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**ON RECTIFIABLE OSCILLATION  
OF EMDEN–FOWLER EQUATIONS**

**Abstract.** We are interested in the oscillatory behavior of solutions of Emden–Fowler equation

$$y'' + f(x)|y|^{\gamma-1}y = 0, \quad (1)$$

where  $\gamma > 0$  and  $\gamma \neq 1$ ,  $f(x) \in C^1(0, 1]$  and  $f(x) > 0$  for  $x \in (0, 1]$ . A solution  $y(x)$  is rectifiable oscillatory if the solution curve  $\{(x, y(x)) : x \in (0, 1]\}$  has a finite arc-length. When the arc-length of the solution curve is infinite, the solution  $y(x)$  is said to be unrectifiable oscillatory. We prove integral criteria in terms of  $f(x)$  which are necessary and sufficient for both rectifiable and unrectifiable oscillations of all solutions of (1). For a discussion on rectifiable oscillation of the linear differential equation, i.e. the equation (1) when  $\gamma = 1$ , we refer to Pašić [15], Wong [17].

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**რეზიუმე. ნამართლად განსივლილია ემდენ-ფაულერის**

$$y'' + f(x)|y|^{\gamma-1}y = 0, \quad (1)$$

განტოლების ამონახსნის ოსცილაციური ყოფიერება, სადაც  $\gamma > 0$ ,  $\gamma \neq 1$ ,  $f(x) \in C^1(0, 1]$  და  $f(x) > 0$ , როცა  $x \in (0, 1]$ .  $y(x)$  ამონახსნის ეწოდება წრფივად რხევადი, თუ ამონახსნის  $\{(x, y(x)) : x \in (0, 1]\}$  გრაფის სახრული სიგრძე აქვს. თუ ამონახსნის გრაფის სიგრძე უსაზღვროა,  $y(x)$  ამონახსნის ეწოდება არაწრფივად რხევადი. დგინდება  $f(x)$ -ის ტერმინებში ჩამოყალიბებული ინტეგრალური კრიტერიუმები, რომლებიც აუცილებელი და საჭიროაა ოსცილაციის, რომ (1) განტოლების ყველა ამონახსნი იყოს წრფივად ან არაწრფივად რხევადი. წრფივი და არაწრფივად რხევადობის, ანუ (1) განტოლების, როცა  $\gamma = 1$ , წრფივად რხევადობის განსივლილის მითითებულია პაშიჩის [15] და ვონგის [17] შრომებში.

1. INTRODUCTION

We study the oscillatory behavior of solutions of the Emden–Fowler equation on a finite interval. Consider the equation

$$y'' + f(x)|y|^{\gamma-1}y = 0, \quad \gamma > 0, \quad \gamma \neq 1, \tag{1}$$

where  $f(x) \in C^2(0, 1]$ ,  $f(x)$  is strictly positive and singular at  $x = 0$ , i.e.  $\lim_{x \rightarrow 0} f(x) = \infty$ . Let  $I_0$  and  $I$  denote the half-open interval  $(0, 1]$  and the closed interval  $[0, 1]$  respectively. Under these assumptions, it is known from results on the semi-infinite interval  $[0, \infty)$  that any solution  $y(x)$  of (1) with prescribed initial conditions  $y(x_0)$  and  $y'(x_0)$  at some  $x_0 \in I_0$  can be extended throughout the entire interval  $I_0$ , see, e.g., Hastings [8], Coffman and Wong [3]. A solution  $y(x)$  of (1) is said to be *oscillatory* if it has an infinite number of zeros in  $I = [0, 1]$  and *non-oscillatory* if it has only a finite number of zeros in  $I$ . If any one of the solutions of the equation (1) is oscillatory in  $I$ , then  $f(x)$  must be singular. The reverse is not necessarily true as can be seen in the Euler equation,

$$y'' + \lambda x^{-2}y = 0, \quad \lambda \leq \frac{1}{4}. \tag{2}$$

Here the coefficient is singular and the general solution of the equation (2) is given by

$$y(x) = c_1\sqrt{x}\cos(\rho \log x) + c_2\sqrt{x}\sin(\rho \log x), \tag{3}$$

where  $\rho^2 = \lambda - \frac{1}{4}$ . When  $\rho^2 \leq 0$ , the solution given by (3) is nonoscillatory.

We are interested in the graph  $G(y)$  of a solution  $y(x)$  of the equation (1) where  $G(y) = \{(x, y(x)) : 0 \leq x \leq 1\} \subseteq \mathbb{R}^2$  is a curve in the plane. The arc-length of the solution curve  $G(y)$ , denoted by  $L_G(y)$ , is defined by

$$L_G(y) = \sup \left\{ \sum_{i=1}^m \|(x_i, y(x_i)) - (x_{i-1}, y(x_{i-1}))\|_2 \right\},$$

where the supremum is taken over all partitions  $0 = x_0 < x_1 < \dots < x_m = 1$  of  $I$ ,  $\|\cdot\|_2$  denotes the Euclidean norm in  $\mathbb{R}^2$  and  $m$  is any finite number. A convenient formula of computing  $L_G(y)$  is the following:

$$L_G(y) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \sqrt{1 + y'^2(x)} dx. \tag{4}$$

It is clear that any continuous function on a finite interval can have finite or infinite length, for example  $y(x) = x \sin \frac{1}{x}$  is a solution of  $y'' + x^{-4}y = 0$  and its arc-length  $L_G(y)$ , according to the formula (4), is infinite. The well known example by Weiesstrass of a continuous but nowhere differentiable function defined on any finite interval also has infinite arc-length .

A curve in  $\mathbb{R}^2$  is called *rectifiable* if it has finite arc-length and is called *unrectifiable* if it has infinite arc-length. A solution  $y(x)$  of the equation (1) is called *rectifiable oscillatory* if its graph  $G(y)$  has infinite length, i.e.

$L_G(y)$  is finite, and it is called *unrectifiable oscillatory* if  $L_G(y) = \infty$ . The equation (1) is *rectifiable oscillatory* if all its solutions are. Likewise for *unrectifiable oscillation*.

It is therefore natural to ask under what conditions imposed on  $a(x)$ , the solutions of the equation (1) are *rectifiable oscillatory*, and if not, *unrectifiable oscillatory*. In the linear case, the Euler-type differential equation

$$y'' + \lambda x^{-\alpha} y = 0, \quad \lambda > 0, \quad \alpha > 0 \quad (5)$$

has been studied by Pašić [15] and Wong [17]. Their main result is the following.

**Theorem A.** *Solutions of the equation (5) are*

- (a) *rectifiable oscillatory if  $2 < \alpha < 4$ , and*
- (b) *unrectifiable oscillatory if  $\alpha \geq 4$ .*

More recently, Kwong, Pašić and Wong [13] improved the above result by proving

**Theorem B.** *For the linear equation (1) with  $\gamma = 1$ , if the coefficient function  $f(x)$  satisfies the Hartman–Wintner condition*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{-\frac{1}{4}}(x) \left| \left( f^{-\frac{1}{4}}(x) \right)'' \right| dx < \infty \quad (H - W)$$

(see [4] and [7]), then all solutions of (1) with  $\gamma = 1$  are

- (i) *rectifiable oscillatory if*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{1/4}(x) dx < \infty, \quad (6)$$

and

- (ii) *rectifiable oscillatory if*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{\frac{1}{4}}(x) dx = \infty. \quad (7)$$

Theorem B is a significant improvement of Theorem A. The question whether Theorem B can be extended to the more general case of the Emden–Fowler equation (1) when  $\gamma > 0, \gamma \neq 1$ , is quite natural, particularly in the light of the fact that the Sturm Comparison Theorem does not hold for the Emden–Fowler equation (1) but was used extensively in the proof of Theorem A. Fortunately, the approach of using asymptotic representation in proving Theorem B can be modified to the nonlinear Emden–Fowler equation (1). The purpose of this paper is to prove an analogue of Theorem B for the equation (1):

**Theorem 1.** *Suppose that  $f(x) \in C^2(0, 1]$ ,  $f(x) > 0$  for  $x \in (0, 1]$ ,  $\lim_{x \rightarrow 0} f(x) = \infty$ , and satisfies*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{-1}(x) f'_+(x) dx = K_0 < \infty, \tag{A_0}$$

where  $f_+(x) = \max(f(x), 0)$ . If, in addition,  $f(x)$  satisfies for  $0 < \gamma < 1$

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \left| d \left( f'(x) f^{-3/2}(x) \right) \right| < \infty, \tag{8}$$

and for  $\gamma > 1$

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \left| d \left( f'(x) f^{-(\gamma+2)/\gamma+1}(x) \right) \right| < \infty, \tag{9}$$

then the equation (1) is (i) *rectifiable oscillatory if*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{1/\gamma+3}(x) dx < \infty, \tag{10}$$

and (ii) *unrectifiable oscillatory if*

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{1/\gamma+3}(x) dx = \infty. \tag{11}$$

In the special case of the Euler-type coefficient  $f(x) = \lambda x^{-\alpha}$ ,  $\lambda > 0$ , the Emden–Fowler equation

$$y'' + \lambda x^{-\alpha} |y|^{\gamma-1} y = 0, \quad \lambda > 0, \quad \gamma > 0, \tag{1_0}$$

$x \in (0, 1]$ , admits the following

**Corollary.**

- (i) *Equation (1<sub>0</sub>) is rectifiable oscillatory if  $2 < \alpha < \gamma + 3$  when  $0 < \gamma < 1$  and  $\gamma + 1 < \alpha < \gamma + 3$  when  $\gamma > 1$ ;*
- (ii) *Equation (1<sub>0</sub>) is unrectifiable oscillatory if  $\alpha \geq \gamma + 3$ .*

Clearly Corollary above reduces to Theorem A when  $\gamma = 1$ . Theorem B however does not follow as a corollary of the main Theorem because the  $(H - W)$  condition is implied by (8) or (9), see Lemma 2 below.

The basis of our proof of the main result is the asymptotic representation formula for the Emden–Fowler equations developed by Kiguradze [9], [10], [11] (see Kiguradze and Chanturia [12; pp. 270–275], and Chanturia [5]). Kiguradze’s results like that of Wintner in the linear case were given on the semi-infinite interval  $[0, \infty)$ . When compared with the linear case, Kiguradze’s results require the additional assumption on the monotonicity of

$f(x)$  near  $x = 0$ , which is provided here by the assumption  $(A_0)$ , a condition somewhat weaker than  $f'(x) \leq 0$ ,  $x \in (0, \varepsilon]$  for some  $\varepsilon > 0$ .

## 2. AUXILIARY LEMMAS

We first introduce a Lyapunov function  $V(x)$  for a given solution  $y(x)$  of the equation (1) by

$$V(x) = f^{-2/\gamma+3}(x) |y'(x)|^2 + \frac{2}{\gamma+1} f(x)^{(\gamma+1)/\gamma+3} |y(x)|^{\gamma+1}. \quad (12)$$

Introduce the function  $g(x) = f'(x)[f(x)]^{-(\gamma+5)/\gamma+3}$  which is related to  $V(x)$  by the following identity by (1):

$$\begin{aligned} V(x) = V(x_0) + \frac{2}{\gamma+3} [g(x_0)y(x_0)y'(x_0) - g(x)y(x)y'(x)] - \\ - \frac{2}{\gamma+3} \int_x^{x_0} y(s)y'(s)dg(s). \end{aligned} \quad (13)$$

From (12), we have

$$|y(x)| \leq \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1} f(x)^{-1/\gamma+3} V(x)^{1/\gamma+1} \quad (14)$$

and

$$|y'(x)| \leq f(x)^{1/\gamma+3} V(x)^{1/2}. \quad (15)$$

To develop an asymptotic representation of solutions of (1), Kiguradze introduces a condition similar to the Hartman–Wintner condition which we label as (K):

$$g(x) = [f(x)]^{-(\gamma+5)/\gamma+3} f'(x) \in BV(0, 1), \text{ and } \lim_{x \rightarrow 0} g(x) = 0, \quad (\text{K})$$

which is implied by an obvious extension of  $(H - W)$  condition, namely,

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f^{-1/\gamma+3}(x) \left| \left( f^{-\frac{1}{\gamma+3}}(x) \right)'' \right| dx < \infty. \quad (*)$$

**Lemma 1.** *The condition (\*) implies the condition (K).*

*Proof.* Let  $k(x) = f^{-\frac{1}{\gamma+3}}(x)$ . It is easy to verify the identity

$$-\frac{1}{\gamma+3} g'(x) = (k(x)k'(x))' = k(x)k''(x) + k'(x)^2. \quad (16)$$

The condition (\*) implies for all  $s \in (0, 1]$

$$\int_s^1 k(x)k''(x)dx = k(1)k'(1) - k(s)k'(s) - \int_s^1 k'(x)^2 dx \geq -C_0, \quad (17)$$

where  $C_0 = \int_0^1 |k(x)k''(x)|dx$ . Since  $k(0) = 0$  and  $k(x) > 0$  for all  $x \in (0, 1]$ , it follows from the mean value theorem that there exists a sequence of points  $s_n \geq 0$ ,  $s_n \rightarrow 0$ , such that  $k'(s_n) > 0$ . Using this in (17), we have

$$\int_{s_n}^1 k'^2(x)dx \leq C_0 + |k(1)k'(1)| - k(s_n)k'(s_n) \leq c_1, \tag{18}$$

where  $C_1 = C_0 + |k(1)k'(1)|$ . Letting  $s_n \rightarrow 0$  in (18), we conclude that  $k'(x) \in L^2(0, 1)$ . Returning to (16) and noting the condition (14), i.e.  $k(x)k''(x) \in L_1(0, 1)$ , we deduce that  $g'(x) \in L_1(0, 1)$  which implies  $\lim_{x \rightarrow 0} g(x) = c$ .

From (17) we note that for all  $s \in (0, 1)$  we have  $k(s)k'(s) \leq C_1$ , which upon integrating from  $s = 0$  to  $s = x$  gives  $k^2(x) \leq 2C_1x$ . Observe that

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{1}{k^2(x)} dx = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{1}{2C_1x} dx = \infty. \tag{19}$$

Now if  $\lim_{x \rightarrow 0} g(x) = k(x)k'(x) = c \neq 0$ , then there is a neighborhood  $[0, x_1]$  of  $x = 0$ ,  $x_1 > 0$ , such that  $|k(s)k'(s)|^{-1} \leq B_0$  for  $s \in [0, x_1]$ . Now from (19) we obtain

$$\infty = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{x_1} \frac{ds}{k^2(s)} = \int_0^{x_1} \frac{(k'(s))^2}{(k(s)k'(s))^2} ds \leq B_0^2 \int_0^{x_1} (k'(s))^2 ds,$$

which contradicts the fact that  $k'(x) \in L^2(0, 1)$ . Thus  $\lim_{x \rightarrow 0} g(x) = 0$ , proving that the condition (\*) implies (K). □

We note that the condition (\*) alone is not sufficient to prove the required asymptotic representation formula for the equation (1) as in the case of the linear equation. Results of Kiguradze suggested that the stronger conditions (8) in case  $0 < \gamma < 1$  and (9) in case  $\gamma > 1$  plus the monotonicity condition ( $A_0$ ) would suffice. We will prove these observations in the lemmas to follow.

**Lemma 2.** *The condition (8) in case  $0 < \gamma < 1$  and the condition (9) in case  $\gamma > 1$  imply the condition (\*) hence the condition (K).*

*Proof.* The condition (8) implies that the limit  $f^{-3/2}(x)f'(x)$  as  $x \rightarrow 0$  exist and is finite, so for  $0 < \gamma < 1$  the expression  $g(x) = f^{-3/2}(x)f'(x)[f(x)]^{\frac{\gamma-1}{2(\gamma+3)}}$  tends to zero as  $x \rightarrow 0$ . Now consider the identity

$$\int_x^1 dg(s) = \int_x^1 f(s)^{(\gamma-1)/2(\gamma+3)} d\left(f'(s)f^{-3/2}(s)\right) +$$

$$\begin{aligned}
& + \frac{(1-\gamma)}{2(\gamma+3)} \int_x^1 |f^{-\frac{(\gamma+4)}{\gamma+3}}(s) f'(s)|^2 ds = \\
& = I_1(x) + \frac{1-\gamma}{2(\gamma+3)} I_2(x), \tag{20}
\end{aligned}$$

where  $I_1(x)$ ,  $I_2(x)$  denote the first and second integrals on the right hand side of (20), respectively. The integral  $I_1(x)$  converges as  $x \rightarrow 0$  because of (8) and  $\lim_{x \rightarrow 0} g(x) = 0$ , so the integral  $I_2(x)$  also converges as  $x \rightarrow 0$ . Now the condition (8) implies that there exists  $B_1 > 0$  such that  $|I_1(x)| \leq B_1$ . We can now estimate (20) and obtain

$$\int_x^1 |dg(s)| \leq |I_1(x)| + (1-\gamma)/2(\gamma+3) I_2(x) \leq B_1 + \frac{(1-\gamma)}{2(\gamma+3)} I_2(x). \tag{21}$$

The convergence of  $I_2(x)$  as  $x \rightarrow 0$  shows by (21) that  $g(x) \in BV(0, 1)$ . Since  $f(x) \in C^2(0, 1]$ , this proves  $g'(x) \in L_1(0, 1)$ , so the condition (7) implies (K). Returning to (16), we note that  $k'(x) \in L^2(0, 1)$  from  $|I_2(x)| \leq B_1$ . This together with  $g'(x) \in L_1(0, 1)$  implies by (16) the validity of (\*).

We assume that the condition (9) holds for  $\gamma > 1$ . Consider instead of (20) the following identity

$$\begin{aligned}
\int_x^1 dg(s) & = \int_x^1 f(s)^{(1-\gamma)/(\gamma+1)(\gamma+3)} d\left(f'(s) f^{-\frac{\gamma+2}{\gamma+1}}(s)\right) + \\
& + \frac{(\gamma-1)}{(\gamma+1)(\gamma+3)} \int_x^1 |f^{-\frac{(\gamma+4)}{\gamma+3}}(s) f'(s)|^2 ds = \\
& = I_3(x) + \frac{\gamma-1}{(\gamma+1)(\gamma+3)} I_2(x), \tag{22}
\end{aligned}$$

where  $I_3(x)$  denotes the first integral of the right hand side of (22) and  $I_2(x)$  denotes the second integral of the right hand side of (22) which is the same as the second integral of the right hand side of (20). The condition (9) implies that  $I_3(x)$  converges as  $x \rightarrow 0$ . Also write

$$g(x) = f^{-(\gamma+5)/\gamma+3}(x) f'(x) = f^{-\frac{\gamma+2}{\gamma+1}}(x) f'(x) f^{(1-\gamma)/(\gamma+1)(\gamma+3)}(x). \tag{23}$$

the condition (9) implies that  $|f^{-(\gamma+2)/\gamma+1}(x) f'(x)|$  is bounded, so  $\gamma > 1$  in (23) implies that  $\lim_{x \rightarrow 0} g(x) = 0$ . We estimate (22) by the following inequality

$$\int_x^1 |dg(s)| \leq |I_3(x)| + \frac{\gamma-1}{(\gamma+1)(\gamma+3)} I_2(x). \tag{24}$$

The condition (9) implies the convergence of  $I_3(x)$  which in turn implies by (22) that  $I_2(x)$  also converges and hence is bounded. These two statements prove that  $g(x) \in BV(0, 1)$  or  $g'(x) \in L_1(0, 1)$ . Thus the condition (9)

implies (K). Now the boundedness of  $I_2(x)$ , which is similar to  $k'(x) \in L^2(0,1)$ , plus  $g'(x) \in L_1(0,1)$  prove by (16) that  $k(x)k''(x) \in L^1(0,1)$ . Hence the condition (8) implies (\*).  $\square$

**Lemma 3.** *Assume that the condition (K) holds. Then for every solution  $y(x)$  of the equation (1) the following hold:*

- (i) For  $\gamma > 1$ ,  $V(x)$  is bounded and  $\lim_{x \rightarrow 0} V(x) = c_0 \geq 0$ ,
- (ii) For  $0 < \gamma < 1$ , if  $V(x)$  is bounded, then  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ .

*Proof.* (i) Suppose that  $V(x)$  is unbounded. Then there exists a sequence  $\{s_k : \lim_{k \rightarrow \infty} s_k = 0\}$  such that  $V(s_k) = \sup_{s_k \leq x \leq 1} V(x)$  form a non-decreasing sequence and  $\lim_{k \rightarrow \infty} V(s_k) = \infty$  as  $k \rightarrow \infty$ . By the condition (K),  $g'(x) \in L_1(0,1)$ , so we can estimate  $V(s_k)$  using (13), (14) and (15):

$$V(s_k) \leq V(x_0) + \frac{2}{\gamma + 3} \left(\frac{\gamma + 1}{2}\right)^{1/\gamma+1} V^{(\gamma+3)/2(\gamma+1)}(s_k) \times \left\{ |g(x_0)| + |g(s_k)| + \int_{s_k}^{x_0} |dg(s)| \right\}. \tag{25}$$

For  $\gamma > 1$ ,  $(\gamma + 3)/2(\gamma + 1) < 1$ , so (25) yields that  $V(x)$  is bounded. By (13),  $\lim_{x \rightarrow 0} V(x) = c_0$  exists and is non-negative by (12).

(ii) For  $0 < \gamma < 1$ , we assume that  $V(x)$  is bounded but  $\lim_{x \rightarrow 0} V(x) = 0$ . We want to show that the condition (K) leads to a contradiction. For any  $x_1 \in (0, 1]$ , we can find  $x_3 < x_2 \leq x_1$ , such that

$$\begin{cases} 2V(x_3) = V(x_2) = V(x_1), \\ V(x_3) \leq V(x) \leq V(x_2), \text{ where } x \in (x_3, x_2). \end{cases} \tag{26}$$

Now putting  $x = x_3$  and  $x_0 = x_2$  in (13), we find

$$\frac{1}{2}V(x_3) \leq \frac{2}{\gamma + 3} \left(\frac{\gamma + 1}{2}\right)^{1/\gamma+1} \times \left\{ \int_{x_3}^{x_2} |dg(s)| + |g(x_3)| + |g(x_2)| \right\} V(x_2)^{\gamma+3/2(\gamma+1)},$$

from which it follows

$$V(x_1)^{(\gamma-1)/2(\gamma+3)} \leq \frac{12}{\gamma + 3} \left(\frac{\gamma + 1}{2}\right)^{1/\gamma+1} \int_0^{x_1} |dg(s)| < \infty, \tag{27}$$

because  $g'(x) \in L_1(0,1)$ . Since  $x_1$  is chosen arbitrarily and  $0 < \gamma < 1$ , so  $\lim_{x \rightarrow 0} V(x) = 0$  implies that the left hand side of (27) tends to  $\infty$  as  $x_1 \rightarrow 0$ . This contradiction proves that  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ .  $\square$

Lemma 2 shows that Kiguradze's condition (8) for  $0 < \gamma < 1$  and condition (9) for  $\gamma > 1$  are stronger than the corresponding Hartman–Wintner's condition (\*). Indeed, Lemma 3 shows that if only the condition (K) holds, then  $V(x)$  may tend to 0 as  $x \rightarrow 0$  in case  $\gamma > 1$  and  $V(x)$  may be unbounded in case  $0 < \gamma < 1$ . However, under the monotonicity assumption ( $A_0$ ), i.e. when  $f(x)$  is basically non-increasing, then conditions the (8) and (9) imply that the Lyapunov function  $V(x)$  of any nontrivial solution satisfies  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$  as  $x \rightarrow 0$  as given by Lemma 4 and Lemma 5 below.

**Lemma 4.** *For  $0 < \gamma < 1$ , if  $f(x)$  satisfies ( $A_0$ ) and the condition (8), then for every nontrivial solution  $y(x)$  of (1), its associated Lyapunov function  $V(x)$  is bounded.*

*Proof.* Suppose that  $V(x)$  is unbounded. Then for any  $x_1 \in (0, 1]$  we can find  $x_3 < x_2 \leq x_1$  such that

$$\begin{cases} \frac{1}{2}V(x_3) = V(x_2) = V(x_1), \\ V(x_2) \leq V(x) \leq V(x_3), \text{ where } x \in (x_3, x_2). \end{cases} \quad (28)$$

Substituting  $x_2$  for  $x_0$  and  $x_3$  for  $x$  in (13), we obtain similarly to (27) the following estimate

$$V(x_3)^{(\gamma-1)/2(\gamma+1)} \leq \frac{12}{\gamma+3} \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1} \int_{x_3}^{x_2} |dg(s)|. \quad (29)$$

Using (28), (29) and  $0 < \gamma < 1$ , we can find a lower bound for  $V(x_1)$ :

$$V(x_1) \geq M_\gamma \left( \int_0^{x_1} |dg(s)| \right)^{2(\gamma+1)/\gamma-1}, \quad (30)$$

where  $M_\gamma = 2(12/\gamma + 3)^{2(\gamma+3)/\gamma-1} \left( \frac{\gamma+1}{2} \right)^{2/\gamma-1}$ . Since  $x_1$  is arbitrary, (30) holds for all  $x \in (0, 1]$ .

Define  $W(x) = f(x)^{-(\gamma+1)/\gamma+3} V(x) = f^{-1}(x) y'^2(x) + \frac{2}{\gamma+1} |y(x)|^{\gamma+1}$ . Using (1), we find

$$W'(x) = -f^{-2}(x) f'(x) y'^2(x) \geq -f^{-1}(x) f'_+(x) W(x). \quad (31)$$

Integrating (31) from  $x$  to  $x_0 \in (0, 1]$ , we obtain by ( $A_0$ )

$$W(x) \leq W(x_0) \exp \left( \int_x^{x_0} f^{-1}(s) f'_+(s) ds \right) \leq e^{K_0} W(x_0). \quad (32)$$

Lemma 2 allows us to rewrite (20) as

$$g(x) = \int_0^x dg(s) = \int_0^x f(s)^{(\gamma-1)/2(\gamma+3)} d \left( f'(s) f^{-3/2}(s) \right) +$$

$$\begin{aligned}
 & + \frac{1-\gamma}{2(\gamma+3)} \int_0^x |f^{-(\gamma+4)/\gamma+3}(s) f'(s)|^2 ds = \\
 & = J_1(x) + \frac{1-\gamma}{2(\gamma+3)} J_2(x), \tag{33}
 \end{aligned}$$

where  $J_1(x)$  and  $J_2(x)$  denote the first and second integrals on the right hand side of (33). We can estimate (33) from above by

$$\int_0^x |dg(s)| \leq f(x)^{(\gamma-1)/2(\gamma+3)} \int_0^x \left| d\left(f'(s) f^{-3/2}(s)\right) \right| + \frac{1-\gamma}{2(\gamma+3)} J_2(x). \tag{34}$$

Using (33) in (34), we obtain

$$\int_0^x |dg(s)| \leq 2(f(x))^{(\gamma-1)/2(\gamma+3)} \int_0^x \left| d\left(f'(x) f^{-3/2}(x)\right) \right| + g(x). \tag{35}$$

Note that

$$|g(x)| \leq (f(x))^{(\gamma-1)/2(\gamma+3)} \int_0^x \left| d\left(f'(x) f^{-3/2}(x)\right) \right|. \tag{36}$$

Substituting (36) into (35), we find

$$\left( \int_0^x |dg(s)| \right)^{2(\gamma+1)/\gamma-1} \geq f(x)^{\frac{\gamma+1}{\gamma+3}} \left( 3 \int_0^x \left| d\left(f^{-3/2}(s) f'(s)\right) \right| \right)^{\frac{2(\gamma+1)}{\gamma-1}}. \tag{37}$$

Replace  $x_1$  by  $x$  in (30) since  $x_1$  is arbitrary, and use the definition  $W(x) = f(x)^{-(\gamma+1)/\gamma+3} V(x)$  to obtain from (30)

$$W(x) \geq M_\gamma \left( 3 \int_0^x \left| d\left(f^{-3/2}(s) f'(s)\right) \right| \right)^{2(\gamma+1)/\gamma-1}. \tag{38}$$

Because of the condition (8) and  $0 < \gamma < 1$ , the right hand side of (38) tends to  $\infty$  as  $x \rightarrow 0$ , which contradicts (32). This proves Lemma 4.  $\square$

Lemma 4 and Lemma 3 (ii) together show that for every nontrivial solution  $y(x)$  its Lyapunov function  $V(x)$  as defined by (11) satisfies  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ . We now wish to show that the condition (9) in the case where  $\gamma > 1$  also gives the same conclusion.

**Lemma 5.** *For  $\gamma > 1$ , if  $f(x)$  satisfies  $(A_0)$  and the condition (9), then for every nontrivial solution  $y(x)$  of (1) its associated Lyapunov function  $V(x)$  satisfies  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ .*

*Proof.* Lemma 3 has already established the fact that  $V(x)$  is bounded and  $\lim_{x \rightarrow 0} V(x)$  exists as a non-negative number  $c_0$ . To show that  $c_0 > 0$ , we assume that  $V(x) \rightarrow 0$  as  $x \rightarrow 0$ . Then for any  $x_1 \in (0, 1]$  we can find  $x_3 < x_2 \leq x_1$  such that

$$\begin{cases} 2V(x_3) = V(x_2) = V(x_1), \\ V(x_3) \leq V(x) \leq V(x_2), \text{ where } x \in (x_3, x_2). \end{cases} \quad (39)$$

Now putting  $x = x_3$  and  $x_0 = x_2$  in (13), we can estimate  $V(x_2)$  by (39) as follows

$$\frac{1}{2}V(x_2) \leq \frac{2}{\gamma+3} \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1} \left\{ \int_{x_3}^{x_2} |dg(s)| + |g(x_3)| + |g(x_2)| \right\} V(x_2)^{\frac{\gamma+3}{2(\gamma+1)}},$$

which reduces to

$$V(x)^{(\gamma-1)/2(\gamma+3)} \leq \frac{12}{\gamma+3} \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1} \int_0^x |dg(s)|. \quad (40)$$

Since  $\lim_{x \rightarrow 0} g(x) = 0$  and  $g'(x) \in L_1(0, 1)$ , we rewrite (22) as

$$\begin{aligned} g(x) &= \int_0^x dg(s) = \int_0^x f(s)^{(1-\gamma)/(\gamma+1)(\gamma+3)} d \left( f'(s) f(s)^{-\frac{\gamma+2}{\gamma+1}} \right) + \\ &\quad + \frac{\gamma-1}{(\gamma+1)(\gamma+3)} \int_0^x \left| f'(s) f(s)^{-\frac{\gamma+4}{\gamma+3}} \right|^2 ds = \\ &= J_3(x) + \frac{\gamma-1}{(\gamma+1)(\gamma+3)} J_2(x), \end{aligned} \quad (41)$$

where  $J_3(x)$  is the first integral on the right hand side of (41) and  $J_2(x)$  represents the second integral. Note that  $J_2(x)$  is the same as the second integral of (33). Using an argument similar to (33), (34), (35), (36) and (37), we obtain

$$\int_0^x |dg(s)| \leq f(x)^{(1-\gamma)/(\gamma+1)(\gamma+3)} \int_0^x \left| d \left( f'(s) f(s)^{-\frac{\gamma+2}{\gamma+1}} \right) \right|. \quad (42)$$

Define  $H(x) = f^{2/\gamma+3}(x)V(x) = y'^2(x) + \frac{2}{\gamma+1}f(x)|y|^{\gamma+1}$ . By (1), we find

$$H'(x) = \frac{2}{\gamma+1}f'(x)|y|^{\gamma+1} \leq f'_+(x)f^{-1}(x)H(x). \quad (43)$$

Integrating (43) we obtain  $H(x) \geq H(x_0)e^{-K_0}$  for all  $x \leq x_0$ ,  $x_0 \in (0, 1]$ , with  $K_0$  given by (A<sub>0</sub>). Returning to (40) and noting that  $H(x)$  is bounded

below by a positive constant, we obtain by (42)

$$H(x) = f^{2/\gamma+3}(x)V(x) \leq K_\gamma \frac{2(\gamma+3)}{\gamma-1} \left\{ \int_0^x \left| d \left( f'(s) f^{-\frac{\gamma+2}{\gamma+1}}(s) \right) \right| \right\}^{\frac{2(\gamma+3)}{\gamma-1}}, \quad (44)$$

where  $K_\gamma = \frac{12K_0}{\gamma+3} \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1}$ . By (9) and  $\gamma > 1$ , we note that the right hand side of (44) tends to 0 as  $x \rightarrow 0$  but  $H(x)$  is bounded below by the positive constant  $H(x_0)e^{-K_0}$ . This is a contradiction proving  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

Under the assumption  $(A_0)$ , (8) for  $0 < \gamma < 1$  and (9) for  $\gamma > 1$  of the Theorem, we know that the condition (K) holds. For every nontrivial solution  $y(x)$  of the equation (1), the associated Lyapunov function  $V(x)$  given by (11) satisfies  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$  which is proved in Lemma 4 and Lemma 5. The asymptotic representation formula developed by Kiguradze [12] on the semi-infinite interval  $[0, \infty)$  can also be formulated on the finite interval  $(0, 1]$  as described below. Given a nontrivial solution  $y(x)$ , we represent it and its derivative  $y'(x)$  by the introduction of two functions  $h(x)$  for  $x \in (0, 1]$  and  $w(t)$  for  $t \in [0, \infty)$  where  $t = h(x)$ , namely

$$y(x) = f(x)^{-\frac{1}{\gamma+3}} V^{\frac{1}{\gamma+1}}(x) w(h(x)) \quad (45)$$

$$y'(x) = -f(x)^{\frac{1}{\gamma+3}} V^{\frac{1}{2}}(x) w'(h(x)). \quad (46)$$

The functions  $h(x)$  and  $w(t)$  satisfy  $\lim_{x \rightarrow 0} h(x) = \infty$  and

$$\ddot{w}(t) + |w(t)|^{\gamma-1} w(t) = 0, \quad t \in [0, \infty), \quad (47)$$

where “dot” denotes differentiation with respect to  $t$  and  $w(t)$  satisfies the initial condition  $w(0) = 0, \dot{w}(0) = 1$ .

Note that the equation (47) is in the form of the equation (1) with  $f(x) \equiv 1$  but is defined over the semi-infinite interval  $[0, \infty)$ . All solutions of (47) are periodic and have the period  $T$  given by

$$T = 4 \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1} \int_0^1 \frac{d\xi}{\sqrt{1-\xi^{\gamma+1}}}. \quad (48)$$

Denote the zeros of  $w(t)$  and  $\dot{w}(t)$  by  $\{t_n\}$  and  $\{\tau_n\}$  which satisfy for all  $n$

$$w(t_n) = \dot{w}(\tau_n) = 0 \quad \text{and} \quad |w(\tau_n)| = \left( \frac{\gamma+1}{2} \right)^{1/\gamma+1}. \quad (49)$$

To show the validity of the asymptotic representation formulas (45), (46), we need to establish the existence of  $h(x)$  such that upon differentiating (45)

we would obtain (46). This means that  $h(x)$  must satisfy the following

$$\begin{aligned} & w'(h(x)) \left[ h'(x)f(x)^{-2/\gamma+3} + V(x)^{(\gamma-1)/2(\gamma+1)} \right] = \\ & = w(h(x)) \left[ \frac{1}{\gamma+3} f(x)^{-(\gamma+5)/\gamma+3} f'(x) - \frac{1}{\gamma+1} f(x)^{-2/\gamma+3} V^{-1}(x)V^1(x) \right] = \\ & = \frac{w(h(x))}{\gamma+3} \left( f(x)^{-(\gamma+5)/\gamma+3} f'(x) \right) \times \\ & \quad \times \left\{ 1 + \left( \frac{2}{\gamma+1} \right) \left[ w'^2(h(x)) - w^{\gamma+1}(h(x)) \right] \right\}. \end{aligned} \quad (50)$$

Note that  $w(t)$  satisfies the identity

$$w'^2(t) + \frac{2}{\gamma+1} w^{\gamma+1}(t) = 1. \quad (51)$$

Thus when  $w'(h(x)) \neq 0$ , (50) implies by (51) that  $h(x)$  satisfies the differential equation

$$\begin{aligned} h'(x) & = -f(x)^{2/\gamma+3} V(x)^{(\gamma-1)/2(\gamma+1)} + \\ & \quad + \frac{1}{\gamma+1} f^{-1}(x) f'(x) w(h(x)) w'(h(x)). \end{aligned} \quad (52)$$

On the other hand, when  $w'(h(x)) = 0$ , we know that  $|w^{\gamma+1}(h(x))| = \frac{\gamma+1}{2}$  by (49), which shows that the last term under brackets in (50) is also zero. This establishes the validity of (50), hence that of (52).

Turning to the differential equation (52), we can rewrite it as

$$-h'(x) = f(x)^{2/\gamma+3} \left\{ V(x)^{(\gamma-1)/2(\gamma+1)} - \frac{1}{\gamma+3} g(x) w(h(x)) w'(h(x)) \right\}. \quad (53)$$

Since  $g(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ , we obtain from (53)

$$t = h(x) \geq \frac{1}{2} c_0^{(\gamma-1)/2(\gamma+1)} \int_x^1 f^{2/\gamma+3}(s) ds \rightarrow \infty \quad (54)$$

as  $x \rightarrow 0$ . Here we set  $h(1) = 0$ . The existence of a solution  $h(x)$  of the first order nonlinear differential equation (52) (or (53)) satisfying the initial condition  $h(1) = 0$  follows from classical existence results except to note that the initial conditions are set at the right hand end point of the interval. This proves the validity of the asymptotic representation formula (45), (46). The divergence of  $\int_x^1 f(s)^{2/\gamma+3} ds$  follows from (19) in Lemma 1.

Let  $y(x)$  be a nontrivial oscillatory solution of the equation (1) and  $\{a_k\}$  be the decreasing sequence of consecutive zeros of  $y(x)$ , i.e.  $y(a_k) = 0$ ,  $a_{k+1} < a_k$ ,  $k = 0, 1, \dots$ ,  $\lim_{k \rightarrow \infty} a_k = 0$  and  $a_0 \in (0, 1]$ . Consider the segment

of the solution curve  $\Gamma_k = \{(x, y(x)) : a_{k+1} \leq x \leq a_k\}$ . Denote the arc-length of  $\Gamma_k$  by  $L(\Gamma_k)$  and let  $s_k$  be the extremum point of  $y(x)$  between  $a_{k+1}$  and  $a_k$ , i.e.  $y'(s_k) = 0$ . Note that

$$2|y(s_k)| \leq L(\Gamma_k) \leq 2|y(s_k)| + (a_k - a_{k+1}). \tag{55}$$

Piecing together the segments  $\Gamma_k$  over the entire interval  $(0, 1]$ , we note that the arc-length of the graph of the solution curve satisfies

$$L_G(y) = \sum_{k=0}^{\infty} L(\Gamma_k) + \text{arc-length} \{(x, y(x)) : a_0 \leq x \leq 1\}. \tag{56}$$

Combining (55) and (56), we obtain

$$2 \sum_{k=0}^{\infty} |y(s_k)| \leq L_G(y) \leq 2 \sum_{k=0}^{\infty} |y(s_k)| + M_0 + a_0, \tag{57}$$

where  $M_0$  is the arc-length of the solution curve on the interval  $[a_0, 1]$  which is finite. We note also that

$$\int_0^1 |y'(x)| dx = \sum_{k=0}^{\infty} |y(s_k)| + \int_{a_0}^1 |y'(x)| dx, \tag{58}$$

so the solution is rectifiable oscillatory or unrectifiable oscillatory depending whether or not  $y'(x) \in L_1(0, 1)$ .

We return to (46) and recall that  $w(t)$  is a periodic hence bounded function. Furthermore,  $\lim_{x \rightarrow 0} V(x) = c_0 > 0$ , so the condition (10) implies that  $y(x)$  is rectifiable oscillatory.

On the other hand, we denote by  $\tau_k$  the zeros of  $\dot{w}(t)$  which correspond to the zeros  $s_k$  of  $y'(x)$  by the asymptotic formula (46). We first note that for  $x$  close to 0  $h'(x)$  is strictly negative by (53), so  $t = h(x)$  has an inverse  $x = h^{-1}(t)$  on the sub-interval  $(0, a_{k_0})$ , where  $k_0$  is sufficiently large. Now we use the asymptotic formula (45) to estimate

$$\sum_{k=k_0}^{\infty} |y(h^{-1}(\tau_k))| \geq \frac{1}{2} c_0 \sum_{k=k_0}^{\infty} f^{-\frac{1}{\gamma+3}}(h^{-1}(\tau_k)) |w(\tau_k)|. \tag{59}$$

By (57), we will show that  $y(x)$  is unrectifiable oscillatory if we will show that the series in (59) is divergent. The periodicity of  $w(t)$  shows by (49) that  $|w(\tau_k)| = (\frac{\gamma+1}{2})^{1/\gamma+1}$ , so the series on the left hand side of (59) will be divergent if we show that  $\sum_{k=k_0}^{\infty} f^{-1/\gamma+3}(h^{-1}(\tau_k))$  is divergent.

By assumption  $(A_0)$ , we note that for  $0 < x_1 < x_2 \leq 1$  we have

$$\log \frac{f(x_2)}{f(x_1)} = \int_{x_1}^{x_2} \frac{f'(s)}{f(s)} ds \leq \int_0^1 \frac{f'_+(s)}{f(s)} ds = K_0. \tag{60}$$

Denote  $F(t) = f^{-1/\gamma+3}(h^{-1}(t))$  and  $J_k = [\tau_k, \tau_{k+1}]$  which for each  $k$  has the fixed length  $T/2$  by (48). Since  $h^{-1}(t) \geq h^{-1}(\tau_{k+1})$  for all  $t \in [\tau_k, \tau_{k+1}]$ ,  $h^{-1}(\tau_k) = s_k$  and  $y'(s_k) = 0$ , by (60) and (53) we obtain

$$\begin{aligned} \frac{2}{T} \sum_{k=k_0}^n F(\tau_k) &\geq e^{-K_0} \sum_{k=k_0}^n \int_{h^{-1}(\tau_k)}^{h^{-1}(\tau_{k+1})} F(t) dt = \\ &= e^{-K_0} \sum_{k=k_0}^n \int_{s_k}^{s_{k+1}} f^{-1/\gamma+3}(s) h'(s) ds \geq \\ &\geq c_0^{(\gamma-1)/2(\gamma+1)} e^{-K_0} \int_{s_{n+1}}^{s_{k_0}} f^{1/\gamma+3}(s) ds. \end{aligned} \quad (61)$$

Now  $s_n$  tends to zero as  $n \rightarrow \infty$ , so the condition (11), i.e.  $\int_0^1 f^{1/\gamma+3}(x) dx = \infty$ , implies by (61) that  $y(x)$  is unrectifiable oscillatory. This completes the proof of the theorem.

#### 4. REMARKS AND OPEN PROBLEMS

We begin with comments about the assumptions  $(A_0)$ , (8) and (9) and key steps of the proof of the main Theorem.

1. let  $f(x) = x^{-4}(2 + \sin x)$ . It is easy to verify that  $f'_+(x)/f(x) \leq (\cos x)_+$  which is integrable on  $(0, 1)$ , so the assumption  $(A_0)$  is satisfied.
2. Both conditions (8) and (9) reduce to  $g_1(x) = f^{-3/2}(x)f'(x)$  when setting  $\gamma = 1$ . In this case the condition  $(K)$  is equivalent to the original Hartman–Wintner condition  $(H - W)$ .
3. It is interesting to note that the proof of the asymptotic formula (45), (46) is not valid when  $\gamma = 1$ . The key estimates (37), (38) when  $0 < \gamma < 1$  and (40), (43) when  $\gamma > 1$  become meaningless if  $\gamma = 1$ . The original proof of Hartman–Wintner asymptotic formula in the linear case was based upon variation of parameters formula which is valid only for linear equations.
4. We refer to Corollary (i) stated in Section 1 and in particular to the lower bounds for  $\alpha > 0$  which are different for  $0 < \gamma < 1$  and for  $\gamma > 1$ . These requirements are imposed to ensure that the equation (10) is oscillatory. When  $\gamma = 1$ , this is the same as Theorem  $A(a)$  where the oscillation follows from the Sturm Comparison Theorem. For the Emden–Fowler equation (1), we can use the integral oscillation criteria of Atkinson [1] and Belohorec [2] which give necessary and sufficient conditions for the oscillation of all solutions of (1). Their results, when converted to the finite interval  $[0, 1]$ , are as follows:

(i) (Belohorec) For  $0 < \gamma < 1$ ,  $\int_0^1 xf(x)dx = \infty$ ,

(ii) (Atkinson) For  $\gamma > 1$ ,  $\int_0^1 x^\gamma f(x)dx = \infty$ .

When  $f(x) = \lambda x^{-\alpha}$ ,  $\lambda > 0$ , the above criteria require (i)  $\alpha > 2$  for  $0 < \gamma < 1$  and (ii)  $\alpha > \gamma + 1$  for  $\gamma > 1$ . In fact, (i) and (ii) also follow directly from the conditions (7) and (8).

We conclude our discussion by posing three open problems which are of independent interest:

- (a) When  $\gamma = 1$ , the linear equation (1) can have a coefficient  $f(x)$  which allows co-existence of both rectifiable and unrectifiable oscillatory solutions. The proof depends on the Wronskian of two linearly independent solutions, see [13]. An example showing co-existence of rectifiable and unrectifiable oscillatory solutions for the nonlinear equation (1) is still at large;
- (b) Curves confined in a bounded domain in  $\mathbb{R}^2$  and having infinite arc-length are known as fractals. In the special case of the equation (5), it is known that when  $\alpha \geq 4$ , the fractal dimension is  $\frac{3}{2} - \frac{2}{\alpha}$ , see Falconer [6], Pašić [14], [16]. Here the proof depends heavily on the Sturm Comparison Theorem which is not available for the Emden–Fowler equation (1). It will be of great interest to prove similar results for the equation (1<sub>0</sub>) with  $\lambda > 0$ ,  $\alpha \geq \gamma + 3$ .
- (c) The linear equation (5) when  $\alpha \leq 2$  may possess both oscillatory and non-oscillatory solutions for different values of  $\alpha$ , but we know that in both cases all solutions are rectifiable. For the Emden–Fowler equation (1) with  $f(x) = \lambda x^{-\alpha}$ ,  $\lambda > 0$ , when  $0 < \alpha \leq 2$ , we also know that it may possess both oscillatory and nonoscillatory solutions even for the same  $\alpha$ . Clearly, bounded non-oscillatory solutions are rectifiable but we have no knowledge about those oscillatory ones for the Emden–Fowler equation (1). Must they also be rectifiable?

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