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ON CORRECT SOLVABILITY OF A PROBLEM WITH LOADED BOUNDARY CONDITIONS FOR A FOURTH ORDER PSEUDOPARABOLIC EQUATION

Abstract. It is known that different conditions can be found for correct solvability of the non-local problem. In this sense, in the paper we find correct solvability conditions of the non-local problem with loaded boundary conditions in the integral form for a fourth order pseudoparabolic equation.

1. Introduction

Recently a great interest to non-local problems for partial differential equations has shown itself [1]–[4]. This is connected with their rise in various problems of physical character. Various classes of problems of non-local character arise, for example, while studying problems of moisture transfer in soils [5], heat transfer in heterogeneous media [6], diffusion of thermal neutrons in inhibitors [7], [8], simulation of various biological processes, phenomena etc. [9].

Pseudoparabolic equations are used for sufficiently adequate description of majority of real processes occurring in nature, technology and so on. In particular, many processes arising in the theory of fluid filtration in cracked media are described by pseudoparabolic equations with non-smooth coefficients [10], [11].

Notice that in this paper the considered equation is a generalization of many model equations of some processes (e.g., telegraph equation, string vibration equation etc.).

Therefore, there arises a very urgent question on investigation of problems of correct solvability for non-local problems connected with pseudoparabolic equations with dominating derivatives, generally speaking, with non-smooth variable coefficients.

2. Of the Statement Problem

Let us consider the equation

$$(V_{3,1}u)(x,t) \equiv D_x^3 D_t u(x,t) + a_{2,1}(x,t)D_x^2 D_t u(x,t) + a_{3,0}(x,t)D_x^3 u(x,t) +$$

$$+ \sum_{\substack{i+j<3 \\ i=\overline{0,2};\, j=\overline{0,1}}} a_{ij}(x,t)D_x^i D_t^j u(x,t) = \varphi_{3,1}(x,t) \in L_p(G), \quad (2.1)$$

under the following loaded boundary conditions

$$\begin{cases} V_{0,0}u \equiv u(x_0,t_0) = \varphi_{0,0} \in R; \\ V_{1,0}u \equiv D_x u(x_0,t_0) = \varphi_{1,0} \in R; \\ V_{2,0}u \equiv D_x^2 u(x_0,t_0) = \varphi_{2,0} \in R; \\ (V_{3,0}u)(x) \equiv D_x^3 u(x,t_0) + \nu(x)D_x^3 u(x,\overline{t}_0) = \varphi_{3,0}(x) \in L_p(G_1); \\ (V_{0,1}u)(t) \equiv D_t u(x_0,t) = \varphi_{0,1}(t) \in L_p(G_2); \\ (V_{1,1}u)(t) \equiv D_x D_t u(x_0,t) + \mu(t)D_x D_t u(\overline{x}_0,t) = \varphi_{1,1}(t) \in L_p(G_2); \\ (V_{2,1}u)(t) \equiv D_x^2 D_t u(x_0,t) + \sigma(t)D_x^2 D_t u(\widetilde{x}_0,t) = \varphi_{2,1}(t) \in L_p(G_2), \end{cases} \quad (2.2)$$

where $\varphi_{i,0}$, $i = \overline{0,2}$, are given constants, and the other $\varphi_{i,j}$ are given measurable functions; $D_t = \frac{\partial}{\partial t}$ is a generalized differentiation operator in S. L. Sobolev's sense. Besides, the above-given $a_{i,j}(x,t)$ are measurable functions on $G = G_1 \times G_2$; $G_1 = (x_0,x_1)$, $G_2 = (t_0,t_1)$ and satisfy only the

following conditions:

$$a_{i,0}(x,t) \in L_p(G), \;\; a_{i,1}(x,t) \in L_{p,\infty}^{x,t}(G), \;\; i = \overline{0,2};$$
$$a_{3,0}(x,t) \in L_{\infty,p}^{x,t}(G).$$

Notice that here we assume $\nu(x) \in L_\infty(G_1)$, $\mu(t) \in L_\infty(G_2)$, $\sigma(t) \in L_\infty(G_2)$, where $\overline{t}_0 \in [t_0, t_1]$, $\overline{x}_0 \in [x_0, x_1]$ and $\widetilde{x}_0 \in [x_0, x_1]$ are fixed points.

Under the imposed conditions, the solution $u(x,t)$ of the problem (2.1), (2.2) is to be found in the S. L. Sobolev space:

$$W_p^{(3,1)}(G) \equiv \left\{ u(x,t) : \; D_x^i D_t^j u(x,t) \in L_p(G), \; i = \overline{0,3}, \; j = \overline{0,1} \right\},$$
$$1 \le p \le \infty.$$

We will define the norm in the space $W_p^{(3,1)}(G)$ by the equality

$$\|u(x,t)\|_{W_p(3,1)(G)} = \sum_{\substack{i=\overline{0,3} \\ j=0,1}} \|D_x^i D_t^j u(x,t)\|_{L_p(G)}.$$

We will investigate the problem $(2.1),(2.2)$ by the operator equations method and follow the scheme of the paper [12]. We will preliminarily write the problem $(2.1),(2.2)$ in the form of the operator equation

$$Vu = \varphi, \tag{2.3}$$

where V is a vector operator defined by the equality

$$V = (V_{0,0}, V_{1,0}, V_{2,0}, V_{3,0}, V_{0,1}, V_{1,1}, V_{2,1}, V_{3,1}) : W_p^{(3,1)}(G) \longrightarrow E_p^{(3,1)}$$

and φ is a given vector element of the form

$$\varphi = \big(\varphi_{0,0}, \varphi_{1,0}, \varphi_{2,0}, \varphi_{3,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{2,1}, \varphi_{3,1}\big)$$

from the space

$$E_p^{(3,1)} \equiv R \times R \times R \times L_p(G_1) \times L_p(G_2) \times L_p(G_2) \times L_p(G_2) \times L_p(G).$$

Notice that in the space $E_p^{(3,1)}$ we will define the norm in the natural way, by means of the equality

$$\|\varphi\|_{E_p^{(3,1)}} = \sum_{i=0}^{2} \|\varphi_{i,0}\|_R + \|\varphi_{3,0}(x)\|_{L_p(G_1)} +$$
$$+ \sum_{i=0}^{2} \|\varphi_{i,1}(t)\|_{L_p(G_2)} + \|\varphi_{3,1}(x,t)\|_{L_p(G)}.$$

Definition 1. If the problem $(2.1),(2.2)$ for any

$$\varphi = (\varphi_{0,0}, \varphi_{1,0}, \varphi_{2,0}, \varphi_{3,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{2,1}, \varphi_{3,1}) \in E_p^{(3,1)}$$

has a unique solution $u(x,t) \in W_p^{(3,1)}(G)$ such that

$$\|u\|_{W_p^{(3,1)}(G)} \le M \|\varphi\|_{E_p^{(3,1)}},$$

we will say that the operator V of the problem $(2.1), (2.2)$ (or of the equation (2.3)) is a homeomorphism from $W_p^{(3,1)}(G)$ on $E_p^{(3,1)}$ or the problem $(2.1), (2.2)$ is everywhere correctly solvable. Here M is a constant independent of φ.

Obviously, if the operator V of the problem $(2.1), (2.2)$ is a homeomorphism from $W_p^{(3,1)}(G)$ on $E_p^{(3,1)}$, then there exists the bounded inverse operator:

$$V^{-1} : E_p^{(3,1)} \longrightarrow W_p^{(3,1)}(G).$$

But if $\nu(x) = \mu(t) = \sigma(t) \equiv 0$, then the conditions (2.2) will take the form

$$\begin{cases} V_{0,0}u \equiv u(x_0,t_0) = \varphi_{0,0} \in R; \\ V_{1,0}u \equiv D_x u(x_0,t_0) = \varphi_{1,0} \in R; \\ V_{2,0}u \equiv D_x^2 u(x_0,t_0) = \varphi_{2,0} \in R; \\ (V_{3,0}u)(x) \equiv D_x^3 u(x,t_0) = \varphi_{3,0}(x) \in L_p(G_1); \\ (V_{0,1}u)(t) \equiv D_t u(x_0,t) = \varphi_{0,1}(t) \in L_p(G_2); \\ (V_{1,1}u)(t) \equiv D_x D_t u(x_0,t) = \varphi_{1,1}(t) \in L_p(G_2); \\ (V_{2,1}u)(t) \equiv D_x^2 D_t u(x_0,t) = \varphi_{2,1}(t) \in L_p(G_2). \end{cases}$$

These may be considered as Goursat conditions. Such a formulation of the Goursat problem has a number of advantages:

1) no additional agreement conditions are required in this formulation;

2) namely, such a formulation generates a homeomorphism between two Banach spaces $W_p^{(3,1)}(G)$ and $E_p^{(3,1)}$;

3) we can consider this problem as a problem formulated by traces in S. L. Sobolev spaces $W_p^{(3,1)}(G)$.

It should be particularly noted that various aspects of Goursat type boundary value problems were studied in detail in the papers [13-19] and etc.

3. Equivalent System of Integral Equations for Non-Local Problem with Loaded Boundary Conditions

We will study the problem $(2.1), (2.2)$ by means of the following integral representation of the functions $u(x,t) \in W_p^{(3,1)}(G)$ [20]:

$$u(x,t) = u(x_0,t_0) + (x - x_0)D_x u(x_0,t_0) + \frac{(x - x_0)^2}{2} D_x^2 u(x_0,t_0) +$$

$$+ \frac{1}{2} \int_{x_0}^{x} (x - \tau)^2 D_x^3 u(\tau,t_0)\, d\tau + \int_{t_0}^{t} D_t u(x_0,\xi)\, d\xi +$$

$$+ (x - x_0) \int_{t_0}^{t} D_x D_t u(x_0,\xi)\, d\xi + \frac{(x - x_0)^2}{2} \int_{t_0}^{t} D_x^2 D_t u(x_0,\xi)\, d\xi +$$

$$+ \frac{1}{2} \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} (x-\tau)^2 D_x^3 D_t u(\tau,\xi) \, d\tau \, d\xi. \tag{3.1}$$

The relation (3.1) shows that the function $u(x,t) \in W_p^{(3,1)}(G)$ satisfying the conditions $(2.2)_1$, $(2.2)_2$, $(2.2)_3$ and $(2.2)_5$ is of the form

$$u(x,t) = B_0(x,t) + \frac{1}{2} \int\limits_{x_0}^{x} (x-\tau)^2 b_{3,0}(\tau) \, d\tau + (x-x_0) \int\limits_{t_0}^{t} b_{1,1}(\xi) \, d\xi +$$

$$+ \frac{(x-x_0)^2}{2} \int\limits_{t_0}^{t} b_{2,1}(\xi) \, d\xi + \frac{1}{2} \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} (x-\tau)^2 b_{3,1}(\tau,\xi) \, d\tau \, d\xi, \tag{3.2}$$

where

$$B_0(x,t) = \sum_{i=0}^{2} \frac{(x-x_0)^i}{i!} \, \varphi_{i,0} + \int\limits_{t_0}^{t} \varphi_{0,1}(\xi) d\xi.$$

Here $b_{3,0}(x) \in L_p(G_1)$, $b_{1,1}(t) \in L_p(G_2)$, $b_{2,1}(t) \in L_p(G_2)$ and $b_{3,1}(x,t) \in L_p(G)$ are unknown functions. Now we will choose the functions $b_{3,0}(x)$, $b_{1,1}(t), b_{2,1}(t)$ and $b_{3,1}(x,t)$ in such a way that the conditions $(2.2)_4$, $(2.2)_6$ and $(2.2)_7$ be fulfilled. To this end, we calculate the derivatives $D_x^3 u(x,t)$, $D_x D_t u(x,t)$ and $D_x^2 D_t u(x,t)$ of the function (3.2):

$$D_x^3 u(x,t) = \frac{\partial^3 B_0(x,t)}{\partial x^3} + b_{3,0}(x) + \int\limits_{t_0}^{t} b_{3,1}(x,\xi) \, d\xi,$$

$$D_x D_t u(x,t) = \frac{\partial^2 B_0(x,t)}{\partial x \partial t} + b_{1,1}(t) + (x-x_0)b_{2,1}(t) + \int\limits_{x_0}^{x} (x-\tau)b_{3,1}(\tau,t) \, d\tau$$

and

$$D_x^2 D_t u(x,t) = \frac{\partial^3 B_0(x,t)}{\partial x^2 \partial t} + b_{2,1}(t) + \int\limits_{x_0}^{x} b_{3,1}(\tau,t) \, d\tau.$$

The expressions for these derivatives show that the following equalities are valid

$$D_x^3 u(x,t_0) = b_{3,0}(x),$$

$$D_x^3 u(x,\overline{t}_0) = b_{3,0}(x) + \int\limits_{G_2} b_{3,1}(x,\xi)\theta(\overline{t}_0 - \xi) \, d\xi,$$

$$D_x D_t u(x_0,t) = b_{1,1}(t),$$

$$D_x D_t u(\overline{x}_0,t) = b_{1,1}(t) + (\overline{x}_0 - x_0)b_{2,1}(t) + \int\limits_{G_1} (\overline{x}_0 - \tau)b_{3,1}(\tau,t)\theta(\overline{x}_0 - \tau) \, d\tau,$$

$$D_x^2 D_t u(x_0, t) = b_{2,1}(t),$$

$$D_x^2 D_t u(\widetilde{x}_0, t) = b_{2,1}(t) + \int\limits_{G_1} b_{3,1}(\tau, t)\theta(\widetilde{x}_0 - \tau)\, d\tau,$$

where $\theta(\tau)$ is the Heaviside function on R, i.e.

$$\theta(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau \le 0. \end{cases}$$

Therefore, we can write the conditions $(2.2)_4$, $(2.2)_6$ and $(2.2)_7$ in the form

$$b_{3,0}(x)[1 + \nu(x)] = \varphi_{3,0}(x) - \nu(x) \int\limits_{G_2} b_{3,1}(x, \xi)\theta(\overline{t}_0 - \xi)d\xi, \quad x \in G_1, \quad (3.3)$$

$$b_{1,1}(t)[1 + \mu(t)] + (\overline{x}_0 - x_0)\mu(t)b_{2,1}(t) =$$

$$= \varphi_{1,1}(t) - \mu(t) \int\limits_{G_1} (\overline{x}_0 - \tau)b_{3,1}(\tau, t)\theta(\overline{x}_0 - \tau)\, d\tau, \quad t \in G_2, \qquad (3.4)$$

$$b_{2,1}(t)[1 + \sigma(t)] = \varphi_{2,1}(t) - \sigma(t) \int\limits_{G_1} b_{3,1}(\tau, t)\theta(\widetilde{x}_0 - \tau)\, d\tau, \quad t \in G_2. \quad (3.5)$$

Now require the function (3.2) to be a solution of the equation (2.1). For this, we take into account the expression (3.2) of this function in the equation (2.1). Then, after the substitution

$$u(x, t) = B_0(x, t) + \widetilde{u}(x, t),$$

where

$$\widetilde{u}(x, t) = \frac{1}{2} \int\limits_{x_0}^{x} (x - \tau)^2 b_{3,0}(\tau)\, d\tau + (x - x_0) \int\limits_{t_0}^{t} b_{1,1}(\xi)\, d\xi +$$

$$+ \frac{(x - x_0)^2}{2} \int\limits_{t_0}^{t} b_{2,1}(\xi)\, d\xi + \frac{1}{2} \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} (x - \tau)^2 b_{3,1}(\tau, \xi)\, d\tau\, d\xi,$$

we can write the equation (2.1) in the form

$$(V_{3,1}\widetilde{u})(x, t) = \widetilde{R}(x, t), \qquad (3.6)$$

where

$$\widetilde{R}(x, t) = \varphi_{3,1}(x, t) - (V_{3,1}B_0)(x, t).$$

Obviously, we can calculate the derivatives of the function $\widetilde{u}$ by means of the equalities

$$D_x \widetilde{u}(x, t) = \int\limits_{x_0}^{x} (x - \tau)b_{3,0}(\tau)\, d\tau + \int\limits_{t_0}^{t} b_{1,1}(\xi)\, d\xi +$$

$$+ (x - x_0) \int\limits_{t_0}^{t} b_{2,1}(\xi) \, d\xi + \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} (x - \tau) b_{3,1}(\tau, \xi) \, d\tau \, d\xi,$$

$$D_x^2 \widetilde{u}(x,t) = \int\limits_{x_0}^{x} b_{3,0}(\tau) \, d\tau + \int\limits_{t_0}^{t} b_{2,1}(\xi) \, d\xi + \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} b_{3,1}(\tau, \xi) \, d\tau \, d\xi,$$

$$D_x^3 \widetilde{u}(x,t) = b_{3,0}(x) + \int\limits_{t_0}^{t} b_{3,1}(x, \xi) \, d\xi,$$

$$D_t \widetilde{u}(x,t) = (x - x_0) b_{1,1}(t) + \frac{(x - x_0)^2}{2} b_{2,1}(t) + \frac{1}{2} \int\limits_{x_0}^{x} (x - \tau)^2 b_{3,1}(\tau, t) \, d\tau,$$

$$D_x D_t \widetilde{u}(x,t) = b_{1,1}(t) + (x - x_0) b_{2,1}(t) + \int\limits_{x_0}^{x} (x - \tau) b_{3,1}(\tau, t) \, d\tau,$$

$$D_x^2 D_t \widetilde{u}(x,t) = b_{2,1}(t) + \int\limits_{x_0}^{x} b_{3,1}(\tau, t) \, d\tau,$$

$$D_x^3 D_t \widetilde{u}(x,t) = b_{3,1}(x, t).$$

Therefore, we can reduce the equation (3.6) to the form

$$(Nh)(x,t) \equiv b_{3,1}(x,t) +$$

$$+ \int\limits_{x_0}^{x} \left\{ \left[a_{2,1}(x,t) + (x - \tau) a_{1,1}(x,t) + \frac{(x - \tau)^2}{2} a_{0,1}(x,t) \right] b_{3,1}(\tau, t) + \right.$$

$$\left. + \left[a_{2,0}(x,t) + (x - \tau) a_{1,0}(x,t) + \frac{(x - \tau)^2}{2} a_{0,0}(x,t) \right] b_{3,0}(\tau) \right\} d\tau +$$

$$+ \int\limits_{t_0}^{t} \left\{ a_{3,0}(x,t) b_{3,1}(x, \xi) + \left[a_{1,0}(x,t) + (x - x_0) a_{0,0}(x,t) \right] b_{1,1}(\xi) + \right.$$

$$\left. + \left[a_{2,0}(x,t) + (x - x_0) a_{1,0}(x,t) + \frac{(x - x_0)^2}{2} a_{0,0}(x,t) \right] b_{2,1}(\xi) \right\} d\xi +$$

$$+ \int\limits_{x_0}^{x} \int\limits_{t_0}^{t} \left[a_{2,0}(x,t) + (x - \tau) a_{1,0}(x,t) + \frac{(x - \tau)^2}{2} a_{0,0}(x,t) \right] b_{3,1}(\tau, \xi) \, d\tau \, d\xi +$$

$$+ \left[a_{1,1}(x,t) + (x - x_0) a_{0,1}(x,t) \right] b_{1,1}(t) +$$

$$+ \left[a_{2,1}(x,t) + (x - x_0) a_{1,1}(x,t) + \frac{(x - x_0)^2}{2} a_{0,1}(x,t) \right] b_{2,1}(t) +$$

$$+ a_{3,0}(x,t) b_{3,0}(x) = \widetilde{R}(x,t), \quad (x,t) \in G, \tag{3.7}$$

where

$$h = (b_{3,0}(x), b_{1,1}(t), b_{2,1}(t), b_{3,1}(x,t)) \in \widetilde{E}_p^{(3,1)} \equiv$$
$$\equiv L_p(G_1) \times L_p(G_2) \times L_p(G_2) \times L_p(G)$$

is the unknown quadruple.

Thus, solution of the problem (2.1), (2.2) in the spaces $W_p^{(3,1)}(G)$ is reduced to solution of the system of integral equations (3.3), (3.4), (3.5), (3.7) with respect to the quadruple of unknowns $h(x,t) = (b_{3,0}(x), b_{1,1}(t), b_{2,1}(t), b_{3,1}(x,t))$ in the space $\widetilde{E}_p^{(3,1)}$.

If $1 + \nu(x) \neq 0$ and $\lambda(x) = (1 + \nu(x))^{-1}$ is bounded on G_1, then $b_{3,0}(x)$ may be determined from the equality (3.3):

$$b_{3,0}(x) = \lambda(x)\varphi_{3,0}(x) - \int_{G_2} \nu(x)\lambda(x)b_{3,1}(x,\xi)\theta(\bar{t}_0 - \xi)d\xi. \qquad (3.8)$$

If $1 + \sigma(t) \neq 0$ and $\rho(t) = (1 + \sigma(t))^{-1}$ is bounded on G_2, then $b_{2,1}(t)$ may be determined from the equality (3.5):

$$b_{2,1}(t) = \rho(t)\varphi_{2,1}(t) - \int_{G_1} \rho(t)\sigma(t)b_{3,1}(\tau,t)\theta(\widetilde{x}_0 - \tau)d\tau. \qquad (3.9)$$

If $1 + \mu(t) \neq 0$ and $l(t) = (1 + \mu(t))^{-1}$ is bounded on G_2, then $b_{1,1}(t)$ may be determined from the equality (3.4) in the following way:

$$b_{1,1}(t) = l(t)\varphi_{1,1}(t) -$$

$$- \int_{G_1} l(t)\mu(t)(\bar{x}_0 - \tau)b_{3,1}(\tau,t)\theta(\bar{x}_0 - \tau)\,d\tau - (\bar{x}_0 - x_0)l(t)\mu(t)b_{2,1}(t) =$$

$$= l(t)\varphi_{1,1}(t) - \int_{G_1} l(t)\mu(t)(\bar{x}_0 - \tau)b_{3,1}(\tau,t)\theta(\bar{x}_0 - \tau)\,d\tau -$$

$$-(\bar{x}_0 - x_0)l(t)\mu(t)\left[\rho(t)\varphi_{2,1}(t) - \int_{G_1} \rho(t)\sigma(t)b_{3,1}(\tau,t)\theta(\widetilde{x}_0 - \tau)\,d\tau\right]. \quad (3.10)$$

Thus under the above-imposed conditions we have proved

Theorem 3.1. *For everywhere correct solvability of the problem* (2.1), (2.2) *in the space* $W_p^{(3,1)}(G)$ *it is necessary and sufficient that the system of integral equations* (3.7), (3.8), (3.9), (3.10) *be everywhere correctly solvable in the space* $\widetilde{E}_p^{(3,1)}$.

Remark 3.1. Obviously, taking into account the expressions (3.8), (3.9) and (3.10) for the unknowns $b_{3,0}(x)$, $b_{2,1}(t)$ and $b_{1,1}(t)$, we can reduce the equation (3.7) to an independent integral equation with the unknown function $b_{3,1}(x,t)$:

$$(Nb_{3,1})(x,t) \equiv b_{3,1}(x,t)+$$

$$+\int\limits_{x_0}^{x}\left\{\left[a_{2,1}(x,t)+(x-\tau)a_{1,1}(x,t)+\frac{(x-\tau)^2}{2}\,a_{0,1}(x,t)\right]b_{3,1}(\tau,t)+\right.$$

$$+\left[a_{2,0}(x,t)+(x-\tau)a_{1,0}(x,t)+\frac{(x-\tau)^2}{2}\,a_{0,0}(x,t)\right]\times$$

$$\left.\times\left[\lambda(\tau)\varphi_{3,0}(\tau)-\int\limits_{G_2}\nu(\tau)\lambda(\tau)b_{3,1}(\tau,\xi)\theta(\overline{t}_0-\xi)\,d\xi\right]\right\}d\tau+$$

$$+\int\limits_{t_0}^{t}\left\{a_{3,0}(x,t)b_{3,1}(x,\xi)+\left[a_{1,0}(x,t)+(x-x_0)a_{0,0}(x,t)\right]\times\right.$$

$$\times\left[l(\xi)\varphi_{1,1}(\xi)-\int\limits_{G_1}l(\xi)\mu(\xi)(\overline{x}_0-\tau)b_{3,1}(\tau,\xi)\theta(\overline{x}_0-\tau)\,d\tau-\right.$$

$$-(\overline{x}_0-x_0)l(\xi)\mu(\xi)\left(\rho(\xi)\varphi_{2,1}(\xi)-\int\limits_{G_1}\rho(\xi)\sigma(\xi)b_{3,1}(\tau,\xi)\theta(\widetilde{x}_0-\tau)\,d\tau\right)\right]+$$

$$+\left[a_{2,0}(x,t)+(x-x_0)a_{1,0}(x,t)+\frac{(x-x_0)^2}{2}\,a_{0,0}(x,t)\right]\times$$

$$\left.\times\left[\rho(\xi)\varphi_{2,1}(\xi)-\int\limits_{G_1}\rho(\xi)\sigma(\xi)b_{3,1}(\tau,\xi)\theta(\widetilde{x}_0-\tau)\,d\tau\right]\right\}d\xi+$$

$$+\int\limits_{x_0}^{x}\int\limits_{t_0}^{t}\left[a_{2,0}(x,t)+(x-\tau)a_{1,0}(x,t)+\frac{(x-\tau)^2}{2}\,a_{0,0}(x,t)\right]b_{3,1}(\tau,\xi)\,d\tau\,d\xi+$$

$$+\left[a_{1,1}(x,t)+(x-x_0)a_{0,1}(x,t)\right]\times$$

$$\times\left[l(t)\varphi_{1,1}(t)-\int\limits_{G_1}l(t)\mu(t)(\overline{x}_0-\tau)b_{3,1}(\tau,t)\theta(\overline{x}_0-\tau)\,d\tau-\right.$$

$$-(\overline{x}_0-x_0)l(t)\mu(t)\left(\rho(t)\varphi_{2,1}(t)-\int\limits_{G_1}\rho(t)\sigma(t)b_{3,1}(\tau,t)\theta(\widetilde{x}_0-\tau)\,d\tau\right)\right]+$$

$$+\left[a_{2,1}(x,t)+(x-x_0)a_{1,1}(x,t)+\frac{(x-x_0)^2}{2}\,a_{0,1}(x,t)\right]\times$$

$$\times\left[\rho(t)\varphi_{2,1}(t)-\int\limits_{G_1}\rho(t)\sigma(t)b_{3,1}(\tau,t)\theta(\widetilde{x}_0-\tau)\,d\tau\right]+$$

$$+a_{3,0}(x,t)b_{3,0}(x)=\widetilde{R}(x,t),\ \ (x,t)\in G.$$

References

1. A. A. Kilbas and O. A. Repin, An analogue of the Bitsadze-Samarskiĭ problem for an equation of mixed type with a fractional derivative. (Russian) *Differ. Uravn.* **39** (2003), No. 5, 638–644; English transl.: *Differ. Equ.* **39** (2003), No. 5, 674–680.

2. L. S.Pul'kina, A nonlocal problem with integral conditions for a hyperbolic equation. (Russian) *Differ. Uravn.* **40** (2004), No. 7, 887–892; English transl.: *Differ. Equ.* **40** (2004), No. 7, 947–953

3. I. G. Mamedov, Generalization of multipoint boundary-value problems of Bitsadze-Samarski and Samarski-Ionkin type for fourth order loaded hyperbolic integro-differential equations and their operator generalization. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **23** (2005), 77–84.

4. I. G. Mamedov, Three-dimensional nonlocal boundary-value problem with integral conditions for loaded Volterra-hyperbolic integro-differential equations. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **24** (2006), 153–162.

5. A. M. Nakhushev, Nonlocal and Goursat problems for a loaded equation of hyperbolic type and their application to the prediction of soil moisture. (Russian) *Dokl. Akad. Nauk SSSR* **242** (1978), No. 5, 1008–1011.

6. L. I. Rubinšteǐn, On a question about the propagation of heat in heterogeneous media. (Russian) *Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz.* **12** (1948), 27–45.

7. N. I. Ionkin and E. N. Moiseev, A problem for a heat equation with two-point boundary conditions. (Russian) *Differentsial'nye Uravneniya* **15** (1979), No. 7, 1284–1295.

8. L. I. Kamynin, A boundary-value problem in the theory of heat conduction with non-classical boundary conditions. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** (1964), 1006–1024.

9. N. Bayli, Mathematics in biology and medicine. (Russian) *Mir, Moscow,* 1970.

10. G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks (strata). (Russian) *Prikl. Mat. Mekh.* **24** (1960), 852–864; English transl.: *PMM, J. Appl. Math. Mech.* **24** (1961), 1286–1303.

11. M. Kh. Shkhanukov, Some boundary value problems for a third-order equation that arise in the modeling of the filtration of a fluid in porous media. (Russian) *Differentsial'nye Uravneniya* **18** (1982), No. 4, 689–699.

12. I. G. Mamedov, Nonlocal problem for one hyperbolic integro-differential equation with loadings in boundary conditions. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **23** (2003), No. 4, *Math. Mech.,* 69–78.

13. T. L. Kiguradze, Periodic boundary value problems for linear hyperbolic equations. I. (Russian) *Differentsial'nye Uravneniya* **29** (1993), No. 2, 281–297; English transl.: *Differential Equations* **29** (1993), No. 2, 231–245.

14. S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Math. Phys.* **4** (1995), 1–127.

15. O. M. Jokhadze, Boundary value problems for higher order linear equations and systems. (Russian) *Doctoral thesis, Tbilisi,* 1999.

16. S. S. Akhiev, Representations of solutions of some linear operator equations. (Russian) *Dokl. Akad. Nauk SSSR* **251** (1980), No. 5, 1037–1040.

17. S. S. Akhiev, Fundamental solutions of some local and nonlocal boundary value problems and their representations. (Russian) *Dokl. Akad. Nauk SSSR* **271** (1983), No. 2, 265–269.

18. O. M. Jokhadze, On the three-dimensional generalized Goursat problem for a third-order equation, and related general two-dimensional Volterra integral equations of the first kind. (Russian) *Differ. Uravn.* **42** (2006), No. 3, 385–394; English transl.: *Differ. Equ.* **42** (2006), No. 3, 412–421.

19. B. Midodashvili, Generalized Goursat problem for a spatial fourth order hyperbolic equation with dominated low terms. *Proc. A. Razmadze Math. Inst.* **138** (2005), 43–54.

20. I. G. MAMEDOV, A new type of Goursat problem for loaded fourth-order Volterra-hyperbolic vector integrodifferential equations with nonsmooth matrix coefficients. (Russian) *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **26** (2006), No. 2, *Inform. Control Probl.,* 74–79.

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