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**INTERACTION PROBLEMS OF METALLIC AND
PIEZOELECTRIC MATERIALS WITH REGARD
TO THERMAL STRESSES**

LIST OF NOTATION

- \mathbb{R}^k – k -dimensional space of real numbers;
 \mathbb{C}^k – k -dimensional space of complex numbers;
 $a \cdot b = \sum_{j=1}^k a_j \bar{b}_j$ – the scalar product of two vectors $a = (a_1, \dots, a_k)$,
 $b = (b_1, \dots, b_k) \in \mathbb{C}^k$;
 $\bar{\Omega}$ – domain occupied by a piezoceramic material;
 $\bar{\Omega}_m$ – domain occupied by a metallic material;
 $\Gamma_m = \partial\Omega_m \cap \Omega$ – contact interface surface between metallic and piezoceramic parts;
 $\bar{\Pi} := \bar{\Omega} \cup \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \dots \cup \bar{\Omega}_{2N}$ – domain occupied by a composite structure;
 $n = (n_1, n_2, n_3)$ – unit normal vector to $\partial\Omega$ and $\partial\Omega_m$;
 $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$ – partial derivatives with respect to the spatial and time variables;
 ρ , $\rho^{(m)}$ – mass densities;
 c_{ijkl} , $c_{ijkl}^{(m)}$ – elastic constants;
 $\lambda^{(m)}$, $\mu^{(m)}$ – Lamé constants;
 e_{kij} – piezoelectric constants;
 ε_{kj} , ε – dielectric (permittivity) constants;
 γ_{kj} , $\gamma_{kj}^{(m)}$, $\gamma^{(m)}$ – thermal strain constants;
 \varkappa_{kj} , $\varkappa_{kj}^{(m)}$, $\varkappa^{(m)}$ – thermal conductivity constants;
 \tilde{c} , $\tilde{c}^{(m)}$ – specific heat per unit mass;
 T_0 , $T_0^{(m)}$ – initial (reference) temperature (temperature in the natural state, i.e., in the absence of deformation and electromagnetic fields);
 $\alpha := \rho\tilde{c}$, $\alpha^{(m)} := \rho^{(m)}\tilde{c}^{(m)}$ – thermal material constants;
 g_i ($i = 1, 2, 3$) – constants characterizing the relation between thermodynamic processes and piezoelectric effect (pyroelectric constants);
 $X = (X_1, X_2, X_3)^\top$, $X^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)})^\top$ – mass force densities;
 X_4 , $X_4^{(m)}$ – heat source densities;
 X_5 – charge density;
 $u = (u_1, u_2, u_3)^\top$, $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ – displacement vectors;
 φ – electric potential;
 $E := -\text{grad } \varphi$ – electric field vector;
 D – electric displacement vector;
 $\vartheta = T - T_0$, $\vartheta^{(m)} = T^{(m)} - T_0^{(m)}$ – relative temperature (temperature increment);
 $q = (q_1, q_2, q_3)$, $q^{(m)} = (q_1^{(m)}, q_2^{(m)}, q_3^{(m)})$ – heat flux vector;
 $s_{kj} = s_{kj}(u) := \frac{1}{2}(\partial_k u_j + \partial_j u_k)$, $s^{(m)} = s_{kj}^{(m)}(u^{(m)}) := \frac{1}{2}(\partial_k u_j^{(m)} + \partial_j u_k^{(m)})$ – strain tensors;
 $\sigma_{kj}^{(m)} = \sigma_{kj}^{(m)}(u^{(m)}, \vartheta^{(m)})$ – mechanical stress tensor in the theory of thermoelasticity;

$\sigma_{kj} = \sigma_{kj}(u, \vartheta, \varphi)$ – mechanical stress tensor in the theory of thermoelectroelasticity;
 $\mathcal{S}, \mathcal{S}^{(m)}$ – entropy densities;
 $U^{(m)} := (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top$ with $u_4^{(m)} = \vartheta^{(m)}$;
 $U := (u_1, u_2, u_3, u_4, u_5)^\top$ with $u_4 = \vartheta$ and $u_5 = \varphi$;
 $\mathbf{U} := (U^{(1)}, \dots, U^{(2N)}, U)$;
 L_p, W_p^r , and H_p^s ($r \geq 0, s \in \mathbb{R}, 1 < p < \infty$) – the Lebesgue, Sobolev–Slobodetski, and Bessel potential spaces;
 $\mathcal{W}_N^1 := [W_2^1(\Omega_1)]^4 \times \dots \times [W_2^1(\Omega_{2N})]^4 \times [W_2^1(\Omega)]^5$;
 $r_{\mathcal{M}}$ – restriction operator on a set \mathcal{M} ;
 $\{\cdot\}_{\partial\Omega}^\pm, \{\cdot\}_{\partial\Omega_m}^\pm$ – trace operators on $\partial\Omega$ and $\partial\Omega_m$;
 $H_2^1(\Omega, \mathcal{M}) := \{w \in H_2^1(\Omega) : r_{\mathcal{M}}\{w\}_{\partial\Omega}^+ = 0, \mathcal{M} \subset \partial\Omega\}$;
 $\mathcal{V}_N^1 := [H_2^1(\Pi, \Sigma_3^-)]^4 \times H_2^1(\Omega, S^- \cup S^+), \Sigma_3^- \subset \partial\Pi, S^- \cup S^+ \subset \partial\Omega$;
 $\tilde{H}_p^s(\mathcal{M}) := \{f : f \in H_p^s(\mathcal{M}_0), \text{supp } f \subset \overline{\mathcal{M}}\}$ for $\mathcal{M} \subset \mathcal{M}_0$;
 $H_p^s(\mathcal{M}) := \{r_{\mathcal{M}}f : f \in H_p^s(\mathcal{M}_0)\}$ – space of restrictions on $\mathcal{M} \subset \mathcal{M}_0$;
 $\|\cdot\|_B$ – norm in a Banach space B ;
 B^* – dual Banach space to B ;
 $\langle \cdot, \cdot \rangle$ – duality pairing between the Banach spaces B and B^* ;
 τ – complex wave number;
 $A^{(m)}(\partial, \tau)$ – 4×4 matrix differential operator of thermoelasticity (see Appendix A);
 $A(\partial, \tau)$ – 5×5 matrix differential operator of thermopiezoelectricity (see Appendix A);
 $\mathcal{T}^{(m)}(\partial, n)$ – 4×4 matrix stress operator of thermoelasticity (see Appendix A);
 $\mathcal{T}(\partial, n)$ – 5×5 matrix stress operator of thermopiezoelectricity (see Appendix A).

1. INTRODUCTION

The interaction of electrical and mechanical fields yields the well known piezo-effects in piezoelectric materials. Due to this properties, they are widely used in electro-mechanical devices and many technical equipments, in particular, in sensors and actuators.

The corresponding mathematical problems for *homogeneous media*, based on W. Voigt's model [41], were considered by many authors.

In their works R. Toupin and R. Mindlin suggested new, more refined models of an elastic medium, where a polarization vector occurs [38], [39], [23], [24]. Furthermore, effects caused by thermal field and hysteresis effects are considered in [22], [34], [16] (see also [31], [32], [35]). We refer also to the book [36] (see also the references therein), where the distribution of stresses near crack tips in the ceramics are studied in the two-dimensional case.

To our knowledge, only few results are known for composed complex structures consisting of piezoelectric and metallic parts (see [37], [7] and [42]).

In [12], [12], and [5] two- and three-dimensional models for composites are derived and analysed for the static case without the influence of temperature fields.

In the present paper we study a linear mathematical model for *piezoelastic and metallic composites* (in particular, stack actuators) and, in addition, we take into consideration the influence of thermal effects. In this case, driving forces are given by electrical charges at electrodes which are embedded as metallic plates in the ceramic matrix. Note that here we have different dimensional unknown fields in the metallic and ceramic sub-domains. This leads to an additional complexity of the model.

Thus, it was challenging to formulate the mathematical model by a coupled system of linear partial differential equations which are completed by appropriate boundary and transmission conditions.

The paper presented now can be considered as a continuation and extension of [12] and [5].

The main goals of this paper are:

- Mathematical formulation of the boundary-transmission problem for a metallic-piezoceramic composite structure (see Figure 1) in an efficient way.
- Derivation of existence and uniqueness results by variational and potential methods.
- Numerical algorithms for computations of the electric and thermo-mechanical fields, visualization of the influence of temperature.

The paper is organized as follows:

In Section 2 we give the mathematical formulation of the mixed boundary transmission problem (MBTP) describing the interaction of metallic and piezoelectric materials with regard to thermal stresses and prove the corresponding uniqueness theorem.

In Sections 3 we introduce the sesquilinear form related to the weak formulation of our mixed boundary transmission problem and show its coercivity in an appropriate function space. Further, we prove the unique solvability of the weak formulated MBTP.

In Section 4 by the potential method we reduce the MBTP to the equivalent system of boundary integral equations. We show that the corresponding boundary integral operator has Fredholm properties and prove its invertibility. As a consequence of these results, we obtain an existence theorem for the MBTP on the one hand and representation formulas of the corresponding solutions by the layer potentials on the other hand.

In Section 5 we establish the standard finite element approximation of solutions to the boundary transmission problem.

For the reader's convenience, in the beginning of the paper we exhibit the *list of notation* used in the text. In Appendix A we collect the field equations of the linear theory of thermoelasticity and thermopiezoelasticity. Here we

introduce also the corresponding matrix partial differential operators generated by the field equations and the generalized matrix boundary stress operators. Various versions of Green's formulas needed in the main text are gathered in Appendix B. In Appendix C we construct explicitly the fundamental matrices of equations of thermoelasticity and thermopiezoelasticity.

2. MATHEMATICAL FORMULATION OF THE BOUNDARY-TRANSMISSION PROBLEM AND UNIQUENESS THEOREM

We will consider a composed piecewise homogeneous multi-structure which models a multi-layer stack actuator (for detailed description of multi-layer actuators see, e.g., [12] and the references therein).

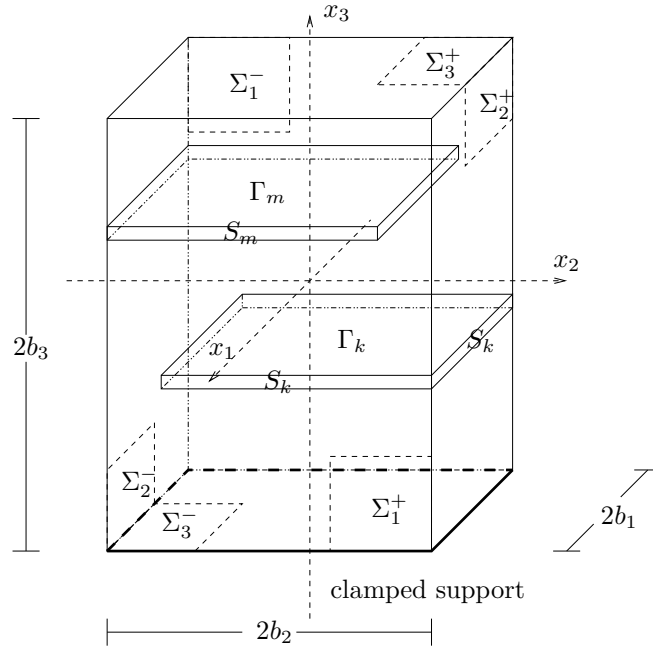


FIGURE 1. The parallelepiped Π occupied by the composed body (Γ_m - interface submanifold between metallic and piezoelectric media)

By $\bar{\Pi}$ we denote a rectangular parallelepiped in \mathbb{R}^3 which is occupied by a composed multi-structure consisting of metallic electrodes and a piezoelectric ceramic matrix (see Figure 1):

$$\bar{\Pi} := \{ -b_1 < x_1 < b_1, -b_2 < x_2 < b_2, -b_3 < x_3 < b_3 \},$$

whose faces are

$$\begin{aligned}\tilde{\Sigma}_1^- &:= \{x_1 = -b_1, -b_2 < x_2 < b_2, -b_3 < x_3 < b_3\}, \\ \tilde{\Sigma}_1^+ &:= \{x_1 = b_1, -b_2 < x_2 < b_2, -b_3 < x_3 < b_3\}, \\ \tilde{\Sigma}_2^- &:= \{x_2 = -b_2, -b_1 < x_1 < b_1, -b_3 < x_3 < b_3\}, \\ \tilde{\Sigma}_2^+ &:= \{x_2 = b_2, -b_1 < x_1 < b_1, -b_3 < x_3 < b_3\}, \\ \tilde{\Sigma}_3^- &:= \{x_3 = -b_3, -b_1 < x_1 < b_1, -b_2 < x_2 < b_2\}, \\ \tilde{\Sigma}_3^+ &:= \{x_3 = b_3, -b_1 < x_1 < b_1, -b_2 < x_2 < b_2\}.\end{aligned}$$

Let $\overline{\Omega}_m$ ($m = \overline{1, 2N}$) be an even number of rectangular parallelepipeds occupied by some metallic medium (electrodes) where alternating negative (for $m = \overline{1, N}$) and positive (for $m = \overline{N+1, 2N}$) charges are applied:

$$\begin{aligned}\Omega_m &:= \{-b_1 < x_1 < b_1, -b_2 < x_2 < b_{2,m}, b'_{3,m} < x_3 < b''_{3,m}\}, \quad m = \overline{1, N}, \\ \Omega_m &:= \{-b_1 < x_1 < b_1, b_{2,m} < x_2 < b_2, b'_{3,m} < x_3 < b''_{3,m}\}, \quad m = \overline{N+1, 2N};\end{aligned}$$

here $-b_2 < b_{2,m} < b_2$, $m = \overline{1, 2N}$, and

$$\begin{aligned}-b_3 < b'_{3,1} < b''_{3,1} < b'_{3,N+1} < b''_{3,N+1} < b'_{3,2} < b''_{3,2} < b'_{3,N+2} < b''_{3,N+2} < \\ < \cdots < b'_{3,N} < b''_{3,N} < b'_{3,2N} < b''_{3,2N} < b_3.\end{aligned}$$

Note that the polarization direction is alternating too.

Further, by Ω we denote the connected sub-domain of Π occupied by a ceramic medium

$$\Omega := \Pi \setminus \left[\bigcup_{m=1}^{2N} \overline{\Omega}_m \right].$$

For the boundaries of the above domains we introduce the following decomposition:

$$\partial\Omega_m := \overline{S}_m \cup \overline{\Gamma}_m, \quad \partial\Omega := \left[\bigcup_{k=1}^3 \overline{\Sigma}_k^- \right] \cup \left[\bigcup_{k=1}^3 \overline{\Sigma}_k^+ \right] \cup \left[\bigcup_{m=1}^{2N} \overline{\Gamma}_m \right], \quad (2.1)$$

where Γ_m is an interface submanifold between metallic (Ω_m) and the piezoelectric (Ω) subdomains,

$$\begin{aligned}\Gamma_m &:= \partial\Omega_m \cap \Pi, \quad S_m := \partial\Omega_m \setminus \overline{\Gamma}_m, \\ \Sigma_k^- &:= \tilde{\Sigma}_k^- \setminus \left[\bigcup_{m=1}^{2N} \partial\Omega_m \right], \quad \Sigma_k^+ := \tilde{\Sigma}_k^+ \setminus \left[\bigcup_{m=1}^{2N} \partial\Omega_m \right].\end{aligned}$$

Note that $\Sigma_3^- = \tilde{\Sigma}_3^-$ and $\Sigma_3^+ = \tilde{\Sigma}_3^+$, and they represent the lower and upper basis of the parallelepiped Π .

It is evident that the metallic and ceramic bodies interact with each other along the surfaces Γ_m . Moreover, in the “metallic” domain Ω_m we consider a usual four-dimensional thermoelastic field described by the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and the temperature $\vartheta^{(m)}$, while in the

piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$, the temperature ϑ , and the electric potential φ . Here and in what follows the superscript \top denotes transposition.

Throughout the paper we employ the Einstein summation convention (with summation from 1 to 3) over repeated indices, unless stated otherwise. Also, to avoid some misunderstanding related to the directions of normal vectors on the contact surfaces Γ_m , throughout the paper we assume that *the normal vector to $\partial\Omega_m$ is directed outward, while on $\partial\Omega$ it is directed inward*. Further, the symbol $\{\cdot\}^+$ denotes the interior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega_m$) from Ω (respectively Ω_m). Similarly, $\{\cdot\}^-$ denotes the exterior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega_m$) from the exterior of Ω (respectively Ω_m). We will use also the notation $\{\cdot\}_{\partial\Omega}^\pm$ and $\{\cdot\}_{\partial\Omega_m}^\pm$ for the trace operators on $\partial\Omega$ and $\partial\Omega_m$.

2.1. Formulation of the boundary transmission problem. By L_p , W_p^r , and H_p^s (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$) we denote the well-known Lebesgue, Sobolev–Slobodetski, and Bessel potential function spaces, respectively (see, e.g., [40], [20], [21]). We recall that $H_2^r = W_2^r$ and $H_p^k = W_p^k$ for any $r \geq 0$ and for any non-negative integer k .

Let \mathcal{M}_0 be a surface without boundary. For a submanifold $\mathcal{M} \subset \mathcal{M}_0$, by $\tilde{H}_p^s(\mathcal{M})$ we denote the subspace of $H_p^s(\mathcal{M}_0)$: $\tilde{H}_p^s(\mathcal{M}) = \{g : g \in H_p^s(\mathcal{M}_0), \text{supp } g \subset \overline{\mathcal{M}}\}$, while $H_p^s(\mathcal{M})$ denotes the space of restrictions on \mathcal{M} of the functions from $H_p^s(\mathcal{M}_0)$: $H_p^s(\mathcal{M}) = \{r_{\mathcal{M}}f : f \in H_p^s(\mathcal{M}_0)\}$, where $r_{\mathcal{M}}$ stands for the restriction operator on \mathcal{M} .

We will use the notation introduced in Appendix A and consider the following model mixed boundary-transmission problem:

Find the vector-functions

$$U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top : \Omega_m \rightarrow \mathbb{C}^4$$

$$\text{with } u^{(m)} := (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}), \quad u_4^{(m)} := \vartheta^{(m)}, \quad m = \overline{1, 2N},$$

and

$$U = (u_1, u_2, u_3, u_4, u_5)^\top : \Omega \rightarrow \mathbb{C}^5 \quad \text{with } u := (u_1, u_2, u_3), \quad u_4 := \vartheta, \quad u_5 := \varphi,$$

belonging to the spaces $[W_2^1(\Omega_m)]^4$ and $[W_2^1(\Omega)]^5$, respectively, and satisfying

(i) *the systems of partial differential equations:*

$$[A^{(m)}(\partial, \tau)U^{(m)}]_j = 0 \quad \text{in } \Omega_m, \quad j = \overline{1, 4}, \quad m = \overline{1, 2N}, \quad (2.2)$$

$$[A(\partial, \tau)U]_k = 0 \quad \text{in } \Omega, \quad k = \overline{1, 5}, \quad (2.3)$$

that is (see Appendix A),

$$\left. \begin{aligned} c_{ijkl}^{(m)} \partial_i \partial_l u_k^{(m)} - \rho^{(m)} \tau^2 u_j^{(m)} - \gamma_{ij}^{(m)} \partial_i \vartheta^{(m)} &= 0, \quad j = 1, 2, 3, \\ -\tau T_0^{(m)} \gamma_{il}^{(m)} \partial_l u_i^{(m)} + \varkappa_{il}^{(m)} \partial_i \partial_l \vartheta^{(m)} - \tau \alpha^{(m)} \vartheta^{(m)} &= 0, \end{aligned} \right\} \quad (2.4)$$

in $\Omega_m, \quad m = \overline{1, 2N},$

and

$$\left. \begin{aligned} c_{ijkl} \partial_i \partial_l u_k - \rho \tau^2 u_j - \gamma_{ij} \partial_i \vartheta + e_{lij} \partial_l \partial_i \varphi &= 0, \quad j = 1, 2, 3, \\ -\tau T_0 \gamma_{il} \partial_l u_i + \varkappa_{il} \partial_i \partial_l \vartheta - \tau \alpha \vartheta + \tau T_0 g_i \partial_i \varphi &= 0, \\ -e_{ikl} \partial_i \partial_l u_k - g_i \partial_i \vartheta + \varepsilon_{il} \partial_i \partial_l \varphi &= 0, \end{aligned} \right\} \quad \text{in } \Omega, \quad (2.5)$$

(ii) the boundary conditions:

$$\{[\mathcal{T}^{(m)} U^{(m)}]_j\}^+ = 0 \quad \text{on } S_m, \quad j = \overline{1, 4}, \quad m = \overline{1, 2N}, \quad (2.6)$$

$$\{[\mathcal{T}U]_j\}^+ = 0 \quad \text{on } \Sigma_1^\pm \cup \Sigma_2^\pm \cup \Sigma_3^+, \quad j = \overline{1, 4}, \quad (2.7)$$

$$\beta_1 \{[\mathcal{T}U]_5\}^+ + \beta_2 \{u_5\}^+ = 0 \quad \text{on } \Sigma_1^\pm \cup \Sigma_3^\pm, \quad (2.8)$$

$$\{u_5\}^+ = -\Phi_0 \quad \text{on } \Sigma_2^- \cup \left[\bigcup_{m=1}^N \Gamma_m \right], \quad (2.9)$$

$$\{u_5\}^+ = +\Phi_0 \quad \text{on } \Sigma_2^+ \cup \left[\bigcup_{m=N+1}^{2N} \Gamma_m \right], \quad (2.10)$$

$$\{u_j\}^+ = 0 \quad \text{on } \Sigma_3^-, \quad j = \overline{1, 4}, \quad (2.11)$$

(iii) the transmission conditions ($m = \overline{1, 2N}$):

$$\{u_j^{(m)}\}^+ - \{u_j\}^+ = 0 \quad \text{on } \Gamma_m, \quad j = \overline{1, 4}, \quad (2.12)$$

$$\{[\mathcal{T}^{(m)} U^{(m)}]_l\}^+ - \{[\mathcal{T}U]_l\}^+ = 0 \quad \text{on } \Gamma_m, \quad l = \overline{1, 3}, \quad (2.13)$$

$$\left\{ \frac{1}{T_0^{(m)}} [\mathcal{T}^{(m)} U^{(m)}]_4 \right\}^+ - \left\{ \frac{1}{T_0} [\mathcal{T}U]_4 \right\}^+ = 0 \quad \text{on } \Gamma_m, \quad (2.14)$$

where the differential operators $A^{(m)}(\partial, \tau)$, $A(\partial, \tau)$, $\mathcal{T}^{(m)}(\partial, n)$, and $\mathcal{T}(\partial, n)$ are defined in Appendix A,

$$\mathcal{T}^{(m)} U^{(m)} = (\sigma_{i1}^{(m)} n_i, \sigma_{i2}^{(m)} n_i, \sigma_{i3}^{(m)} n_i, -q_i^{(m)} n_i)^\top,$$

$$\mathcal{T}(\partial, n) U = (\sigma_{i1} n_i, \sigma_{i2} n_i, \sigma_{i3} n_i, -q_i n_i, -D_i n_i)^\top,$$

Φ_0 is a constant, β_1 and β_2 are sufficiently smooth real functions, and from now on throughout the paper we assume that

$$|\beta_1| \geq \beta_0 > 0, \quad \beta_1 \beta_2 \leq 0 \quad (2.15)$$

with some positive constant β_0 .

The boundary conditions (2.6)–(2.11) can be interpreted as follows. The lower basis Σ_3^- of the composed parallelepiped Π is mechanically fixed

(clamped) along a dielectric basement, and the remaining part of its boundary is mechanically traction free. Moreover, the temperature distribution is given on the lower basis, and on the remaining part of the boundary the heat flux influence is neglected. On the mutually opposite lateral faces Σ_2^- and Σ_2^+ , which are assumed to be covered with a vaporized thin metallic film having no mechanical influence, electric voltage is applied. Mathematically this is described by the nonhomogeneous boundary conditions (2.9) and (2.10) for the electric potential function $u_5 = \varphi$. The boundary conditions (2.8) for the electric field given on the faces $\Sigma_1^\pm \cup \Sigma_3^\pm$ show a relationship between the electric potential function and the electric displacement vector. An alternative formulation of this relationship by more complicated analytical functions, however, can be found in the corresponding literature (see, e.g., [18], [12], [13]).

Finally, the transmission conditions (2.12)–(2.14) show the usual continuity of the mechanical displacement vector, mechanical stress vector, temperature distribution and heat flux along the interface surfaces Γ_m .

A vector function

$$\mathbf{U} := (U^{(1)}, \dots, U^{(2N)}, U) \in \mathcal{W}_N^1 := [W_2^1(\Omega_1)]^4 \times \dots \times [W_2^1(\Omega_{2N})]^4 \times [W_2^1(\Omega)]^5$$

will be referred to as *a weak solution to the boundary-transmission problem* (2.2)–(2.14). Here and in what follows the symbol \times denotes the direct product of spaces, unless stated otherwise.

The pseudo-oscillation differential equations (2.2) and (2.3) in Ω_m and Ω , respectively, are understood in the distributional sense, in general. However, we remark that in the case of homogeneous equations actually we have $U^{(m)} \in [W_2^1(\Omega_m)]^4 \cap [C^\infty(\Omega_m)]^4$ and $U \in [W_2^1(\Omega)]^5 \cap [C^\infty(\Omega)]^5$ due to the ellipticity of the corresponding differential operators. In fact, $U^{(m)}$ and U are analytic vectors of the real spatial variables x_1, x_2, x_3 in Ω_m and Ω , respectively.

The above boundary and transmission conditions involving boundary limiting values of the vectors $U^{(m)}$ and U are understood in the usual trace sense, while the conditions involving boundary limiting values of the vectors $\mathcal{T}^{(m)}U^{(m)}$ and $\mathcal{T}U$ are understood in the functional sense defined by the relations (related to Green's formulae, see (B.2), (B.6))

$$\begin{aligned} & \left\langle \{T^{(m)}U^{(m)}\}^+, \{V^{(m)}\}^+ \right\rangle_{\partial\Omega_m} := \int_{\Omega_m} A^{(m)}(\partial, \tau)U^{(m)} \cdot V^{(m)} dx + \\ & + \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{v^{(m)}}) + \varrho^{(m)}\tau^2 u^{(m)} \cdot v^{(m)} + \varkappa_j^{(m)} \partial_j u_4^{(m)} \overline{\partial_l v_4^{(m)}} + \right. \\ & \quad \left. + \tau \alpha^{(m)} u_4^{(m)} \cdot \overline{v_4^{(m)}} + \gamma_{jl}^{(m)} (\tau T_0^{(m)} \partial_j u_i^{(m)} \overline{v_4^{(m)}} - u_4^{(m)} \overline{\partial_j v_l^{(m)}}) \right] dx, \quad (2.16) \\ & \left\langle \{\mathcal{T}U\}^+, \{V\}^+ \right\rangle_{\partial\Omega} := - \int_{\Omega} A(\partial, \tau)U \cdot V dx - \int_{\Omega} \left[E(u, \overline{v}) + \varrho\tau^2 u \cdot v + \right. \end{aligned}$$

$$\begin{aligned}
& +\gamma_{jl}(\tau T_0 \partial_j u_l \overline{v_4} - u_4 \overline{\partial_j v_l}) + \varkappa_{jl} \partial_j u_4 \overline{\partial_l v_4} + e_{lij}(\partial_l u_5 \overline{\partial_i v_j} - \partial_i u_j \overline{\partial_l v_5}) + \\
& + \tau \alpha u_4 \overline{v_4} - g_l(\tau T_0 \partial_l u_5 \overline{v_4} + u_4 \overline{\partial_l v_5}) + \varepsilon_{jl} \partial_j u_5 \overline{\partial_l v_5} \Big] dx, \quad (2.17)
\end{aligned}$$

where $V^{(m)} \in [W_2^1(\Omega_m)]^4$ and $V \in [W_2^1(\Omega)]^5$ are arbitrary vector-functions. Here $\langle \cdot, \cdot \rangle_{\partial\Omega_m}$ (respectively $\langle \cdot, \cdot \rangle_{\partial\Omega}$) denotes the duality between the spaces $[H_2^{-1/2}(\partial\Omega_m)]^4$ and $[H_2^{1/2}(\partial\Omega_m)]^4$ (respectively $[H_2^{-1/2}(\partial\Omega)]^5$ and $[H_2^{1/2}(\partial\Omega)]^5$), which extends the usual L_2 -scalar product:

$$\langle f, g \rangle_{\mathcal{M}} = \int_{\mathcal{M}} \sum_{j=1}^M f_j \overline{g_j} d\mathcal{M} \text{ for } f, g \in [L_2(\mathcal{M})]^M, \mathcal{M} \in \{\partial\Omega_m, \partial\Omega\}.$$

By standard arguments it can easily be shown that the functionals $\{\mathcal{T}^{(m)}(\partial, n)U^{(m)}\}^+ \in [H_2^{-1/2}(\partial\Omega_m)]^4$ and $\{\mathcal{T}(\partial, n)U\} \in [H_2^{-1/2}(\partial\Omega)]^5$ are correctly determined by the above relations, provided that $A^{(m)}(\partial, \tau)U^{(m)} \in [L_2(\Omega_m)]^4$ and $A(\partial, \tau)U \in [L_2(\Omega)]^5$.

2.2. Uniqueness theorem. There holds the following uniqueness

Theorem 2.1. *Let $\tau = \sigma + i\omega$, and either $\sigma > 0$ or $\tau = 0$.*

The homogeneous version of the boundary–transmission problem (2.2)–(2.14) ($\Phi_0 = 0$) has then only the trivial solution in the space \mathcal{W}_N^1 .

Proof. Let $\mathbf{U} \in \mathcal{W}_N^1$ be a solution to the homogeneous boundary–transmission problem (2.2)–(2.13).

Green’s formulae (B.4) and (B.8) with $V^{(m)} = U^{(m)}$ and $V = U$ along with the homogeneous boundary and transmission conditions then imply

$$\begin{aligned}
& \sum_{m=1}^{2N} \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \varrho^{(m)} \tau^2 |u^{(m)}|^2 + \right. \\
& \left. + \frac{\tau}{|\tau|^2 T_0^{(m)}} \varkappa_{ij}^{(m)} \partial_i u_4^{(m)} \overline{\partial_j u_4^{(m)}} + \frac{\alpha^{(m)}}{T_0^{(m)}} |u_4^{(m)}|^2 \right] dx + \\
& + \int_{\Omega} \left[E(u, \overline{u}) + \varrho \tau^2 |u|^2 + \frac{\alpha}{T_0} |u_4|^2 + \varepsilon_{jl} \partial_l u_5 \overline{\partial_j u_5} + \frac{\tau}{|\tau|^2 T_0} \varkappa_{jl} \partial_l u_4 \overline{\partial_j u_4} - \right. \\
& \left. - 2\Re\{g_l u_4 \overline{\partial_l u_5}\} \right] dx - \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} \frac{\beta_2}{\beta_1} |\{u_5\}^+|^2 dS = 0. \quad (2.18)
\end{aligned}$$

Note that due to the relations (A.7), (A.39), and (A.40) and the positive definiteness of the matrix ε_{lj} we have

$$\begin{aligned}
E^{(m)}(u^{(m)}, \overline{u^{(m)}}) &= c_{ijkl}^{(m)} \partial_i u_j^{(m)} \overline{\partial_k u_l^{(m)}} \geq 0, \\
E(u, \overline{u}) &= c_{ijkl} \partial_i u_j \overline{\partial_k u_l} \geq 0, \quad (2.19) \\
\varkappa_{ij}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} &\geq 0, \quad \varkappa_{jl} \partial_l u_4 \overline{\partial_j u_4} \geq 0, \quad \varepsilon_{jl} \partial_l u_5 \overline{\partial_j u_5} \geq 0
\end{aligned}$$

with the equality only for complex rigid displacement vectors, constant temperature distributions and a constant potential field:

$$u^{(m)} = a^{(m)} \times x + b^{(m)}, \quad u_4^{(m)} = a_4^{(m)}, \quad u = a \times x + b, \quad u_4 = a_4, \quad u_5 = a_5, \quad (2.20)$$

where $a^{(m)}, b^{(m)}, a, b \in \mathbb{C}^3$, $a_4^{(m)}, a_4, a_5 \in \mathbb{C}$, and \times denotes the usual cross product of two vectors.

Take into account the above inequalities and separate the real and imaginary parts of (2.18) to obtain

$$\begin{aligned} & \sum_{m=1}^{2N} \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \varrho^{(m)}(\sigma^2 - \omega^2)|u^{(m)}|^2 + \frac{\alpha^{(m)}}{T_0^{(m)}} |u_4^{(m)}|^2 + \right. \\ & \quad \left. + \frac{\sigma}{|\tau|^2 T_0^{(m)}} \varkappa_{ij}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} \right] dx + \\ & + \int_{\Omega} \left[E(u, \bar{u}) + \varrho(\sigma^2 - \omega^2)|u|^2 + \frac{\alpha}{T_0} |u_4|^2 + \frac{\sigma}{|\tau|^2 T_0} \varkappa_{jl} \partial_l u_4 \overline{\partial_j u_4} - \right. \\ & \quad \left. - 2\Re\{g_l u_4 \overline{\partial_l u_5}\} + \varepsilon_{ji} \partial_l u_5 \overline{\partial_j u_5} \right] dx - \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} \frac{\beta_2}{\beta_1} |\{u_5\}^+|^2 dS = 0, \quad (2.21) \end{aligned}$$

$$\begin{aligned} & \sum_{m=1}^{2N} \int_{\Omega_m} \left[2\varrho^{(m)} \sigma \omega |u^{(m)}|^2 + \frac{\omega}{|\tau|^2 T_0^{(m)}} \varkappa_{ij}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} \right] dx + \\ & + \int_{\Omega} \left[2\varrho \sigma \omega |u|^2 + \frac{\omega}{|\tau|^2 T_0} \varkappa_{jl} \partial_l u_4 \overline{\partial_j u_4} \right] dx = 0. \quad (2.22) \end{aligned}$$

First, let us assume that $\sigma > 0$ and $\omega \neq 0$. With the help of the homogeneous boundary and transmission conditions we easily derive from (2.22) that $u_j^{(m)} = 0$ ($j = \overline{1, 4}$) in Ω_m and $u_j = 0$ ($j = \overline{1, 4}$) in Ω . From (2.21) we then conclude that

$$\int_{\Omega} \varepsilon_{jl} \partial_l u_5 \overline{\partial_j u_5} dx = 0,$$

whence $u_5 = 0$ in Ω follows due to (2.19) and the homogeneous boundary condition on Γ_m .

Thus $U^{(m)} = 0$ in Ω_m and $U = 0$ in Ω .

The proof for the case $\sigma > 0$ and $\omega = 0$ is quite similar. The only difference is that now, in addition to the above relations, we have to apply the inequality in (A.41) as well.

For $\tau = 0$, by adding the relations (B.9) and (B.10) with $c/T_0^{(m)}$ and c/T_0 for c_1 and c , respectively, we arrive at the equality

$$\sum_{m=1}^{2N} \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \frac{c}{T_0^{(m)}} \varkappa_{ij}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} - \gamma_{jl}^{(m)} u_4^{(m)} \overline{\partial_j u_j^{(m)}} \right] dx +$$

$$\begin{aligned}
& + \int_{\Omega} \left[E(u, \bar{u}) + \frac{c}{T_0} \varkappa_{jl} \partial_l u_4 \overline{\partial_j u_4} - \gamma_{jl} u_4 \overline{\partial_l u_j} - g_l \bar{u}_4 \partial_l u_5 + \varepsilon_{jl} \partial_l u_5 \overline{\partial_j u_5} \right] dx - \\
& - \int_{\Sigma_1^{\pm} \cup \Sigma_3^{\pm}} \frac{\beta_2}{\beta_1} |\{u_5\}^+|^2 dS = 0, \quad (2.23)
\end{aligned}$$

where c is an arbitrary constant parameter.

Dividing the equality by c and sending c to infinity we conclude that $u_4^{(m)} = 0$ in Ω_m and $u_4 = 0$ in Ω due to the homogeneous boundary and transmission conditions for the temperature distributions. This easily yields in view of (2.23) that $U^{(m)} = 0$ in Ω_m and $U = 0$ in Ω due to the homogeneous boundary conditions on Σ_3^- . Thus $\mathbf{U} = 0$. \square

Note that for $\tau = i\omega$ (i.e., for $\sigma = 0$ and $\omega \neq 0$) the homogeneous problem may possess a nontrivial solution, in general.

3. WEAK FORMULATION OF THE BOUNDARY-TRANSMISSION PROBLEM AND EXISTENCE RESULTS

In this section we give a weak formulation of the transmission problem (2.2)–(2.14). To this end, it is convenient to reduce the nonhomogeneous Dirichlet type boundary conditions (2.9) and (2.10) to the homogeneous ones preserving at the same time the continuity property (2.12).

Below we will consider only the case $\Re\tau > 0$. However, we remark that the case $\tau = 0$ can be treated quite similarly (and is even a simpler case).

Denote by S^- and S^+ the subsurfaces where the electric potential function φ is prescribed and takes on constant values $-\Phi_0$ and $+\Phi_0$, respectively:

$$\overline{S^-} := \overline{\Sigma_2^-} \cup \left[\bigcup_{m=1}^N \overline{\Gamma_m} \right], \quad \overline{S^+} := \overline{\Sigma_2^+} \cup \left[\bigcup_{m=N+1}^{2N} \overline{\Gamma_m} \right], \quad S^{\pm} = \overline{S^{\pm}} \setminus \partial \overline{S^{\pm}}. \quad (3.1)$$

Further, let $B_{4\delta}^-$ and $B_{4\delta}^+$ be the following spatial disjoint neighbourhoods of S^- and S^+ :

$$B_{4\delta}^- := \bigcup_{x \in \overline{S^-}} B(x, 2\delta), \quad B_{4\delta}^+ := \bigcup_{x \in \overline{S^+}} B(x, 2\delta), \quad B_{4\delta}^- \cap B_{4\delta}^+ = \emptyset, \quad (3.2)$$

where $B(x, 2\delta)$ is a ball centered at x and of radius 2δ with sufficiently small $\delta > 0$.

Choose a real function $u_5^* \in C^\nu(\mathbb{R}^3)$ ($\nu \geq 3$) with a compact support such that

$$r_{B_{2\delta}^-} u_5^* = -\Phi_0, \quad r_{B_{2\delta}^+} u_5^* = +\Phi_0, \quad r_{\mathbb{R}^3 \setminus [B_{4\delta}^+ \cup B_{4\delta}^-]} u_5^* = 0. \quad (3.3)$$

We set

$$U^* := (0, 0, 0, 0, u_5^*)^\top. \quad (3.4)$$

In view of (A.36), (A.45), and (A.46) we get

$$\begin{aligned} A(\partial, \tau)U^* &= \left([e_{lij}\partial_l\partial_i u_5^*]_{j=1}^3, \tau T_0 g_i \partial_i u_5^*, \varepsilon_{il}\partial_i \partial_l u_5^* \right)^\top \in [C^{\nu-2}(\mathbb{R}^3)]^5, \\ \{\mathcal{T}U^*\}^+ &= \left\{ \left([e_{lij}n_i \partial_l u_5^*]_{j=1}^3, 0, \varepsilon_{il}n_i \partial_l u_5^* \right)^\top \right\}^+ \text{ on } \partial\Omega. \end{aligned} \quad (3.5)$$

Due to the property (3.3) of the function u_5^* , we see that

$$r_{[S^-\cup S^+]} \{\mathcal{T}U^*\}^+ = 0 \quad (3.6)$$

and, moreover, the set $\text{supp}\{\mathcal{T}U^*\}^+$ is a proper part of $\Sigma_1^\pm \cup \Sigma_3^\pm$, i.e.,

$$\text{supp}\{\mathcal{T}U^*\}^+ \cap \partial[\overline{\Sigma_1^\pm} \cup \overline{\Sigma_3^\pm}] = \emptyset.$$

Now, assuming $\mathbf{U} := (U^{(1)}, \dots, U^{(2N)}, U) \in \mathcal{W}_N^1$ to be a solution to the transmission problem (2.2)–(2.14), we can reformulate the problem for the vectors $U^{(m)}$ and $\tilde{U} := U - U^*$ as follows: *Find the vector-functions*

$$\begin{aligned} U^{(m)} &= (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top : \Omega_m \rightarrow \mathbb{C}^4 \\ \text{with } u^{(m)} &:= (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)}) := \vartheta^{(m)}, \quad m = \overline{1, 2N}, \end{aligned}$$

and

$$\begin{aligned} \tilde{U} &= (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4, \tilde{u}_5)^\top : \Omega \rightarrow \mathbb{C}^5 \\ \text{with } \tilde{u} &:= (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3), \quad \tilde{u}_l = u_l, \quad l = 1, 2, 3, \\ &\quad \tilde{u}_4 := u_4 = \vartheta, \quad \tilde{u}_5 := u_5 - u_5^* = \varphi - u_5^*, \end{aligned}$$

belonging to the spaces $[W_2^1(\Omega_m)]^4$ and $[W_2^1(\Omega)]^5$, respectively, and satisfying

the differential equations:

$$[A^{(m)}(\partial, \tau)U^{(m)}]_j = 0 \text{ in } \Omega_m, \quad j = \overline{1, 4}, \quad m = \overline{1, 2N}, \quad (3.7)$$

$$[A(\partial, \tau)\tilde{U}]_k = X_k^* \text{ in } \Omega, \quad k = \overline{1, 5}, \quad (3.8)$$

the boundary conditions:

$$\{[\mathcal{T}^{(m)}U^{(m)}]_j\}^+ = 0 \text{ on } S_m, \quad j = \overline{1, 4}, \quad m = \overline{1, 2N}, \quad (3.9)$$

$$\{[\mathcal{T}\tilde{U}]_l\}^+ = F_l^* \text{ and } \{[\mathcal{T}\tilde{U}]_4\}^+ = 0 \text{ on } \Sigma_1^\pm \cup \Sigma_2^\pm \cup \Sigma_3^\pm, \quad l = \overline{1, 3}, \quad (3.10)$$

$$\beta_1 \{[\mathcal{T}\tilde{U}]_5\}^+ + \beta_2 \{\tilde{u}_5\}^+ = F_5^* \text{ on } \Sigma_1^\pm \cup \Sigma_3^\pm, \quad (3.11)$$

$$\{\tilde{u}_5\}^+ = 0 \text{ on } \Sigma_2^- \cup \left[\bigcup_{m=1}^N \Gamma_m \right], \quad (3.12)$$

$$\{\tilde{u}_5\}^+ = 0 \text{ on } \Sigma_2^+ \cup \left[\bigcup_{m=N+1}^{2N} \Gamma_m \right], \quad (3.13)$$

$$\{\tilde{u}_j\}^+ = 0 \text{ on } \Sigma_3^-, \quad j = \overline{1, 4}, \quad (3.14)$$

the transmission conditions ($m = \overline{1, 2N}$):

$$\{u_j^{(m)}\}^+ - \{\tilde{u}_j\}^+ = 0 \text{ on } \Gamma_m, \quad j = \overline{1, 4}, \quad (3.15)$$

$$\{[\mathcal{T}^{(m)}U^{(m)}]_l\}^+ - \{[\mathcal{T}\tilde{U}]_l\}^+ = 0 \text{ on } \Gamma_m, \quad l = \overline{1, 3}, \quad (3.16)$$

$$\left\{\frac{1}{T_0^{(m)}}[\mathcal{T}^{(m)}U^{(m)}]_4\right\}^+ - \left\{\frac{1}{T_0}[\mathcal{T}\tilde{U}]_4\right\}^+ = 0 \text{ on } \Gamma_m, \quad (3.17)$$

where

$$\begin{aligned} X_k^* &:= -[A(\partial, \tau)U^*]_k \text{ in } \Omega, \quad k = \overline{1, 5}, \\ F_j^* &:= -\{[\mathcal{T}U^*]_j\}^+ \text{ on } \Sigma_1^\pm \cup \Sigma_2^\pm \cup \Sigma_3^+, \quad j = \overline{1, 3}, \\ F_5^* &:= -\beta_1 \{[\mathcal{T}U^*]_5\}^+ - \beta_2 \{u_5^*\}^+ \text{ on } \Sigma_1^\pm \cup \Sigma_3^\pm. \end{aligned} \quad (3.18)$$

In accordance with (3.5) and (3.6), we have

$$\begin{aligned} X_k^* &:= C^{\nu-2}(\overline{\Omega}), \quad k = \overline{1, 5}, \\ F_j^* &:= -\{[\mathcal{T}U^*]_j\}^+ \in \tilde{H}^{-1/2}(\partial\Omega \setminus [S^- \cup S^+]), \quad j = 1, 2, 3, 5. \end{aligned} \quad (3.19)$$

Remark 3.1. It is evident that if $\tilde{\mathbf{U}} := (U^{(1)}, \dots, U^{(2N)}, \tilde{U}) \in \mathcal{W}_N^1$ solves the boundary transmission problem (3.7)–(3.17), then

$$\mathbf{U} := (U^{(1)}, \dots, U^{(2N)}, \tilde{U} + U^*) \in \mathcal{W}_N^1$$

solves the original transmission problem (2.2)–(2.14).

In what follows, we give a weak formulation of the problem (3.7)–(3.17) and show its solvability by the standard Hilbert space method.

First, let us introduce the sesquilinear forms:

$$\begin{aligned} \mathcal{E}^{(m)}(U^{(m)}, V^{(m)}) &:= \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{v^{(m)}}) + \varrho^{(m)} \tau^2 u^{(m)} \cdot v^{(m)} + \right. \\ &\quad \left. + \frac{\alpha^{(m)}}{T_0^{(m)}} u_4^{(m)} \overline{v_4^{(m)}} + \frac{1}{\tau T_0^{(m)}} \varkappa_{jl}^{(m)} \partial_j u_4^{(m)} \overline{\partial_l v_4^{(m)}} + \right. \\ &\quad \left. + \gamma_{jl}^{(m)} (\partial_j u_l^{(m)} \overline{v_4^{(m)}} - u_4^{(m)} \overline{\partial_j v_l^{(m)}}) \right] dx, \quad (3.20) \end{aligned}$$

$$\begin{aligned} \mathcal{E}(U, V) &:= \int_{\Omega} \left[E(u, \overline{v}) + \varrho \tau^2 u \cdot v + \frac{1}{\tau T_0} \varkappa_{jl} \partial_j u_4 \overline{\partial_l v_4} + \frac{\alpha}{T_0} u_4 \overline{v_4} + \right. \\ &\quad \left. + \varepsilon_{jl} \partial_j u_5 \overline{\partial_l v_5} + \gamma_{jl} (\partial_j u_l \overline{v_4} - u_4 \overline{\partial_j v_l}) + \right. \\ &\quad \left. + e_{lij} (\partial_l u_5 \overline{\partial_i v_j} - \partial_i u_j \overline{\partial_l v_5}) - g_l (\partial_l u_5 \overline{v_4} + u_4 \overline{\partial_l v_5}) \right] dx. \quad (3.21) \end{aligned}$$

These forms coincide with the volume integrals in the right-hand side of Green's formulae (B.2) and (B.6) if we put there the functions $v_4^{(m)}$ and v_4 multiplied by $[\overline{\tau T_0^{(m)}}]^{-1}$ and $[\overline{\tau T_0}]^{-1}$, respectively. Note that we have the same type factors in the transmission conditions (3.17) (see also (2.14)).

We put

$$\mathcal{A}(\mathbf{U}, \mathbf{V}) := \sum_{m=1}^{2N} \mathcal{E}^{(m)}(U^{(m)}, V^{(m)}) + \mathcal{E}(U, V). \quad (3.22)$$

We apply Green's formulae (B.2), (B.6), and the above introduced forms to write:

$$\begin{aligned} \mathcal{A}(\mathbf{U}, \mathbf{V}) = & - \sum_{m=1}^{2N} \int_{\Omega_m} \left\{ \sum_{j=1}^3 [A^{(m)}U^{(m)}]_j \overline{v_j^{(m)}} + \frac{1}{\tau T_0^{(m)}} [A^{(m)}U^{(m)}]_4 \overline{v_4^{(m)}} \right\} dx - \\ & - \int_{\Omega} \left\{ \sum_{j=1}^3 [AU]_j \overline{v_j} + \frac{1}{\tau T_0} [AU]_4 \overline{v_4} + [AU]_5 \overline{v_5} \right\} dx + \\ & + \sum_{m=1}^{2N} \int_{\partial\Omega_m} \left\{ \sum_{j=1}^3 \{[T^{(m)}U^{(m)}]_j\}^+ \{v_j^{(m)}\}^+ + \right. \\ & \left. + \frac{1}{\tau T_0^{(m)}} \{[T^{(m)}U^{(m)}]_4\}^+ \{v_4^{(m)}\}^+ \right\} dS - \\ & - \int_{\partial\Omega} \left\{ \sum_{j=1}^3 \{[TU]_j\}^+ \{\overline{v_j}\}^+ + \frac{1}{\tau T_0} \{[TU]_4\}^+ \{\overline{v_4}\}^+ + \{[TU]_5\}^+ \{\overline{v_5}\}^+ \right\} dS, \quad (3.23) \end{aligned}$$

where we assume that

$$\begin{aligned} \mathbf{U} & := (U^{(1)}, \dots, U^{(2N)}, U), \quad \mathbf{V} := (V^{(1)}, \dots, V^{(2N)}, V), \quad \mathbf{U}, \mathbf{V} \in \mathcal{W}_N^1, \\ & \text{and } A^{(m)}(\partial, \tau)U^{(m)} \in [L_2(\Omega_m)]^4, \quad A(\partial, \tau)U \in [L_2(\Omega)]^5. \end{aligned} \quad (3.24)$$

Taking into consideration the Dirichlet type homogeneous boundary and transmission conditions of the problem (3.7)–(3.17), we define the following closed subspace of \mathcal{W}_N^1 :

$$\begin{aligned} \mathcal{V}_N^1 & := \left\{ \mathbf{V} \in \mathcal{W}_N^1 : \{v_5\}^+ = 0 \text{ on } S^-, \{v_5\}^+ = 0 \text{ on } S^+, \right. \\ & \left. \{v_j\}^+ = 0 \text{ on } \Sigma_3^-, \{v_j^{(m)}\}^+ - \{v_j\}^+ = 0 \text{ on } \Gamma_m, j = \overline{1, 4}, m = \overline{1, 2N} \right\}, \end{aligned} \quad (3.25)$$

where S^- and S^+ are as in (3.1), and Σ_3^- and Γ_m are defined in the beginning of Section 2 (see Figure 1).

Further, for any Lipschitz domain $D \subset \mathbb{R}^3$ and any sub-manifold $\mathcal{M} \subset \partial D$ with Lipschitz boundary $\partial\mathcal{M}$ we set

$$\begin{aligned} H_2^1(D, \mathcal{M}) & := \left\{ w \in H_2^1(D) : r_{\mathcal{M}} \{w\}_{\partial D}^+ = 0, \mathcal{M} \subset \partial D \right\} = \\ & = \left\{ w \in H_2^1(D) : \{w\}_{\partial D}^+ \in \widetilde{H}_2^{1/2}(\partial D \setminus \mathcal{M}) \right\}. \end{aligned}$$

If for arbitrary $\mathbf{V} \in \mathcal{V}_N^1$ we define $\mathbb{V} := (\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4, \mathbb{V}_5)^\top$ with

$$\mathbb{V}_j := \begin{cases} V_j^m & \text{in } \Omega_m, \quad m = \overline{1, 2N}, \quad j = \overline{1, 4}, \\ V_j & \text{in } \Omega, \quad j = \overline{1, 4}, \end{cases} \quad \mathbb{V}_5 := V_5 \text{ in } \Omega,$$

then actually we have $\mathbb{V} \in [H_2^1(\Pi, \Sigma_3^-)]^4 \times H_2^1(\Omega, S^+ \cup S^-)$.

Therefore, we can write (in the sense just described)

$$\mathcal{V}_N^1 = [H_2^1(\Pi, \Sigma_3^-)]^4 \times H_2^1(\Omega, S^+ \cup S^-). \quad (3.26)$$

Clearly, any solution to the problem (3.7)–(3.17) belongs to the space \mathcal{V}_N^1 .

It is evident that \mathcal{W}_N^1 and \mathcal{V}_N^1 are Hilbert spaces with the standard scalar product and the corresponding norm associated with the Sobolev spaces W_2^1 :

$$\begin{aligned} (\mathbf{U}, \mathbf{V})_{\mathcal{W}_N^1} &:= \sum_{m=1}^{2N} (U^{(m)}, V^{(m)})_{[W_2^1(\Omega_m)]^4} + (U, V)_{[W_2^1(\Omega)]^5}, \\ \|\mathbf{U}\|_{\mathcal{W}_N^1}^2 &:= \sum_{m=1}^{2N} \|U^{(m)}\|_{[W_2^1(\Omega_m)]^4}^2 + \|U\|_{[W_2^1(\Omega)]^5}^2. \end{aligned} \quad (3.27)$$

For $\mathbf{V} \in \mathcal{V}_N^1$ we have the evident equality

$$\begin{aligned} \|\mathbf{V}\|_{\mathcal{V}_N^1}^2 &:= \|\mathbf{V}\|_{\mathcal{W}_N^1}^2 = \sum_{m=1}^{2N} \|V^{(m)}\|_{[W_2^1(\Omega_m)]^4}^2 + \|V\|_{[W_2^1(\Omega)]^5}^2 = \\ &= \sum_{j=1}^4 \|\mathbb{V}_j\|_{H_2^1(\Pi)}^2 + \|\mathbb{V}_5\|_{H_2^1(\Omega)}^2. \end{aligned}$$

Now, having in hand the relation (3.23), we are in the position to formulate the weak setting of the above transmission problem (3.7)–(3.17):

Find a vector $\tilde{\mathbf{U}} \in \mathcal{V}_N^1$ such that

$$\mathcal{A}(\tilde{\mathbf{U}}, \mathbf{V}) + \mathcal{B}(\tilde{\mathbf{U}}, \mathbf{V}) = \mathcal{F}(\mathbf{V}) \quad \text{for all } \mathbf{V} \in \mathcal{V}_N^1, \quad (3.28)$$

where \mathcal{A} is defined by (3.22),

$$\mathcal{B}(\tilde{\mathbf{U}}, \mathbf{V}) := - \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} \frac{\beta_2}{\beta_1} \{\tilde{u}_5\}^+ \{\bar{v}_5\}^+ dS, \quad (3.29)$$

$$\begin{aligned} \mathcal{F}(\mathbf{V}) &:= - \sum_{l=1}^3 \int_{\Sigma_1^\pm \cup \Sigma_3^+} F_l^* \{\bar{v}_l\}^+ dS - \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} \frac{1}{\beta_1} F_5^* \{\bar{v}_5\}^+ dS \\ &\quad - \int_{\Omega} \left\{ \sum_{j=1}^3 X_j^* \bar{v}_j + \frac{1}{\tau T_0} X_4^* \bar{v}_4 + X_5^* \bar{v}_5 \right\} dx. \end{aligned} \quad (3.30)$$

All the integrals in the right-hand side of (3.29) and (3.30) are well defined and, moreover, the anti-linear functional $\mathcal{F} : \mathcal{V}_N^1 \rightarrow \mathbb{C}$ is continuous, since the functions involved belong to appropriate spaces.

With the help of the equality (3.23) by standard arguments it can easily be shown that the transmission problem (3.7)–(3.17) is equivalent to the variational equation (3.28).

Moreover, due to the relations (A.7), (A.39), (A.40), (A.41), and Korn's inequality ([30], [17]) from (3.20)-(3.22), (3.25), and (2.15) it follows that

$$\Re[\mathcal{A}(\mathbf{U}, \mathbf{U}) + \mathcal{B}(\mathbf{U}, \mathbf{U})] \geq C_1 \|\mathbf{U}\|_{\mathcal{V}_N^1}^2 - C_2 \|\mathbf{U}\|_{\mathcal{V}_N^0}^2 \quad \text{for all } \mathbf{U} \in \mathcal{V}_N^1. \quad (3.31)$$

It is also evident that the sesquilinear form $\mathcal{A} + \mathcal{B} : \mathcal{V}_N^1 \times \mathcal{V}_N^1 \rightarrow \mathbb{C}$ is continuous.

Theorem 3.2. *Let $\tau = \sigma + i\omega$ with $\sigma > 0$. Then the variational problem (3.28) possesses a unique solution.*

Proof. On the one hand, since for the sesquilinear form $\mathcal{A} + \mathcal{B}$ there holds the coerciveness property (3.31), due to the well known results from the theory of variational equations in Hilbert spaces we conclude that the operator

$$\mathcal{P} : \mathcal{V}_N^1 \rightarrow \{\mathcal{V}_N^1\}^* \quad (3.32)$$

corresponding to the variational problem (3.28) is Fredholm with zero index (see, e.g., [21]). Therefore, the uniqueness implies the existence of a solution.

On the other hand, by the word for word arguments as in the proof of Theorem 2.1 we can show that the homogeneous variational equation possesses only the trivial solution for arbitrary $\tau = \sigma + i\omega$ with $\sigma > 0$. Thus the nonhomogeneous equation (3.28) is uniquely solvable. \square

Due to Theorem 3.2 and the above mentioned equivalence, we conclude that the modified problem (3.7)-(3.17), and, consequently, the original transmission problem (2.2)-(2.14) are uniquely solvable. The relation between these solutions is described in Remark 3.1.

Remark 3.3. As it can be seen from (3.20), (3.21), (3.22), and (3.29), if

$$\sigma > 0 \quad \text{and} \quad \sigma \geq |\omega|, \quad (3.33)$$

then we can take $C_2 = 0$ in (3.31) and the real part of the sesquilinear form $\mathcal{A}(\cdot, \cdot) + \mathcal{B}(\cdot, \cdot)$ becomes strictly positive definite, i.e., it satisfies the conditions of the Lax-Milgram theorem. However, Theorem 3.1 gives a wider range for the parameter τ yielding the unique solvability of the equation (3.28).

Remark 3.4. As we have mentioned above, the electric boundary conditions are still debated (see, e.g. [36]) and in the literature one can find different versions. For example, instead of the above considered Robin type linear boundary operator relating the normal component of electric displacement vector and the corresponding electric potential function on $\Sigma_1^\pm \cup \Sigma_3^\pm$, in [12] and [13], the following nonlinear boundary operator is considered

$$R(U) := \{[T\tilde{U}]_5\}^+ + \beta_2(\{\tilde{u}_5\}^+), \quad (3.34)$$

where

$$\beta_2 : \tilde{H}_2^{1/2}(\Sigma_1^\pm \cup \Sigma_3^\pm) \rightarrow H_2^{-1/2}(\Sigma_1^\pm \cup \Sigma_3^\pm)$$

is a well defined monotone operator.

4. BOUNDARY INTEGRAL EQUATIONS METHOD

Here we investigate the boundary transmission problem formulated in Section 2 (see (2.2)-(2.14)) by the potential method. To this end, first we present basic mapping and jump properties of potential type operators, and reduce the transmission problem under consideration to an equivalent system of integral equations. Next we show the invertibility of the corresponding matrix integral operators and prove the existence results for the original boundary transmission problem. At the same time, we obtain that the solutions can be represented by surface potentials.

4.1. Properties of potentials of thermoelasticity. Here we collect the well-known properties of the single layer, double layer, and volume potentials of the theory of thermoelasticity and the corresponding boundary integral operators (for details see [14] and [15]; see also [9], [8], [26], [21], [27], [1], [2]).

Denote by $\Psi^{(m)}(\cdot, \tau) := [\Psi_{kj}^{(m)}(\cdot, \tau)]_{4 \times 4}$ a fundamental matrix of the differential operator $A^{(m)}(\partial, \tau)$:

$$A^{(m)}(\partial, \tau)\Psi^{(m)}(x, \tau) = I_4\delta(x), \quad (4.1)$$

where $\delta(\cdot)$ is Dirac's distribution. The explicit expressions of $\Psi^{(m)}(\cdot, \tau)$ and their properties for the general anisotropic and isotropic cases are given in Appendix C.

Let us introduce the following surface and volume potentials

$$V_\tau^{(m)}(\ell^{(m)})(x) := \int_{\partial\Omega_m} \Psi^{(m)}(x-y, \tau)\ell^{(m)}(y) dS_y, \quad (4.2)$$

$$W_\tau^{(m)}(h^{(m)})(x) := \int_{\partial\Omega_m} \left\{ \tilde{\mathcal{T}}^{(m)}(\partial_y, n(y), \bar{\tau})[\Psi^{(m)}(x-y, \tau)]^\top \right\}^\top h^{(m)}(y) dS_y, \quad (4.3)$$

$$N_\tau^{(m)}(\Phi^{(m)})(x) := \int_{\Omega_m} \Psi^{(m)}(x-y, \tau)\Phi^{(m)}(y) dS_y, \quad (4.4)$$

where

$$\begin{aligned} \ell &= (\ell_1^{(m)}, \dots, \ell_4^{(m)})^\top, & h^{(m)} &= (h_1^{(m)}, \dots, h_4^{(m)})^\top, \\ \Phi^{(m)} &= (\Phi_1^{(m)}, \dots, \Phi_4^{(m)})^\top \end{aligned}$$

are density vectors. The matrix differential operator $\tilde{\mathcal{T}}^{(m)}(\partial, n), \bar{\tau}$ is given by (A.21)–(A.22).

With the help of Green's identity (B.1), by standard arguments we obtain the following integral representation formula for $x \in \Omega_m$ (see, for example, [14], [21])

$$\begin{aligned} U^{(m)}(x) &= W_\tau^{(m)}([U^{(m)}]^+)(x) - \\ &\quad - V_\tau^{(m)}([\mathcal{T}^{(m)}U^{(m)}]^+)(x) + N_\tau^{(m)}(A^{(m)}U^{(m)})(x), \end{aligned} \quad (4.5)$$

where n is the exterior normal to $\partial\Omega_m$.

We assume here that Ω_m is a Lipschitz domain with piecewise smooth compact boundary and $U^{(m)} \in [W_2^1(\Omega_m)]^4$ with $A^{(m)}(\partial, \tau)U^{(m)} \in [L_2(\Omega_m)]^4$.

Further we introduce the boundary operators on $\partial\Omega_m$ generated by the above potentials

$$\begin{aligned}\mathcal{H}_\tau^{(m)}\ell^{(m)}(x) &:= \int_{\partial\Omega_m} \Psi^{(m)}(x-y, \tau)\ell^{(m)}(y) dS_y, \\ \tilde{\mathcal{K}}_\tau^{(m)*}h^{(m)}(x) &:= \int_{\partial\Omega_m} \left\{ \tilde{\mathcal{T}}^{(m)}(\partial_y, n(y), \bar{\tau})[\Psi^{(m)}(x-y, \tau)]^\top \right\}^\top h^{(m)}(y) dS_y, \\ \mathcal{K}_\tau^{(m)}\ell^{(m)}(x) &:= \int_{\partial\Omega_m} \left\{ \mathcal{T}^{(m)}(\partial_x, n(x))\Psi^{(m)}(x-y, \tau) \right\} \ell^{(m)}(y) dS_y,\end{aligned}$$

where $x \in S = \partial\Omega_m$.

In contrast to the classical elasticity theory, the operator $\mathcal{H}_\tau^{(m)}$ is not self-adjoint, and neither $\tilde{\mathcal{K}}_\tau^{(m)*}$ nor $\mathcal{K}_\tau^{(m)}$ are mutually adjoint.

The basic mapping and jump properties of the potentials are give by the following

Theorem 4.1. *Let $\partial\Omega_m$ be a Lipschitz surface and n be its exterior normal. Then*

(i) *the single and double layer potentials have the following mapping properties*

$$\begin{aligned}V_\tau^{(m)} : [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^1(\Omega_m)]^4, \\ W_\tau^{(m)} : [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^1(\Omega_m)]^4;\end{aligned}$$

(ii) *for any $\ell^{(m)} \in [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4$ and any $h^{(m)} \in [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4$ there hold the jump relations*

$$\begin{aligned}[V_\tau^{(m)}(\ell^{(m)})]^+ &= [V_\tau^{(m)}(\ell^{(m)})]^- = \mathcal{H}_\tau^{(m)}\ell^{(m)}, \\ [W_\tau^{(m)}(h^{(m)})]^\pm &= [\pm 2^{-1}I_4 + \tilde{\mathcal{K}}_\tau^{(m)*}]h^{(m)}, \\ [\mathcal{T}^{(m)}(\partial, n)V_\tau^{(m)}(\ell^{(m)})]^\pm &= [\mp 2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]\ell^{(m)}, \\ [\mathcal{T}^{(m)}(\partial, n)W_\tau^{(m)}(h^{(m)})]^+ &= [\mathcal{T}^{(m)}(\partial, n)W_\tau^{(m)}(h^{(m)})]^- =: \mathcal{L}_\tau^{(m)}h^{(m)};\end{aligned}$$

(iii) *the above introduced boundary operators have the following mapping properties*

$$\begin{aligned}\mathcal{H}_\tau^{(m)} : [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4, \\ \pm 2^{-1}I_4 + \tilde{\mathcal{K}}_\tau^{(m)*} : [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mp 2^{-1}I_4 + \mathcal{K}_\tau^{(m)} : [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mathcal{L}_\tau^{(m)} : [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4 &\rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4\end{aligned}$$

for $\Re\tau = \sigma > 0$ all these operators are isomorphisms;

(iv) for arbitrary $\Phi^{(m)} \in [L_2(\Omega_m)]^4$ the volume potential $N_\tau^{(m)}(\Phi^{(m)})$ belongs to the space $[W_2^2(\Omega_m)]^4$ and

$$A^{(m)}(\partial, \tau)N_\tau^{(m)}(\Phi^{(m)}) = \Phi^{(m)} \text{ in } \Omega_m;$$

(v) the following operator equalities hold in the corresponding function spaces:

$$\begin{aligned} \tilde{\mathcal{K}}_\tau^{(m)*}\mathcal{H}_\tau^{(m)} &= \mathcal{H}_\tau^{(m)}\mathcal{K}_\tau^{(m)}, & \mathcal{K}_\tau^{(m)}\mathcal{L}_\tau^{(m)} &= \mathcal{L}_\tau^{(m)}\tilde{\mathcal{K}}_\tau^{(m)*}, \\ \mathcal{L}_\tau^{(m)}\mathcal{H}_\tau^{(m)} &= [-4^{-1}I_4 + (\mathcal{K}_\tau^{(m)})^2], & \mathcal{H}_\tau^{(m)}\mathcal{L}_\tau^{(m)} &= [-4^{-1}I_4 + (\tilde{\mathcal{K}}_\tau^{(m)*})^2]; \end{aligned}$$

(vi) for the corresponding Steklov–Poincaré type operator

$$\mathcal{A}_\tau^{(m)} := [-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}] [\mathcal{H}_\tau^{(m)}]^{-1} = [\mathcal{H}_\tau^{(m)}]^{-1} [-2^{-1}I_4 + \tilde{\mathcal{K}}_\tau^{(m)*}]$$

and for arbitrary $h^{(m)} \in [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4$ we have the following inequality

$$\Re \langle \mathcal{A}_\tau^{(m)} h^{(m)}, h^{(m)} \rangle \geq c' \|h^{(m)}\|_{[H_2^{\frac{1}{2}}(\partial\Omega_m)]^4}^2 - c'' \|h^{(m)}\|_{[H_2^2(\partial\Omega_m)]^4}^2$$

with some positive constants c' and c'' independent of $h^{(m)}$.

We only note here that the injectivity of the operators in item (iii) of Theorem 4.1 and their adjoint ones follows from the uniqueness results for the corresponding homogeneous Dirichlet and Neumann boundary value problems for the domains $\Omega_m^+ := \Omega_m$ and $\Omega_m^- := \mathbb{R}^3 \setminus \bar{\Omega}_m$. Fredholm properties with index equal to zero then follow since the ranges of these operators in the corresponding function spaces are closed (for details see, e.g., [14], [21]).

4.2. Properties of potentials of thermopiezoelasticity. In this subsection we collect the well-known properties of the single layer, double layer and volume potentials of the theory of thermopiezoelasticity and the corresponding boundary integral operators (for details see [3]; see also [21], [1], [2]).

Denote by $\Psi(\cdot, \tau) := [\Psi_{kj}(\cdot, \tau)]_{5 \times 5}$ a fundamental matrix of the differential operator $A(\partial, \tau)$: $A(\partial, \tau)\Psi(x, \tau) = I_5\delta(x)$. The explicit expressions of $\Psi(\cdot, \tau)$ for the general anisotropic and transversally isotropic cases and their properties are given in Appendix C.

Let us introduce the corresponding surface and volume potentials

$$V_\tau(\ell)(x) := \int_{\partial\Omega} \Psi(x-y, \tau)\ell(y) dS_y, \quad (4.6)$$

$$W_\tau(h)(x) := \int_{\partial\Omega} \left\{ \tilde{\mathcal{T}}(\partial_y, n(y), \bar{\tau}) [\Psi(x-y, \tau)]^\top \right\}^\top h(y) dS_y, \quad (4.7)$$

$$N_\tau(\Phi)(x) := \int_{\Omega} \Psi(x-y, \tau)\Phi(y) dS_y, \quad (4.8)$$

where $\ell = (\ell_1, \dots, \ell_5)^\top$, $h = (h_1, \dots, h_5)^\top$, and $\Phi = (\Phi_1, \dots, \Phi_5)^\top$ are density vectors.

Recall that due to our agreement, on $\partial\Omega$ the normal vector n is directed inward.

With the help of Green's identity (B.5), by standard arguments we obtain the following integral representation formula

$$U(x) = -W_\tau([U]^+)(x) + V_\tau([\mathcal{T}(\partial, n)U]^+)(x) + N_\tau(A(\partial, \tau)U)(x), \quad x \in \Omega, \quad (4.9)$$

We assume here that Ω is a Lipschitz domain with a piecewise smooth compact boundary and $U \in [H^1(\Omega)]^5$ with $A(\partial, \tau)U \in [L_2(\Omega)]^5$.

Further we introduce the boundary operators on $\partial\Omega$ generated by the above potentials

$$\begin{aligned} \mathcal{H}_\tau \ell(x) &:= \int_{\partial\Omega} \Psi(x-y, \tau) \ell(y) dS_y, \\ \tilde{\mathcal{K}}_\tau^* h(x) &:= \int_{\partial\Omega} \left\{ \tilde{\mathcal{T}}(\partial_y, n(y), \bar{\tau}) [\Psi(x-y, \tau)]^\top \right\}^\top h(y) dS_y, \\ \mathcal{K}_\tau \ell(x) &:= \int_{\partial\Omega} \left\{ \mathcal{T}(\partial_x, n(x)) \Psi(x-y, \tau) \right\} \ell(y) dS_y, \end{aligned}$$

where $x \in S = \partial\Omega$.

The basic mapping and jump properties of the potentials are given by the following

Theorem 4.2. *Let $\partial\Omega$ be a Lipschitz surface and n be its interior normal. Then*

(i) *the single and double layer potentials have the following mapping properties*

$$V_\tau : [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^1(\Omega)]^5, \quad W_\tau : [H_2^{\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^1(\Omega)]^5;$$

(ii) *for any $h \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$ and any $\ell \in [H_2^{\frac{1}{2}}(\partial\Omega)]^5$ there hold the jump relations*

$$\begin{aligned} [V_\tau(\ell)]^+ &= [V_\tau(\ell)]^- =: \mathcal{H}_\tau \ell, \quad [\mathcal{T}(\partial, n)V_\tau(\ell)]^\pm =: [\pm 2^{-1}I_5 + \mathcal{K}_\tau] \ell, \\ [W_\tau(h)]^\pm &= [\mp 2^{-1}I_5 + \tilde{\mathcal{K}}_\tau^*] h, \quad [\mathcal{T}(\partial, n)W_\tau(h)]^+ = [\mathcal{T}(\partial, n)W_\tau(h)]^- =: \mathcal{L}_\tau h; \end{aligned}$$

(iii) *the above introduced boundary operators have the following mapping properties*

$$\begin{aligned} \mathcal{H}_\tau &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5, \\ \pm 2^{-1}I_5 + \tilde{\mathcal{K}}_\tau^* &: [H_2^{\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5, \\ \mp 2^{-1}I_5 + \mathcal{K}_\tau &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^5, \end{aligned}$$

$$\mathcal{L}_\tau : [H_2^{\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$$

the operator \mathcal{H}_τ is an isomorphism for $\Re\tau = \sigma > 0$;

(iv) for arbitrary $\Phi \in [L_2(\Omega)]^5$ the volume potential $N_\tau(\Phi)$ belongs to the space $[W_2^2(\Omega)]^5$ and

$$A(\partial, \tau)N_\tau(\Phi) = \Phi \text{ in } \Omega;$$

(v) the following operator equalities hold in the corresponding function spaces:

$$\begin{aligned} \tilde{\mathcal{K}}_\tau^* \mathcal{H}_\tau &= \mathcal{H}_\tau \mathcal{K}_\tau, & \mathcal{K}_\tau \mathcal{L}_\tau &= \mathcal{L}_\tau \tilde{\mathcal{K}}_\tau^*, \\ \mathcal{L}_\tau \mathcal{H}_\tau &= [-4^{-1}I_5 + (\mathcal{K}_\tau)^2], & \mathcal{H}_\tau \mathcal{L}_\tau &= [-4^{-1}I_5 + (\tilde{\mathcal{K}}_\tau^*)^2]; \end{aligned}$$

(vi) for the corresponding Steklov–Poincaré type operator

$$\mathcal{A}_\tau := -[2^{-1}I_5 + \mathcal{K}_\tau] \mathcal{H}_\tau^{-1} = -\mathcal{H}_\tau^{-1} [2^{-1}I_5 + \tilde{\mathcal{K}}_\tau^*]$$

and for arbitrary $h \in [H_2^{\frac{1}{2}}(\partial\Omega)]^5$ we have the following inequality

$$\Re \langle \mathcal{A}_\tau h, h \rangle \geq c' \|h\|_{[H_2^{\frac{1}{2}}(\partial\Omega)]^5}^2 - c'' \|h\|_{[H_2^0(\partial\Omega)]^5}^2$$

with some positive constants c' and c'' independent of h .

We only note here that the injectivity of the operator \mathcal{H}_τ and its adjoint one follows from the uniqueness results for the corresponding homogeneous Dirichlet boundary value problems for the domains $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^3 \setminus \bar{\Omega}$. Fredholm properties with index equal to zero then follow since the range of the operator in the corresponding function space is closed (for details see, e.g., [3], [21]).

4.3. Representation formulas for solutions. Throughout this subsection we assume that $\Re\tau = \sigma > 0$. Here we consider *two auxiliary problems* needed for our further purposes.

4.3.1. *Auxiliary problem I.* Find a vector function

$$U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top : \Omega_m \rightarrow \mathbb{C}^4$$

which belongs to the space $[W_2^1(\Omega_m)]^4$ and satisfies the following differential equation and boundary conditions:

$$A^{(m)}(\partial, \tau)U^{(m)} = 0 \text{ in } \Omega_m, \quad (4.10)$$

$$\{\mathcal{T}^{(m)}U^{(m)}\}^+ = \ell^{(m)} \text{ on } \partial\Omega_m, \quad (4.11)$$

where $\ell^{(m)} = (\ell_1^{(m)}, \ell_2^{(m)}, \ell_3^{(m)}, \ell_4^{(m)})^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4$. With the help of Green's formulae it can easily be shown that the homogeneous version of this auxiliary BVP I possesses only the trivial solution.

Recall that on $\partial\Omega_m$ the normal vector n is directed outward.

From Theorem 4.1 and the above mentioned uniqueness result for the BVP (4.10)-(4.11) immediately follows

Corollary 4.3. *Let $\Re\tau = \sigma > 0$. An arbitrary solution $U^{(m)} \in [W_2^1(\Omega_m)]^4$ to the homogeneous equation (4.10) can be uniquely represented by the single layer potential*

$$U^{(m)}(x) = V_\tau^{(m)} \left(\left[-2^{-1}I_4 + \mathcal{K}_\tau^{(m)} \right]^{-1} \ell^{(m)} \right) (x), \quad x \in \Omega_m,$$

where the density vector $\ell^{(m)}$ satisfies the relation $\ell^{(m)} = [\mathcal{T}^{(m)}U^{(m)}]^+$ on $\partial\Omega_m$.

4.3.2. *Auxiliary problem II.* Find a vector function $U = (u_1, u_2, u_3, u_4, u_5)^\top : \Omega \rightarrow \mathbb{C}^5$ which belongs to the space $[W_2^1(\Omega)]^5$ and satisfies the following conditions:

$$A(\partial, \tau)U = 0 \quad \text{in } \Omega, \quad (4.12)$$

$$\{[\mathcal{T}U]_j\}^+ = \ell_j \quad \text{on } \partial\Omega, \quad j = \overline{1, 4}, \quad (4.13)$$

$$\{U_5\}^+ = \ell_5 \quad \text{on } \partial\Omega, \quad (4.14)$$

where $\ell_j \in H_2^{-\frac{1}{2}}(\partial\Omega)$ for $j = \overline{1, 4}$, and $\ell_5 \in H_2^{\frac{1}{2}}(\partial\Omega)$.

Denote $\ell' := (\ell_1, \ell_2, \ell_3, \ell_4)^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^4$.

By the same arguments as in the proof of Theorem 2.1, we can easily show that the homogeneous version of this boundary value problem possesses only the trivial solution.

We look for a solution to the auxiliary BVP II as a single layer potential, $U(x) = V_\tau(f)(x)$, where $f = (f_1, f_2, f_3, f_4, f_5)^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$ is a sought density.

The boundary conditions (4.13) and (4.14) lead then to the system of equations:

$$\begin{aligned} [(2^{-1}I_5 + \mathcal{K}_\tau)f]_{\overline{j}} &= \ell_j \quad \text{on } \partial\Omega, \quad j = \overline{1, 4}, \\ [\mathcal{H}_\tau f]_5 &= \ell_5 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.15)$$

Denote the operator generated by the left hand side expressions of these equations by \mathcal{P}_τ and rewrite the system as

$$\mathcal{P}_\tau f = \ell,$$

where

$$\mathcal{P}_\tau := \begin{bmatrix} [(2^{-1}I_5 + \mathcal{K}_\tau)_{jk}]_{4 \times 5} \\ [(\mathcal{H}_\tau)_{5k}]_{1 \times 5} \end{bmatrix}$$

and $\ell = (\ell', \ell_5)^\top \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega)$.

Lemma 4.4. *Let $\Re\tau = \sigma > 0$. The operator*

$$\mathcal{P}_\tau : [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega)$$

is an isomorphism.

Proof. The injectivity of the operator \mathcal{P}_τ follows from the uniqueness result of the auxiliary BVP II.

To show that \mathcal{P}_τ is surjective, we proceed as follows.

Due to Theorem 4.2(iii) the operator $\mathcal{H}_\tau : [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5$ is invertible and we can introduce a new unknown vector function $h = (h_1, h_2, h_3, h_4, h_5)^\top$ by the relation $h := \mathcal{H}_\tau f \in [H_2^{\frac{1}{2}}(\partial\Omega)]^5$. From the equations (4.15) we then have:

$$\begin{aligned} [(2^{-1}I_5 + \mathcal{K}_\tau)\mathcal{H}_\tau^{-1}h]_{\overline{j}} &= \ell_j \quad \text{on } \partial\Omega, \quad j = \overline{1,4}, \\ h_5 &= \ell_5 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.16)$$

Recall that for the SteklovPoincaré operator

$$\mathcal{A}_\tau = [(\mathcal{A}_\tau)_{jk}]_{5 \times 5} := -[2^{-1}I_5 + \mathcal{K}_\tau]\mathcal{H}_\tau^{-1} \quad (4.17)$$

there holds the following inequality (see Theorem 4.2, item (v))

$$\Re\langle \mathcal{A}_\tau h^*, h^* \rangle \geq c' \|h^*\|_{[H_2^{\frac{1}{2}}(\partial\Omega)]^5}^2 - c'' \|h^*\|_{[H_2^0(\partial\Omega)]^5}^2, \quad (4.18)$$

for all $h^* \in [H_2^{\frac{1}{2}}(\partial\Omega)]^5$ with positive constants c' and c'' .

Set $h' := (h_1, h_2, h_3, h_4)^\top$. Take into consideration that $h_5 = \ell_5$ and rewrite the first four equations in (4.16) as follows:

$$\tilde{\mathcal{A}}_\tau h' = \ell^*, \quad (4.19)$$

where $\ell^* = (\ell_1^*, \ell_2^*, \ell_3^*, \ell_4^*)^\top$, and where $\ell_j^* := -\ell_j - [(\mathcal{A}_\tau)_{j5}\ell_5] \in H_2^{-\frac{1}{2}}(\partial\Omega)$, $j = \overline{1,4}$, are known functions. Here $\tilde{\mathcal{A}}_\tau := [(\mathcal{A}_\tau)_{jk}]_{4 \times 4}$, where $(\mathcal{A}_\tau)_{jk}$ are the entries of the Steklov–Poincaré operator (4.17).

The equation (4.19) can be written componentwise as

$$\sum_{k=1}^4 (\mathcal{A}_\tau)_{jk} h_k = \ell_j^*, \quad j = \overline{1,4}.$$

If in (4.18) we substitute $h^* = (h', 0)^\top$ with arbitrary $h' \in [H_2^{\frac{1}{2}}(\partial\Omega)]^4$, then we get

$$\Re\langle \tilde{\mathcal{A}}_\tau h', h' \rangle \geq c' \|h'\|_{[H_2^{\frac{1}{2}}(\partial\Omega)]^4}^2 - c'' \|h'\|_{[H_2^0(\partial\Omega)]^4}^2. \quad (4.20)$$

Therefore, the operator

$$\tilde{\mathcal{A}}_\tau : [H_2^{\frac{1}{2}}(\partial\Omega)]^4 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \quad (4.21)$$

is a Fredholm operator with index zero (see, e.g., [21], Ch. 2).

Clearly, to show the invertibility it suffices to prove that the equation

$$\tilde{\mathcal{A}}_\tau h' = 0, \quad \text{i.e., } [\mathcal{A}_\tau \tilde{h}]_j = 0, \quad j = \overline{1,4} \quad (4.22)$$

with $\tilde{h} := (h', 0)^\top$ has only the trivial solution.

Assume that $h' \in [H_2^{\frac{1}{2}}(\partial\Omega)]^4$ solves the equation (4.22) and $f \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$ is a vector such that

$$\mathcal{H}_\tau f = \tilde{h}, \quad \text{i.e., } f = \mathcal{H}_\tau^{-1} \tilde{h}. \quad (4.23)$$

From (4.22) and (4.23) we have

$$[(2^{-1}I_5 + \mathcal{K}_\tau)f]_j = 0, \quad j = \overline{1,4}, \quad [\mathcal{H}_\tau f]_5 = 0,$$

i.e., $\mathcal{P}_\tau f = 0$.

Since \mathcal{P}_τ is injective, we conclude that $f = 0$ and, consequently, $h' = 0$ in accordance to (4.23).

Thus we have shown that the operator (4.21) is invertible, which in its turn implies that the operator

$$\mathcal{P}_\tau : [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \quad (4.24)$$

is invertible as well. \square

Corollary 4.5. *Let $\Re\tau = \sigma > 0$. An arbitrary solution $U \in [W_2^1(\Omega)]^5$ to the homogeneous equation (4.12) can be uniquely represented by the single layer potential for $x \in \Omega$: $U(x) = V_\tau(\mathcal{P}_\tau^{-1}\ell)(x)$, where*

$$\ell = \left([\mathcal{T}U]_1^+, [\mathcal{T}U]_2^+, [\mathcal{T}U]_3^+, [\mathcal{T}U]_4^+, [U]_5^+ \right)^\top \text{ on } \partial\Omega.$$

4.4. Existence results. Here we again assume that $\tau = \sigma + i\omega$ with $\Re\tau = \sigma > 0$.

Let us look for a solution of the transmission problem (2.2)–(2.14) in the form of single layer potentials:

$$U^{(m)}(x) = V_\tau^{(m)}([-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]^{-1}\ell^{(m)})(x), \quad x \in \Omega_m, \quad m = \overline{1, 2N}, \quad (4.25)$$

$$U(x) = V_\tau(\mathcal{P}_\tau^{-1}\ell)(x), \quad x \in \Omega, \quad (4.26)$$

where the unknown densities $\ell^{(m)}$ and ℓ have the following properties (due to Corollaries 4.3 and 4.5)

$$\ell^{(m)} = \{\mathcal{T}^{(m)}U^{(m)}\}^+ \text{ on } \partial\Omega_m, \quad m = \overline{1, 2N}, \quad (4.27)$$

$$\ell^{(m)} = (\ell_1^{(m)}, \ell_2^{(m)}, \ell_3^{(m)}, \ell_4^{(m)})^\top \in [\tilde{H}_2^{-\frac{1}{2}}(\Gamma_m)]^4, \quad m = \overline{1, 2N}, \quad (4.28)$$

$$\ell = \left(\{\mathcal{T}U\}_1^+, \{\mathcal{T}U\}_2^+, \{\mathcal{T}U\}_3^+, \{\mathcal{T}U\}_4^+, \{U\}_5^+ \right) \text{ on } \partial\Omega, \quad (4.29)$$

$$\ell = (\ell', \ell_5)^\top, \quad \ell' = (\ell_1, \ell_2, \ell_3, \ell_4)^\top \in [\tilde{H}_2^{-\frac{1}{2}}(\Sigma_3^- \cup \Gamma)]^4, \quad (4.30)$$

$$\Gamma = \bigcup_{m=1}^{2N} \Gamma_m. \quad (4.31)$$

Note that the unknown function ℓ_5 can be represented in the form

$$\ell_5 = \psi_5 + \Phi_5 \text{ with } \psi_5 \in \tilde{H}_2^{\frac{1}{2}}(\Sigma_1^\pm \cup \Sigma_3^\pm) \text{ and } \Phi_5 \in H_2^{\frac{1}{2}}(\partial\Omega), \quad (4.32)$$

where Φ_5 is a fixed extension onto the whole boundary $\partial\Omega$ of the constant functions $+\Phi_0$ (on S^+) and $-\Phi_0$ (on S^-), i.e., $r_{S^\pm} \Phi_5 = \pm\Phi_0$.

It is evident that

$$\{U^{(m)}\}^+ = \mathcal{R}_\tau^{(m)} \ell^{(m)} \quad \text{on } \partial\Omega_m, \quad m = \overline{1, 2N}, \quad (4.33)$$

$$\{\mathcal{T}^{(m)}U^{(m)}\}^+ = \ell^{(m)} \quad \text{on } \partial\Omega_m, \quad m = \overline{1, 2N}, \quad (4.34)$$

$$\{U\}^+ = \mathcal{R}_\tau \ell = \left([\mathcal{R}_\tau \ell]_1, [\mathcal{R}_\tau \ell]_2, [\mathcal{R}_\tau \ell]_3, [\mathcal{R}_\tau \ell]_4, \ell_5 \right)^\top \quad \text{on } \partial\Omega, \quad (4.35)$$

$$\{\mathcal{T}U\}^+ = \mathcal{B} \ell = (\ell_1, \ell_2, \ell_3, \ell_4, [\mathcal{B}_\tau \ell]_5)^\top \quad \text{on } \partial\Omega, \quad (4.36)$$

where

$$\begin{aligned} \mathcal{R}_\tau^{(m)} &:= \mathcal{H}_\tau^{(m)} [-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]^{-1} = [\mathcal{A}_\tau^{(m)}]^{-1}, \\ \mathcal{R}_\tau &:= \mathcal{H}_\tau \mathcal{P}_\tau^{-1}, \quad \mathcal{B}_\tau := [2^{-1}I_5 + \mathcal{K}_\tau] \mathcal{P}_\tau^{-1}. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{B}_\tau &= [(\mathcal{B}_\tau)_{jk}]_{5 \times 5}, \quad (\mathcal{B}_\tau)_{jk} = 0, \quad j \neq k, \quad j = \overline{1, 4}, \quad k = \overline{1, 5}, \\ (\mathcal{B}_\tau)_{jj} &= 1, \quad j = \overline{1, 4}, \quad (\mathcal{B}_\tau)_{5k} = [2^{-1}I_5 + \mathcal{K}_\tau]_{5l} (\mathcal{P}_\tau^{-1})_{lk}, \quad k = \overline{1, 5}. \end{aligned}$$

Clearly, the operators

$$\begin{aligned} \mathcal{R}_\tau^{(m)} &: [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mathcal{R}_\tau &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5, \end{aligned}$$

are invertible due to Theorems 4.1, 4.2, and Lemma 4.4, while the operator

$$\mathcal{B}_\tau : [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^5$$

is bounded.

We assume that the restrictions of the unknown densities $\ell^{(m)}$ and ℓ on the interface Γ_m satisfy the following conditions

$$r_{\Gamma_m} \ell_j^{(m)} = r_{\Gamma_m} \ell_j, \quad j = 1, 2, 3, \quad m = \overline{1, 2N}, \quad (4.37)$$

$$[T_0^{(m)}]^{-1} r_{\Gamma_m} \ell_4^{(m)} = [T_0]^{-1} r_{\Gamma_m} \ell_4, \quad m = \overline{1, 2N}. \quad (4.38)$$

Let us introduce new unknown vectors

$$\psi^{(m)} = (\psi_1^{(m)}, \psi_2^{(m)}, \psi_3^{(m)}, \psi_4^{(m)})^\top \in [\tilde{H}_2^{-\frac{1}{2}}(\Gamma_m)]^4, \quad m = \overline{1, 2N}, \quad (4.39)$$

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^\top \in [\tilde{H}_2^{-\frac{1}{2}}(\Sigma_3^-)]^4, \quad (4.40)$$

where $\psi^{(m)}$ is defined on $\partial\Omega_m \cup \partial\Omega$, while ψ is defined on $\partial\Omega$, and

$$r_{\Gamma_m} \psi_j^{(m)} := r_{\Gamma_m} \ell_j^{(m)} = r_{\Gamma_m} \ell_j, \quad j = 1, 2, 3, \quad m = \overline{1, 2N}, \quad (4.41)$$

$$r_{\Gamma_m} \psi_4^{(m)} := [T_0^{(m)}]^{-1} r_{\Gamma_m} \ell_4^{(m)} = [T_0]^{-1} r_{\Gamma_m} \ell_4, \quad m = \overline{1, 2N}, \quad (4.42)$$

$$r_{\Sigma_3^-} \psi_j := r_{\Sigma_3^-} \ell_j, \quad j = \overline{1, 4}. \quad (4.43)$$

The original unknown vectors then can be written as

$$\begin{aligned} \ell^{(m)} &= (\psi_1^{(m)}, \psi_2^{(m)}, \psi_3^{(m)}, T_0^{(m)}\psi_4^{(m)})^\top = \mathcal{I}_4(T_0^{(m)})\psi^{(m)}, \quad m = \overline{1, 2N}, \\ \ell &= (\ell', \ell_5)^\top = \left(\sum_{m=1}^{2N} \mathcal{I}_4(T_0)\psi^{(m)} + \psi, \psi_5 + \Phi_5 \right)^\top. \end{aligned} \quad (4.44)$$

Here $\mathcal{I}_4(a) := \text{diag}[1, 1, 1, a]$.

The potentials (4.25) and (4.26) can be rewritten as follows

$$U^{(m)}(x) = V_\tau^{(m)} \left([-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]^{-1} \mathcal{I}_4(T_0^{(m)})\psi^{(m)} \right)(x), \quad (4.45)$$

$$x \in \Omega_m, \quad m = \overline{1, 2N},$$

$$U(x) = V_\tau \left(\mathcal{P}_\tau^{-1} \left[\sum_{m=1}^{2N} \mathcal{I}_4(T_0)\psi^{(m)} + \psi, \psi_5 + \Phi_5 \right]^\top \right)(x), \quad x \in \Omega. \quad (4.46)$$

They have to satisfy the conditions of the boundary-transmission problem (2.2)–(2.14).

It can easily be shown that the conditions (2.2)–(2.7) are satisfied automatically.

The boundary condition (2.8) leads to the equation

$$\begin{aligned} \beta_1 \{\mathcal{T}U\}_5^+ + \beta_2 \{u\}_5^+ &\equiv \\ &\equiv \beta_1 \left([2^{-1}I_5 + \mathcal{K}_\tau] \mathcal{P}_\tau^{-1} \left[\sum_{m=1}^{2N} \mathcal{I}_4(T_0)\psi^{(m)} + \psi, \psi_5 + \Phi_5 \right]^\top \right)_5 + \\ &\quad + \beta_2 (\psi_5 + \Phi_5) = 0 \quad \text{on } \Sigma_1^\pm \cup \Sigma_3^\pm. \end{aligned} \quad (4.47)$$

The conditions (2.9) and (2.10) are also satisfied automatically. The condition (2.11) implies

$$\begin{aligned} \{u\}_j^+ &\equiv \left(\mathcal{H}_\tau \mathcal{P}_\tau^{-1} \left[\sum_{m=1}^{2N} \mathcal{I}_4(T_0)\psi^{(m)} + \psi, \psi_5 + \Phi_5 \right]^\top \right)_j = 0 \quad (4.48) \\ &\quad \text{on } \Sigma_3^-, \quad j = \overline{1, 4}. \end{aligned}$$

The condition (2.12) gives

$$\begin{aligned} \{u^{(m)}\}_j^+ - \{u\}_j^+ &\equiv \left(\mathcal{H}_\tau^{(m)} [-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]^{-1} \mathcal{I}_4(T_0^{(m)})\psi^{(m)} \right)_j - \\ &\quad - \left(\mathcal{H}_\tau^{(m)} \mathcal{P}_\tau^{-1} \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0)\psi^{(l)} + \psi, \psi_5 + \Phi_5 \right]^\top \right)_j = 0 \quad (4.49) \\ &\quad \text{on } \Gamma_m, \quad m = \overline{1, 2N}, \quad j = \overline{1, 4}. \end{aligned}$$

Finally, the conditions (2.13) and (2.14) are also automatically satisfied.

Thus we have to find the unknown vector functions $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(2N)}, \psi$, and the scalar function ψ_5 satisfying the equations (4.47), (4.48), (4.4). We

can rewrite these equations as the following system

$$\begin{aligned} r_{\Gamma_m} (\mathcal{R}_\tau^{(m)} \mathcal{I}_4(T_0^{(m)}) \psi^{(m)})_j - r_{\Gamma_m} \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_j &= \\ &= F_j^{(m)} \text{ on } \Gamma_m, \quad m = \overline{1, 2N}, \quad j = \overline{1, 4}, \end{aligned} \quad (4.50)$$

$$r_{\Sigma_3^-} \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_j = F_j \text{ on } \Sigma_3^-, \quad j = \overline{1, 4}, \quad (4.51)$$

$$r_{\Sigma_1^\pm \cup \Sigma_3^\pm} \left\{ \left(\mathcal{B}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_5 + \beta \psi_5 \right\} = F_5 \text{ on } \Sigma_1^\pm \cup \Sigma_3^\pm, \quad (4.52)$$

where (for $j = \overline{1, 4}$, $m = \overline{1, 2N}$)

$$F_j^{(m)} := r_{\Gamma_m} [\mathcal{R}_\tau \tilde{\Phi}]_j, \quad F^{(m)} = (F_1^{(m)}, F_2^{(m)}, F_3^{(m)}, F_4^{(m)}) \in [H_2^{\frac{1}{2}}(\Gamma_m)]^4,$$

$$F_j := -r_{\Sigma_3^-} [\mathcal{R}_\tau \tilde{\Phi}]_j,$$

$$F_5 := -r_{\Sigma_1^\pm \cup \Sigma_3^\pm} \left\{ [(2^{-1}I_5 + \mathcal{K}_\tau) \mathcal{P}_\tau^{-1} \tilde{\Phi}]_5 + \beta \Phi_5 \right\}, \quad \beta = \beta_2 \beta_1^{-1},$$

$$F := (F_1, F_2, F_3, F_4) \in [H_2^{\frac{1}{2}}(\Sigma_3^-)]^4, \quad \tilde{\Phi} := (0, 0, 0, 0, \Phi_5)^\top \in [H_2^{\frac{1}{2}}(\partial\Omega)]^5.$$

Denote by \mathcal{N}_τ the linear operator generated by the left-hand side expressions in the system (4.4)–(4.52) and rewrite the latter in matrix form as

$$\mathcal{N}_\tau \Psi = \mathbf{F}, \quad (4.53)$$

where

$$\Psi := (\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(2N)}, \psi, \psi_5)^\top \quad (4.54)$$

is an $8N + 5$ dimensional sought vector function and

$$\mathbf{F} := (F^{(1)}, F^{(2)}, \dots, F^{(2N)}, F, F_5)^\top \quad (4.55)$$

is an $8N + 5$ dimensional known vector function.

Let us introduce the following $8N + 5$ dimensional function space \mathbf{X} and its dual space \mathbf{X}^* ,

$$\mathbf{X} := [\tilde{H}_2^{-\frac{1}{2}}(\Gamma_1)]^4 \times \dots \times [\tilde{H}_2^{-\frac{1}{2}}(\Gamma_{2N})]^4 \times [\tilde{H}_2^{-\frac{1}{2}}(\Sigma_3^-)]^4 \times \tilde{H}_2^{\frac{1}{2}}(\Sigma_1^\pm \cup \Sigma_3^\pm),$$

$$\mathbf{X}^* := [H_2^{\frac{1}{2}}(\Gamma_1)]^4 \times \dots \times [H_2^{\frac{1}{2}}(\Gamma_{2N})]^4 \times [H_2^{\frac{1}{2}}(\Sigma_3^-)]^4 \times H_2^{-\frac{1}{2}}(\Sigma_1^\pm \cup \Sigma_3^\pm).$$

Note that \mathbf{X} is a reflexive Hilbert space: $(\mathbf{X}^*)^* = \mathbf{X}$.

Due to the properties of the surface potentials and its inverses involved in the left hand side expressions of the system (4.4)–(4.52), the operator \mathcal{N}_τ has the mapping property $\mathcal{N}_\tau : \mathbf{X} \rightarrow \mathbf{X}^*$.

Now we investigate the solvability of the system (4.53) (i.e., (4.4)–(4.52)).

Theorem 4.6. *The operator*

$$\mathcal{N}_\tau : \mathbf{X} \rightarrow \mathbf{X}^* \quad (4.56)$$

is invertible.

Proof. From the uniqueness result (see Theorem 2.1) and the invertibility of the operators $\mathcal{H}_\tau^{(m)}$, $(-2^{-1}I_4 + \mathcal{K}_\tau^{(m)})$, \mathcal{H}_τ , and \mathcal{P}_τ , it immediately follows that the linear bounded operator (4.56) is injective for arbitrary τ with $\Re\tau = \sigma > 0$.

Further we show that it is surjective, $\mathcal{N}_\tau(\mathbf{X}) = \mathbf{X}^*$.

First we show the invertibility of the operator (4.56) when $\sigma^2 - \omega^2$ is a positive number and afterwards we consider the general case.

We prove this in several steps.

Step 1. Let us apply Green's formulae (B.3) and (B.7) to the potentials (4.25) and (4.26), where $\ell^{(m)}$ and ℓ are related to the vectors-functions $\psi^{(m)}$, ψ , and ψ_5 by the equations (4.32) and (4.44) with $\Phi_5 = 0$. We get

$$\begin{aligned} & \sum_{m=1}^{2N} \int_{\partial\Omega_m} \left[\sum_{j=1}^3 \{\mathcal{T}^{(m)}U^{(m)}\}_j^+ \{\overline{u_j^{(m)}}\}^+ + \frac{1}{\tau T_0^{(m)}} \{\mathcal{T}^{(m)}U^{(m)}\}_4^+ \{\overline{u_4^{(m)}}\}^+ \right] dS - \\ & - \int_{\partial\Omega} \left[\sum_{j=1}^3 \{\mathcal{T}U\}_j^+ \{\overline{u_j}\}^+ + \frac{1}{\tau T_0} \{\mathcal{T}U\}_4^+ \{\overline{u_4}\}^+ + \{\overline{\mathcal{T}U}\}_5^+ \{u_5\}^+ \right] dS = Q_1 + iQ_2, \end{aligned}$$

where

$$\begin{aligned} Q_1 + iQ_2 := & \sum_{m=1}^{2N} \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \varrho^{(m)}\tau^2 |u^{(m)}|^2 + \frac{\alpha^{(m)}}{T_0^{(m)}} |u_4^{(m)}|^2 + \right. \\ & \left. + \frac{\overline{\tau}}{|\tau|^2 T_0^{(m)}} \varkappa_{jl}^{(m)} \partial_j u_4^{(m)} \overline{\partial_l u_4^{(m)}} + \gamma_{jl}^{(m)} (\partial_j u_l^{(m)} \overline{u_4^{(m)}} - u_4^{(m)} \overline{\partial_j u_l^{(m)}}) \right] dx + \\ & + \int_{\Omega} \left[E(u, \overline{u}) + \varrho\tau^2 |u|^2 + \gamma_{jl} (\partial_j u_l \overline{u_4} - u_4 \overline{\partial_j u_l}) + \right. \\ & \left. + \frac{\overline{\tau}}{|\tau|^2 T_0} \varkappa_{jl} \partial_j u_4 \overline{\partial_l u_4} + \frac{\alpha}{T_0} |u_4|^2 + e_{lij} (\partial_l u_5 \overline{\partial_i u_j} - \overline{\partial_i u_j} \partial_l u_5) - \right. \\ & \left. - \ell_l (\partial_l u_5 \overline{u_4} + \overline{u_4} \partial_l u_5) + \varepsilon_{jl} \overline{\partial_j u_5} \partial_l u_5 \right] dx. \end{aligned}$$

The left-hand side expression involving the surface integrals can be represented in terms of the operators $\mathcal{R}_\tau^{(m)}$, \mathcal{R}_τ and \mathcal{B}_τ , and the vectors $\ell^{(m)}$ and ℓ (see (4.25)–(4.36)):

$$\begin{aligned} & \sum_{m=1}^{2N} \int_{\partial\Omega_m} \left[\sum_{j=1}^3 \ell_j^{(m)} (\overline{\mathcal{R}_\tau^{(m)} \ell^{(m)}})_j + \frac{1}{\tau T_0^{(m)}} \ell_4^{(m)} (\overline{\mathcal{R}_\tau^{(m)} \ell^{(m)}})_4 \right] dS - \\ & - \int_{\partial\Omega} \left[\sum_{j=1}^3 \ell_j (\overline{\mathcal{R}_\tau \ell})_j + \frac{1}{\tau T_0} \ell_4 (\overline{\mathcal{R}_\tau \ell})_4 + (\overline{\mathcal{B}_\tau \ell})_5 \ell_5 \right] dS = Q_1 + iQ_2. \end{aligned}$$

Apply (4.44) and take the complex conjugate of this equation to obtain

$$\begin{aligned}
& Q_1 - iQ_2 = \\
& = \sum_{m=\overline{1,2N}}^{2N} \int \left[\sum_{j=1}^3 \left\{ (\mathcal{R}_\tau^{(m)} I_4(T_0^{(m)}) \psi^{(m)})_j - \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_j \right\} \overline{\psi_j^{(m)}} + \right. \\
& + \frac{1}{\tau} \left\{ (\mathcal{R}_\tau^{(m)} I_4(T_0^{(m)}) \psi^{(m)})_4 - \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_4 \right\} \overline{\psi_4^{(m)}} \Big] dS \\
& - \int_{\Sigma_3^-} \left[\sum_{j=1}^3 \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_j \overline{\psi_j} + \right. \\
& \left. + \frac{1}{\tau T_0} \left(\mathcal{R}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_4 \overline{\psi_4} \right] dS - \\
& - \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} \left[\left(\mathcal{B}_\tau \left[\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right]^\top \right)_5 + \beta \psi_5 \right] \overline{\psi_5} dS + \beta \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} |\psi_5|^2 dS.
\end{aligned}$$

This equality can be rewritten in the form

$$\langle \tilde{\mathcal{N}}_\tau \Psi, \Psi \rangle + \beta \int_{\Sigma_1^\pm \cup \Sigma_3^\pm} |\psi_5|^2 dS = Q_1 - iQ_2, \quad (4.57)$$

where the operator $\tilde{\mathcal{N}}_\tau$ is generated by the expressions in the left-hand side of the system (4.4)–(4.52) if we multiply the fourth equations in (4.4) and (4.51) by the numbers $\frac{1}{\tau}$ and $\frac{1}{\tau T_0}$ respectively. That is, the equation $\tilde{\mathcal{N}}_\tau \Psi = \tilde{F}$, where $\Psi \in \mathbf{X}$ and $\tilde{F} \in \mathbf{X}^*$, corresponds to the system (cf. (4.4)–(4.52))

$$\begin{aligned}
& r_{\Gamma_m} \left[\mathcal{R}_\tau^{(m)} \mathcal{I}_4(T_0^{(m)}) \psi^{(m)} \right]_j - r_{\Gamma_m} \left[\mathcal{R}_\tau \left(\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right)^\top \right]_j = \tilde{F}_j^{(m)} \\
& \quad \text{on } \Gamma_m, \quad m = \overline{1, 2N}, \quad j = 1, 2, 3, \\
& r_{\Gamma_m} \left[\frac{1}{\tau} \mathcal{R}_\tau^{(m)} \mathcal{I}_4(T_0^{(m)}) \psi^{(m)} \right]_4 - r_{\Gamma_m} \left[\frac{1}{\tau} \mathcal{R}_\tau \left(\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right)^\top \right]_4 = \tilde{F}_4^{(m)} \\
& \quad \text{on } \Gamma_m, \quad m = \overline{1, 2N}, \\
& r_{\Sigma_3^-} \left[\mathcal{R}_\tau \left(\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right)^\top \right]_j = \tilde{F}_j \quad \text{on } \Sigma_3^-, \quad j = 1, 2, 3, \\
& r_{\Sigma_3^-} \left[\frac{1}{\tau T_0} \mathcal{R}_\tau \left(\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right)^\top \right]_4 = \tilde{F}_4 \quad \text{on } \Sigma_3^-,
\end{aligned}$$

$$r_{\Sigma_1^\pm \cup \Sigma_3^\pm} \left\{ \left[\mathcal{B}_\tau \left(\sum_{l=1}^{2N} \mathcal{I}_4(T_0) \psi^{(l)} + \psi, \psi_5 \right)^\top \right]_5 + \beta \psi_5 \right\} = \tilde{F}_5 \quad \text{on } \Sigma_1^\pm \cup \Sigma_3^\pm.$$

Therefore, the mapping and Fredholm properties of the operators \mathcal{N}_τ and $\tilde{\mathcal{N}}_\tau$ are absolutely the same. In particular, the invertibility of $\tilde{\mathcal{N}}_\tau$ yields the same property for the operator \mathcal{N}_τ .

Step 2. Here we give the estimate from below for Q_1 . Let us note that for the function u_5 the $H_2^1(\Omega)$ norm is equivalent to the semi-norm [30]

$$\sum_{j=1}^3 \|\partial_j u_5\|_{L_2(\Omega)}$$

since the support of u_5 is contained in $\Sigma_1^\pm \cup \Sigma_3^\pm$ which is a proper submanifold of $\partial\Omega$.

Note also that the vector

$$\tilde{u} := \begin{cases} u^{(m)} & \text{in } \Omega_m, \quad m = \overline{1, 2N} \\ u & \text{in } \Omega \end{cases}$$

belongs to the space $[H_2^1(\Pi)]^3$ due to the transmission conditions (2.12) on Γ_m and vanishes on Σ_3^- .

Therefore, due to the inequalities (A.7), (A.39), (A.40), (A.41), and the well known Korn's inequality we derive

$$\begin{aligned} Q_1 = \Re\{Q_1 + iQ_2\} &= \sum_{m=1}^{2N} \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \varrho^{(m)}(\sigma^2 - \omega^2)|u^{(m)}|^2 + \right. \\ &\quad \left. + \frac{\sigma}{|\tau|^2 T_0^{(m)}} \varkappa_{jl}^{(m)} \partial_j u_4^{(m)} \overline{\partial_l u_4^{(m)}} + \frac{\alpha^{(m)}}{T_0^{(m)}} |u_4^{(m)}|^2 \right] dx + \\ &\quad + \int_{\Omega} \left[E(u, \bar{u}) + \varrho(\sigma^2 - \omega^2)|u|^2 + \frac{\sigma}{|\tau|^2 T_0} \varkappa_{jl} \partial_j u_4 \overline{\partial_l u_4} + \frac{\alpha}{T_0} |u_4|^2 - \right. \\ &\quad \left. - 2\Re\{\ell_l \partial_l u_5 \bar{u}_4\} + \varepsilon_{jl} \overline{\partial_j u_5} \partial_l u_5 \right] dx \geq \\ &\geq c_1(\tau) \left[\sum_{m=1}^{2N} \|U^{(m)}\|_{[H_2^1(\Omega_m)]^4} + \|U\|_{[H_2^1(\Omega)]^5} \right] + \\ &\quad + \varrho^{(m)}(\sigma^2 - \omega^2) \sum_{m=1}^{2N} \int_{\Omega_m} |u^{(m)}|^2 dx + \varrho(\sigma^2 - \omega^2) \int_{\Omega} |u|^2 dx. \end{aligned}$$

Here and in what follows $c_k(\tau)$ are some positive constants independent of $U^{(m)}$ and U .

Provided that $\sigma^2 - \omega^2 \geq 0$ we arrive at the relation

$$Q_1 \geq c_1(\tau) \left[\sum_{m=1}^{2N} \|U^{(m)}\|_{[H_2^1(\Omega_m)]^4} + \|U\|_{[H_2^1(\Omega)]^5} \right].$$

Whence by the trace theorem

$$Q_1 \geq c_2(\tau) \left[\sum_{m=1}^{2N} \|\{U^{(m)}\}^+\|_{[H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4} + \|\{U\}^+\|_{[H_2^{-\frac{1}{2}}(\partial\Omega)]^5} \right].$$

From this inequality, with the help of the representations (4.45) and (4.46) with $\Phi_5 = 0$ and the invertibility properties of the operators $\mathcal{H}_\tau^{(m)}$, \mathcal{H}_τ , \mathcal{P}_τ , and $-\frac{1}{2}I_4 + \mathcal{K}_\tau^{(m)}$ (see Theorems 4.1, 4.2 and Lemma 4.4), we conclude

$$Q_1 \geq c_3(\tau) \|\Psi\|_{\mathbf{X}}^2. \quad (4.58)$$

Step 3. Taking into account that $\beta < 0$, from the relations (4.57) and (4.58) we finally get

$$\Re \langle \tilde{\mathcal{N}}_\tau \Psi, \Psi \rangle \geq Q_1 \geq c_3(\tau) \|\Psi\|_{\mathbf{X}}^2 \text{ for all } \Psi \in \mathbf{X}. \quad (4.59)$$

From this inequality it follows that the operator

$$\tilde{\mathcal{N}}_\tau : \mathbf{X} \rightarrow \mathbf{X}^* \quad (4.60)$$

is invertible.

Thus, the operator (4.56) is also invertible for $\sigma^2 - \omega^2 \geq 0$ due to the above mentioned equivalence of the operators $\tilde{\mathcal{N}}_\tau$ and \mathcal{N}_τ .

Step 4. Now let τ be an arbitrary complex number with $\Re \tau = \sigma > 0$. It can easily be seen that the entries of the differences of the fundamental matrices $\Psi^{(m)}(x-y, \tau) - \Psi^{(m)}(x-y, \tau_0)$ and $\Psi(x-y, \tau) - \Psi(x-y, \tau_0)$ either have a logarithmic singularity or are bounded for arbitrary τ_0 with $\Re \tau_0 = \sigma_0 > 0$ (see Appendix C). Therefore, the operators

$$\begin{aligned} \mathcal{H}_\tau^{(m)} - \mathcal{H}_{\tau_0}^{(m)} &: [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mathcal{K}_\tau^{(m)} - \mathcal{K}_{\tau_0}^{(m)} &: [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mathcal{H}_\tau - \mathcal{H}_{\tau_0} &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5, \\ \mathcal{K}_\tau - \mathcal{K}_{\tau_0} &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^5, \\ \mathcal{P}_\tau - \mathcal{P}_{\tau_0} &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \end{aligned}$$

are compact.

Clearly the differences of the corresponding inverse operators (when they exist) are also compact. Indeed, if the operators $\mathcal{A}, \mathcal{B} : \mathbf{B}_1 \rightarrow \mathbf{B}_2$ are invertible and $\mathcal{A} - \mathcal{B} : \mathbf{B}_1 \rightarrow \mathbf{B}_2$ is compact, then the compactness of the operator $\mathcal{A}^{-1} - \mathcal{B}^{-1} \equiv \mathcal{A}^{-1}(\mathcal{B} - \mathcal{A})\mathcal{B}^{-1} : \mathbf{B}_2 \rightarrow \mathbf{B}_1$ follows immediately.

From these results it immediately follows that the operators

$$\begin{aligned} \mathcal{R}_\tau^{(m)} - \mathcal{R}_{\tau_0}^{(m)} &: [H_2^{-\frac{1}{2}}(\partial\Omega_m)]^4 \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega_m)]^4, \\ \mathcal{R}_\tau - \mathcal{R}_{\tau_0} &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \rightarrow [H_2^{\frac{1}{2}}(\partial\Omega)]^5, \\ \mathcal{B}_\tau - \mathcal{B}_{\tau_0} &: [H_2^{-\frac{1}{2}}(\partial\Omega)]^4 \times H_2^{\frac{1}{2}}(\partial\Omega) \rightarrow [H_2^{-\frac{1}{2}}(\partial\Omega)]^5 \end{aligned}$$

are compact.

Having these results in hand, we can easily conclude that the operator $\mathcal{N}_\tau - \mathcal{N}_{\tau_0} : \mathbf{X} \rightarrow \mathbf{X}^*$ is compact. Now let us choose $\tau_0 = \sigma_0 + i\omega_0$ such that $\sigma_0^2 - \omega_0^2 \geq 0$. The operator $\mathcal{N}_{\tau_0} : \mathbf{X} \rightarrow \mathbf{X}^*$ is then invertible and the operator $\mathcal{N}_\tau = \mathcal{N}_{\tau_0} + (\mathcal{N}_\tau - \mathcal{N}_{\tau_0})$ represents a compact perturbation of the invertible operator. Therefore, its index equals to zero and the injectivity implies the surjectivity. This shows that $\mathcal{N}_\tau(\mathbf{X}) = \mathbf{X}^*$ and, consequently, (4.56) is invertible. The proof is complete. \square

Now from Theorem 4.6 the existence result for the original mixed boundary transmission problem follows directly.

Theorem 4.7. *The MBTP (2.2)–(2.14) has a unique solution*

$$(U^{(1)}, U^{(2)}, \dots, U^{(2N)}, U)$$

which can be represented in the form of single layer potentials (4.45)–(4.46), where the unknown densities $(\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(2N)}, \psi, \psi_5)$ solve the system (4.4)–(4.52).

5. NUMERICAL ALGORITHMS

In this section we describe the standard finite element approximation of solutions to the boundary transmission problem (2.2)–(2.14). Our consideration relies on the weak formulation of the problem given in Section 3.

5.1. Finite element approximation. Let us recall the weak setting of the mixed boundary transmission problem given by the equation (3.28). Under the notation introduced in Section 3, it reads as follows:

Find a vector $\tilde{\mathbf{U}} \in \mathcal{V}_N^1$ such that

$$\mathcal{A}(\tilde{\mathbf{U}}, \mathbf{V}) + \mathcal{B}(\tilde{\mathbf{U}}, \mathbf{V}) = \mathcal{F}(\mathbf{V}) \quad \text{for all } \mathbf{V} \in \mathcal{V}_N^1, \quad (5.1)$$

where \mathcal{A} , \mathcal{B} , and \mathcal{F} are defined by (3.22), (3.29) and (3.30), respectively.

If $\tau = \sigma + i\omega$ and $\sigma > 0$, then this problem possesses a unique solution due to Theorem 3.2.

Now we describe the discrete counterpart of the problem.

Let us divide the parallelepiped Π into the small parallelepipeds (elements) Π_α of dimension $l_{\alpha_1} \times l_{\alpha_2} \times l_{\alpha_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. We assume that for some $p > 0$

$$p^{-1} \leq l_{\alpha_i} / l_{\alpha_j} \leq p, \quad i, j = 1, 2, 3.$$

Denote $h := \max_i \sup_{\alpha_i} l_{\alpha_i}$.

Let $\mathcal{V}_{N,h}^1 \subset C(\Pi)$ be the subspace of \mathcal{V}_N^1 consisting of the continuous functions whose restrictions on each element Π_α represent a linear combination of first order polynomials. It can be easily proved that $\bigcup_h \mathcal{V}_{N,h}^1$ is dense in \mathcal{V}_N^1 .

Consider the equation (3.28) in the finite-dimensional space $\mathcal{V}_{N,h}^1$:

$$\mathcal{A}(\tilde{\mathbf{U}}_h, \mathbf{V}_h) + \mathcal{B}(\tilde{\mathbf{U}}_h, \mathbf{V}_h) = \mathcal{F}(\mathbf{V}_h) \quad \text{for all } \mathbf{V}_h \in \mathcal{V}_{N,h}^1. \quad (5.2)$$

Theorem 5.1. *The equation (5.1) has the unique solution $\tilde{\mathbf{U}}_h \in \mathcal{V}_{N,h}^1$ for all $h > 0$. This solution converges in \mathcal{V}_N^1 to the solution $\tilde{\mathbf{U}}$ of (5.1) as $h \rightarrow 0$.*

Proof. Let us replace the fifth equation of (2.3) by its complex conjugate in the mixed boundary-transmission problem formulation (2.2)–(2.14) and repeat word for word the considerations adduced in Section 3. Instead of (3.28) (that is, (5.1)) we arrive then at the relation:

$$\mathcal{A}_c(\tilde{\mathbf{U}}_c, \mathbf{V}) + \mathcal{B}_c(\tilde{\mathbf{U}}_c, \mathbf{V}) = \mathcal{F}_c(\mathbf{V}) \text{ for all } \mathbf{V} \in \mathcal{V}_N^1. \quad (5.3)$$

Here \mathcal{A}_c , \mathcal{B}_c , and \mathcal{F}_c are properly modified expressions \mathcal{A} , \mathcal{B} , and \mathcal{F} , respectively.

The equations (5.1) and (5.3) are equivalent in the following sense:

$$\tilde{\mathbf{U}} := (U_1^{(1)}, \dots, U_4^{(1)}, \dots, U_1^{(2N)}, \dots, U_4^{(2N)}, U_1, U_2, U_3, U_4, U_5)$$

is a solution of the equation (5.1) if and only if

$$\tilde{\mathbf{U}}_c := (U_{c1}^{(1)}, \dots, U_{c4}^{(1)}, \dots, U_{c1}^{(2N)}, \dots, U_{c4}^{(2N)}, U_{c1}, U_{c2}, U_{c3}, \overline{U_{c4}}, U_{c5})$$

is a solution of the equation (5.3).

Note that the equation (5.2) is not linear due to the complex conjugation operation involved.

For each τ with $\Re\tau > 0$ we can choose a positive number $c_1(\tau)$ such that

$$c_1^{-1} \|\tilde{\mathbf{U}}_c\|_{\mathcal{V}_N^1}^2 \leq \mathcal{A}_c(\tilde{\mathbf{U}}_c, \tilde{\mathbf{U}}_c) + \mathcal{B}_c(\tilde{\mathbf{U}}_c, \tilde{\mathbf{U}}_c) \text{ for all } \tilde{\mathbf{U}}_c \in \mathcal{V}_N^1. \quad (5.4)$$

This inequality can be proved in the same way as the inequality (3.31).

Let $\tilde{\mathbf{U}}_h$ be the solution of the homogeneous equation (5.2):

$$\mathcal{A}(\tilde{\mathbf{U}}_h, \mathbf{V}_h) + \mathcal{B}(\tilde{\mathbf{U}}_h, \mathbf{V}_h) = 0 \text{ for all } \mathbf{V}_h \in \mathcal{V}_{N,h}^1.$$

Then due to (5.4) $\|\tilde{\mathbf{U}}_{ch}\|_{\mathcal{V}_N^1} = 0$ and $\tilde{\mathbf{U}}_h = 0$. Therefore the equation (5.2) has a unique solution which due to (5.4), (5.3) satisfies the inequality

$$\begin{aligned} c_1^{-1} \|\tilde{\mathbf{U}}_h\|_{\mathcal{V}_N^1}^2 &= c_1^{-1} \|\tilde{\mathbf{U}}_{ch}\|_{\mathcal{V}_N^1}^2 \leq \mathcal{A}_c(\tilde{\mathbf{U}}_{ch}, \tilde{\mathbf{U}}_{ch}) + \mathcal{B}_c(\tilde{\mathbf{U}}_{ch}, \tilde{\mathbf{U}}_{ch}) = \\ &= \mathcal{F}_c(\tilde{\mathbf{U}}_{ch}) \leq c_2 \|\tilde{\mathbf{U}}_{ch}\|_{\mathcal{V}_N^1} = c_2 \|\tilde{\mathbf{U}}_h\|_{\mathcal{V}_N^1} \end{aligned}$$

for some positive c_2 independent of h .

Hence the sequence $\{\|\tilde{\mathbf{U}}_h\|_{\mathcal{V}_N^1}\}$ is bounded and we can extract a subsequence $\{\tilde{\mathbf{U}}_{h_k}\}$ which weakly converges to some $\mathbf{W} \in \mathcal{V}_N^1$.

Let us take arbitrary $\mathbf{V} \in \mathcal{V}_N^1$ and for each $h > 0$ choose $\mathbf{V}_h \in \mathcal{V}_{N,h}^1$ such that $\mathbf{V}_h \rightarrow \mathbf{V}$ in \mathcal{V}_N^1 . From (5.2) we then have

$$\mathcal{A}(\mathbf{W}, \mathbf{V}) + \mathcal{B}(\mathbf{W}, \mathbf{V}) = \mathcal{F}(\mathbf{V}).$$

Hence \mathbf{W} solves (5.1). Note that since each subsequence converges weakly to the same solution \mathbf{W} , the whole sequence $\{\tilde{\mathbf{U}}_h\}$ also converges weakly to $\mathbf{W} = \tilde{\mathbf{U}}$. Now let us prove that it converges in the space \mathcal{V}_N^1 .

Denote $\mathcal{A}_c^{(1)}(\mathbf{U}, \mathbf{V}) := \mathcal{A}_c(\mathbf{U}, \mathbf{V}) + \mathcal{B}_c(\mathbf{U}, \mathbf{V})$. Due to (5.1)–(5.4), we have:

$$\begin{aligned}
c_1^{-1} \|\tilde{\mathbf{U}}_h - \tilde{\mathbf{U}}\|_{\mathcal{V}_N^1}^2 &= c_1^{-1} \|\tilde{\mathbf{U}}_{ch} - \tilde{\mathbf{U}}_c\|_{\mathcal{V}_N^1}^2 \leq |\mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_{ch} - \tilde{\mathbf{U}}_c, \tilde{\mathbf{U}}_{ch} - \tilde{\mathbf{U}}_c)| = \\
&= \left| \mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_{ch}, \tilde{\mathbf{U}}_{ch}) - \mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_{ch}, \tilde{\mathbf{U}}_c) - \mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_c, \tilde{\mathbf{U}}_{ch} - \tilde{\mathbf{U}}_c) \right| = \\
&= \left| \mathcal{F}_c(\tilde{\mathbf{U}}_{ch}) - \mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_{ch}, \tilde{\mathbf{U}}_c) - \mathcal{F}_c(\tilde{\mathbf{U}}_{ch} - \tilde{\mathbf{U}}_c) \right| \rightarrow \\
&\quad \rightarrow |\mathcal{F}_c(\tilde{\mathbf{U}}_c) - \mathcal{A}_c^{(1)}(\tilde{\mathbf{U}}_c, \tilde{\mathbf{U}}_c)| = 0,
\end{aligned}$$

which completes the proof. \square

6. APPENDIX A: FIELD EQUATIONS

6.1. Thermoelastic field equations in Ω_m . Here we collect the field equations of the linear theory of thermoelasticity and introduce the corresponding matrix partial differential operators (cf. [33], [19]).

6.1.1. General anisotropy. The basic governing equations of the classical thermoelasticity read as follows (see the list of notation and take into consideration the symmetry condition (A.6) below):

Constitutive relations:

$$\sigma_{ij}^{(m)} = \sigma_{ji}^{(m)} = c_{ijkl}^{(m)} s_{lk}^{(m)} - \gamma_{ij}^{(m)} \vartheta^{(m)} = c_{ijkl}^{(m)} \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \vartheta^{(m)}, \quad (\text{A.1})$$

$$\mathcal{S}^{(m)} = \gamma_{ij}^{(m)} s_{ij}^{(m)} + \alpha^{(m)} [T_0^{(m)}]^{-1} \vartheta^{(m)}; \quad (\text{A.2})$$

Fourier Law:

$$q_j^{(m)} = -\varkappa_{jl}^{(m)} \partial_l T^{(m)}; \quad (\text{A.3})$$

Equations of motion:

$$\partial_i \sigma_{ij}^{(m)} + X_j^{(m)} = \rho^{(m)} \partial_t^2 u_j^{(m)}; \quad (\text{A.4})$$

Equation of the entropy balance:

$$T^{(m)} \partial_t \mathcal{S}^{(m)} = -\partial_j q_j^{(m)} + X_4^{(m)}. \quad (\text{A.5})$$

Constants involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)}, \quad \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)}, \quad \varkappa_{ij}^{(m)} = \varkappa_{ji}^{(m)}, \quad i, j, k, l = 1, 2, 3. \quad (\text{A.6})$$

Moreover, we assume that there are positive constants c_0 and c_1 such that

$$c_{ijkl}^{(m)} \xi_{ij} \xi_{kl} \geq c_0 \xi_{ij} \xi_{ij}, \quad \varkappa_{ij}^{(m)} \xi_i \xi_j \geq c_1 \xi_i \xi_i \quad (\text{A.7})$$

for all $\xi_{ij} = \xi_{ji}$, $\xi_j \in \mathbb{R}$, $i, j = 1, 2, 3$.

In particular, the first inequality implies that the density of potential energy corresponding to the displacement vector $u^{(m)}$,

$$E^{(m)}(u^{(m)}, u^{(m)}) = c_{ijkl}^{(m)} s_{ij}^{(m)} s_{lk}^{(m)} \quad (\text{A.8})$$

is positive definite with respect to the symmetric components of the strain tensor

$$s_{lk}^{(m)} = 2^{-1} (\partial_l u_k^{(m)} + \partial_k u_l^{(m)}).$$

Substitution of (A.1) into (A.4) leads to the equation:

$$c_{ijkl}^{(m)} \partial_i \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \partial_i \vartheta^{(m)} + X_j^{(m)} = \varrho^{(m)} \partial_t^2 u_j^{(m)}, \quad j = 1, 2, 3. \quad (\text{A.9})$$

Taking into account the Fourier law (A.3) and the relation (A.2) from the equation of the entropy balance (A.5), we obtain the heat transfer equation

$$\begin{aligned} \varkappa_{jl}^{(m)} \partial_j \partial_l \vartheta^{(m)} + X_4^{(m)} &= \\ &= T^{(m)} (\gamma_{jl}^{(m)} \partial_t s_{jl}^{(m)} + \alpha^{(m)} [T_0^{(m)}]^{-1} \partial_t \vartheta^{(m)}), \quad j = 1, 2, 3. \end{aligned} \quad (\text{A.10})$$

Assuming that $|\vartheta^{(m)}/T_0^{(m)}| \ll 1$ and taking into consideration the equality $T^{(m)} = T_0^{(m)}(1 + \vartheta^{(m)}/T_0^{(m)})$, we can linearize the equation (A.10):

$$\varkappa_{il}^{(m)} \partial_i \partial_l \vartheta^{(m)} - \alpha^{(m)} \partial_t \vartheta^{(m)} - T_0^{(m)} \gamma_{il}^{(m)} \partial_t \partial_l u_i^{(m)} + X_4^{(m)} = 0. \quad (\text{A.11})$$

The simultaneous equations (A.9) and (A.11) represent the basic system of dynamics of the theory of thermoelasticity. If all the functions involved in these equations are harmonic time dependent with the multiplier $\exp\{\tau t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the theory of thermoelasticity. If $\tau = i\omega$ is a pure imaginary number, with the so called oscillation parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$, we get the *equations of statics*.

Combining all these cases, we arrive at the equations (cf. (A.9) and (A.11))

$$c_{ijkl}^{(m)} \partial_i \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \partial_i \vartheta^{(m)} + X_j^{(m)} = \begin{cases} \varrho^{(m)} \partial_t^2 u_j^{(m)}, \\ \varrho^{(m)} \tau^2 u_j^{(m)}, \\ 0, \end{cases} \quad j = 1, 2, 3, \quad (\text{A.12})$$

$$\varkappa_{il}^{(m)} \partial_i \partial_l \vartheta^{(m)} + X_4^{(m)} = \begin{cases} \alpha^{(m)} \partial_t \vartheta^{(m)} + T_0^{(m)} \gamma_{il}^{(m)} \partial_t \partial_l u_i^{(m)}, \\ \tau \alpha^{(m)} \vartheta^{(m)} + \tau T_0^{(m)} \gamma_{il}^{(m)} \partial_l u_i^{(m)}, \\ 0. \end{cases} \quad (\text{A.13})$$

We will consider the system of pseudo-oscillations

$$\begin{aligned} c_{ijkl}^{(m)} \partial_i \partial_l u_k^{(m)} - \varrho^{(m)} \tau^2 u_j^{(m)} - \gamma_{ij}^{(m)} \partial_i \vartheta^{(m)} + X_j^{(m)} &= 0, \quad j = 1, 2, 3, \\ -\tau T_0^{(m)} \gamma_{il}^{(m)} \partial_l u_i^{(m)} + \varkappa_{il}^{(m)} \partial_i \partial_l \vartheta^{(m)} - \tau \alpha^{(m)} \vartheta^{(m)} + X_4^{(m)} &= 0, \end{aligned} \quad (\text{A.14})$$

which in matrix form can be rewritten as

$$A^{(m)}(\partial_x, \tau) U^{(m)}(x) + \tilde{X}^{(m)}(x) = 0 \quad \text{in } \Omega_m, \quad (\text{A.15})$$

where $U^{(m)} := (u^{(m)}, \vartheta^{(m)})^\top$ is the sought vector,

$$\tilde{X}^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)}, X_4^{(m)})^\top, \quad X^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)})^\top$$

is a given mass force density, $X_4^{(m)}$ is a given heat source density, $A^{(m)}(\partial_x, \tau)$ is the matrix differential operator generated by the equations (A.14)

$$\begin{aligned} A^{(m)}(\partial_x, \tau) &= [A_{jk}^{(m)}(\partial_x, \tau)]_{4 \times 4}, \quad A_{jk}^{(m)}(\partial_x, \tau) = c_{ijkl}^{(m)} \partial_i \partial_l - \varrho^{(m)} \tau^2 \delta_{jk}, \\ A_{4k}^{(m)}(\partial_x, \tau) &= -\tau T_0^{(m)} \gamma_{kl}^{(m)} \partial_l, \quad A_{j4}^{(m)}(\partial_x, \tau) = -\gamma_{ij}^{(m)} \partial_i, \end{aligned} \quad (\text{A.16})$$

$$A_{44}^{(m)}(\partial_x, \tau) = \varkappa_{il}^{(m)} \partial_i \partial_l - \alpha^{(m)} \tau,$$

where $j, k = 1, 2, 3$, and δ_{jk} is the Kronecker delta.

By $[A^{(m)}(\partial, \tau)]^*$ we denote the 4×4 matrix differential operator formally adjoint to $A^{(m)}(\partial, \tau)$: $[A^{(m)}(\partial, \tau)]^* = \overline{[A^{(m)}(-\partial, \tau)]^\top}$, where the over-bar denotes the complex conjugation.

With the help of the inequalities (A.7) it can easily be shown that $A^{(m)}(\partial, \tau)$ is an elliptic operator with a positive definite principal homogeneous symbol matrix.

Components of the mechanical thermostress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ read as follows

$$\sigma_{ij}^{(m)} n_i = c_{ijkl}^{(m)} n_i \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} n_i \vartheta^{(m)}, \quad j = 1, 2, 3, \quad (\text{A.17})$$

while the normal component of the heat flux vector (with opposite sign) has the form

$$-q_i^{(m)} n_i = \varkappa_{il}^{(m)} n_i \partial_l \vartheta^{(m)}. \quad (\text{A.18})$$

We introduce the following generalized thermostress operator

$$\mathcal{T}^{(m)}(\partial, n) = [\mathcal{T}_{jk}^{(m)}(\partial, n)]_{4 \times 4}, \quad (\text{A.19})$$

where (for $j, k = 1, 2, 3$)

$$\begin{aligned} \mathcal{T}_{jk}^{(m)}(\partial, n) &= c_{ijkl}^{(m)} n_i \partial_l, & \mathcal{T}_{j4}^{(m)}(\partial, n) &= -\gamma_{ij}^{(m)} n_i, \\ \mathcal{T}_{4k}^{(m)}(\partial, n) &= 0, & \mathcal{T}_{44}^{(m)}(\partial, n) &= \varkappa_{il}^{(m)} n_i \partial_l. \end{aligned}$$

For the four-vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$ we have

$$\mathcal{T}^{(m)} U^{(m)} = (\sigma_{i1}^{(m)} n_i, \sigma_{i2}^{(m)} n_i, \sigma_{i3}^{(m)} n_i, -q_i^{(m)} n_i)^\top. \quad (\text{A.20})$$

Clearly, the components of the vector $\mathcal{T}^{(m)} U^{(m)}$ given by (A.20) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelasticity, while the fourth one is the normal component of the heat flux vector (with opposite sign).

We introduce also the boundary operator associated with the adjoint operator $[A^{(m)}(\partial, \tau)]^*$ which appears in Green's formulae:

$$\tilde{\mathcal{T}}^{(m)}(\partial, n, \tau) = [\tilde{\mathcal{T}}_{jk}^{(m)}(\partial, n, \tau)]_{4 \times 4}, \quad (\text{A.21})$$

where (for $j, k = 1, 2, 3$)

$$\begin{aligned} \tilde{\mathcal{T}}_{jk}^{(m)}(\partial, n, \tau) &= c_{ijkl}^{(m)} n_i \partial_l, & \tilde{\mathcal{T}}_{j4}^{(m)}(\partial, n, \tau) &= \bar{\tau} \Gamma_0^{(m)} \gamma_{ij}^{(m)} n_i, \\ \tilde{\mathcal{T}}_{4k}^{(m)}(\partial, n, \tau) &= 0, & \tilde{\mathcal{T}}_{44}^{(m)}(\partial, n, \tau) &= \varkappa_{il}^{(m)} n_i \partial_l. \end{aligned} \quad (\text{A.22})$$

6.1.2. *Isotropic bodies.* For an isotropic medium the thermomechanical coefficients are

$$\begin{aligned} c_{ijkl}^{(m)} &= \lambda^{(m)} \delta_{ij} \delta_{lk} + \mu^{(m)} (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}), \\ \gamma_{ij}^{(m)} &:= \gamma^{(m)} \delta_{ij}, \quad \varkappa_{ij}^{(m)} = \varkappa^{(m)} \delta_{ij}, \end{aligned} \quad (\text{A.23})$$

and the basic equations (A.14) of the theory of thermoelasticity are written in the form (see, e.g., [33], [19]):

$$\begin{aligned} \mu^{(m)} \Delta u^{(m)} + (\lambda^{(m)} + \mu^{(m)}) \text{grad div } u^{(m)} - \\ - \gamma^{(m)} \text{grad } \vartheta^{(m)} - \tau^2 \varrho^{(m)} u^{(m)} + X^{(m)} &= 0, \\ \varkappa^{(m)} \Delta \vartheta^{(m)} - \tau \alpha^{(m)} \vartheta - \tau T_0^{(m)} \gamma^{(m)} \text{div } u^{(m)} + X_4^{(m)} &= 0, \end{aligned} \quad (\text{A.24})$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

The matrix differential operator generated by these equations is (cf. (6.1.1))

$$\begin{aligned} A^{(m)}(\partial, \tau) &= [A_{jk}^{(m)}(\partial, \tau)]_{4 \times 4}, \\ A_{jk}^{(m)}(\partial, \tau) &= \delta_{jk} (\mu^{(m)} \Delta - \varrho^{(m)} \tau^2) + (\lambda^{(m)} + \mu^{(m)}) \partial_j \partial_k, \\ A_{j4}^{(m)}(\partial, \tau) &= -\gamma^{(m)} \partial_j, \\ A_{4k}^{(m)}(\partial, \tau) &= -\tau T_0^{(m)} \gamma^{(m)} \partial_k, \quad A_{44}^{(m)}(\partial, \tau) = \varkappa^{(m)} \Delta - \tau \alpha^{(m)}, \end{aligned} \quad (\text{A.25})$$

for $j, k = 1, 2, 3$, while the corresponding thermostress operator reads as

$$\mathcal{T}^{(m)}(\partial, n) = [\mathcal{T}_{jk}^{(m)}(\partial, n)]_{4 \times 4} \quad (\text{A.26})$$

with

$$\begin{aligned} \mathcal{T}_{jk}^{(m)}(\partial, n) &= \lambda^{(m)} n_j \partial_k + \mu^{(m)} n_k \partial_j + \mu^{(m)} \delta_{jk} \partial_n, \quad \mathcal{T}_{4k}^{(m)}(\partial, n) = 0, \\ \mathcal{T}_{j4}^{(m)}(\partial, n) &= -\gamma^{(m)} n_j, \quad \mathcal{T}_{44}^{(m)}(\partial, n) = \varkappa^{(m)} \partial_n, \quad j, k = 1, 2, 3. \end{aligned} \quad (\text{A.27})$$

Here $\partial_n = n_i \partial_i$ denotes the usual normal derivative.

Clearly, in this case we have $\tilde{\mathcal{T}}^{(m)}(\partial, n, \tau) = [\tilde{\mathcal{T}}_{jk}^{(m)}(\partial, n, \tau)]_{4 \times 4}$, where (cf. (A.21)–(A.22))

$$\begin{aligned} \tilde{\mathcal{T}}_{jk}^{(m)}(\partial, n, \tau) &= \lambda^{(m)} n_j \partial_k + \mu^{(m)} n_k \partial_j + \mu^{(m)} \delta_{jk} \partial_n, \quad \tilde{\mathcal{T}}_{4k}^{(m)}(\partial, n, \tau) = 0, \\ \tilde{\mathcal{T}}_{j4}^{(m)}(\partial, n, \tau) &= \bar{\tau} T_0^{(m)} \gamma^{(m)} n_j, \quad \tilde{\mathcal{T}}_{44}^{(m)}(\partial, n, \tau) = \varkappa^{(m)} \partial_n, \quad j, k = 1, 2, 3. \end{aligned}$$

6.2. Thermopiezoelastic field equations in Ω . In this subsection we collect the field equations of the linear theory of thermopiezoelectricity and introduce the corresponding matrix partial differential operators (cf. [31], [36]).

6.2.1. *General anisotropy.* In the thermopiezoelasticity we have the following governing equations (see the list of notation):

Constitutive relations:

$$\begin{aligned}\sigma_{ij} &= \sigma_{ji} = c_{ijkl}s_{kl} - e_{lij}E_l - \gamma_{ij}\vartheta = \\ &= c_{ijkl}\partial_l u_k + e_{lij}\partial_l \varphi - \gamma_{ij}\vartheta, \quad i, j = 1, 2, 3,\end{aligned}\quad (\text{A.28})$$

$$\mathcal{S} = \gamma_{ij}s_{ij} + g_l E_l + \alpha[T_0]^{-1}\vartheta, \quad (\text{A.29})$$

$$\begin{aligned}D_j &= e_{jkl}s_{kl} + \varepsilon_{jl}E_l + g_j\vartheta = \\ &= e_{jkl}\partial_l u_k - \varepsilon_{jl}\partial_l \varphi + g_j\vartheta, \quad j = 1, 2, 3.\end{aligned}\quad (\text{A.30})$$

Fourier Law:

$$q_i = -\varkappa_{il}\partial_l T, \quad i = 1, 2, 3. \quad (\text{A.31})$$

Equations of motion:

$$\partial_i \sigma_{ji} + X_j = \rho \partial_t^2 u_j, \quad j = 1, 2, 3. \quad (\text{A.32})$$

Equation of the entropy balance:

$$T \partial_t \mathcal{S} = -\partial_j q_j + X_4. \quad (\text{A.33})$$

Equation of static electric field:

$$\partial_i D_i - X_5 = 0. \quad (\text{A.34})$$

From the relations (A.28)–(A.34) we derive the linear system of dynamics of the theory of thermopiezoelasticity:

$$\begin{aligned}c_{ijkl}\partial_i \partial_l u_k - \gamma_{ij}\partial_i \vartheta + e_{lij}\partial_l \partial_i \varphi + X_j &= \rho \partial_t^2 u_j, \quad j = 1, 2, 3, \\ -T_0 \gamma_{il} \partial_t \partial_l u_i + \varkappa_{il} \partial_i \partial_l \vartheta - \alpha \partial_t \vartheta + T_0 g_i \partial_t \partial_i \varphi + X_4 &= 0, \\ -e_{ikl} \partial_i \partial_l u_k - g_i \partial_i \vartheta + \varepsilon_{il} \partial_i \partial_l \varphi + X_5 &= 0.\end{aligned}\quad (\text{A.35})$$

In particular, the corresponding pseudo-oscillation equations read as

$$\begin{aligned}c_{ijkl}\partial_i \partial_l u_k - \rho \tau^2 u_j - \gamma_{ij}\partial_i \vartheta + e_{lij}\partial_l \partial_i \varphi + X_j &= 0, \quad j = 1, 2, 3, \\ -\tau T_0 \gamma_{il} \partial_l u_i + \varkappa_{il} \partial_i \partial_l \vartheta - \tau \alpha \vartheta + \tau T_0 g_i \partial_i \varphi + X_4 &= 0, \\ -e_{ikl} \partial_i \partial_l u_k - g_i \partial_i \vartheta + \varepsilon_{il} \partial_i \partial_l \varphi + X_5 &= 0,\end{aligned}\quad (\text{A.36})$$

or in matrix form

$$A(\partial, \tau)U(x) + \tilde{X}(x) = 0 \quad \text{in } \Omega, \quad (\text{A.37})$$

where $U := (u, \vartheta, \varphi)^\top$, $\tilde{X} = (X_1, X_2, X_3, X_4, X_5)^\top$, $X = (X_1, X_2, X_3)^\top$ is a given mass force density, X_4 is a given heat source density, X_5 is a given charge density, $A(\partial, \tau)$ is the matrix differential operator generated by the equations (A.36)

$$\begin{aligned}A(\partial, \tau) &= [A_{jk}(\partial, \tau)]_{5 \times 5}, \quad A_{jk}(\partial, \tau) = c_{ijkl}\partial_i \partial_l - \rho \tau^2 \delta_{jk}, \\ A_{j4}(\partial, \tau) &= -\gamma_{ij}\partial_i, \quad A_{j5}(\partial, \tau) = e_{lij}\partial_l \partial_i, \quad A_{4k}(\partial, \tau) = -\tau T_0 \gamma_{kl} \partial_l, \\ A_{44}(\partial, \tau) &= \varkappa_{il} \partial_i \partial_l - \alpha \tau, \quad A_{45}(\partial, \tau) = \tau T_0 g_i \partial_i, \quad A_{5k}(\partial, \tau) = -e_{ikl} \partial_i \partial_l, \\ A_{54}(\partial, \tau) &= -g_i \partial_i, \quad A_{55}(\partial, \tau) = \varepsilon_{il} \partial_i \partial_l, \quad j, k = 1, 2, 3.\end{aligned}\quad (\text{A.38})$$

Clearly, from (A.36)–(6.2.1) we obtain the equations and operators of statics if $\tau = 0$.

The constants involved in these equations satisfy the symmetry conditions:

$$\begin{aligned} c_{ijkl} &= c_{jikl} = c_{klij}, & e_{ijk} &= e_{ikj}, & \varepsilon_{ij} &= \varepsilon_{ji}, \\ \gamma_{ij} &= \gamma_{ji}, & \varkappa_{ij} &= \varkappa_{ji}, & i, j, k, l &= 1, 2, 3. \end{aligned}$$

Moreover, from the physical considerations it follows that (see, e.g., [31]):

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij} \quad \text{for all } \xi_{ij} = \xi_{ji} \in \mathbb{R}, \quad (\text{A.39})$$

$$\varepsilon_{ij}\eta_i\eta_j \geq c_1|\eta|^2, \quad \varkappa_{ij}\eta_i\eta_j \geq c_2|\eta|^2 \quad \text{for all } \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3, \quad (\text{A.40})$$

where c_0 , c_1 , and c_2 are positive constants. In addition, we require that (cf., e.g., [31])

$$\varepsilon_{ij}\eta_i\bar{\eta}_j + \frac{\alpha}{T_0}|\zeta|^2 - 2\Re(\zeta g_l \bar{\eta}_l) \geq c_3(|\zeta|^2 + |\eta|^2) \quad \text{for all } \zeta \in \mathbb{C} \text{ and } \eta \in \mathbb{C}^3 \quad (\text{A.41})$$

with a positive constant c_3 . A sufficient condition for (A.41) to be satisfied reads as follows

$$\frac{\alpha c_1}{3T_0} - g^2 > 0, \quad (\text{A.42})$$

where $g = \max\{|g_1|, |g_2|, |g_3|\}$ and c_1 is the constant involved in (A.40).

By $A^*(\partial, \tau)$ we denote the operator formally adjoint to $A(\partial, \tau)$: $A^*(\partial, \tau) = \overline{[A(-\partial, \tau)]}^\top$.

With the help of the inequalities (A.39) and (A.40), it can easily be shown that the principal part of the operator $A(\partial, \tau)$ is strongly elliptic, but not self-adjoint.

In the theory of thermopiezoelectricity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{ij}n_i = c_{ijkl}n_i\partial_l u_k + e_{ij}n_i\partial_l \varphi - \gamma_{ij}n_i\vartheta \quad \text{for } j = 1, 2, 3, \quad (\text{A.43})$$

while the normal components of the electric displacement vector and the heat flux vector (with opposite sign) read as

$$-D_i n_i = -e_{ikl}n_i\partial_l u_k + \varepsilon_{il}n_i\partial_l \varphi - g_i n_i \vartheta, \quad -q_i n_i = \varkappa_{il}n_i\partial_l \vartheta. \quad (\text{A.44})$$

Let us introduce the following matrix differential operator

$$\mathcal{T}(\partial, n) = [\mathcal{T}_{jk}(\partial, n)]_{5 \times 5}, \quad (\text{A.45})$$

where (for $j, k = 1, 2, 3$)

$$\begin{aligned} \mathcal{T}_{jk}(\partial, n) &= c_{ijkl}n_i\partial_l, & \mathcal{T}_{j4}(\partial, n) &= -\gamma_{ij}n_i, & \mathcal{T}_{j5}(\partial, n) &= e_{lij}n_i\partial_l, \\ \mathcal{T}_{4k}(\partial, n) &= 0, & \mathcal{T}_{44}(\partial, n) &= \varkappa_{il}n_i\partial_l, & \mathcal{T}_{45}(\partial, n) &= 0, \\ \mathcal{T}_{5k}(\partial, n) &= -e_{ikl}n_i\partial_l, & \mathcal{T}_{54}(\partial, n) &= -g_i n_i, & \mathcal{T}_{55}(\partial, n) &= \varepsilon_{il}n_i\partial_l. \end{aligned} \quad (\text{A.46})$$

For a vector $U = (u, \varphi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = (\sigma_{i1}n_i, \sigma_{i2}n_i, \sigma_{i3}n_i, -q_i n_i, -D_i n_i)^\top. \quad (\text{A.47})$$

Clearly, the components of the vector TU given by (A.47) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelectroelasticity, the fourth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

In Green's formulas there appear also the following boundary operator associated with the differential operator $A^*(\partial, \tau)$:

$$\tilde{T}(\partial, n, \tau) = [\tilde{T}_{jk}(\partial, n, \tau)]_{5 \times 5}, \quad (\text{A.48})$$

where (for $j, k = 1, 2, 3$)

$$\begin{aligned} \tilde{T}_{jk}(\partial, n, \tau) &= c_{ijkl} n_i \partial_l, & \tilde{T}_{j4}(\partial, n, \tau) &= \bar{\tau} T_0 \gamma_{ij} n_i, \\ \tilde{T}_{j5}(\partial, n, \tau) &= -e_{lij} n_i \partial_l, \\ \tilde{T}_{4k}(\partial, n, \tau) &= 0, & \tilde{T}_{44}(\partial, n, \tau) &= \varkappa_{il} n_i \partial_l, & \tilde{T}_{45}(\partial, n, \tau) &= 0, \\ \tilde{T}_{5k}(\partial, n, \tau) &= e_{ikl} n_i \partial_l, & \tilde{T}_{54}(\partial, n, \tau) &= -\bar{\tau} T_0 g_i n_i, & \tilde{T}_{55}(\partial, n, \tau) &= \varepsilon_{il} n_i \partial_l. \end{aligned} \quad (\text{A.49})$$

6.2.2. *Special classes of anisotropy (transversally isotropic case).* Consider a thermopiezoelectric medium with crystal symmetry of the class $\mathbf{C}_{6\nu} = \mathbf{6mm}$. In particular, piezoceramic materials belong to this class [10]. To simplify the notation, we introduce the standard two-index symbols:

$$c_{f(ij)f(kl)} := c_{ijkl}, \quad e_{if(kl)} := e_{ikl}, \quad \gamma_{f(ij)} := \gamma_{ij},$$

where

$$\begin{aligned} f(11) &= 1, & f(22) &= 2, & f(33) &= 3, \\ f(23) &= f(32) = 4, & f(13) &= f(31) = 5, & f(12) &= f(21) = 6. \end{aligned}$$

For crystals of the class $\mathbf{6mm}$ the following relations then hold:

$$\begin{aligned} c_{11} &= c_{22}, & c_{13} &= c_{23}, & c_{44} &= c_{55}, \\ c_{11} - c_{66} &= c_{66} + c_{12}, & c_{ij} &= 0 \text{ for } i \neq j \text{ and } i, j = 4, 5, 6; \\ e_{24} &= e_{15}, & e_{31} &= e_{32}, & e_{1i} &= e_{2j} = e_{3k} = 0 \text{ for } i \neq 5, j \neq 4, k > 3; \\ \varepsilon_{11} &= \varepsilon_{22}, & \varepsilon_{12} &= \varepsilon_{13} = \varepsilon_{23} = 0; \\ \varkappa_{11} &= \varkappa_{22}, & \varkappa_{12} &= \varkappa_{13} = \varkappa_{23} = 0; \\ \gamma_1 &= \gamma_2, & \gamma_4 &= \gamma_5 = \gamma_6 = 0; & g_1 &= g_2 = 0. \end{aligned}$$

In this case the equations (A.36) have the form:

$$\begin{aligned} (c_{11} \partial_1^2 + c_{66} \partial_2^2 + c_{44} \partial_3^2) u_1 &+ (c_{11} - c_{66}) \partial_1 \partial_2 u_2 + \\ &+ (c_{13} + c_{44}) \partial_1 \partial_3 u_3 - \gamma_1 \partial_1 \vartheta + \\ &+ (e_{31} + e_{15}) \partial_1 \partial_3 \varphi - \varrho \tau^2 u_1 + X_1 = 0, \\ (c_{11} - c_{66}) \partial_2 \partial_1 u_1 &+ (c_{66} \partial_1^2 + c_{11} \partial_2^2 + c_{44} \partial_3^2) u_2 + \\ &+ (c_{13} + c_{44}) \partial_2 \partial_3 u_3 - \gamma_1 \partial_2 \vartheta + \\ &+ (e_{31} + e_{15}) \partial_2 \partial_3 \varphi - \varrho \tau^2 u_2 + X_2 = 0, \end{aligned} \quad (\text{A.50})$$

$$\begin{aligned}
& (c_{13} + c_{44})\partial_3\partial_1u_1 + (c_{13} + c_{44})\partial_3\partial_2u_2 + \\
& + (c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2)u_3 - \gamma_3\partial_3\vartheta + \\
& + (e_{15}\partial_1^2 + e_{15}\partial_2^2 + e_{33}\partial_3^2)\varphi - \varrho\tau^2u_3 + X_3 = 0, \\
& -\tau T_0(\gamma_1\partial_1u_1 + \gamma_1\partial_2u_2 + \gamma_3\partial_3u_3) + (\varkappa_{11}\partial_1^2 + \varkappa_{11}\partial_2^2 + \varkappa_{33}\partial_3^2)\vartheta - \\
& -\tau\alpha\vartheta + \tau T_0g_3\partial_3\varphi + X_4 = 0, \\
& -(e_{31} + e_{15})\partial_1\partial_3u_1 - (e_{31} + e_{15})\partial_2\partial_3u_2 - \\
& - (e_{15}\partial_1^2 + e_{15}\partial_2^2 + e_{33}\partial_3^2)u_3 - g_3\partial_3\vartheta + \\
& + (\varepsilon_{11}\partial_1^2 + \varepsilon_{11}\partial_2^2 + \varepsilon_{33}\partial_3^2)\varphi + X_5 = 0.
\end{aligned}$$

The matrix operators \mathcal{T} and $\tilde{\mathcal{T}}$ defined by the relations (A.45)–(A.46) and (A.48)–(A.49) in this particular case read as follows

$$\mathcal{T}(\partial, n) = [\mathcal{T}_{jk}(\partial, n)]_{5 \times 5}, \quad \tilde{\mathcal{T}}(\partial, n, \tau) = [\tilde{\mathcal{T}}_{jk}(\partial, n, \tau)]_{5 \times 5}, \quad (\text{A.51})$$

where

$$\begin{aligned}
\mathcal{T}_{11}(\partial, n) &= \tilde{\mathcal{T}}_{11}(\partial, n, \tau) = c_{11}n_1\partial_1 + c_{66}n_2\partial_2 + c_{44}n_3\partial_3, \\
\mathcal{T}_{12}(\partial, n) &= \tilde{\mathcal{T}}_{12}(\partial, n, \tau) = (c_{11} - 2c_{66})n_1\partial_2 + c_{66}n_2\partial_1, \\
\mathcal{T}_{13}(\partial, n) &= \tilde{\mathcal{T}}_{13}(\partial, n, \tau) = c_{13}n_1\partial_3 + c_{44}n_3\partial_1, \\
\mathcal{T}_{14}(\partial, n) &= -[\bar{\tau}T_0]^{-1}\tilde{\mathcal{T}}_{14}(\partial, n, \tau) = -\gamma_1n_1, \\
\mathcal{T}_{15}(\partial, n) &= -\tilde{\mathcal{T}}_{15}(\partial, n, \tau) = e_{15}n_3\partial_1 + e_{31}n_1\partial_3, \\
\mathcal{T}_{21}(\partial, n) &= \tilde{\mathcal{T}}_{21}(\partial, n, \tau) = c_{66}n_1\partial_2 + (c_{11} - 2c_{66})n_2\partial_1, \\
\mathcal{T}_{22}(\partial, n) &= \tilde{\mathcal{T}}_{22}(\partial, n, \tau) = c_{66}n_1\partial_1 + c_{11}n_2\partial_2 + c_{44}n_3\partial_3, \\
\mathcal{T}_{23}(\partial, n) &= \tilde{\mathcal{T}}_{23}(\partial, n, \tau) = c_{13}n_2\partial_3 + c_{44}n_3\partial_2, \\
\mathcal{T}_{24}(\partial, n) &= -[\bar{\tau}T_0]^{-1}\tilde{\mathcal{T}}_{24}(\partial, n, \tau) = -\gamma_1n_2, \\
\mathcal{T}_{25}(\partial, n) &= -\tilde{\mathcal{T}}_{25}(\partial, n, \tau) = e_{15}n_3\partial_2 + e_{31}n_2\partial_3, \\
\mathcal{T}_{31}(\partial, n) &= \tilde{\mathcal{T}}_{31}(\partial, n, \tau) = c_{44}n_1\partial_3 + c_{13}n_3\partial_1, \\
\mathcal{T}_{32}(\partial, n) &= \tilde{\mathcal{T}}_{32}(\partial, n, \tau) = c_{44}n_2\partial_3 + c_{13}n_3\partial_2, \\
\mathcal{T}_{33}(\partial, n) &= \tilde{\mathcal{T}}_{33}(\partial, n, \tau) = c_{44}n_1\partial_1 + c_{44}n_2\partial_2 + c_{33}n_3\partial_3, \\
\mathcal{T}_{34}(\partial, n) &= -[\bar{\tau}T_0]^{-1}\tilde{\mathcal{T}}_{34}(\partial, n, \tau) = -\gamma_3n_3, \\
\mathcal{T}_{35}(\partial, n) &= -\tilde{\mathcal{T}}_{35}(\partial, n, \tau) = e_{15}(n_1\partial_1 + n_2\partial_2) + e_{33}n_3\partial_3, \\
\mathcal{T}_{4j}(\partial, n) &= \tilde{\mathcal{T}}_{4j}(\partial, n, \tau) = 0 \text{ for } j = 1, 2, 3, \\
\mathcal{T}_{44}(\partial, n) &= \tilde{\mathcal{T}}_{44}(\partial, n, \tau) = \varkappa_{11}(n_1\partial_1 + n_2\partial_2) + \varkappa_{33}n_3\partial_3, \\
\mathcal{T}_{45}(\partial, n) &= \tilde{\mathcal{T}}_{45}(\partial, n, \tau) = 0, \\
\mathcal{T}_{51}(\partial, n) &= -\tilde{\mathcal{T}}_{51}(\partial, n, \tau) = -e_{15}n_1\partial_3 - e_{31}n_3\partial_1,
\end{aligned} \quad (\text{A.52})$$

$$\begin{aligned}
\mathcal{T}_{52}(\partial, n) &= -\tilde{\mathcal{T}}_{52}(\partial, n, \tau) = -e_{15}n_2\partial_3 - e_{31}n_3\partial_2, \\
\mathcal{T}_{53}(\partial, n) &= -\tilde{\mathcal{T}}_{53}(\partial, n, \tau) = -e_{15}(n_1\partial_1 + n_2\partial_2) - e_{33}n_3\partial_3, \\
\mathcal{T}_{54}(\partial, n) &= [\bar{\tau}T_0]^{-1}\tilde{\mathcal{T}}_{54}(\partial, n, \tau) = -g_3n_3, \\
\mathcal{T}_{55}(\partial, n) &= \tilde{\mathcal{T}}_{55}(\partial, n, \tau) = \varepsilon_{11}(n_1\partial_1 + n_2\partial_2) + \varepsilon_{33}n_3\partial_3.
\end{aligned}$$

7. APPENDIX B: GREEN'S FORMULAE

As it has been mentioned in the introduction of Section 2, to avoid some misunderstanding related to the directions of normal vectors on the contact surfaces Γ_m we assume that *the normal vector to $\partial\Omega_m$ is directed outward, while on $\partial\Omega$ it is directed inward.*

Here we recall Green's formulae for the differential operators $A^{(m)}(\partial, \tau)$ and $A(\partial, \tau)$ in Ω_m and Ω , respectively (see, e.g., [14], [15], [6], [3]).

Let Ω_m and Ω be smooth domains and

$$\begin{aligned}
U^{(m)} &= (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top \in [C^2(\bar{\Omega}_m)]^4, \quad u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top, \\
V^{(m)} &= (v_1^{(m)}, v_2^{(m)}, v_3^{(m)}, v_4^{(m)})^\top \in [C^2(\bar{\Omega}_m)]^4, \quad v^{(m)} = (v_1^{(m)}, v_2^{(m)}, v_3^{(m)})^\top.
\end{aligned}$$

Then we have the following integral identities (Green's formulae) related to the differential equations and the boundary operators of the thermoelasticity theory:

$$\begin{aligned}
&\int_{\Omega_m} \left[A^{(m)}(\partial, \tau)U^{(m)} \cdot V^{(m)} - U^{(m)} \cdot A^{(m)*}(\partial, \tau)V^{(m)} \right] dx = \\
&= \int_{\partial\Omega_m} \left[\{\mathcal{T}^{(m)}U^{(m)}\}^+ \cdot \{V^{(m)}\}^+ - \{U^{(m)}\}^+ \cdot \{\tilde{\mathcal{T}}^{(m)}V^{(m)}\}^+ \right] dS, \quad (\text{B.1})
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_m} A^{(m)}(\partial, \tau)U^{(m)} \cdot V^{(m)} dx = \int_{\partial\Omega_m} \{\mathcal{T}^{(m)}U^{(m)}\}^+ \cdot \{V^{(m)}\}^+ dS - \\
&\quad - \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{v^{(m)}}) + \varrho^{(m)}\tau^2 u^{(m)} \cdot v^{(m)} + \varkappa_{jl}^{(m)} \partial_j u_4^{(m)} \overline{\partial_l v_4^{(m)}} + \right. \\
&\quad \left. + \tau\alpha^{(m)} u_4^{(m)} \overline{v_4^{(m)}} + \gamma_{jl}^{(m)} (\tau r T_0^{(m)} \partial_j u_l^{(m)} \overline{v_4^{(m)}} - u_4^{(m)} \overline{\partial_j v_l^{(m)}}) \right] dx, \quad (\text{B.2})
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_m} \left[\sum_{j=1}^3 [A^{(m)}(\partial, \tau)U^{(m)}]_j \overline{v_j^{(m)}} + \frac{1}{\tau T_0^{(m)}} [A^{(m)}(\partial, \tau)U^{(m)}]_4 \overline{v_4^{(m)}} \right] dx = \\
&= \int_{\partial\Omega_m} \left[\sum_{j=1}^3 \{\mathcal{T}^{(m)}U^{(m)}\}_j^+ \overline{\{v_j^{(m)}\}^+} + \frac{1}{\tau T_0^{(m)}} \{\mathcal{T}^{(m)}U^{(m)}\}_4^+ \overline{\{v_4^{(m)}\}^+} \right] dS - \\
&\quad - \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{v^{(m)}}) + \varrho^{(m)}\tau^2 u^{(m)} \cdot \overline{v^{(m)}} + \frac{1}{\tau T_0^{(m)}} \varkappa_{jl}^{(m)} \partial_j u_4^{(m)} \overline{\partial_l v_4^{(m)}} + \right.
\end{aligned}$$

$$+ \frac{\alpha^{(m)}}{T_0^{(m)}} u_4^{(m)} \overline{v_4^{(m)}} + \gamma_{jl}^{(m)} (\partial_j u_l^{(m)} \overline{v_4^{(m)}} - u_4^{(m)} \overline{\partial_j v_l^{(m)}}) dx, \quad (\text{B.3})$$

$$\begin{aligned} & \int_{\Omega_m} \left[\sum_{j=1}^3 [A^{(m)}(\partial, \tau) U^{(m)}]_j \overline{u_j^{(m)}} + \frac{\tau}{|\tau|^2 T_0^{(m)}} [A^{(m)}(\partial, \tau) U^{(m)}]_4 u_4^{(m)} \right] dx = \\ & = - \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + \varrho^{(m)} \tau^2 |u^{(m)}|^2 + \frac{\alpha^{(m)}}{T_0^{(m)}} |u_4^{(m)}|^2 + \right. \\ & \quad \left. + \frac{\tau}{|\tau|^2 T_0^{(m)}} \varkappa_{ij}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} \right] dx \\ & + \int_{\partial\Omega_m} \left[\sum_{j=1}^3 \{T^{(m)} U^{(m)}\}_j^+ \{\overline{u_j^{(m)}}\}^+ + \frac{\tau}{|\tau|^2 T_0^{(m)}} \{\overline{T^{(m)} U^{(m)}}\}_4^+ \{u_4^{(m)}\}^+ \right] dS, \quad (\text{B.4}) \end{aligned}$$

where $E^{(m)}(u^{(m)}, \overline{u^{(m)}}) = c_{ijkl}^{(m)} \partial_i u_j^{(m)} \overline{\partial_l v_k^{(m)}}$ and the operators $A^{(m)}$, $A^{(m)*}$, $T^{(m)}$ and $\tilde{T}^{(m)}$ are defined in Appendix A.

For arbitrary vector-functions

$$\begin{aligned} U &= (u_1, u_2, u_3, u_4, u_5)^\top \in [C^2(\overline{\Omega})]^5, \quad u = (u_1, u_2, u_3)^\top, \\ V &= (v_1, v_2, v_3, v_4, v_5)^\top \in [C^2(\overline{\Omega})]^5, \quad v = (v_1, v_2, v_3)^\top, \end{aligned}$$

we have the similar Green formulae related to the differential equations and boundary operators of the thermoelastoelectricity theory:

$$\begin{aligned} & \int_{\Omega} [A(\partial, \tau) U \cdot V - U \cdot A^*(\partial, \tau) V] dx = \\ & = - \int_{\partial\Omega} [\{TU\}^+ \cdot \{V\}^+ - \{U\}^+ \cdot \{\tilde{T}V\}^+] dS, \quad (\text{B.5}) \\ & \int_{\Omega} A(\partial, \tau) U \cdot V dx = - \int_{\partial\Omega} \{TU\}^+ \cdot \{V\}^+ dS - \\ & - \int_{\Omega} [E(u, \overline{v}) + \varrho \tau^2 u \cdot v + \gamma_{jl}(\tau T_0 \partial_j u_l \overline{v_4} - u_4 \overline{\partial_j v_l}) + \\ & + \varkappa_{jl} \partial_j u_4 \overline{\partial_l v_4} + \tau \alpha u_4 \overline{v_4} + e_{lij}(\partial_l u_5 \overline{\partial_i v_j} - \partial_i u_j \overline{\partial_l v_5}) - \\ & - g_l(\tau T_0 \partial_l u_5 \overline{v_4} + u_4 \overline{\partial_l v_5}) + \varepsilon_{jl} \partial_j u_5 \overline{\partial_l v_5}] dx, \quad (\text{B.6}) \\ & \int_{\Omega} \left[\sum_{j=1}^3 [A(\partial, \tau) U]_j \overline{v_j} + \frac{1}{\tau T_0} [A(\partial, \tau) U]_4 \overline{v_4} + [\overline{A(\partial, \tau) U}]_5 v_5 \right] dx = \\ & = - \int_{\partial\Omega} \left[\sum_{j=1}^3 \{TU\}_j^+ \{\overline{v_j}\}^+ + \frac{1}{\tau T_0} \{TU\}_4^+ \{\overline{v_4}\}^+ + \{\overline{TU}\}_5^+ \{v_5\}^+ \right] dS \end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega} \left[E(u, \bar{v}) + \varrho \tau^2 u \cdot v + \gamma_{jl} (\partial_j u_l \bar{v}_4 - u_4 \bar{\partial}_j v_l) + \frac{1}{\tau T_0} \varkappa_{jl} \partial_j u_4 \bar{\partial}_l v_4 + \frac{\alpha}{T_0} u_4 \bar{v}_4 + \right. \\
& \left. + e_{ij} (\partial_l u_5 \bar{\partial}_i v_j - \bar{\partial}_i u_j \partial_l v_5) - g_l (\partial_l u_5 \bar{v}_4 + \bar{u}_4 \partial_l v_5) + \varepsilon_{jl} \bar{\partial}_j u_5 \partial_l v_5 \right] dx, \quad (\text{B.7}) \\
& \int_{\Omega} \left[\sum_{j=1}^3 [A(\partial, \tau) U]_j \bar{u}_j + \frac{\tau}{|\tau|^2 T_0} [\overline{A(\partial, \tau) U}]_4 u_4 + [\overline{A(\partial, \tau) U}]_5 u_5 \right] dx \\
& = - \int_{\Omega} \left[E(u, \bar{u}) + \varrho \tau^2 |u|^2 + \frac{\alpha}{T_0} |u_4|^2 + \frac{\tau}{|\tau|^2 T_0} \varkappa_{jl} \partial_l u_4 \bar{\partial}_j u_4 - \right. \\
& \quad \left. - 2\Re\{g_l u_4 \bar{\partial}_l u_5\} + \varepsilon_{jl} \partial_l u_5 \bar{\partial}_j u_5 \right] dx - \\
& - \int_{\partial\Omega} \left[\sum_{j=1}^3 \{TU\}_j^+ \{\bar{u}_j\}^+ + \frac{\tau}{|\tau|^2 T_0} \{\overline{TU}\}_4^+ \{u_4\}^+ + \{\overline{TU}\}_5^+ \{u_5\}^+ \right] dS, \quad (\text{B.8})
\end{aligned}$$

where $E(u, \bar{v}) = c_{ijkl} \partial_i u_j \bar{\partial}_l v_k$ and the operators A , A^* , \mathcal{T} , and $\tilde{\mathcal{T}}$ are defined in Appendix A.

Note that in front of the surface integrals in the formulas (B.5)–(B.8) the minus sign appeared due to the inward direction of the normal vector on $\partial\Omega$.

For $\tau = 0$, Green's formulae (B.1), (B.2), (B.6), and (B.5) remain valid and, in addition, there hold the following identities

$$\begin{aligned}
& \int_{\Omega_m} \left[\sum_{j=1}^3 [A^{(m)}(\partial) U^{(m)}]_j \overline{u_j^{(m)}} + c_1 [A^{(m)}(\partial) U^{(m)}]_4 \overline{u_4^{(m)}} \right] dx = \\
& = - \int_{\Omega_m} \left[E^{(m)}(u^{(m)}, \overline{u^{(m)}}) + c_1 \varkappa_{lj}^{(m)} \partial_l u_4^{(m)} \overline{\partial_j u_4^{(m)}} - \gamma_{jl}^{(m)} u_4^{(m)} \overline{\partial_j u_l^{(m)}} \right] dx + \\
& + \int_{\partial\Omega_m} \left[\sum_{j=1}^3 [\mathcal{T}^{(m)}(\partial, n) U^{(m)}]_j^+ [\overline{u_j^{(m)}}]^+ + c_1 [\mathcal{T}^{(m)}(\partial, n) U^{(m)}]_4^+ [\overline{u_4^{(m)}}]^+ \right] dS, \quad (\text{B.9}) \\
& \int_{\Omega} \left[\sum_{j=1}^3 [A(\partial) U]_j \bar{u}_j + c [A(\partial) U]_4 \bar{u}_4 + [\overline{A(\partial) U}]_5 u_5 \right] dx = \\
& = - \int_{\Omega} \left[E(u, \bar{u}) + c \varkappa_{jl} \partial_l u_4 \bar{\partial}_j u_4 - \gamma_{jl} u_4 \bar{\partial}_l u_j - g_l \bar{u}_4 \partial_l u_5 + \varepsilon_{jl} \partial_l u_5 \bar{\partial}_j u_5 \right] dx - \\
& - \int_{\partial\Omega} \left[\sum_{j=1}^3 \{TU\}_j^+ \{\bar{u}_j\}^+ + c \{TU\}_4^+ \{\bar{u}_4\}^+ + \{\overline{TU}\}_5^+ \{u_5\}^+ \right] dS, \quad (\text{B.10})
\end{aligned}$$

where $A^{(m)}(\partial) := A^{(m)}(\partial, 0)$ and $A(\partial) := A(\partial, 0)$, and c_1 and c are arbitrary constants.

Remark that the above Green's formulae (B.2), (B.4), (B.6), and (B.8) by standard limiting procedure can be generalized to the Lipschitz domains and to the vector-functions

$$U^{(m)} \in [W_p^1(\Omega_m)]^4, \quad V^{(m)} \in [W_{p'}^1(\Omega_m)]^4, \quad U \in [W_p^1(\Omega)]^5, \quad V \in [W_{p'}^1(\Omega)]^5$$

with

$$A^{(m)}(\partial, \tau)U^{(m)} \in [L_p(\Omega_m)]^4, \quad A(\partial, \tau)U \in [L_p(\Omega)]^5, \quad 1/p + 1/p' = 1.$$

Moreover, in addition, if $A^{(m)*}(\partial, \tau)V^{(m)} \in [L_{p'}(\Omega_m)]^4$, $A^*(\partial, \tau)V \in [L_{p'}(\Omega)]^5$, then the formulae (B.1) and (B.5) hold true as well (for details see [30], [26], [11]).

8. APPENDIX C: FUNDAMENTAL MATRICES

Here we present the explicit expressions of the fundamental matrices of the differential operators $A^{(m)}(\partial, \tau)$ and $A(\partial, \tau)$ for the general anisotropic case as well as for the isotropic and transversally isotropic cases.

8.1. Fundamental matrix of thermoelasticity: general anisotropy.

Denote by $\Psi^{(m)}(\cdot, \tau) := [\Psi_{kj}^{(m)}(\cdot, \tau)]_{4 \times 4}$ a fundamental matrix of the differential operator $A^{(m)}(\partial, \tau)$,

$$A^{(m)}(\partial, \tau)\Psi^{(m)}(x, \tau) = I_4\delta(x), \quad (\text{C.1})$$

where $\delta(\cdot)$ is Dirac's distribution.

Denote by $A^{(m,0)}(\partial)$ the principal homogeneous part of the operator $A^{(m)}(\partial, \tau)$

$$A^{(m,0)}(\partial_x) = \begin{bmatrix} [c_{ijkl}^{(m)}\partial_i\partial_l]_{3 \times 3} & [0]_{3 \times 1} \\ [0]_{1 \times 3} & [\varkappa_{il}^{(m)}\partial_i\partial_l]_{4 \times 4} \end{bmatrix}. \quad (\text{C.2})$$

Clearly, $A^{(m,0)}(\partial)$ is a strongly elliptic formally self-adjoint operator. We the corresponding symbol matrices denote by $A^{(m)}(-i\xi, \tau)$ and $A^{(m,0)}(-i\xi) = -A^{(m,0)}(\xi)$, respectively. Note that $A^{(m,0)}(\xi)$ is a positive definite matrix for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$ due to the relations (A.7). The following assertions can easily be checked with the help of (6.1.1) (for details see [14], Lemma 1.1).

Lemma 8.1. *Let $\tau = \sigma + i\omega$, $\sigma > 0$, and $\omega \in \mathbb{R}$. Then*

(i) *for arbitrary $\xi \in \mathbb{R}^3$*

$$\det A^{(m)}(-i\xi, \tau) \neq 0; \quad (\text{C.3})$$

(ii) *for sufficiently large $|\xi|$ there holds the asymptotic relation*

$$\det A^{(m)}(-i\xi, \tau) = a(\tilde{\xi})|\xi|^8 + \mathcal{O}(|\xi|^6), \quad (\text{C.4})$$

where $\tilde{\xi} = \xi/|\xi|$ and $0 < a_1 \leq a(\tilde{\xi}) \leq a_2$ for arbitrary $\xi \neq 0$ with positive constants a_1 and a_2 depending only on the material constants;

(iii) *for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$ there holds the equality*

$$\det A^{(m,0)}(-i\xi) = a(\tilde{\xi})|\xi|^8$$

with the same $a(\tilde{\xi})$ as in (C.4); the entries of the matrix $[A^{(m,0)}(\xi)]^{-1}$ are $C^\infty(\mathbb{R}^3 \setminus \{0\})$ -regular homogeneous functions of order -2 ;

(iv) the entries of the inverse matrix $[A^{(m)}(\cdot, \tau)]^{-1}$ are rational, $C^\infty(\mathbb{R}^3)$ -regular functions and belong to the space $L_2(\mathbb{R}^3)$.

By $\mathcal{F}_{x \rightarrow \xi}$ and $\mathcal{F}_{\xi \rightarrow x}^{-1}$ we denote the generalized Fourier and inverse Fourier transforms which for summable functions on \mathbb{R}^n are defined as follows

$$\mathcal{F}_{x \rightarrow \xi}[f] = \int_{\mathbb{R}^n} f(x) e^{ix\xi} dx, \quad \mathcal{F}_{\xi \rightarrow x}^{-1}[g] = (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{-ix\xi} d\xi.$$

Due to the equation (C.1) we can represent $\Psi^{(m)}(x, \tau)$ by the Fourier integral

$$\begin{aligned} \Psi^{(m)}(x, \tau) &= \mathcal{F}_{\xi \rightarrow x}^{-1}([A^{(m)}(-i\xi, \tau)]^{-1}) = \\ &= (2\pi)^{-3} \lim_{R \rightarrow \infty} \int_{|\xi| < R} [A^{(m)}(-i\xi, \tau)]^{-1} e^{-ix\xi} d\xi. \end{aligned} \quad (\text{C.5})$$

From Lemma 8.1 and properties of the Fourier transform it follows that the entries of the matrix $\Psi^{(m)}(x, \tau)$ together with all derivatives decrease more rapidly than any negative power of $|x|$ as $|x| \rightarrow +\infty$.

In a neighbourhood of the origin (say $|x| < 1/2$) the matrix $\Psi^{(m)}(x, \tau)$ has a singularity of the type $\mathcal{O}(|x|^{-1})$ and its principal singular part $\Psi^{(m,0)}(x)$, which is independent of τ , can be written explicitly (for details see [29], [14], Lemma 2.1)

$$\Psi^{(m,0)}(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}([A^{(m,0)}(\xi)]^{-1}) = -\frac{1}{8\pi^2|x|} \int_0^{2\pi} [A^{(m,0)}(a(x)\eta)]^{-1} d\phi, \quad (\text{C.6})$$

where $x \in \mathbb{R}^3 \setminus \{0\}$, $a(x) = [a_{kj}(x)]_{3 \times 3}$ is an orthogonal matrix with property $a^\top(x)x^\top = (0, 0, |x|)^\top$, $\eta = (\cos \phi, \sin \phi, 0)^\top$. Clearly, $\Psi^{(m,0)}(\cdot)$ is the fundamental matrix of the operator $A^{(m,0)}(\partial)$ whose entries are homogeneous functions of order -1 (note that by this homogeneity property $\Psi^{(m,0)}(\cdot)$ is defined uniquely). Moreover, $\Psi(x) = \Psi(-x) = [\Psi(x)]^\top$.

There is a positive constant $c_0 > 0$ (depending on the material constants and on the parameter τ) such that in a neighbourhood of the origin (say $|x| < 1/2$) there hold the estimates

$$\begin{aligned} |\Psi_{kj}^{(m)}(x, \tau) - \Psi_{kj}^{(m,0)}(x)| &\leq c_0 \log |x|^{-1}, \\ |\partial^\alpha [\Psi_{kj}^{(m)}(x, \tau) - \Psi_{kj}^{(m,0)}(x)]| &\leq c_0 |x|^{-|\alpha|} \text{ for } |\alpha| = 1, 2 \text{ and } k, j = \overline{1, 4}, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Note that if by $\Psi^{(m)*}(\cdot, \tau)$ we denote the fundamental matrix of the adjoint operator $A^{(m)*}(\partial, \tau)$, represented by the Fourier integral similar to (C.5), then we have the evident equalities:

$$\Psi^{(m)*}(x, \tau) = [\Psi^{(m)}(x, \bar{\tau})]^\top, \quad \Psi^{(m)}(-x, \bar{\tau}) = \overline{\Psi^{(m)}(x, \tau)},$$

$$\Psi^{(m)}(x, \tau) = [\overline{\Psi^{(m)*}(-x, \tau)}]^\top.$$

8.2. Fundamental matrix of thermoelasticity: isotropic case. The entries of the fundamental matrix $\Psi^{(m)}(x, \tau) := [\Psi_{kj}^{(m)}(x, \tau)]_{4 \times 4}$ for the isotropic case (see the Appendix A, Subsection 6.1.2) read as follows (see [19], Ch. II)

$$\begin{aligned} \Psi_{kj}^{(m)}(x, \tau) &= -\frac{1}{2} \sum_{l=1}^3 \left\{ (1 - \delta_{k4})(1 - \delta_{j4}) [(2\pi\mu^{(m)})^{-1} \delta_{kj} \delta_{l3} - a_l^{(m)} \partial_k \partial_j] \right. \\ &\quad \left. - b_l^{(m)} \left[\tau \delta_{k4} \frac{T_0^{(m)} \gamma^{(m)}}{\varkappa^{(m)}} (1 - \delta_{j4}) \partial_j + \gamma^{(m)} \delta_{j4} (1 - \delta_{k4}) \partial_k \right] - \delta_{j4} \delta_{k4} c_l^{(m)} \right\} \times \\ &\quad \times \frac{\exp(i d_l^{(m)} |x|)}{|x|}, \quad k, j = \overline{1, 4}, \\ a_l^{(m)} &= \frac{(-1)^l (\varkappa^{(m)} [d_l^{(m)}]^2 + \tau) (\delta_{1l} + \delta_{2l})}{2\pi (\lambda^{(m)} + 2\mu^{(m)}) \varkappa^{(m)} [d_l^{(m)}]^2 ([d_2^{(m)}]^2 - [d_1^{(m)}]^2)} + \frac{\delta_{3l}}{2\pi \varrho^{(m)} \tau^2}, \\ b_l^{(m)} &= \frac{(-1)^l (\delta_{1l} + \delta_{2l})}{2\pi (\lambda^{(m)} + 2\mu^{(m)}) ([d_2^{(m)}]^2 - [d_1^{(m)}]^2)}, \\ c_l^{(m)} &= \frac{(-1)^l ([d_l^{(m)}] - [k_1^{(m)}]^2) (\delta_{1l} + \delta_{2l})}{2\pi ([d_2^{(m)}]^2 - [d_1^{(m)}]^2)}, \\ \sum_{l=1}^3 a_l^{(m)} &= 0, \quad \sum_{l=1}^3 b_l^{(m)} = 0, \quad \sum_{l=1}^3 c_l^{(m)} = 1, \\ [k_1^{(m)}]^2 &= -\frac{\varrho^{(m)} \tau^2}{\lambda^{(m)} + 2\mu^{(m)}}, \quad [k_2^{(m)}]^2 = [d_3^{(m)}]^2 = -\frac{\varrho^{(m)} \tau^2}{\mu^{(m)}}, \\ [d_1^{(m)}]^2 [d_2^{(m)}]^2 &= -\frac{\tau [k_1^{(m)}]^2 \alpha^{(m)}}{\varkappa^{(m)}}, \\ [d_2^{(m)}]^2 + [d_1^{(m)}]^2 &= -\frac{\tau \alpha^{(m)}}{\varkappa^{(m)}} - \frac{\tau T_0^{(m)} (\gamma^{(m)})^2}{\varkappa^{(m)} (\lambda^{(m)} + 2\mu^{(m)})} + [k_1^{(m)}]^2. \end{aligned}$$

Here we assume that $[d_2^{(m)}]^2 \neq [d_1^{(m)}]^2$. The case $[d_2^{(m)}]^2 = [d_1^{(m)}]^2$ can be obtained from the above formulas by the limiting procedure $([d_2^{(m)}]^2 \rightarrow [d_1^{(m)}]^2)$.

Remark that by the limiting procedure as $\tau \rightarrow 0$ we obtain the fundamental matrix $\Psi^{(m)}(x, 0) := [\Psi_{kj}^{(m)}(x, 0)]_{4 \times 4}$ of the operator of thermoelastostatics $A^{(m)}(\partial, 0)$:

$$\begin{aligned} \Psi_{kj}^{(m)}(x, 0) &= (1 - \delta_{k4})(1 - \delta_{j4}) \left[\frac{\lambda^{(m)*} \delta_{kj}}{|x|} + \frac{\mu^{(m)*} x_k x_j}{|x|^3} \right] - \\ &\quad - \frac{\gamma^{(m)} \delta_{j4} (1 - \delta_{k4})}{8\pi (\lambda^{(m)} + 2\mu^{(m)})} \frac{x_k}{|x|} - \frac{\delta_{k4} \delta_{j4}}{4\pi} \frac{1}{|x|}, \quad k, j = \overline{1, 4}, \end{aligned}$$

$$\lambda^{(m)*} = -\frac{\lambda^{(m)} + 3\mu^{(m)}}{8\pi\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})},$$

$$\mu^{(m)*} = -\frac{\lambda^{(m)} + \mu^{(m)}}{8\pi\mu^{(m)}(\lambda^{(m)} + 2\mu^{(m)})}.$$

8.3. Fundamental matrix of thermopiezoelectricity: general anisotropic case. We denote by $\Psi(\cdot, \tau) := [\Psi_{kj}(\cdot, \tau)]_{5 \times 5}$ the fundamental matrix of the differential operator $A(\partial, \tau)$, i.e., $A(\partial, \tau)\Psi(x, \tau) = \delta(x)I_5$. Applying here the Fourier transform we get $A(-i\xi, \tau)\widehat{\Psi}(\xi, \tau) = I_5$, where $\widehat{\Psi}(\xi, \tau)$ is the Fourier transform of $\Psi(x, \tau)$ and $A(-i\xi, \tau) = [A_{kj}(-i\xi, \tau)]_{5 \times 5}$ is the symbol matrix of the operator $A(\partial, \tau)$:

$$A(-i\xi, \tau) := \begin{bmatrix} [-c_{ijkl}\xi_i\xi_l - \rho\tau^2\delta_{jk}]_{3 \times 3} & [i\gamma_{jl}\xi_l]_{3 \times 1} & [-e_{lkj}\xi_l\xi_k]_{3 \times 1} \\ [i\tau T_0\gamma_{kl}\xi_l]_{1 \times 3} & -\varkappa_{kl}\xi_k\xi_l - \alpha\tau & -i\tau T_0g_k\xi_k \\ [e_{jkl}\xi_j\xi_l]_{1 \times 3} & ig_k\xi_k & -\varepsilon_{kl}\xi_k\xi_l \end{bmatrix}_{5 \times 5}. \quad (\text{C.7})$$

Denote by $A^{(0)}(\partial)$ the principal homogeneous part of the operator $A(\partial, \tau)$. Then the symbol matrix $A^{(0)}(-i\xi)$ is the principal homogeneous symbol matrix of the operator $A(\partial, \tau)$,

$$A^{(0)}(-i\xi) := \begin{bmatrix} [-c_{ijkl}\xi_i\xi_l]_{3 \times 3} & [0]_{3 \times 1} & [-e_{lkj}\xi_l\xi_k]_{3 \times 1} \\ [0]_{1 \times 3} & -\varkappa_{kl}\xi_k\xi_l & 0 \\ [e_{jkl}\xi_j\xi_l]_{1 \times 3} & 0 & -\varepsilon_{kl}\xi_k\xi_l \end{bmatrix}_{5 \times 5}. \quad (\text{C.8})$$

Note that

$$\det A^{(0)}(-i\xi) = -\varkappa_{jl}\xi_j\xi_l \det \widetilde{A}(-i\xi) \neq 0 \quad \text{for } |\xi| = 1, \quad (\text{C.9})$$

where

$$\widetilde{A}(-i\xi) := \begin{bmatrix} [-c_{ijkl}\xi_i\xi_l]_{3 \times 3} & [-e_{lkj}\xi_l\xi_k]_{3 \times 1} \\ [e_{jkl}\xi_j\xi_l]_{1 \times 3} & -\varepsilon_{kl}\xi_k\xi_l \end{bmatrix}_{4 \times 4}$$

is the symbol matrix of the strongly elliptic operator $\widetilde{A}(\partial)$ generated by the equations of statics of piezoelectricity.

It can be shown that the fundamental matrix of the operator $A^{(0)}(\partial)$ is representable in the form (cf. [29])

$$\Psi^{(0)}(x) := \mathcal{F}_{\xi \rightarrow x}^{-1}([A^{(0)}(-i\xi)]^{-1}) = -\frac{1}{8\pi^2|x|} \int_0^{2\pi} [A^{(0)}(a(x)\eta)]^{-1} d\phi \quad (\text{C.10})$$

with the same $a(x)$ and η as in (C.6). The entries of this matrix are homogeneous functions of order -1 .

We start the study of the near and far field properties of the fundamental matrix $\Psi(\cdot, \tau)$ by the following auxiliary lemma.

Lemma 8.2. *Let $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then*

$$\det A(-i\xi, \tau) \neq 0 \quad (\text{C.11})$$

for arbitrary $\xi \in \mathbb{R}^3$, $\xi \neq 0$.

Proof. Let $\zeta = (\zeta_1, \dots, \zeta_5)$ be a solution of the homogeneous system of linear equations

$$\sum_{k=1}^5 A_{jk}(-i\xi, \tau)\zeta_k = 0, \quad j = \overline{1, 5}. \quad (\text{C.12})$$

Consider the expression

$$\begin{aligned} E &:= \sum_{j=1}^3 \left(\sum_{k=1}^5 A_{jk}(-i\xi, \tau)\zeta_k \right) \bar{\zeta}_j - \\ &- \frac{1}{\bar{\tau}T_0} \sum_{k=1}^5 \overline{A_{4k}(-i\xi, \tau)\zeta_k} \zeta_4 + \sum_{k=1}^5 \overline{A_{5k}(-i\xi, \tau)\zeta_k} \zeta_5 = \\ &= - \sum_{n,j,l,k=1}^3 c_{njlk} \xi_n \xi_l \zeta_k \bar{\zeta}_j - \sum_{j=1}^3 \rho \tau^2 \zeta_j \bar{\zeta}_j - \\ &- \frac{1}{\bar{\tau}T_0} \sum_{j,l=1}^3 \varkappa_{jl} \xi_j \xi_l |\zeta_4|^2 - \frac{\alpha}{T_0} |\zeta_4|^2 + \\ &+ i \sum_{j=1}^3 g_j \xi_j (\zeta_4 \bar{\zeta}_5 - \bar{\zeta}_4 \zeta_5) - \sum_{j,l=1}^3 \varepsilon_{jl} \xi_j \xi_l |\zeta_5|^2, \end{aligned} \quad (\text{C.13})$$

which in view of (C.12) equals to zero. Taking into account (A.39)–(A.40), from (C.13) we get

$$\Im E = -2\rho\sigma\omega \sum_{j=1}^3 \zeta_j \bar{\zeta}_j - \frac{\omega}{|\tau|^2 T_0} \sum_{j,l=1}^3 \varkappa_{jl} \xi_j \xi_l |\zeta_4|^2 = 0. \quad (\text{C.14})$$

If $\omega \neq 0$ and $\sigma > 0$, $\xi \neq 0$, from (C.14) it follows $\zeta_j = 0$, $j = 1, \dots, 4$. Then from (C.13) we easily get $\zeta_5 = 0$.

If $\omega = 0$, then from (C.13), (A.39), (A.40), and (A.41) we obtain

$$-\Re E = \rho\sigma^2 \sum_{j=1}^3 \zeta_j \bar{\zeta}_j + \frac{1}{\sigma T_0} \sum_{j,l=1}^3 \varkappa_{jl} \xi_j \xi_l |\zeta_4|^2 + E_1 + E_2 = 0, \quad (\text{C.15})$$

where

$$\begin{aligned} E_1 &= \sum_{n,j,l,k=1}^3 c_{njlk} \xi_n \xi_l \zeta_k \bar{\zeta}_j \geq 0, \\ E_2 &= \frac{\alpha}{T_0} |\zeta_4|^2 - i \sum_{j=1}^3 g_j \xi_j (\zeta_4 \bar{\zeta}_5 - \bar{\zeta}_4 \zeta_5) + \sum_{j,l=1}^3 \varepsilon_{jl} \xi_j \xi_l |\zeta_5|^2 \geq 0. \end{aligned}$$

If $\sigma > 0$ and $\xi \neq 0$, then we conclude that $\zeta_j = 0$, $j = \overline{1, 5}$. Hence the system (C.12) has only the trivial solution and (C.11) holds. \square

Lemma 8.2 enables us to represent the matrix $\Psi(x, \tau)$ in the form

$$\begin{aligned} \Psi(x, \tau) &= \mathcal{F}_{\xi \rightarrow x}^{-1}([A(-i\xi, \tau)]^{-1}) = \\ &= (2\pi)^{-3} \lim_{R \rightarrow \infty} \int_{|\xi| < R} [A(-i\xi, \tau)]^{-1} e^{-ix\xi} d\xi. \end{aligned} \quad (\text{C.16})$$

Note that the matrix $\Psi^*(x, \tau) := [\Psi(x, \bar{\tau})]^\top$ represents the fundamental matrix of the adjoint operator $A^*(\partial, \tau)$, and $\Psi(x, \tau) = [\overline{\Psi^*(-x, \tau)}]^\top$, since $\Psi(-x, \bar{\tau}) = \overline{\Psi(x, \tau)}$.

Theorem 8.3. *Let $\tau = \sigma + i\omega$, with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then the fundamental matrix $\Psi(\cdot, \tau)$ in a neighbourhood of the origin (say $|x| < 1/2$) can be represented as*

$$\Psi(x, \tau) = \Psi^{(0)}(x) + \Psi^{(r)}(x, \tau), \quad (\text{C.17})$$

where $\Psi^{(0)}(x)$ is the fundamental matrix of the operator $A^{(0)}(\partial)$ given by (C.10), and the following estimates hold

$$|\Psi^{(r)}(x, \tau)| \leq c_0 \log(|x|^{-1}), \quad |\partial^\alpha \Psi^{(r)}(x, \tau)| \leq c_0 |x|^{-|\alpha|}, \quad |\alpha| = 1, 2 \quad (\text{C.18})$$

with some constant $c_0 > 0$.

Proof. Note, that $\det A(-i\xi, \tau)$ can be written as the sum of the homogeneous functions with respect to ξ ,

$$\Lambda(\xi, \tau) := \det A(-i\xi, \tau) = \sum_{n=1}^5 \Lambda^{(2n)}(\xi, \tau), \quad (\text{C.19})$$

where

$$\Lambda^{(2n)}(t\xi, \tau) = t^{2n} \Lambda^{(2n)}(\xi, \tau), \quad t \in \mathbb{R}, \quad k = 1, \dots, 5.$$

In particular,

$$\Lambda^{(2)}(\xi, \tau) = -\rho^3 \tau^7 \sum_{j,l=1}^3 (\alpha \varepsilon_{jl} - T_0 g_j g_l) \xi_j \xi_l, \quad (\text{C.20})$$

$$\Lambda^{(10)}(\xi, \tau) = \Lambda(\xi, 0) = \det A(-i\xi, 0) = \det A^{(0)}(-i\xi), \quad (\text{C.21})$$

and there is a positive constant c such that

$$|\Lambda^{(2)}(\xi, \tau)| \geq c |\tau|^7 |\xi|^2, \quad (\text{C.22})$$

$$|\Lambda^{(10)}(\xi, \tau)| \geq c |\xi|^{10}. \quad (\text{C.23})$$

The equalities (C.19)-(C.21) can be checked directly. To derive (C.22), it suffices to insert

$$\eta_l = \xi_l, \quad \zeta = \frac{T_0}{\alpha} \sum_{l=1}^3 g_l \xi_l$$

into (A.41). The relation (C.23) follows from the equality (C.9).

Expansions similar to (C.19) hold also for the co-factors $\Lambda_{kj}(\xi, \tau)$ of the matrix $A(-i\xi, \tau)$:

$$\Lambda_{kj}(\xi, \tau) = \sum_{n=0}^8 \Lambda_{kj}^{(n)}(\xi, \tau), \quad k, j = 1, \dots, 5, \quad (\text{C.24})$$

where the functions $\Lambda_{kj}^{(n)}(\xi, \tau)$ are homogeneous with respect to ξ of order n , and

$$\Lambda_{kj}^{(n)}(\xi, \tau) = 0, \quad n=0, 1, \quad k, j = \overline{1, 5}, \quad k+j \neq 10, \quad \Lambda_{kj}^{(8)}(\xi, \tau) = \tilde{\Lambda}_{kj}^{(8)}(\xi); \quad (\text{C.25})$$

here $\tilde{\Lambda}_{kj}^{(8)}(\xi)$ are the co-factors of the corresponding entries of the matrix $A^{(0)}(-i\xi)$.

Now we derive some asymptotic expansions of the matrix $A^{-1}(-i\xi, \tau)$ at infinity. To this end, note that if $P = Q + R$, then

$$\frac{1}{P} = \frac{1}{R} + \sum_{l=1}^s \frac{(-1)^l Q^l}{R^{l+1}} + (-1)^{s+1} \frac{Q^{s+1}}{P R^{s+1}}. \quad (\text{C.26})$$

If we insert here $s = 1$, $P = \Lambda(\xi, \tau)$, $Q = Q^{(1)} := \sum_{n=1}^4 \Lambda^{(2n)}(\xi, \tau)$, $R = \Lambda^{(10)}(\xi, \tau)$, and multiply both sides by $\Lambda_{kj}(\xi, \tau)$, we get

$$\begin{aligned} A_{jk}^{-1}(-i\xi, \tau) &= \frac{\Lambda_{kj}(\xi, \tau)}{\Lambda(\xi, 0)} - \frac{\Lambda_{kj}(\xi, \tau) Q^{(1)}(\xi, \tau)}{[\Lambda(\xi, 0)]^2} + \frac{\Lambda_{kj}(\xi, \tau) [Q^{(1)}(\xi, \tau)]^2}{\Lambda(\xi, \tau) [\Lambda(\xi, 0)]^2} = \\ &= [A^{(0)}(-i\xi)]_{jk}^{-1} + \sum_{n=0}^7 \frac{\Lambda_{kj}^{(n)}(\xi, \tau)}{\Lambda(\xi, 0)} - \frac{\Lambda_{kj}(\xi, \tau) Q^{(1)}(\xi, \tau)}{[\Lambda(\xi, 0)]^2} + \\ &+ \frac{\Lambda_{kj}(\xi, \tau) [Q^{(1)}(\xi, \tau)]^2}{\Lambda(\xi, \tau) [\Lambda(\xi, 0)]^2}. \end{aligned}$$

By the homogeneity property of the functions $\Lambda^{(2n)}(\xi, \tau)$ and $\Lambda_{kj}^{(n)}(\xi, \tau)$, and the relations (C.19), (C.21), (C.24) and (C.25), we can rewrite the last equality as follows

$$A_{jk}^{-1}(-i\xi, \tau) = [A^{(0)}(-i\xi)]_{jk}^{-1} + \sum_{n=-3}^{-5} f_{jk}^{(n)}(\xi, \tau) + f_{jk}^{(-6)}(\xi, \tau), \quad (\text{C.27})$$

where $f_{jk}^{(n)}(\xi, \tau)$ for $n = -3, \dots, -5$, are homogeneous functions of order n with respect to ξ and

$$|f_{jk}^{(-6)}(\xi, \tau)| \leq c(\tau) |\xi|^{-6}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \quad (\text{C.28})$$

Let $\nu(\cdot)$ be some cut-off function: $\nu \in C^\infty(\mathbb{R}^3)$, $\nu(\xi) = 1$ for $|\xi| \leq 1$ and $\nu(\xi) = 0$ for $|\xi| \geq 2$. Multiply both sides of (C.27) by $(-i\xi)^\alpha$ and apply the inverse Fourier transform to obtain

$$\begin{aligned} \partial_x^\alpha \Psi_{jk}(x, \tau) &= \\ &= \partial_x^\alpha \Psi_{jk}^{(0)}(x) + \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\nu(\xi) (-i\xi)^\alpha \left([A_{jk}(\xi, \tau)]^{-1} - [A_{jk}(\xi, 0)]^{-1} \right) \right) (x) + \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=-3}^{-5} \mathcal{F}_{\xi \rightarrow x}^{-1} \left((1 - \nu(\xi)) (-i\xi)^\alpha f_{jk}^{(n)}(\xi, \tau) \right) (x) + \\
& + \mathcal{F}_{\xi \rightarrow x}^{-1} \left((1 - \nu(\xi)) (-i\xi)^\alpha f_{jk}^{(-6)}(\xi, \tau) \right) (x), \quad (\text{C.29})
\end{aligned}$$

where $\Psi(x, \tau)$ and $\Psi^{(0)}(x)$ are the fundamental matrices of the operators $A(\partial, \tau)$ and $A^{(0)}(\partial)$, respectively (see (C.16), (C.10)).

Note that the expression $\nu(\xi) (-i\xi)^\alpha (A_{jk}^{-1}(\xi, \tau) - A_{jk}^{-1}(\xi, 0))$ has a compact support and therefore its inverse Fourier transform belongs to $C^\infty(\mathbb{R}^3)$.

The summand $\mathcal{F}_{\xi \rightarrow x}^{-1} \left((1 - \nu(\xi)) (-i\xi)^\alpha f_{jk}^{(n)}(\xi, \tau) \right) (x)$ can be rewritten as

$$\mathcal{F}_{\xi \rightarrow x}^{-1} \left((1 - \nu(\xi)) \chi(\xi) (-i\xi)^\alpha f_{jk}^{(n)}(\xi, \tau) \right) (x) + F_{jk}^{(n)}(x, \tau), \quad (\text{C.30})$$

where $\chi(\xi)$ is the indicator function of the set $|\xi| \leq 2$ and

$$F_{jk}^{(n)}(x, \tau) = (2\pi)^{-3} \int_{|\xi| \geq 2} e^{-ix\xi} (-i\xi)^\alpha f_{jk}^{(n)}(\xi, \tau) d\xi.$$

The first summand in (C.30) belongs to $C^\infty(\mathbb{R}^3)$ as the inverse Fourier transform of a distribution with compact support. For the second summand the following estimates hold (see, e.g., [28]):

$$\begin{aligned}
|F_{jk}^{(n)}(x, \tau)| & \leq c(\tau) |x|^{-(n+|\alpha|+3)}, \quad \text{if } |\alpha| > -n - 3, \\
|F_{jk}^{(n)}(x, \tau)| & \leq c(\tau) \log(|x|^{-1}), \quad \text{if } |\alpha| = -n - 3, \\
|F_{jk}^{(n)}(x, \tau)| & \leq c(\tau), \quad \text{if } |\alpha| < -n - 3.
\end{aligned} \quad (\text{C.31})$$

Provided that $|\alpha| \leq 2$, the last summand in (8.3) is continuous since it is the inverse Fourier transform of the summable function $(1 - \nu(\xi)) (-i\xi)^\alpha f_{jk}^{(-6)}(\xi, \tau)$. This completes the proof. \square

Theorem 8.4. *For arbitrary multi-index α the fundamental matrix $\Psi(x, \tau)$ admits the following estimates at infinity (as $|x| \rightarrow \infty$)*

$$|\partial^\alpha \Psi_{kj}(x, \tau)| \leq c_1 |x|^{-3-|\alpha|}, \quad k, j = 1, \dots, 5, \quad k + j \neq 10, \quad (\text{C.32})$$

$$|\partial^\alpha \Psi_{55}(x, \tau)| \leq c_1 |x|^{-1-|\alpha|} \quad (\text{C.33})$$

with some constant $c_1 > 0$ depending on the material constants, the multi-index α and the parameter τ .

Proof. To prove the theorem we need the following representation of $[A(-i\xi, \tau)]^{-1}$ in a neighbourhood of the origin:

$$[A_{jk}(-i\xi, \tau)]^{-1} = \sum_{n=-2}^0 g_{jk}^{(n)}(\xi, \tau) + g_{jk}^{(1)}(\xi, \tau), \quad (\text{C.34})$$

where $g_{jk}^{(n)}(\xi, \tau)$ for $n = -2, -1, 0$, are homogeneous functions of order n with respect to ξ and $g_{jk}^{(n)}(\xi, \tau) = 0$ for $n = -2, -1$, and $k + j \neq 10$.

Moreover, the estimates

$$\begin{aligned} |\partial_\xi^\alpha g_{jk}^{(1)}(\xi, \tau)| &\leq c(\tau)|\xi|^{1-|\alpha|} \text{ for } |\xi| < 1 \text{ and} \\ |\partial_\xi^\alpha g_{jk}^{(1)}(\xi, \tau)| &\leq c(\tau)|\xi|^{N-|\alpha|} \text{ for } |\xi| \geq 1 \end{aligned}$$

hold for some N .

The representation (C.34) can be obtained from (C.26) with

$$P = \Lambda(\xi, \tau), \quad Q = Q^{(2)} := \sum_{n=2}^5 \Lambda^{(2n)}(\xi, \tau), \quad R = \Lambda^{(1)}(\xi, \tau).$$

Applying the inverse Fourier transform to both sides of (C.34), we get

$$\begin{aligned} \Psi_{jk}(x, \tau) &= \sum_{n=-2}^0 \mathcal{F}_{\xi \rightarrow x}^{-1}(g_{jk}^{(n)}(\xi, \tau))(x) + \mathcal{F}_{\xi \rightarrow x}^{-1}(\nu(\xi)g_{jk}^{(1)}(\xi, \tau))(x) + \\ &\quad + \mathcal{F}_{\xi \rightarrow x}^{-1}\left((1 - \nu(\xi))g_{jk}^{(1)}(\xi, \tau)\right)(x), \end{aligned} \quad (\text{C.35})$$

where $\nu(\cdot)$ is the cut-off function introduced in the proof of Theorem 8.3.

It is evident that the expressions $\mathcal{F}_{\xi \rightarrow x}^{-1}(g_{jk}^{(n)}(\xi, \tau))(x)$ are inverse Fourier transforms of homogeneous functions of order n and hence are homogeneous of order $-3 - n$; note that $\mathcal{F}_{\xi \rightarrow x}^{-1}(g_{jk}^{(n)}(\xi, \tau))(x) = 0$ for $n = -1, -2$, and $k + j \neq 10$.

If $|\beta| \leq |\alpha| + 3$, then $(-i\xi)^\alpha \partial_\xi^\beta (\nu(\xi)g_{jk}^{(1)}(\xi, \tau))$ belongs to $L_1(\mathbb{R}^3)$ and $x^\beta \partial_x^\alpha \mathcal{F}_{\xi \rightarrow x}^{-1}(\nu(\xi)g_{jk}^{(1)}(\xi, \tau))$ vanishes at infinity, i.e.,

$$\left| \partial_x^\alpha \mathcal{F}_{\xi \rightarrow x}^{-1}(\nu(\xi)g_{jk}^{(1)}(\xi, \tau)) \right| \leq c|x|^{-3-|\alpha|}.$$

For the last summand in (C.35) we have

$$\left| \partial_x^\alpha \mathcal{F}_{\xi \rightarrow x}^{-1}\left((1 - \nu(\xi))g_{jk}^{(1)}(\xi, \tau)\right)(x) \right| \leq c'|x|^{-N}, \quad c' = \text{const} > 0, \quad (\text{C.36})$$

for sufficiently large $|x|$ and for arbitrary α and N . To prove this, note that if $|\beta| \geq N + 4 + |\alpha|$, then $(-i\xi)^\alpha \partial_\xi^\beta ((1 - \nu(\xi))g_{jk}^{(1)}(\xi, \tau)) \in L_1(\mathbb{R}^3)$. Hence

$$\left| \partial_x^\alpha \mathcal{F}_{\xi \rightarrow x}^{-1}\left((1 - \nu(\xi))g_{jk}^{(1)}(\xi, \tau)\right) \right| \leq c''|x|^{-|\beta|}, \quad c'' = \text{const} > 0.$$

This completes the proof. \square

8.4. Fundamental matrices of statics of thermopiezoelectricity:

transversally isotropic case. The equation of thermopiezoelectricity in matrix form for a transversally isotropic medium can be rewritten as $A(\partial)U = F$, where $A(\partial) = A(\partial, 0) = [A_{ik}(\partial)]_{5 \times 5}$ with

$$\begin{aligned} A_{11}(\partial) &= c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2, & A_{12}(\partial) &= A_{21}(\partial) = (c_{11} - c_{66})\partial_1\partial_2, \\ A_{13}(\partial) &= A_{31}(\partial) = (c_{13} + c_{44})\partial_1\partial_3, & A_{14}(\partial) &= -\gamma_1\partial_1, \\ A_{15}(\partial) &= -A_{51}(\partial) = (e_{15} + e_{31})\partial_1\partial_3, & A_{22}(\partial) &= c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2, \\ A_{23}(\partial) &= A_{32}(\partial) = (c_{13} + c_{44})\partial_2\partial_3, & A_{24}(\partial) &= -\gamma_1\partial_2, \end{aligned}$$

$$\begin{aligned}
A_{25}(\partial) &= -A_{52}(\partial) = (e_{15} + e_{31})\partial_2\partial_3, & A_{33}(\partial) &= c_{44}(\partial_1^2 + \partial_2^2) + c_{33}\partial_3^2, \\
A_{34}(\partial) &= -\gamma_3\partial_3, & A_{35}(\partial) &= -A_{53}(\partial) = e_{15}(\partial_1^2 + \partial_2^2) + e_{33}\partial_3^2, \\
A_{4k}(\partial) &= 0, \quad k = 1, 2, 3, & A_{44}(\partial) &= \varkappa_{11}(\partial_1^2 + \partial_2^2) + \varkappa_{33}\partial_3^2, \\
A_{45}(\partial) &= 0, & A_{54}(\partial) &= -g_3\partial_3, & A_{55}(\partial) &= \varepsilon_{11}(\partial_1^2 + \partial_2^2) + \varepsilon_{33}\partial_3^2.
\end{aligned}$$

Denote by $A(-i\xi)$ the symbol matrix of the operator $A(\partial)$. We get

$$\begin{aligned}
\det A(-i\xi) &= -[c_{66}(\xi_1^2 + \xi_2^2) + c_{44}\xi_3^2] [\varkappa_{11}(\xi_1^2 + \xi_2^2) + \varkappa_{33}\xi_3^2] \times \\
&\times \left\{ c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}) + (\xi_1^2 + \xi_2^2)^3 + [c_{11}(2e_{15}e_{33} + c_{33}\varepsilon_{11}) - c_{13}^2\varepsilon_{11} - \right. \\
&\quad - 2c_{13}(e_{15}(e_{15} + e_{31}) + c_{44}\varepsilon_{11}) + c_{44}(e_{31}^2 + c_{11}\varepsilon_{33})] (\xi_1^2 + \xi_2^2)^2 \xi_3^2 + \\
&\quad + \left[e_{33}(c_{11}e_{33} - 2c_{44}e_{31} - 2c_{13}(e_{15} + e_{31})) - c_{13}(c_{13} + 2c_{44})\varepsilon_{33} + \right. \\
&\quad \left. + c_{33}((e_{15} + e_{31})^2 + c_{44}\varepsilon_{11} + c_{11}\varepsilon_{33}) \right] (\xi_1^2 + \xi_2^2)\xi_3^4 + \\
&\quad \left. + c_{44}(e_{33}^2 + c_{33}\varepsilon_{33})\xi_3^6 \right\}. \quad (\text{C.37})
\end{aligned}$$

Let a_k ($k = 1, 2, 3$) be the roots of the equation with respect to ζ

$$\begin{aligned}
c_{11}(e_{15}^2 + c_{44}\varepsilon_{11})\zeta^3 - \left\{ -c_{13}^2\varepsilon_{11} + c_{11}(2e_{15}e_{33} + c_{33}\varepsilon_{11}) - \right. \\
\left. - 2c_{13}[e_{15}(e_{15} + e_{31}) + c_{44}\varepsilon_{11}] + c_{44}(e_{31}^2 + c_{11}\varepsilon_{33}) \right\} \zeta^2 + \\
+ \left\{ e_{33}[-2c_{44}e_{31} - 2c_{13}(e_{15} + e_{31}) + c_{11}e_{33}] - \right. \\
\left. - c_{13}(c_{13} + 2c_{44})\varepsilon_{33} + c_{33}[(e_{15} + e_{31})^2 + c_{44}\varepsilon_{11} + c_{11}\varepsilon_{33}] \right\} \zeta - \\
- c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}) = 0, \quad (\text{C.38})
\end{aligned}$$

and let $a_4 = c_{44}/c_{66}$, $a_5 = \varkappa_{33}/\varkappa_{11}$. We can then rewrite (C.37) in the form:

$$\begin{aligned}
\det A(-i\xi) &= \\
&= -a(\rho^2 + a_1\xi_3^2)(\rho^2 + a_2\xi_3^2)(\rho^2 + a_3\xi_3^2)(\rho^2 + a_4\xi_3^2)(\rho^2 + a_5\xi_3^2), \quad (\text{C.39})
\end{aligned}$$

where $a = c_{11}c_{66}\varkappa_{11}(e_{15}^2 + c_{44}\varepsilon_{11})$, $\rho^2 = \xi_1^2 + \xi_2^2$. In what follows, we assume that $a_j \neq a_k$ for $k \neq j$, $k, j = \overline{1, 5}$.

Note that $A(\partial)$ is an elliptic operator: $\det A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$. Therefore, $a_j \in \mathbb{C} \setminus (-\infty, 0]$, $j = \overline{1, 5}$.

We have to find a fundamental matrix $\Psi = [\Psi_{kj}]_{5 \times 5}$ of the operator $A(\partial)$: $A(\partial)\Psi(\cdot) = \delta(\cdot)I_5$, where δ is Dirac's distribution.

To this end, let us find a solution φ of the scalar equation

$$\begin{aligned}
\det A(\partial)\varphi(\cdot) &:= a(\Delta_2 + a_1\partial_3^2)(\Delta_2 + a_2\partial_3^2) \times \\
&\times (\Delta_2 + a_3\partial_3^2)(\Delta_2 + a_4\partial_3^2)(\Delta_2 + a_5\partial_3^2)\varphi(\cdot) = \delta(\cdot) \quad (\text{C.40})
\end{aligned}$$

with $\Delta_2 = \partial_1^2 + \partial_2^2$.

Applying the Fourier transform to both sides of (C.40), we get

$$-a(\rho^2 + a_1\xi_3^2)(\rho^2 + a_2\xi_3^2)(\rho^2 + a_3\xi_3^2)(\rho^2 + a_4\xi_3^2)(\rho^2 + a_5\xi_3^2)\widehat{\varphi} = 1, \quad (\text{C.41})$$

where $\widehat{\varphi}$ is the Fourier transform of φ .

Let $b_j, j = \overline{1, n}$, be different complex numbers ($b_i \neq b_j$ for $i \neq j$) and show that

$$\xi_3^{2(n-1)} = \sum_{k=1}^n d_k (\rho^2 + b_1\xi_3^2)(\rho^2 + b_2\xi_3^2) \cdots (\rho^2 + b_n\xi_3^2) \frac{1}{\rho^2 + b_k\xi_3^2}, \quad (\text{C.42})$$

where

$$d_k = \left[(b_1 - b_k)(b_2 - b_k) \cdots (b_{k-1} - b_k)(b_{k+1} - b_k) \cdots (b_n - b_k) \right]^{-1}. \quad (\text{C.43})$$

We can prove (C.42) as follows. Let $C(r)$ be a circle in the complex λ plane with the radius r which encloses the points $b_0 = -\rho^2/\xi_3^2$ and $b_k, k = \overline{1, n}$. Due to the Cauchy theorem, we have

$$\begin{aligned} \xi_3^{2(n-1)} - \sum_{k=1}^n d_k (\rho^2 + b_1\xi_3^2)(\rho^2 + b_2\xi_3^2) \cdots (\rho^2 + b_n\xi_3^2) \frac{1}{(\rho^2 + b_k\xi_3^2)} &= \\ &= \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C(r)} \frac{(\rho^2 + b_1\xi_3^2) \cdots (\rho^2 + b_n\xi_3^2)}{(\rho^2 + \lambda\xi_3^2)(b_1 - \lambda) \cdots (b_n - \lambda)} d\lambda = 0. \end{aligned}$$

Consider the identity (C.42) for $n = 5$ and $b_k = a_k$, where a_k are the above introduced roots of the equation (C.38). By multiplying the both sides by $\widehat{\varphi}$ and taking into account (C.41), we derive

$$\xi_3^8 \widehat{\varphi} = -\frac{1}{a} \sum_{k=1}^5 \frac{d_k}{\rho^2 + a_k \xi_3^2},$$

whence by the inverse Fourier transform

$$\partial_3^8 \varphi(x) = -\frac{1}{a} \sum_{k=1}^5 d_k F_{\xi \rightarrow x}^{-1} \left[\frac{1}{\xi_1^2 + \xi_2^2 + a_k \xi_3^2} \right] = -\frac{1}{4\pi a} \sum_{k=1}^5 \frac{d_k}{|x|_k}, \quad (\text{C.44})$$

where $|x|_k = \sqrt{a_k(x_1^2 + x_2^2) + x_3^2}$. If a_k is complex, then we choose that branch of \sqrt{z} for which $\sqrt{1} = 1$ ($-\pi < \arg z < \pi$). This implies that $\Re|x|_k \geq 0$ for all $x \in \mathbb{R}^3$.

One of the solutions of the equation (C.44) has the form

$$\varphi = \frac{1}{4\pi a} \sum_{k=1}^5 d_k \varphi_k, \quad (\text{C.45})$$

where

$$\begin{aligned} \varphi_k = \frac{1}{2822400} \left\{ \left[256a_k^3(x_1^2 + x_2^2)^3 - 5175a_k^2(x_1^2 + x_2^2)^2 x_3^2 + \right. \right. \\ \left. \left. + 8132a_k(x_1^2 + x_2^2)x_3^4 - 1452x_3^6 \right] |x|_k + \right. \end{aligned}$$

$$\begin{aligned}
& + 35 \left[-35a_k^3(x_1^2 + x_2^2)^3 + 210a_k^2(x_1^2 + x_2^2)^2 x_3^2 - \right. \\
& \left. - 168a_k(x_1^2 + x_2^2)x_3^4 + 16x_3^6 \right] x_3 \log(|x|_k + x_3) \Big\} = \\
& = |x|_k P(x, a_k) + Q(x, a_k) \log(|x|_k + x_3) \quad (C.46)
\end{aligned}$$

with

$$\begin{aligned}
P(x, a_k) = \frac{1}{2822400} \left[256a_k^3(x_1^2 + x_2^2)^3 - 5175a_k^2(x_1^2 + x_2^2)^2 x_3^2 + \right. \\
\left. + 8132a_k(x_1^2 + x_2^2)x_3^4 - 1452x_3^6 \right], \quad (C.47)
\end{aligned}$$

$$\begin{aligned}
Q(x, a_k) = \frac{35}{2822400} x_3 \left[-35a_k^3(x_1^2 + x_2^2)^3 + 210a_k^2(x_1^2 + x_2^2)^2 x_3^2 - \right. \\
\left. - 168a_k(x_1^2 + x_2^2)x_3^4 + 16x_3^6 \right]. \quad (C.48)
\end{aligned}$$

Here we choose the branch of $\log z$ in accordance with $-\pi < \arg z < \pi$ and $\log 1 = 0$.

It is clear, that the functions φ_k , $k = \overline{1, 5}$, as well as φ are infinitely differentiable functions in the set $D = \{x \in \mathbb{R}^3 : x_3 \notin (-\infty, 0]\}$. Now we prove that the function φ can be extended to an infinitely differentiable function in $\mathbb{R}^3 \setminus \{0\}$. From (C.46) then it follows that φ is a homogeneous function of degree 7 in $\mathbb{R}^3 \setminus \{0\}$.

If $x_1^2 + x_2^2 \neq 0$, then

$$\begin{aligned}
\log(|x|_k + x_3) &= \log \left(\frac{a_k(x_1^2 + x_2^2)}{|x|_k - x_3} \right) = \\
&= \log a_k + \log(x_1^2 + x_2^2) - \log(|x|_k - x_3) + 2n\pi i
\end{aligned}$$

with some integer n . Recalling that $a_k \neq (-\infty, 0]$, we can conclude that actually $n = 0$, i.e.,

$$\log(|x|_k + x_3) = \log \left(\frac{a_k(x_1^2 + x_2^2)}{|x|_k - x_3} \right) = \log a_k + \log(x_1^2 + x_2^2) - \log(|x|_k - x_3).$$

Using this equality, φ can be represented as

$$\begin{aligned}
\varphi(x) &= \frac{1}{4\pi a} \sum_{k=1}^5 d_k |x|_k P(x, a_k) + \frac{1}{4\pi a} \sum_{k=1}^5 d_k Q(x, a_k) \log a_k - \\
&\quad - \frac{1}{4\pi a} \sum_{k=1}^5 d_k Q(x, a_k) \log(|x|_k - x_3) + \frac{1}{4\pi a} \sum_{k=1}^5 d_k Q(x, a_k) \log(x_1^2 + x_2^2).
\end{aligned}$$

All the terms in the right hand side of this equality except the fourth summand are infinitely differentiable functions in a neighborhood of an arbitrary point of the set $\{x \in \mathbb{R}^3 : -\infty < x_3 < 0\}$. As for the last sum, we can prove that

$$\sum_{k=1}^5 d_k Q(x, a_k) = 0. \quad (C.49)$$

Indeed, assume that $b_j, j = 1, \dots, n$, are different complex numbers, d_k are given by (C.43) and the degree of the polynomial Q is less than $n - 1$. Then by Cauchy's theorem

$$\sum_{k=1}^n d_k Q(b_k) = - \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C(r)} \frac{Q(\lambda)}{(b_1 - \lambda) \cdots (b_n - \lambda)} d\lambda = 0, \quad (\text{C.50})$$

whence (C.49) follows due to (C.48).

So we have

$$\begin{aligned} \varphi(x) &= \frac{1}{4\pi a} \sum_{k=1}^5 d_k |x|_k P(x, a_k) + \frac{1}{4\pi a} \sum_{k=1}^5 d_k Q(x, a_k) \log a_k - \\ &\quad - \frac{1}{4\pi a} \sum_{k=1}^5 d_k Q(x, a_k) \log(|x|_k - x_3). \end{aligned} \quad (\text{C.51})$$

The function φ given by (C.51) can be extended onto the half-space $x_3 < 0$ as an infinitely differentiable function. Moreover, (C.51) implies that the extended function (denote it again by φ) is a homogeneous function of degree 7.

It can easily be checked that for $x_3 \notin (-\infty, 0)$

$$\begin{aligned} (\partial_1^2 + \partial_2^2 + a_k \partial_3^2) \varphi_k(x) &= \\ &= \frac{1}{960} a_k x_3 \left[-5a_k^2 (x_1^2 + x_2^2)^2 + 20a_k (x_1^2 + x_2^2) x_3^2 - 8x_3^4 \right]. \end{aligned} \quad (\text{C.52})$$

Therefore, φ is well defined in $\mathbb{R}^3 \setminus \{0\}$ and solves the equation

$$\begin{aligned} \det A(\partial) \varphi &= a(\Delta_2 + a_1 \partial_3^2)(\Delta_2 + a_2 \partial_3^2)(\Delta_2 + a_3 \partial_3^2) \times \\ &\quad \times (\Delta_2 + a_4 \partial_3^2)(\Delta_2 + a_5 \partial_3^2) \varphi = 0 \end{aligned} \quad (\text{C.53})$$

in $\mathbb{R}^3 \setminus \{0\}$. Taking into account that φ is a homogeneous function of degree 7 and that the support of the distribution $\det A(\partial) \varphi$ is an isolated point (the origin), we can conclude that $\det A(\partial) \varphi = c \delta(\cdot)$ with some $c \neq 0$.

From (C.40)–(C.44) it can be deduced that

$$\partial_3^8 \varphi = \frac{1}{4\pi a} \sum_{k=1}^5 d_k F_{\xi \rightarrow x}^{-1} \left[\frac{1}{\xi_1^2 + \xi_2^2 + a_k \xi_3^2} \right] = -\frac{c}{4\pi a} \sum_{k=1}^5 d_k \frac{1}{|x|_k}, \quad (\text{C.54})$$

which together with (C.44) yields $c = 1$. Thus $\det A(\partial) \varphi = \delta(\cdot)$.

Let us set $\Psi := M(\partial)(\varphi I_5)$, where $M(-i\xi)$ is the matrix of co-factors corresponding to the matrix $A(-i\xi)$. This means that $M(\partial)$ is the matrix operator constructed by the formal co-factors of the matrix operator $A(\partial)$, i.e., $A(\partial)M(\partial) = M(\partial)A(\partial) = \det A(\partial)I_5$. Clearly we have

$$A(\partial)\Psi = A(\partial)M(\partial)(\varphi I_5) = (\det A(\partial)\varphi)I_5 = \delta(\cdot)I_5,$$

i.e., Ψ is a fundamental matrix of the operator $A(\partial)$ and it can be written in the form

$$\Psi(x) = \left[\sum_{k=1}^5 d_k M_{ij}(\partial) \varphi_k(x) \right]_{5 \times 5} = -\frac{1}{4\pi a} [\Phi_{ij}]_{5 \times 5}. \quad (\text{C.55})$$

Here

$$\begin{aligned} M_{11}(\partial) = & (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \left\{ - \left[(e_{15} + e_{31})(c_{13}e_{15} - c_{44}e_{31})\Delta + \right. \right. \\ & + [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \partial_3^2 + \\ & + (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \left[- (c_{13} + c_{44})^2 \partial_2^2 \partial_3^2 + (c_{44}\Delta + \right. \\ & + c_{33}\partial_3^2)(c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2) \left. \right] + \\ & + (e_{15}\Delta + e_{33}\partial_3^2) \left[- (c_{13} + c_{44})(e_{15} + e_{31})\partial_2^2 \partial_3^2 + \right. \\ & \left. \left. + (c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2)(e_{15}\Delta + e_{33}\partial_3^2) \right] \right\}, \end{aligned}$$

$$\begin{aligned} M_{12}(\partial) = M_{21}(\partial) = & -(\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \left\{ - (e_{15} + e_{31}) \left[(c_{13}e_{15} - c_{44}e_{31})\Delta - \right. \right. \\ & - [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \partial_3^2 + \\ & + (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \left[- (c_{13} + c_{44})^2 \partial_3^2 + (c_{11} - c_{66})(c_{44}\Delta + c_{33}\partial_3^2) \right] + \\ & + (e_{15}\Delta + e_{33}\partial_3^2) \left[- (c_{13} + c_{44})(e_{15} + e_{31})\partial_3^2 + \right. \\ & \left. \left. + (c_{11} - c_{66})(e_{15}\Delta + e_{33}\partial_3^2) \right] \right\} \partial_1 \partial_2, \end{aligned}$$

$$\begin{aligned} M_{13}(\partial) = M_{31}(\partial) = & -(c_{66}\Delta + c_{44}\partial_3^2) \left[[e_{15}(e_{15} + e_{31}) + (c_{13} + c_{44})\varepsilon_{11}]\Delta + \right. \\ & \left. + [(e_{15} + e_{31})e_{33} + (c_{13} + c_{44})\varepsilon_{33}] \partial_3^2 \right] (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \partial_1 \partial_3, \end{aligned}$$

$$\begin{aligned} M_{14}(\partial) = & (c_{66}\Delta + c_{44}\partial_3^2) \left\{ \left[(c_{13}e_{15} - c_{44}e_{31})\Delta - \right. \right. \\ & - [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \left. \right] g_3 \partial_3^2 + \\ & + (e_{15}\Delta + e_{33}\partial_3^2) \left[e_{15}\gamma_1 \Delta + [e_{33}\gamma_1 - (e_{15} + e_{31})\gamma_3] \partial_3^2 \right] + \\ & \left. + [c_{44}\gamma_1 \Delta + [c_{33}\gamma_1 - (c_{13} + c_{44})\gamma_3] \partial_3^2] (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \right\} \partial_1, \end{aligned}$$

$$\begin{aligned} M_{15}(\partial) = -M_{51}(\partial) = & -(c_{66}\Delta + c_{44}\partial_3^2) \left[(-c_{13}e_{15} + c_{44}e_{31})\Delta + \right. \\ & \left. + [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \right] (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \partial_1 \partial_3, \end{aligned}$$

$$\begin{aligned}
M_{22}(\partial) = (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \Big\{ & - \left[(e_{15} + e_{31})(c_{13}e_{15} - c_{44}e_{31})\Delta + \right. \\
& + \left. [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \right] \partial_1^2 \partial_3^2 + \\
& + (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \left[- (c_{13} + c_{44})^2 \partial_1^2 \partial_3^2 + \right. \\
& + \left. (c_{44}\Delta + c_{33}\partial_3^2)(c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2) \right] + \\
& + (e_{15}\Delta + e_{33}\partial_3^2) \left[- (c_{13} + c_{44})(e_{15} + e_{31}) \partial_1^2 \partial_3^2 + \right. \\
& + \left. (c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2)(e_{15}\Delta + e_{33}\partial_3^2) \right] \Big\},
\end{aligned}$$

$$\begin{aligned}
M_{23}(\partial) = M_{32}(\partial) = -(c_{66}\Delta + c_{44}\partial_3^2) \Big\{ & [e_{15}(e_{15} + e_{31}) + (c_{13} + c_{44})\varepsilon_{11}] \Delta + \\
& + [(e_{15} + e_{31})e_{33} + (c_{13} + c_{44})\varepsilon_{33}] \partial_3^2 \Big\} (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \partial_2 \partial_3,
\end{aligned}$$

$$\begin{aligned}
M_{24}(\partial) = (c_{66}\Delta + c_{44}\partial_3^2) \Big\{ & [(c_{13}e_{15} - c_{44}e_{31})\Delta - \\
& - [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2] g_3 \partial_3^2 + \\
& + (e_{15}\Delta + e_{33}\partial_3^2) [e_{15}\gamma_1 \Delta + [e_{33}\gamma_1 - (e_{15} + e_{31})\gamma_3] \partial_3^2] + \\
& + [c_{44}\gamma_1 \Delta + [c_{33}\gamma_1 - (c_{13} + c_{44})\gamma_3] \partial_3^2] (\varepsilon_{11}\Delta + \varepsilon_{33}\partial_3^2) \Big\} \partial_2,
\end{aligned}$$

$$\begin{aligned}
M_{25}(\partial) = -M_{52}(\partial) = -(c_{66}\Delta + c_{44}\partial_3^2) \Big[& (-c_{13}e_{15} + c_{44}e_{31})\Delta + \\
& + [c_{33}(e_{15} + e_{31}) - (c_{13} + c_{44})e_{33}] \partial_3^2 \Big] (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \partial_2 \partial_3,
\end{aligned}$$

$$\begin{aligned}
M_{33}(\partial) = (c_{66}\Delta + c_{44}\partial_3^2) (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \Big[& c_{11}\varepsilon_{11}\Delta^2 + \\
& + [(e_{15} + e_{31})^2 + c_{44}\varepsilon_{11} + c_{11}\varepsilon_{33}] \Delta \partial_3^2 + c_{44}\varepsilon_{33}\partial_3^4 \Big],
\end{aligned}$$

$$\begin{aligned}
M_{34}(\partial) = -(c_{66}\Delta + c_{44}\partial_3^2) \Big\{ & \gamma_1 [e_{15}(e_{15} + e_{31}) + (c_{13} + c_{44})\varepsilon_{11}] \Delta^2 + \\
& + [(e_{15} + e_{31}) [-e_{33}\gamma_1 + (e_{15} + e_{31})\gamma_3] + \\
& + c_{13} [(e_{15} + e_{31})g_3 - \gamma_1\varepsilon_{33}] + c_{44}(e_{31}g_3 + \gamma_3\varepsilon_{11} - \gamma_1\varepsilon_{33})] \Delta \partial_3^2 - \\
& - c_{44}(e_{33}g_3 - \gamma_3\varepsilon_{33}) \partial_3^4 - c_{11} [(e_{15}g_3 - \gamma_3\varepsilon_{11})\Delta + \\
& + (e_{33}g_3 - \gamma_3\varepsilon_{33}) \partial_3^2] \Delta \Big\} \partial_3,
\end{aligned}$$

$$\begin{aligned}
M_{35}(\partial) = -M_{53}(\partial) = -(c_{66}\Delta + c_{44}\partial_3^2) (\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \Big[& c_{11}e_{15}\Delta^2 - \\
& - [c_{44}e_{31} + c_{13}(e_{15} + e_{31}) - c_{11}e_{33}] \Delta \partial_3^2 + c_{44}e_{33}\partial_3^4 \Big],
\end{aligned}$$

$$\begin{aligned}
M_{41}(\partial) &= M_{42}(\partial) = M_{43}(\partial) = M_{45}(\partial) = 0, \\
M_{44}(\partial) &= (c_{66}\Delta + c_{44}\partial_3^2) \left\{ c_{11}(e_{15}^2 + c_{44}\varepsilon_{11})\Delta^3 + \right. \\
&\quad + \left[-c_{13}^2\varepsilon_{11} + c_{11}(2e_{15}e_{33} + c_{33}\varepsilon_{11}) - \right. \\
&\quad \left. - 2c_{13}[e_{15}(e_{15} + e_{31}) + c_{44}\varepsilon_{11}] + c_{44}(e_{31}^2 + c_{11}\varepsilon_{33}) \right] \Delta^2 \partial_3^2 + \\
&\quad + \left[e_{33}[-2c_{44}e_{31} - 2c_{13}(e_{15} + e_{31}) + c_{11}e_{33}] - c_{13}(c_{13} + 2c_{44})\varepsilon_{33} + \right. \\
&\quad \left. + c_{33}[(e_{15} + e_{31})^2 + c_{44}\varepsilon_{11} + c_{11}\varepsilon_{33}] \right] \Delta \partial_3^4 + c_{44}(e_{33}^2 + c_{33}\varepsilon_{33})\partial_3^6 \left. \right\},
\end{aligned}$$

$$\begin{aligned}
M_{54}(\partial) &= (c_{66}\Delta + c_{44}\partial_3^2) \left\{ [(-c_{13}e_{15} + c_{44}e_{31})\gamma_1 + c_{11}(c_{44}g_3 + e_{15}\gamma_3)]\Delta^2 - \right. \\
&\quad - \left[c_{13}^2g_3 - [c_{33}(e_{15} + e_{31}) - c_{44}e_{33}]\gamma_1 + c_{44}e_{31}\gamma_3 + \right. \\
&\quad \left. + c_{13}[2c_{44}g_3 + e_{33}\gamma_1 + (e_{15} + e_{31})\gamma_3] - c_{11}(c_{33}g_3 + e_{33}\gamma_3) \right] \Delta \partial_3^2 + \\
&\quad \left. + c_{44}(c_{33}g_3 + e_{33}\gamma_3)\partial_3^4 \right\} \partial_3,
\end{aligned}$$

$$\begin{aligned}
M_{55}(\partial) &= (c_{66}\Delta + c_{44}\partial_3^2)(\varkappa_{11}\Delta + \varkappa_{33}\partial_3^2) \times \\
&\quad \times \left[c_{11}c_{44}\Delta^2 - (c_{13}^2 - c_{11}c_{33} + 2c_{13}c_{44})\Delta \partial_3^2 + c_{33}c_{44}\partial_3^4 \right].
\end{aligned}$$

To calculate explicitly the entries of the matrix $M(\partial)(\varphi I_5)$, we note that

$$\begin{aligned}
&(\partial_1^2 + \partial_2^2)\varphi_k(x) = \\
&= -a_k\partial_3^2\varphi_k(x) + \frac{1}{960}a_kx_3[-5a_k^2(x_1^2 + x_2^2)^2 + 20a_k(x_1^2 + x_2^2)x_3^2 - 8x_3^4]
\end{aligned}$$

for $x_3 \notin (-\infty, 0]$. This shows that the sixth order derivatives of $\Delta_2\varphi_k = (\partial_1^2 + \partial_2^2)\varphi_k$ and $-a_k\partial_3^2\varphi_k(x)$ coincide.

We need also to simplify the first and the second order derivatives of the functions

$$\begin{aligned}
\psi_k &= \partial_3^6\varphi_k = x_3 \log(x_3 + \sqrt{a_k(x_1^2 + x_2^2) + x_3^2}) - \\
&\quad - \sqrt{a_k(x_1^2 + x_2^2) + x_3^2}, \quad k = \overline{1, 5}.
\end{aligned}$$

It can easily be shown that

$$\begin{aligned}
\partial_i\psi_k &= -\frac{a_k}{(|x|_k + x_3)}, \quad i = 1, 2, \quad \partial_1^2\psi_k = -\frac{a_k}{(|x|_k + x_3)} + \frac{a_k^2x_1^2}{|x|_k(|x|_k + x_3)^2}, \\
\partial_2^2\psi_k &= -\frac{a_k}{(|x|_k + x_3)} + \frac{a_k^2x_2^2}{|x|_k(|x|_k + x_3)^2}, \quad \partial_1\partial_2\psi_k = \frac{a_k^2x_1x_2}{|x|_k(|x|_k + x_3)^2}, \\
\partial_1\partial_3\psi_k &= \frac{a_kx_1}{|x|_k(|x|_k + x_3)}, \quad \partial_2\partial_3\psi_k = \frac{a_kx_2}{|x|_k(|x|_k + x_3)}.
\end{aligned}$$

Moreover, taking into account that if the degree of a polynomial $Q(z)$ is less than 4, then by virtue of (C.49) we can derive

$$\sum_{k=1}^5 \frac{d_k a_k Q(a_k)}{|x|_k + x_3} = \sum_{k=1}^4 d_k Q(a_k) \frac{|x|_k - |x|_5}{x_1^2 + x_2^2} = \sum_{k=1}^4 \frac{d_k Q(a_k)(a_k - a_5)}{|x|_k + |x|_5}, \quad (\text{C.56})$$

$$\begin{aligned} & \sum_{k=1}^5 \frac{d_k a_k Q(a_k)}{|x|_k (|x|_k + x_3)} = \\ & = \sum_{k=1}^4 \frac{d_k Q(a_k)}{x_1^2 + x_2^2} \left(\frac{|x|_k - x_3}{|x|_k} - \frac{|x|_5 - x_3}{|x|_5} \right) = \sum_{k=1}^4 \frac{d_k Q(a_k)(a_k - a_5)x_3}{|x|_k |x|_5 (|x|_k + |x|_5)}, \quad (\text{C.57}) \end{aligned}$$

$$\begin{aligned} & \sum_{k=1}^5 \frac{d_k a_k^2 Q(a_k)}{|x|_k (|x|_k + x_3)^2} = \\ & = \sum_{k=1}^4 \frac{d_k Q(a_k)}{|x|_k |x|_5 (x_1^2 + x_2^2)^2} \left(|x|_5 (|x|_k - x_3)^2 - |x|_k (|x|_5 - x_3)^2 \right) = \\ & = \sum_{k=1}^4 \frac{d_k Q(a_k)(a_k - a_5)(a_k a_5 (x_1^2 + x_2^2) + (a_k + a_5)x_3^2)}{|x|_k |x|_5 (|x|_k + |x|_5)(|x|_k |x|_5 + x_3^2)}. \quad (\text{C.58}) \end{aligned}$$

Note that the element Φ_{34} can be written in the form

$$\sum_{k=1}^5 d_k Q(a_k) \log(|x|_k + x_3), \quad j = 1, 2, 3, \quad (\text{C.59})$$

where $Q(z)$ is some polynomial of degree less than 4. This kind of summands need a special consideration. Namely, from (C.49) it follows that

$$\sum_{k=1}^5 d_k Q(a_k) \log(|x|_k + x_3) = \sum_{k=1}^4 d_k Q(a_k) \log \left(\frac{|x|_k + x_3}{|x|_5 + x_3} \right). \quad (\text{C.60})$$

Further, the following identity

$$\frac{|x|_k + x_3}{|x|_5 + x_3} = 1 + \frac{(a_k - a_5)(|x|_5 - x_3)}{a_5(|x|_k + |x|_5)}, \quad k = 1, \dots, 4,$$

shows that the expressions under the logarithmic function in the right hand side of (C.60) are bounded and do not vanish for $x \neq 0$.

Taking into account (C.55)-(C.60), we obtain the explicit expressions for the entries of the fundamental matrix

$$\begin{aligned} \Psi &= -\frac{1}{4\pi a} [\Phi_{ij}]_{5 \times 5} : \\ \Phi_{11} &= \sum_{k=1}^4 \left\{ \frac{1}{|x|_k^2 (|x|_k |x|_5 + x_3^2)} \left\{ (a_5 - a_k)(e_{15} + e_{31}) [c_{33}(e_{15} + e_{31}) + \right. \right. \\ & \quad \left. \left. + a_k(c_{13}e_{15} - c_{44}e_{31}) - (c_{13} + c_{44})e_{33} \right\} \times \right. \\ & \quad \left. \times \left(a_5 a_k |x|_5 |x|_k x_1^2 (x_1^2 + x_2^2) + [1 + (a_5 + a_k)x_1^2] |x|_5 |x|_k x_3^2 + x_3^4 \right) \right\} + \end{aligned}$$

$$\begin{aligned}
& + \frac{(a_k e_{15} - e_{33})}{|x|_k (|x|_k |x|_5 + x_3^2)} (a_5 - a_k) (c_{13} + c_{44}) (e_{15} + e_{31}) \times \\
& \times \left\{ a_5 a_k |x|_5 |x|_k x_1^2 (x_1^2 + x_2^2) + |x|_5 |x|_k [1 + (a_5 + a_k) x_1^2] x_3^2 + x_3^4 \right\} + \\
& + \frac{(a_5 - a_k)}{|x|_k (|x|_k |x|_5 + x_3^2)} \left\{ (e_{33} - a_k e_{15} + a_k \varepsilon_{11} - \varepsilon_{33}) \times \right. \\
& \times \left[c_{11} (1 + a_k x_1^2) + c_{66} (1 + a_k x_2^2) x_3^2 + (c_{11} + c_{66}) x_3^4 + \right. \\
& \left. \left. + a_5 |x|_5 |x|_k (c_{11} x_1^2 + c_{66} x_2^2) [a_k (x_1^2 + x_2^2) + x_3^2] \right] \right\} |x|_5 |x|_k \Big\}, \\
\Phi_{12} = \Phi_{21} &= \sum_{k=1}^4 \frac{x_1 x_2 (a_k - a_5) [a_k a_5 (x_1^2 + x_2^2) + (a_k + a_5) x_3^2]}{(|x|_k + |x|_5) (|x|_k |x|_5 + x_3^2)} \times \\
& \times \left\{ (e_{15} + e_{31}) [(c_{13} + c_{44}) e_{33} + a_k (c_{44} c_{31} - c_{13} e_{15}) - c_{33} (e_{15} + e_{31})] + \right. \\
& + (a_k e_{15} - e_{33}) [- (c_{13} + c_{44}) (e_{15} + e_{31}) + (c_{11} - c_{66}) (e_{33} - a_k e_{15})] - \\
& - \left[- (c_{13} + c_{44})^2 + (c_{33} - a_k c_{44})^2 + \right. \\
& \left. \left. + (c_{33} - a_k c_{44}) (c_{11} - c_{66}) \right] (\varepsilon_{33} - a_k \varepsilon_{11}) \right\}, \\
\Phi_{13} = \Phi_{31} &= \sum_{k=1}^4 \frac{x_1 x_3}{|x|_k |x|_5 (|x|_k + |x|_5)} (a_k - a_5) (a_k c_{66} - c_{44}) \times \\
& \times \left\{ (e_{15} + e_{31}) e_{33} - a_k [e_{15} (e_{15} + e_{31}) + (c_{13} + c_{44}) \varepsilon_{11}] + (c_{13} + c_{44}) \varepsilon_{33} \right\}, \\
\Phi_{14} &= \sum_{k=1}^4 \frac{x_1}{|x|_k + |x|_5} (a_5 - a_k) (c_{44} - a_k c_{66}) \times \\
& \times \left\{ [-c_{33} (e_{15} + e_{31}) - a_k (c_{13} e_{15} + c_{44} e_{31}) + (c_{13} + c_{44}) e_{33}] g_3 + \right. \\
& + (a_k e_{15} - e_{33}) [(a_k e_{15} - e_{33}) \gamma_1 + (e_{15} + e_{31}) \gamma_3] + \\
& \left. + [(c_{33} - a_k c_{44}) \gamma_1 - (c_{13} + c_{44}) \gamma_3] (\varepsilon_{33} - a_k \varepsilon_{11}) \right\}, \\
\Phi_{15} = -\Phi_{51} &= \sum_{k=1}^4 \frac{x_1 x_3}{|x|_k |x|_5 (|x|_k + |x|_5)} (a_k - a_5) (c_{44} - a_k c_{66}) \times \\
& \times [-c_{33} (e_{15} + e_{31}) + a_k (c_{44} e_{13} - c_{13} e_{15}) + (c_{13} + c_{44}) e_{33}], \\
\Phi_{22} &= \sum_{k=1}^4 \left\{ \frac{(a_5 - a_k) (e_{15} + e_{31})}{|x|_k^2 (|x|_k |x|_5 + x_3^2)} \times \right. \\
& \times [c_{33} (e_{15} + e_{31}) + a_k (c_{13} e_{15} - c_{44} e_{31}) - (c_{13} + c_{44}) e_{33}] \times \\
& \times \left. \left(a_5 a_k |x|_5 |x|_k x_2^2 (x_1^2 + x_2^2) + [1 + (a_5 + a_k) x_2^2] |x|_5 |x|_k x_3^2 + x_3^4 \right) + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{(a_k e_{15} - e_{33})(a_5 - a_k)(c_{13} + c_{44})(e_{15} + e_{31})}{|x|_k(|x|_k|x|_5 + x_3^2)} \times \\
& \times \left\{ a_5 a_k |x|_5 |x|_k x_2^2 (x_1^2 + x_2^2) + |x|_5 |x|_k [1 + (a_5 + a_k)x_2^2] x_3^2 + x_3^4 \right\} + \\
& + \frac{(a_5 - a_k)}{|x|_k(|x|_k|x|_5 + x_3^2)} \left\{ (e_{33} - a_k e_{15} + a_k \varepsilon_{11} - \varepsilon_{33}) \times \right. \\
& \times \left[c_{11}(1 + a_k x_2^2) + c_{66}(1 + a_k x_1^2)x_3^2 + (c_{11} + c_{66})x_3^4 + \right. \\
& \left. \left. + a_5 |x|_5 |x|_k (c_{11} x_2^2 + c_{66} x_1^2) [a_k (x_1^2 + x_2^2) + x_3^2] \right] \right\} |x|_5 |x|_k \Big\}, \\
\Phi_{23} = \Phi_{32} &= \sum_{k=1}^4 \frac{x_2 x_3}{|x|_k |x|_5 (|x|_k + |x|_5)} (a_k - a_5)(a_k c_{66} - c_{44}) \left\{ (e_{15} + e_{31}) e_{33} - \right. \\
& \left. - a_k [e_{15}(e_{15} + e_{31}) + (c_{13} + c_{44}) \varepsilon_{11}] + (c_{13} + c_{44}) \varepsilon_{33} \right\}, \\
\Phi_{24} &= \sum_{k=1}^4 \frac{x_2 (a_5 - a_k)(c_{44} - a_k c_{66})}{|x|_k + |x|_5} \times \\
& \times \left\{ \left[-c_{33}(e_{15} + e_{31}) - a_k (c_{13} e_{15} - c_{44} e_{31}) + (c_{13} + c_{44}) e_{33} \right] g_3 + \right. \\
& + (a_k e_{15} - e_{33}) \left[(a_k e_{15} - e_{33}) \gamma_1 + (e_{15} + e_{31}) \gamma_3 \right] + \\
& \left. + \left[(c_{33} - a_k c_{44}) \gamma_1 - (c_{13} + c_{44}) \gamma_3 \right] (\varepsilon_{33} - a_k \varepsilon_{11}) \right\}, \\
\Phi_{25} = -\Phi_{52} &= \sum_{k=1}^4 \frac{x_2 x_3}{|x|_k |x|_5 (|x|_k + |x|_5)} (a_k - a_5)(c_{44} - a_k c_{66}) \times \\
& \times \left[-c_{33}(e_{15} + e_{31}) + a_k (c_{44} e_{13} - c_{13} e_{15}) + (c_{13} + c_{44}) e_{33} \right], \\
\Phi_{33} &= \sum_{k=1}^5 \frac{(c_{44} - a_k c_{66})}{|x|_k} \times \\
& \times \left\{ a_k^2 c_{11} \varepsilon_{11} + c_{44} e_{33} - a_k [(e_{15} + e_{31})^2 + c_{44} + c_{11} \varepsilon_{33}] \right\}, \\
\Phi_{34} &= \sum_{k=1}^4 \log \left(\frac{|x|_k + x_3}{|x|_5 + x_3} \right) (c_{44} - a_k c_{66}) \times \\
& \times \left\{ - \left[a_k^2 \gamma_1 (e_{15}(e_{15} + e_{31}) + (c_{13} + c_{44}) \varepsilon_{11}) \right] + \right. \\
& + a_k c_{11} \left[e_{33} g_3 + a_k (-e_{15} g_3 + \gamma_3 \varepsilon_{11}) - \gamma_3 \varepsilon_{33} \right] + \\
& + c_{44} (-e_{33} g_3 + \gamma_3 \varepsilon_{33}) - a_k \left[(e_{15} + e_{31}) [-e_{33} \gamma_1 + (e_{15} + e_{31}) \gamma_3] + \right. \\
& \left. \left. + c_{13} [(e_{15} + e_{31}) g_3 - \gamma_1 \varepsilon_{33}] + c_{44} (e_{31} g_3 + \gamma_3 \varepsilon_{11} - \gamma_1 \varepsilon_{33}) \right] \right\},
\end{aligned}$$

$$\begin{aligned}\Phi_{35} = -\Phi_{53} &= -\sum_{k=1}^5 \frac{1}{|x|_k} (c_{44} - a_k c_{66}) \times \\ &\times \left\{ a_k \left[(a_k c_{11} + c_{13}) e_{15} + (c_{13} + c_{44}) e_{31} \right] - (a_k c_{11} - c_{44}) e_{33} \right\}, \\ \Phi_{41} = \Phi_{42} = \Phi_{43} = \Phi_{45} &= 0, \\ \Phi_{44} &= \frac{1}{\varkappa_{11} |x|_5}, \\ \Phi_{55} &= \sum_{k=1}^5 \frac{1}{|x|_k} a_k c_{44} (c_{44} - a_k c_{66}) (c_{13}^2 - c_{11} + 2a_k c_{13} + c_{33}).\end{aligned}$$

Notice that the entries of the matrix Ψ are complex functions, in general (the equation (C.38) may have complex, mutually conjugate roots a_j). It is evident that in this case the real part $\Re\Psi(x)$ is also a fundamental matrix of the operator $A(\partial)$ since its coefficients are real.

Remark also that if the parameters a_j are not distinct (that is, at list two of them are equal to each other, $a_q = a_p$) then we can apply the usual limiting procedure ($a_q \rightarrow a_p$) in the above expression to construct the corresponding fundamental matrix.

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