(\beta+2\gamma)\frac{d}{dr}+\frac{2\beta}{r}\Big]\widetilde{u}_{mk}(r)-\\
&\quad-\frac{k(k+1)}{r}\,\Big[\delta v_{mk}(r)+\beta\widetilde{v}_{mk}(r)\Big],\\
\widetilde{b}_{mk}(r) &= \frac{1}{r}\,\Big[(\varkappa-\nu)(u_{mk}(r)-v_{mk}(r))+(\gamma-\varepsilon)(\widetilde{u}_{mk}(r)-\widetilde{v}_{mk}(r))\Big]+\\
&\quad+(\varkappa+\nu)\,\frac{d}{dr}\,v_{mk}(r)+(\gamma+\varepsilon)\,\frac{d}{dr}\,\widetilde{v}_{mk}(r)+2\nu\widetilde{\omega}_{mk}(r),\\
\widetilde{c}_{mk}(r) &= \Big[(\varkappa+\nu)\,\frac{d}{dr}-(\varkappa-\nu)\,\frac{1}{r}\Big]\omega_{mk}(r)+\\
&\quad+\Big[(\gamma+\varepsilon)\,\frac{d}{dr}-(\gamma-\varepsilon)\,\frac{1}{r}\Big]\widetilde{\omega}_{mk}(r)-2\nu\widetilde{v}_{mk}(r).
\end{aligned}
$$

Here the vectors $u_{mk}, v_{mk}, \dots, \widetilde{\omega}_{mk}$ are given by (4.5).

Represent the boundary data as the Fourier–Laplace series with respect to the system (4.4)

$$
\begin{aligned}
f^{(j)}(z) &= \alpha_{00}^{(j)}X_{00}(\vartheta,\varphi)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}^{(j)}X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\,\Big[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big]\Big\},\quad j=1,2,
\end{aligned}
$$

where $\alpha^{(j)}_{mk}$, $\sqrt{k(k+1)}\,\beta^{(j)}_{mk}$, $\sqrt{k(k+1)}\,\gamma^{(j)}_{mk}$ are the Fourier coefficients.

In view of the boundary condition (3.2), with the help of the equalities (4.6) we arrive at the system of linear algebraic equations

$$\begin{gathered}
a_{00}(R)=\alpha^{(1)}_{00}, \quad \widetilde{a}_{00}(R)=\alpha^{(2)}_{00};\\
a_{m1}(R)=\alpha^{(1)}_{m1}, \quad b_{m1}(R)=\beta^{(1)}_{m1}, \quad c_{m1}(R)=\gamma^{(1)}_{m1},\\
\widetilde{a}_{m1}(R)=\alpha^{(2)}_{m1}, \quad \widetilde{b}_{m1}(R)=\beta^{(2)}_{m1}, \quad \widetilde{c}_{m1}(R)=\gamma^{(2)}_{m1};\\
a_{mk}(R)=\alpha^{(1)}_{mk}, \quad b_{mk}(R)=\beta^{(1)}_{mk}, \quad c_{mk}(R)=\gamma^{(1)}_{mk},\\
\widetilde{a}_{mk}(R)=\alpha^{(2)}_{mk}, \quad \widetilde{b}_{mk}(R)=\beta^{(2)}_{mk}, \quad \widetilde{c}_{mk}(R)=\gamma^{(2)}_{mk}, \quad k\geq 2.
\end{gathered} \tag{4.7}$$

Let us analyze the solvability of this system. To this end, we prove the following assertion.

Lemma 4.1. *Let the conditions* (4.2) *be fulfilled. Then the vectors u and ω represented the by formulas* (2.3) *vanish identically if and only if Φ_j, $j=1,2,3,4$, and Ψ_j, $j=1,2$, vanish.*

Proof. The sufficiency is trivial. Let us show the necessity. From the equations (2.3) we get

$$\Phi_4(x)=-\frac{1}{\lambda_1^2(\lambda+2\mu)}\operatorname{div}\omega(x),$$

$$\Big(2r\frac{\partial}{\partial r}+3\Big)\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2(x)=\frac{1}{2a}\Big[\operatorname{div}u(x)+\frac{\delta+2\varkappa}{\lambda+2\mu}\operatorname{div}\omega(x)\Big].$$

If u and ω vanish, then by the formulas (4.1) we see that

$$\Phi_2(x)=0, \quad \Phi_4(x)=0. \tag{4.8}$$

Again from (2.3) by (4.8) we have

$$\omega(x)-\frac{1}{2}\operatorname{rot}u(x)=-\sum_{j=1}^{2}\Big(\eta_j+\frac{1}{2}k_j\Big)\Big[\operatorname{rot}\operatorname{rot}(x\Psi_j(x))+k_j\operatorname{rot}(x\Psi_j(x))\Big].$$

Whence, if u and ω vanish, we get

$$\operatorname{rot}\operatorname{rot}(x\Psi_j(x))+k_j\operatorname{rot}(x\Psi_j(x))=0, \quad j=1,2.$$

Taking scalar product of this equation by x leads to

$$\Big[r\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}+1\Big)+r^2k_j^2\Big]\Psi_j(x)=0, \quad j=1,2.$$

Whence by (4.1) and (4.2)

$$\Psi_j(x)=0, \quad j=1,2. \tag{4.9}$$

The equations (4.8) and (4.9) along with (2.3) imply

$$\begin{aligned}
u(x)&=\operatorname{grad}\Phi_1(x)+\operatorname{rot}(x\Phi_3(x)),\\
\omega(x)&=\frac{1}{2}\operatorname{rot}\operatorname{rot}(x\Phi_3(x)).
\end{aligned} \tag{4.10}$$

From (4.10) we get

$$r\,\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_3(x)=2(x\cdot\omega(x)).$$

If we substitute in these equations $\omega(x)=0$ and apply (4.1) and (4.2), we obtain $\Phi_3(x)=0$. In accordance with this equation we get

$$\operatorname{grad}\Phi_1(x)=u(x).$$

Whence $\Phi_1(x)=const$, and by (4.2) we conclude that $\Phi_1(x)=0$. □

Necessary and sufficient conditions for Problem $(\mathrm{II})^+$ to be solvable read as follows

$$\int\limits_{\partial\Omega} f^{(1)}(z)\,ds=0,\quad \int\limits_{\partial\Omega}\big[z\times f^{(1)}(z)+f^{(2)}(z)\big]\,ds=0. \tag{4.11}$$

Substituting $f^{(j)}$, $j=1,2$, into (4.11) and taking into consideration the equations

$$\int\limits_{\partial\Omega} X_{mk}(\vartheta,\varphi)\,ds=\begin{cases}\sqrt{\frac{2\pi}{3}}\,\big[(\delta_{-1m}-\delta_{1m})e_1-i(\delta_{-1m}+\delta_{1m})e_2+\\ \qquad\qquad +\sqrt{2}\,\delta_{0m}e_3\big]R^2,\ \ k=1,\ \ m=\pm1,\\ 0\ \text{ otherwise};\end{cases}$$

$$\int\limits_{\partial\Omega} Y_{mk}(\vartheta,\varphi)\,ds=\begin{cases}2\sqrt{\frac{\pi}{3}}\,\big[(\delta_{-1m}-\delta_{1m})e_1-i(\delta_{-1m}+\delta_{1m})e_2+\\ \qquad\qquad +\sqrt{2}\,\delta_{0m}e_3\big]R^2,\ \ k=1,\ \ m=\pm1,\\ 0\ \text{ otherwise};\end{cases}$$

$$\int\limits_{\partial\Omega} Z_{mk}(\vartheta,\varphi)\,ds=0\ \text{ for all } k \text{ and } m,$$

where δ_{kj} is the Kronecker symbol, $e_1=(1,0,0)^\top$, $e_2=(0,1,0)^\top$, $e_3=(0,0,1)^\top$, we get (4.12)

$$\alpha^{(1)}_{m1}+2\beta^{(1)}_{m1}=0,\quad \alpha^{(2)}_{m1}+2\beta^{(2)}_{m1}+2R\gamma^{(1)}_{m1}=0,\ \ m=0,\pm1. \tag{4.12}$$

Theorem 3.1 and Lemma 4.1 imply that the system (4.7) is solvable. Moreover, from (4.12) it follows that we can define all unknown coefficients but $A^{(1)}_{m1}$ and $A^{(3)}_{m1}$, $m=0,\pm1$. This is natural and reflects the fact that the solution is defined modulo a rigid displacement vector.

In our analysis we need the following technical results [27].

Theorem 4.2. *For $k\geq 0$ the following inequalities are true*

$$\big|X_{mk}(\vartheta,\varphi)\big|\leq\sqrt{\frac{2k+1}{4\pi}}\,,\ \ k\geq 0,$$

$$\big|Y_{mk}(\vartheta,\varphi)\big|<\sqrt{\frac{2k(k+1)}{2k+1}}\,,\ \ k\geq 1,$$

$$\big|Z_{mk}(\vartheta,\varphi)\big|<\sqrt{\frac{2k(k+1)}{2k+1}},\ \ k\geq 1.$$

Theorem 4.3. *If $f^{(j)} \in C^l(\partial\Omega)$, where $\partial\Omega$ is a sphere, then the coefficients $\alpha_{kj}^{(j)}$, $\beta_{kj}^{(j)}$ and $\gamma_{kj}^{(j)}$ admit the bounds*

$$\alpha_{mk}^{(j)} = O(k^{-l}), \ \ \beta_{mk}^{(j)} = O(k^{-l-1}), \ \ \gamma_{mk}^{(j)} = O(k^{-l-1}).$$

Applying the above theorems and the asymptotic behavior of the Bessel functions, we can show that for $x \in \Omega_1$ the series are absolutely and uniformly convergent in the interior domain.

Let now $x \in \partial\Omega_1$, i.e., $r = R$. In this case, due to Theorem 4.2 the majorizing number series for (4.3) and (4.6) is

$$\sum_{k=k_0}^{\infty} k^{\frac{1}{2}} \sum_{j=1}^{2} \Big[|\alpha_{mk}^{(j)}| + k\big(|\beta_{mk}^{(j)}| + |\gamma_{mk}^{(j)}|\big)\Big].$$

By Theorem 4.3 this series is convergent if $f^{(j)} \in C^2(\partial\Omega_1)$.

Thus the series (4.3) and (4.6) are absolutely and uniformly convergent if $f^{(j)} \in C^2(\partial\Omega_1)$.

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