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ON A PERIODIC TYPE BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A POSITIVELY HOMOGENEOUS OPERATOR

Abstract. Consider the problem

$$u'(t) = H(u,u)(t) + Q(u)(t), \quad u(a) - \lambda u(b) = h(u),$$

where $H : C([a,b];R) \times C([a,b];R) \to L([a,b];R)$ is a continuous positively homogeneous operator, $Q : C([a,b];R) \to L([a,b];R)$ is a continuous operator satisfying the Carathéodory condition, $h : C([a,b];R) \to R$ is a continuous functional, and $\lambda \in [0,1[$. The efficient conditions sufficient for the existence of a solution to the problem considered are established.

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რეზიუმე. ნაშრომში განხილულია ამოცანა

$$u'(t) = H(u,u)(t) + Q(u)(t), \quad u(a) - \lambda u(b) = h(u),$$

სადაც $H : C([a,b];R) \times C([a,b];R) \to L([a,b];R)$ უწყვეტი დადებითად ერთგვაროვანი ოპერატორია, $Q : C([a,b];R) \to L([a,b];R)$ უწყვეტი ოპერატორია, რომელიც აკმაყოფილებს კარათეოდორის პირობას, $h : C([a,b];R) \to R$ უწყვეტი ფუნქციაა, ხოლო $\lambda \in [0,1[$. დადგენილია ამოცანის ამოხსნის არსებობის ეფექტური საკმარისი პირობები.

Introduction

Boundary value problems for functional differential equations have been intensively studied by many mathematicians in the last decades. The reason is development of other natural sciences as physics, chemistry, biology, etc., whence more precise mathematical models describing natural processes arise involving continuous operators which are not, in general, of Nemytski type. One of the questions in qualitative theory of functional differential equations is the solvability of the boundary value problem

$$u'(t) = F(u)(t), \quad h(u) = 0,$$

where $F : C([a, b]; R) \to L([a, b]; R)$ is a continuous operator and $h : C([a, b]; R) \to R$ is a continuous functional. In such a form this problem was studied e.g. in [1], [2], [6], [11], [14], [17], [21], [24]– [29], [32], [33], [35], [37], [41], [44]. Of course, more accurate results can be obtained in the case where F and h are of a special form. Thus criteria guaranteeing unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) - \lambda u(b) = c,$$

where $\ell : C([a, b]; R) \to L([a, b]; R)$ is a linear bounded operator, $q \in L([a, b]; R)$, and λ, $c \in R$, can be found e.g. in [5], [7], [8], [12], [13], [15], [16], [18]– [20], [22], [23], [30], [31], [36], [43], [46], [47].

In this paper, we present efficient conditions sufficient for the solvability of the periodic–type boundary value problem for the functional differential equation with a positively homogeneous operator on the right-hand side, i.e. of the problem

$$u'(t) = H(u, u)(t) + Q(u)(t), \tag{0.1}$$

$$u(a) - \lambda u(b) = h(u), \tag{0.2}$$

where $H : C([a, b]; R) \times C([a, b]; R) \to L([a, b]; R)$ is a continuous, positively homogeneous operator, which is nondecreasing in the first argument and nonincreasing in the second one, $Q : C([a, b]; R) \to L([a, b]; R)$ and $h : C([a, b]; R) \to R$ are continuous operators satisfying the Carathéodory condition, and $\lambda \in [0, 1[$. Such types of problems were studied e.g. in [3], [4], [9], [10], [38], [40], [42], [45].

The following notation is used throughout.

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0, +\infty[$, $R_- =] - \infty, 0]$.

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

$C([a, b]; R)$ is the Banach space of continuous functions $u : [a, b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a, b]\}$.

$C([a, b]; \mathcal{D}) = \big\{u \in C([a, b]; R) : u(t) \in \mathcal{D} \text{ for } t \in [a, b]\big\}$, where $\mathcal{D} \subseteq R$.

$\widetilde{C}([a, b]; \mathcal{D})$, where $\mathcal{D} \subseteq R$, is the set of absolutely continuous functions $u : [a, b] \to \mathcal{D}$.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

$L([a,b];D) = \big\{p \in L([a,b];R) : \; p(t) \in \mathcal{D}$ for a.e. $t \in [a,b]\big\}$, where $\mathcal{D} \subseteq R$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$\mathcal{H}_{ab}$ is the set of continuous operators $H : C([a,b];R) \times C([a,b];R) \to L([a,b];R)$ satisfying the following conditions:

(1) for every u, v, $w \in C([a,b];R)$ we have

$$\text{if } u(t) \le v(t) \text{ for } t \in [a,b] \text{ then } H(u,w)(t) \le H(v,w)(t) \text{ for a.e. } t \in [a,b],$$
$$\text{if } u(t) \le v(t) \text{ for } t \in [a,b] \text{ then } H(w,u)(t) \ge H(w,v)(t) \text{ for a.e. } t \in [a,b],$$

(2) for every $(u,v) \in C([a,b];R) \times C([a,b];R)$ and a constant $\alpha > 0$ we have

$$H(\alpha u, \alpha v)(t) = \alpha H(u,v)(t) \text{ for a.e. } t \in [a,b].$$

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathéodory condition, i.e., for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \text{ for a.e. } t \in [a,b], \;\; v \in C([a,b];R), \;\; \|v\|_C \le r.$$

$K([a,b] \times A; B)$, where $A \subseteq R$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i.e., $f(\cdot\,, x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t, \cdot\,) : A \to B$ is a continuous function for almost every $t \in [a,b]$, and for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \le q_r(t) \text{ for a.e. } t \in [a,b], \;\; x \in A, \;\; |x| \le r.$$

Note that since $H \in \mathcal{H}_{ab}$, in view of (1) in the definition of the set $\mathcal{H}_{ab}$, we have that both $H(\cdot\,,0)$ and $H(0,\cdot\,)$ belong to the set $\mathcal{K}_{ab}$.

We will say that $F \in \mathcal{K}_{ab}$ is an a-Volterra operator if for every $c \in \,]a,b[$ and u, $v \in C([a,b];R)$ satisfying

$$u(t) = v(t) \text{ for } t \in [a,c]$$

we have

$$F(u)(t) = F(v)(t) \text{ for a.e. } t \in [a,c].$$

By a solution to the equation (0.1), where $H \in \mathcal{H}_{ab}$ and $Q \in \mathcal{K}_{ab}$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equality (0.1) almost everywhere in $[a,b]$.

Consider the problem on existence of a solution to (0.1) satisfying the condition (0.2), where $\lambda \in [0,1[$ and $h : C([a,b];R) \to R$ is a continuous operator such that for each $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \le M_r \text{ for } v \in C([a,b];R), \;\; \|v\|_C \le r.$$

The general results are formulated in Section 1 through the use of the sets $V_{ab}^+(\lambda;\geq)$, $V_{ab}^+(\lambda;\leq)$, $W_{ab}^+(\lambda;+)$, and $W_{ab}^+(\lambda;-)$ (see Definitions 0.1–0.4 introduced below). In the second part of Section 1, there are established efficient conditions sufficient for the inclusions

$$H\in V_{ab}^+(\lambda;\geq),\quad H\in V_{ab}^+(\lambda;\leq),\quad H\in W_{ab}^+(\lambda;+),\quad \text{and}\quad H\in W_{ab}^+(\lambda;-). \tag{0.3}$$

At the end of Section 1, the results guaranteeing the inclusions (0.3) are illustrated for the special case of the operator H defined by

$$H(u,v)(t) \overset{\text{def}}{=} p(t)\max\big\{u(s):\ \tau_1(t)\leq s\leq \tau_2(t)\big\}- \\ -g(t)\max\big\{v(s):\ \mu_1(t)\leq s\leq \mu_2(t)\big\}\ \text{ for a.e. } t\in[a,b]. \tag{0.4}$$

Here $p,g\in L([a,b];R_+)$, $\tau_i,\,\mu_i\in\mathcal{M}_{ab}$ $(i=1,2)$, and $\tau_1(t)\leq\tau_2(t)$, $\mu_1(t)\leq\mu_2(t)$ for almost every $t\in[a,b]$. Auxiliary propositions necessary to prove the main results are contained together with their proofs in Section 2. Section 3 is devoted to the proofs of theorems on the solvability of (0.1), (0.2), as well as to the proofs of the conditions guaranteeing the inclusions (0.3) in both general and special cases. Illustrative examples are gathered in Section 4.

Definition 0.1. We will say that an operator $H\in\mathcal{H}_{ab}$ belongs to the set $V_{ab}^+(\lambda;\geq)$ if an arbitrary function $u\in\widetilde{C}([a,b];R)$ satisfying

$$u'(t)\geq H(u,0)(t)\ \text{ for a.e. } t\in[a,b],\quad u(a)-\lambda u(b)\geq 0 \tag{0.5}$$

admits the inequality

$$u(t)\geq 0\ \text{ for } t\in[a,b].$$

Definition 0.2. We will say that an operator $H\in\mathcal{H}_{ab}$ belongs to the set $V_{ab}^+(\lambda;\leq)$ if an arbitrary function $u\in\widetilde{C}([a,b];R)$ satisfying

$$u'(t)\leq H(u,0)(t)\ \text{ for a.e. } t\in[a,b],\quad u(a)-\lambda u(b)\leq 0 \tag{0.6}$$

admits the inequality

$$u(t)\leq 0\ \text{ for } t\in[a,b].$$

Remark 0.1. Let us note that if $H\in\mathcal{H}_{ab}$ is a homogeneous operator, then $H\in V_{ab}^+(\lambda;\geq)$ if and only if $H\in V_{ab}^+(\lambda;\leq)$.

Definition 0.3. We will say that an operator $H\in\mathcal{H}_{ab}$ belongs to the set $W_{ab}^+(\lambda;-)$ if for every $\delta\in[0,1]$ and $y\in C([a,b];R_-)$, arbitrary functions $u,\,v\in\widetilde{C}([a,b];R)$ satisfying

$$u'(t)-\delta H(y,u)(t)\geq v'(t)-\delta H(y,v)(t)\ \text{ for a.e. } t\in[a,b], \tag{0.7}$$

$$u(a)-\lambda u(b)\geq v(a)-\lambda v(b) \tag{0.8}$$

admit the inequality

$$u(t)\geq v(t)\ \text{ for } t\in[a,b]. \tag{0.9}$$

Definition 0.4. We will say that an operator $H\in\mathcal{H}_{ab}$ belongs to the set $W_{ab}^+(\lambda;+)$ if for every $\delta\in[0,1]$ and $y\in C([a,b];R_+)$, arbitrary functions $u,\,v\in\widetilde{C}([a,b];R)$ satisfying (0.7) and (0.8) admit the inequality (0.9).

1. Main Results

In what follows, $q \in K([a,b] \times R_+; R_+)$ is a function such that

$$\lim_{x\to+\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0. \tag{1.1}$$

1.1. **Existence Theorems.**

Theorem 1.1. *Let $H \in V_{ab}^+(\lambda;\geq) \cap W_{ab}^+(\lambda;-)$ and let there exist $c \in R_+$ such that for $v \in C([a,b];R_-)$ the inequalities*

$$-q(t,\|v\|_C) \leq Q(v)(t) \leq 0 \;\text{ for a.e. }\; t\in[a,b], \quad -c \leq h(v) \leq 0$$

are fulfilled. Then (0.1), (0.2) *has at least one nonpositive solution.*

Theorem 1.2. *Let $H \in V_{ab}^+(\lambda;\leq) \cap W_{ab}^+(\lambda;+)$ and let there exist $c \in R_+$ such that for $v \in C([a,b];R_+)$ the inequalities*

$$0 \leq Q(v)(t) \leq q(t,\|v\|_C) \;\text{ for a.e. }\; t\in[a,b], \quad 0 \leq h(v) \leq c$$

are fulfilled. Then (0.1), (0.2) *has at least one nonnegative solution.*

Theorem 1.3. *Let $H \in V_{ab}^+(\lambda;\geq) \cap V_{ab}^+(\lambda;\leq)$ and either $H \in W_{ab}^+(\lambda;-)$ or $H \in W_{ab}^+(\lambda;+)$. Let, moreover, there exist $c \in R_+$ such that for $v \in C([a,b];R)$ the inequalities*

$$|Q(v)(t)| \leq q(t,\|v\|_C) \;\text{ for a.e. }\; t\in[a,b], \quad |h(v)| \leq c$$

are fulfilled. Then (0.1), (0.2) *has at least one solution.*

If, in addition,

$$h(v) \leq 0, \quad Q(v)(t) \leq 0 \;\text{ for a.e. }\; t\in[a,b], \;\; v \in C([a,b];R_-), \tag{1.2}$$

then (0.1), (0.2) *has at least one nonpositive solution, respectively if*

$$h(v) \geq 0, \quad Q(v)(t) \geq 0 \;\text{ for a.e. }\; t\in[a,b], \;\; v \in C([a,b];R_+), \tag{1.3}$$

then (0.1), (0.2) *has at least one nonnegative solution.*

1.2. **On the Sets $V_{ab}^+(\lambda;\geq)$, $V_{ab}^+(\lambda;\leq)$, $W_{ab}^+(\lambda;-)$, and $W_{ab}^+(\lambda;+)$.**

Proposition 1.1. *Let $H \in \mathcal{H}_{ab}$. Then $H \in V_{ab}^+(\lambda;\geq)$ if and only if the problem*

$$u'(t) \leq -H(-u,0)(t), \quad u(a) - \lambda u(b) = 0 \tag{1.4}$$

has no nontrivial nonnegative solution.

Theorem 1.4. *Let $H \in \mathcal{H}_{ab}$. Then $H \in V_{ab}^+(\lambda;\geq)$ if and only if there exists $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying*

$$\gamma'(t) \geq -H(-\gamma,0)(t) \;\text{ for a.e. }\; t\in[a,b], \tag{1.5}$$

$$\gamma(a) - \lambda\gamma(b) > 0. \tag{1.6}$$

Corollary 1.1. *Let $H \in \mathcal{H}_{ab}$ and let at least one of the following conditions be fulfilled:*

(a) *$H(\cdot\,,0)$ is an a-Volterra operator and*

$$\lambda \exp\Big(-\int\limits_a^b H(-1,0)(s)\,ds\Big) < 1;$$

(b) *there exist numbers $m,k \in N\cup\{0\}$, $m>k$, and a constant $\alpha \in]0,1[$ such that*

$$\rho_m(t) \leq \alpha\rho_k(t) \quad \text{for } t \in [a,b], \tag{1.7}$$

where $\rho_0 \equiv 1$ and

$$\rho_i(t) = -\frac{\lambda}{1-\lambda}\int\limits_a^b H(-\rho_{i-1},0)(s)\,ds - \\ -\int\limits_a^t H(-\rho_{i-1},0)(s)\,ds \quad \text{for } t \in [a,b], \quad i \in N;$$

(c) *there exists $\widehat{H} \in \mathcal{H}_{ab}$ such that*

$$\lambda \exp\Big(-\int\limits_a^b H(-1,0)(s)\,ds\Big) + \\ + \int\limits_a^b \widehat{H}(1,0)(s)\exp\Big(-\int\limits_s^b H(-1,0)(\xi)\,d\xi\Big)\,ds < 1 \tag{1.8}$$

and on the set $\{v \in C([a,b];R_+) : v(a)-\lambda v(b) = 0\}$ the inequality

$$-H(-\vartheta(v),0)(t)+H(-1,0)(t)\vartheta(v)(t) \leq \widehat{H}(v,0)(t) \quad \text{for a.e. } t \in [a,b] \tag{1.9}$$

holds, where

$$\vartheta(v)(t) \overset{\text{def}}{=} -\frac{\lambda}{1-\lambda}\int\limits_a^b H(-v,0)(s)\,ds - \int\limits_a^t H(-v,0)(s)\,ds \quad \text{for } t\in[a,b]. \tag{1.10}$$

Then $H \in V_{ab}^{+}(\lambda;\geq)$.

Corollary 1.1 (b) with $m=1$ and $k=0$ results in the following assertion.

Corollary 1.2. *Let*

$$\int\limits_a^b |H(-1,0)(s)|\,ds < 1-\lambda.$$

Then $H \in V_{ab}^{+}(\lambda;\geq)$.

Theorem 1.5. *Let $H \in \mathcal{H}_{ab}$, let there exist $\overline{H} \in \mathcal{H}_{ab}$ such that $\overline{H}(0,\cdot\,)$ is an a-Volterra operator, and let for every $y \in C([a,b];R_-)$ and $u, v \in \widetilde{C}([a,b];R)$ satisfying $u(a) - \lambda u(b) \geq v(a) - \lambda v(b)$, the inequality*

$$H(y,u)(t) - H(y,v)(t) \geq \overline{H}(0,u-v)(t) \quad \text{for } t \in [a,b] \tag{1.11}$$

be fulfilled. If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];R_+)$ satisfying

$$\gamma(t) > 0 \quad \text{for } t \in [a,b[\,, \tag{1.12}$$

$$\gamma'(t) \leq \overline{H}(0,\gamma)(t) \quad \text{for a.e. } t \in [a,b], \tag{1.13}$$

then $H \in W^+_{ab}(\lambda;-)$.

Theorem 1.6. *Let $H \in \mathcal{H}_{ab}$ and let there exist $\overline{H} \in \mathcal{H}_{ab}$ such that for every $y \in C([a,b];R_-)$ and $u, v \in \widetilde{C}([a,b];R)$ satisfying $u(a) - \lambda u(b) \geq v(a) - \lambda v(b)$ the inequality (1.11) is fulfilled. Let, moreover, at least one of the following conditions be fulfilled:*

(a)

$$\int_a^b \left|\overline{H}(0,1)(s)\right| ds \leq \lambda; \tag{1.14}$$

(b) $\overline{H}(0,\cdot\,)$ *is an a-Volterra operator and*

$$\int_a^b \left|\overline{H}(0,1)(s)\right| ds \leq 1. \tag{1.15}$$

Then $H \in W^+_{ab}(\lambda;-)$.

Now we formulate the conditions guaranteeing the inclusions $H \in V^+_{ab}(\lambda;\leq)$ and $H \in W^+_{ab}(\lambda;+)$. Their proofs are similar to those of the above-listed assertions and therefore they will be omitted.

Proposition 1.2. *Let $H \in \mathcal{H}_{ab}$. Then $H \in V^+_{ab}(\lambda;\leq)$ if and only if the problem*

$$u'(t) \leq H(u,0)(t), \quad u(a) - \lambda u(b) = 0$$

has no nontrivial nonnegative solution.

Theorem 1.7. *Let $H \in \mathcal{H}_{ab}$. Then $H \in V^+_{ab}(\lambda;\leq)$ if and only if there exists $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying*

$$\gamma'(t) \geq H(\gamma,0)(t) \quad \text{for a.e. } t \in [a,b], \tag{1.16}$$

$$\gamma(a) - \lambda\gamma(b) > 0. \tag{1.17}$$

Corollary 1.3. *Let $H \in \mathcal{H}_{ab}$ and let at least one of the following conditions be fulfilled:*

(a) *$H(\cdot\,,0)$ is an a-Volterra operator and*

$$\lambda \exp\left(\int_a^b H(1,0)(s)\,ds\right) < 1;$$

(b) *there exist numbers $m, k \in N\cup\{0\}$, $m > k$, and a constant $\alpha \in]0,1[$ such that*

$$\rho_m(t) \le \alpha\rho_k(t) \quad \text{for } t \in [a,b],$$

where $\rho_0 \equiv 1$ and

$$\rho_i(t) = \frac{\lambda}{1-\lambda}\int_a^b H(\rho_{i-1},0)(s)\,ds + \int_a^t H(\rho_{i-1},0)(s)\,ds \quad \text{for } t\in[a,b], \quad i\in N;$$

(c) *there exists $\widehat{H} \in \mathcal{H}_{ab}$ such that*

$$\lambda \exp\left(\int_a^b H(1,0)(s)\,ds\right) + \int_a^b \widehat{H}(1,0)(s)\exp\left(\int_s^b H(1,0)(\xi)\,d\xi\right) ds < 1$$

and on the set $\{v \in C([a,b];R_+) : v(a) - \lambda v(b) = 0\}$ the inequality

$$H(\vartheta(v),0)(t) - H(1,0)(t)\vartheta(v)(t) \le \widehat{H}(v,0)(t) \quad \text{for a.e. } t \in [a,b]$$

holds, where

$$\vartheta(v)(t) \overset{\text{def}}{=} \frac{\lambda}{1-\lambda}\int_a^b H(v,0)(s)\,ds + \int_a^t H(v,0)(s)\,ds \quad \text{for } t \in [a,b].$$

Then $H \in V_{ab}^{+}(\lambda;\le)$.

Corollary 1.3 (b) with $m = 1$ and $k = 0$ results in the following assertion.

Corollary 1.4. *Let*

$$\int_a^b H(1,0)(s)\,ds < 1-\lambda.$$

Then $H \in V_{ab}^{+}(\lambda;\le)$.

Theorem 1.8. *Let $H \in \mathcal{H}_{ab}$, let there exist $\overline{H} \in \mathcal{H}_{ab}$ such that $\overline{H}(0,\cdot\,)$ is an a-Volterra operator, and let for every $y \in C([a,b];R_+)$ and $u, v \in \widetilde{C}([a,b];R)$ satisfying $u(a) - \lambda u(b) \ge v(a) - \lambda v(b)$, the inequality* (1.11) *be fulfilled. If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];R_+)$ satisfying* (1.12) *and* (1.13)*, then $H \in W_{ab}^{+}(\lambda;+)$.*

Theorem 1.9. *Let $H \in \mathcal{H}_{ab}$ and let there exist $\overline{H} \in \mathcal{H}_{ab}$ such that for every $y \in C([a,b];R_+)$ and $u, v \in \widetilde{C}([a,b];R)$ satisfying $u(a)-\lambda u(b) \ge v(a)-\lambda v(b)$ the inequality* (1.11) *is fulfilled. Let, moreover, either* (1.14) *hold or $\overline{H}(0,\cdot\,)$ be an a-Volterra operator satisfying* (1.15)*. Then $H \in W_{ab}^{+}(\lambda;+)$.*

Remark 1.1. Let us note that if an operator $H \in \mathcal{H}_{ab}$ is subadditive in the first argument, more precisely, if

$$H(u+v,0)(t) \le H(u,0)(t)+H(v,0)(t) \ \text{ for a.e. } t \in [a,b], \ \ u, v \in C([a,b];R),$$

then we have

$$-H(-u,0)(t) \le H(u,0)(t) \ \text{ for a.e. } t \in [a,b], \ \ u \in C([a,b];R),$$

and if there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying (1.16) and (1.17), then it satisfies also the inequalities (1.5) and (1.6). Therefore, according to Theorems 1.4 and 1.7, in that case the inclusion $H \in V_{ab}^{+}(\lambda;\le)$ implies the inclusion $H \in V_{ab}^{+}(\lambda;\ge)$.

Obviously, if $H \in \mathcal{H}_{ab}$ is a superadditive operator in the first argument, i.e. if

$$H(u+v,0)(t) \ge H(u,0)(t)+H(v,0)(t) \ \text{ for a.e. } t \in [a,b], \ \ u, v \in C([a,b];R),$$

then we have

$$-H(-u,0)(t) \ge H(u,0)(t) \ \text{ for a.e. } t \in [a,b], \ \ u \in C([a,b];R),$$

and, according to Theorems 1.4 and 1.7, in such a case the inclusion $H \in V_{ab}^{+}(\lambda;\ge)$ implies the inclusion $H \in V_{ab}^{+}(\lambda;\le)$.

1.3. Assertions for the Operator with Maxima.

Theorem 1.10. *Let $i \in \{1,2\}$ and let there exist $\varepsilon > 0$ such that*

$$\Big(\frac{\lambda}{1-\lambda}(b-a) + (\tau_i(t)-a) + \varepsilon\Big)p(t) \le 1 \ \text{ for } t \in [a,b]. \tag{1.18}$$

Then the operator H defined by (0.4) *belongs to the set $V_{ab}^{+}(\lambda;\ge)$ if $i=1$ and it belongs to the set $V_{ab}^{+}(\lambda;\le)$ if $i=2$.*

Theorem 1.11. *Let $i \in \{1,2\}$, $p \not\equiv 0$, and let*

$$\operatorname{ess\,sup}\left\{ \int_t^{\tau_i(t)} p(s)\,ds : \ t \in [a,b] \right\} < \eta, \tag{1.19}$$

where

$$\eta = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x\big(1-\lambda\exp\big(x\int_a^b p(s)\,ds\big)\big)}{\exp\big(x\int_a^b p(s)\,ds\big)-1} \right) : \ x \in]0,\omega[\right\}, \tag{1.20}$$

$$\omega = \begin{cases} \dfrac{1}{\|p\|_L} \ln \dfrac{1}{\lambda} & \text{if } \lambda \ne 0 \\ +\infty & \text{if } \lambda = 0 \end{cases}.$$

Then the operator H defined by (0.4) *belongs to the set $V_{ab}^{+}(\lambda;\ge)$ if $i=1$ and it belongs to the set $V_{ab}^{+}(\lambda;\le)$ if $i=2$.*

The following two assertions are consequences of Theorem 1.11 for $\lambda \neq 0$ and $\lambda = 0$, respectively.

Corollary 1.5. *Let $i \in \{1,2\}$, $\lambda \neq 0$, $p \not\equiv 0$, and let*

$$\operatorname{ess\,sup}\bigg\{ \int\limits_t^{\tau_i(t)} p(s)\,ds : \ t \in [a,b] \bigg\} < \frac{\|p\|_L}{|\ln \lambda|} \ln \frac{|\ln \lambda|}{\|p\|_L}\,.$$

Then the operator H defined by (0.4) *belongs to the set $V_{ab}^{+}(\lambda;\geq)$ if $i=1$ and it belongs to the set $V_{ab}^{+}(\lambda;\leq)$ if $i=2$.*

Corollary 1.6. *Let $i \in \{1,2\}$ and let*

$$\operatorname{ess\,sup}\bigg\{ \int\limits_t^{\tau_i(t)} p(s)\,ds : \ t \in [a,b] \bigg\} \leq \frac{1}{e}\,.$$

Then the operator H defined by (0.4) *belongs to the set $V_{ab}^{+}(0;\geq)$ if $i=1$ and it belongs to the set $V_{ab}^{+}(0;\leq)$ if $i=2$.*

Theorem 1.12. *Let $i \in \{1,2\}$ and let at least one of the following conditions be fulfilled:*

(a) *$p(t)\big(\tau_i(t)-t\big) \leq 0$ for almost every $t \in [a,b]$ and*

$$\lambda \exp\bigg(\int\limits_a^b p(s)\,ds \bigg) < 1; \tag{1.21}$$

(b) *there exists $\alpha \in \,]0,1[$ such that*

$$\frac{\lambda}{1-\lambda} \int\limits_a^b p(s) \int\limits_a^{\tau_i(s)} p(\xi)\,d\xi\,ds + \int\limits_a^t p(s) \int\limits_a^{\tau_i(s)} p(\xi)\,d\xi\,ds \leq$$

$$\leq \bigg(\alpha - \frac{\lambda}{1-\lambda} \int\limits_a^b p(s)\,ds \bigg) \bigg(\frac{\lambda}{1-\lambda} \int\limits_a^b p(s)\,ds + \int\limits_a^t p(s)\,ds \bigg) \ \text{ for } t \in [a,b];$$

(c)

$$\lambda \exp\bigg(\int\limits_a^b p(s)\,ds \bigg) + \int\limits_a^b p(s)\sigma_i(s) \bigg(\int\limits_s^{\tau_i(s)} p(\xi)\,d\xi \bigg) \exp\bigg(\int\limits_s^b p(\xi)\,d\xi \bigg)\,ds < 1,$$

where

$$\sigma_i(t) = \frac{1}{2}\big(1 + \operatorname{sgn}(\tau_i(t)-t)\big) \ \text{ for a.e. } t \in [a,b].$$

Then the operator H defined by (0.4) *belongs to the set $V_{ab}^{+}(\lambda;\geq)$ if $i=1$ and it belongs to the set $V_{ab}^{+}(\lambda;\leq)$ if $i=2$.*

Theorem 1.13. *Let*

$$\int_a^b p(s)\,ds < 1 - \lambda.$$

Then the operator H defined by (0.4) belongs to both sets $V_{ab}^{+}(\lambda;\geq)$ and $V_{ab}^{+}(\lambda;\leq)$.

Remark 1.2. Let us note that the operator H defined by (0.4) belongs to the set $W_{ab}^{+}(\lambda;-)$ if and only if it belongs to the set $W_{ab}^{+}(\lambda;+)$. Therefore, the conditions of the following assertions guarantee both inclusions $H \in W_{ab}^{+}(\lambda;-)$ and $H \in W_{ab}^{+}(\lambda;+)$.

Theorem 1.14. *Let $g(t)(\mu_2(t) - t) \leq 0$ for almost every $t \in [a,b]$ and let at least one of the following conditions be fulfilled:*

(a)

$$\int_a^b g(s)\,ds \leq 1;$$

(b)

$$g(t)(b - \mu_1(t)) \leq 1 \ \text{ for a.e. } \ t \in [a,b];$$

(c)

$$\int_a^b g(s) \int_{\mu_1(s)}^{s} g(\xi) \exp\left(\int_{\mu_1(\xi)}^{s} g(\eta)\,d\eta \right) d\xi\,ds \leq 1; \tag{1.22}$$

(d) $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{ \int_{\mu_1(t)}^{t} g(s)\,ds : \ t \in [a,b] \right\} < \vartheta, \tag{1.23}$$

where

$$\vartheta = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\big(x \int_a^b g(s)\,ds\big) - 1} \right) : \ x > 0 \right\}.$$

Then the operator H defined by (0.4) belongs to both sets $W_{ab}^{+}(\lambda;-)$ and $W_{ab}^{+}(\lambda;+)$.

Corollary 1.7. *Let $g(t)(\mu_2(t) - t) \leq 0$ for almost every $t \in [a,b]$ and let*

$$\operatorname{ess\,sup}\left\{ \int_{\mu_1(t)}^{t} g(s)\,ds : \ t \in [a,b] \right\} \leq \frac{1}{e}\,.$$

Then the operator H defined by (0.4) belongs to both sets $W_{ab}^{+}(\lambda;-)$ and $W_{ab}^{+}(\lambda;+)$.

Theorem 1.15. *Let*

$$\int_a^b g(s)\,ds \le \lambda.$$

Then the operator H defined by (0.4) *belongs to both sets $W^+_{ab}(\lambda;-)$ and $W^+_{ab}(\lambda;+)$.*

2. Auxiliary Propositions

First we formulate a result from [29] in a suitable for us form.

Lemma 2.1 (Corollary 2 in [29]). *Let $F \in \mathcal{K}_{ab}$, $c \in R$, and let there exist a number $\rho > 0$ such that for every $\delta \in]0,1[$, an arbitrary function $u \in \widetilde{C}([a,b];R)$ satisfying*

$$u'(t) = \delta F(u)(t) \ \text{ for a.e. } t \in [a,b], \quad u(a) - \lambda u(b) = \delta c, \tag{2.1}$$

admits the estimate

$$\|u\|_C \le \rho. \tag{2.2}$$

Then the problem

$$u'(t) = F(u)(t), \quad u(a) - \lambda u(b) = c$$

has at least one solution.

Lemma 2.2. *Let $H \in \mathcal{H}_{ab}$ and let for every $\delta \in [0,1]$ the problem*

$$u'(t) = \delta H(0,u)(t), \quad u(a) - \lambda u(b) = 0 \tag{2.3}$$

have only the trivial solution. Then for every $\alpha \in [0,1]$, $y \in C([a,b];R)$, $q_0 \in L([a,b];R)$ and $c \in R$ the problem

$$u'(t) = \alpha H(y,u)(t) + q_0(t), \quad u(a) - \lambda u(b) = c$$

has at least one solution.

Proof. Let $\alpha \in [0,1]$, $y \in C([a,b];R)$, $q_0 \in L([a,b];R)$ and $c \in R$ be fixed. Put

$$F(v)(t) \overset{\text{def}}{=} \alpha H(y,v)(t) + q_0(t) \ \text{ for a.e. } t \in [a,b].$$

Then according to Lemma 2.1 it is sufficient to show that for every $\delta \in]0,1[$ an arbitrary function $u \in \widetilde{C}([a,b];R)$ satisfying (2.1) admits the estimate (2.2).

Assume on the contrary that for every $n \in N$ there exist $\delta_n \in]0,1[$ and $u_n \in \widetilde{C}([a,b];R)$ such that

$$u_n'(t) = \delta_n\big[\alpha H(y,u_n)(t) + q_0(t)\big] \ \text{ for a.e. } t \in [a,b], \tag{2.4}$$

$$u_n(a) - \lambda u_n(b) = \delta_n c, \tag{2.5}$$

and

$$\|u_n\|_C > n. \tag{2.6}$$

Put

$$v_n(t) = \frac{u_n(t)}{\|u_n\|_C} \ \text{ for } t \in [a,b], \quad n \in N. \tag{2.7}$$

Then, obviously,

$$\|v_n\|_C = 1 \quad \text{for} \quad n \in N \tag{2.8}$$

and, since $H \in \mathcal{H}_{ab}$, using (2.4) and (2.5) we have

$$v_n'(t) = \delta_n \Big[\alpha H\Big(\frac{y}{\|u_n\|_C}, v_n\Big)(t) + \frac{1}{\|u_n\|_C} q_0(t) \Big] \tag{2.9}$$
$$\text{for a.e. } t \in [a,b], \quad n \in N,$$

$$v_n(a) - \lambda v_n(b) = \frac{\delta_n c}{\|u_n\|_C} \quad \text{for} \quad n \in N. \tag{2.10}$$

Furthermore, by virtue of the assumptions $\delta_n \in]0,1[$ for $n \in N$ and $H \in \mathcal{H}_{ab}$,

$$|v_n(t) - v_n(s)| \leq \Big| \int_s^t H\Big(\frac{y}{\|u_n\|_C}, v_n\Big)(\xi)\, d\xi \Big| +$$
$$+ \Big| \int_s^t q_0(\xi)\, d\xi \Big| \leq \Big| \int_s^t \omega(\xi)\, d\xi \Big| \quad \text{for} \quad s,\, t \in [a,b], \quad n \in N,$$

where $\omega \equiv H(\|y\|_C, -1) + |H(-\|y\|_C, 1)| + |q_0|$. Thus (2.8) and the last inequality guarantee that the sequence of functions $\{v_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous. Therefore, without loss of generality, we can assume that there exist $\delta_0 \in [0,1]$ and $v_0 \in C([a,b]; R)$ such that

$$\lim_{n \to +\infty} \delta_n = \delta_0 \tag{2.11}$$

and

$$\lim_{n \to +\infty} \|v_n - v_0\|_C = 0. \tag{2.12}$$

On the other hand, in view of (2.6), (2.11), (2.12), and the assumptions $\delta_n \in]0,1[$ for $n \in N$ and $H \in \mathcal{H}_{ab}$, we have uniformly on $[a,b]$

$$\lim_{n \to +\infty} \delta_n \alpha \int_a^t H\Big(\frac{y}{\|u_n\|_C}, v_n\Big)(\xi)\, d\xi = \delta_0 \alpha \int_a^t H(0, v_0)(\xi)\, d\xi, \tag{2.13}$$

$$\lim_{n \to +\infty} \frac{\delta_n}{\|u_n\|_C} \int_a^t q_0(\xi)\, d\xi = 0, \tag{2.14}$$

and

$$\lim_{n \to +\infty} \frac{\delta_n c}{\|u_n\|_C} = 0. \tag{2.15}$$

Integration of (2.9) from a to t yields

$$v_n(t) = v_n(a) + \delta_n \alpha \int_a^t H\Big(\frac{y}{\|u_n\|_C}, v_n\Big)(\xi)\, d\xi +$$

$$+\frac{\delta_n}{\|u_n\|_C}\int\limits_a^t q_0(\xi)\,d\xi \ \text{ for } \ t\in[a,b], \ \ n\in N$$

and, consequently, as $n\to+\infty$, with respect to (2.10), (2.12)–(2.15),

$$v_0(t)=v_0(a)+\delta_0\alpha\int\limits_a^t H(0,v_0)(\xi)\,d\xi \ \text{ for } \ t\in[a,b], \tag{2.16}$$

$$v_0(a)-\lambda v_0(b)=0.$$

Moreover, according to (2.8) and (2.12) we have

$$\|v_0\|_C=1. \tag{2.17}$$

However, in view of (2.16), $v_0\in\widetilde{C}([a,b];R)$ and the function v_0 is a nontrivial solution to (2.3) with $\delta=\delta_0\alpha$, a contradiction. □

Lemma 2.3. *Let $H\in\mathcal{H}_{ab}$, $y_n,y_0,u_n\in C([a,b];R)$ $(n\in N)$ be such that*

$$\lim_{n\to+\infty}\|y_n-y_0\|_C=0, \tag{2.18}$$

and the set $\{u_n\}_{n=1}^{+\infty}$ is relatively compact. Then

$$\lim_{n\to+\infty}\|H(y_n,u_n)-H(y_0,u_n)\|_L=0. \tag{2.19}$$

Proof. Suppose that (2.19) does not hold. Then there exist $\varepsilon_0>0$ and subsequences $\{y_{n_k}\}_{k=1}^{+\infty}\subseteq\{y_n\}_{n=1}^{+\infty}$ and $\{u_{n_k}\}_{k=1}^{+\infty}\subseteq\{u_n\}_{n=1}^{+\infty}$ such that

$$\|H(y_{n_k},u_{n_k})-H(y_0,u_{n_k})\|_L\geq\varepsilon_0 \ \text{ for } \ k\in N. \tag{2.20}$$

Obviously, $\{u_{n_k}\}_{k=1}^{+\infty}$ is also a relatively compact set. Therefore there exists a convergent subsequence $\{u_{m_k}\}_{k=1}^{+\infty}\subseteq\{u_{n_k}\}_{k=1}^{+\infty}$. Let $u_0\in C([a,b];R)$ be such that

$$\lim_{k\to+\infty}\|u_{m_k}-u_0\|_C=0. \tag{2.21}$$

According to (2.20) we have

$$\|H(y_{m_k},u_{m_k})-H(y_0,u_{m_k})\|_L\geq\varepsilon_0 \ \text{ for } \ k\in N. \tag{2.22}$$

On the other hand, with respect to (2.18), (2.21) and the assumption $H\in\mathcal{H}_{ab}$, we have

$$\big\|H(y_{m_k},u_{m_k})-H(y_0,u_{m_k})\big\|_L\leq\big\|H(y_{m_k},u_{m_k})-H(y_0,u_0)\big\|_L+$$
$$+\big\|H(y_0,u_0)-H(y_0,u_{m_k})\big\|_L\to0 \ \text{ as } \ k\to+\infty,$$

which contradicts (2.22). □

Now we formulate a result from [48] in a suitable for us form.

Lemma 2.4 (Theorem 2.1 in [48])**.** *Let $T_n: C([a,b];R)\to C([a,b];R)$ $(n\in N\cup\{0\})$, T_0 be a continuous and compact operator. Let, moreover, $u_0\in C([a,b];R)$ be a unique fixed point of T_0 and let there exist $r>0$*

such that for each $n \in N$ there exists at least one fixed point $u_n \in \{v \in C([a,b];R) : \|v - u_0\|_C \le r\}$ of T_n. Then

$$\lim_{n\to+\infty} \|u_n - u_0\|_C = 0 \tag{2.23}$$

if and only if

$$\lim_{n\to+\infty} \|T_n(u_n) - T_0(u_n)\|_C = 0.$$

Lemma 2.5. *Let $H \in W_{ab}^{+}(\lambda;-)$ and let there exist a number $\rho > 0$ such that for every $\delta \in]0,1[$ an arbitrary function $u \in \widetilde{C}([a,b];R_-)$ satisfying*

$$\begin{gathered} u'(t) = \delta\big[H(u,u)(t) + Q(u)(t)\big] \ \text{ for a.e. } t \in [a,b], \\ u(a) - \lambda u(b) = \delta h(u) \end{gathered} \tag{2.24}$$

admits the estimate (2.2). *If, moreover,*

$$Q(v)(t) \le 0 \ \text{ for a.e. } t \in [a,b], \ \ v \in C([a,b];R_-), \tag{2.25}$$

$$h(v) \le 0 \ \text{ for } v \in C([a,b];R_-), \tag{2.26}$$

then the problem (0.1), (0.2) *has at least one nonpositive solution.*

Proof. Put

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \le s \le \rho \\ 2 - \dfrac{s}{\rho} & \text{for } \rho < s < 2\rho \\ 0 & \text{for } 2\rho \le s \end{cases}, \tag{2.27}$$

$$\widetilde{Q}(y)(t) \overset{\text{def}}{=} \chi(\|y\|_C)Q(y)(t) \ \text{ for a.e. } t \in [a,b], \quad \widetilde{h}(y) \overset{\text{def}}{=} \chi(\|y\|_C)h(y), \tag{2.28}$$

and for arbitrarily fixed $y \in C([a,b];R_-)$ consider the problem

$$u'(t) = \chi(\|y\|_C)H(y,u)(t) + \widetilde{Q}(y)(t), \quad u(a) - \lambda u(b) = \widetilde{h}(y). \tag{2.29}$$

Since $H \in W_{ab}^{+}(\lambda;-)$, we have that for every $\delta \in [0,1]$ the problem (2.3) has only the trivial solution and, according to Lemma 2.2, (2.27) and the inclusion $H \in W_{ab}^{+}(\lambda;-)$, the problem (2.29) is uniquely solvable. Moreover, in view of (2.25)–(2.28), from (2.29) we get

$$u'(t) \le \chi(\|y\|_C)H(0,u)(t) \ \text{ for a.e. } t \in [a,b], \quad u(a) - \lambda u(b) \le 0,$$

and, consequently, due to (2.27) and the assumption $H \in W_{ab}^{+}(\lambda;-)$ we have

$$u(t) \le 0 \ \text{ for } t \in [a,b]. \tag{2.30}$$

Denote by Ω the operator which assigns to every $y \in C([a,b];R_-)$ the solution to (2.29). According to (2.27) and (2.28), there exist $q_{2\rho} \in L([a,b];R_+)$ and $M_{2\rho} \in R_+$ such that

$$|\widetilde{Q}(v)(t)| \le q_{2\rho}(t) \ \text{ for a.e. } t \in [a,b], \ \ v \in C([a,b];R), \tag{2.31}$$

$$|\widetilde{h}(v)| \le M_{2\rho} \ \text{ for } v \in C([a,b];R). \tag{2.32}$$

Let $y \in C([a,b];R_-)$, $u = \Omega(y)$. Then (2.30) holds and

$$u(t) = \frac{\widetilde{h}(y)}{1-\lambda} + \frac{1}{1-\lambda}\int_a^t \big[\chi(\|y\|_C)H(y,u)(\xi) + \widetilde{Q}(y)(\xi)\big]\,d\xi +$$

$$+ \frac{\lambda}{1-\lambda}\int_t^b \big[\chi(\|y\|_C)H(y,u)(\xi) + \widetilde{Q}(y)(\xi)\big]\,d\xi \quad \text{for } t \in [a,b],$$

whence in view of (2.30)–(2.32) we have

$$\|u\|_C \le M, \tag{2.33}$$

where

$$M = \frac{1}{1-\lambda}\big(M_{2\rho} + 2\rho\|H(-1,0)\|_L + \|q_{2\rho}\|_L\big).$$

Consequently,

$$|u(t)-u(s)| \le \left|\int_s^t \chi(\|y\|_C)H(y,u)(\xi)\,d\xi\right| + \left|\int_s^t \widetilde{Q}(y)(\xi)\,d\xi\right| \le$$

$$\le \left|\int_s^t \omega(\xi)\,d\xi\right| \quad \text{for } s,\, t \in [a,b], \tag{2.34}$$

where $\omega \equiv MH(0,-1) - 2\rho H(-1,0) + q_{2\rho}$. Therefore, by virtue of (2.30), (2.33) and (2.34), according to Arzelà-Ascoli lemma, the set $\Omega\big(C([a,b];R_-)\big)$ is a relatively compact subset of $C([a,b];R_-)$.

Let now $y_n, y_0 \in C([a,b];R_-)$ be such that the condition (2.18) holds. For every $n \in N \cup \{0\}$ put $u_n = \Omega(y_n)$ and

$$T_n(v)(t) \overset{\text{def}}{=} \frac{\widetilde{h}(y_n)}{1-\lambda} + \frac{1}{1-\lambda}\int_a^t \big[\chi(\|y_n\|_C)H(y_n,v)(\xi) + \widetilde{Q}(y_n)(\xi)\big]\,d\xi +$$

$$+ \frac{\lambda}{1-\lambda}\int_t^b \big[\chi(\|y_n\|_C)H(y_n,v)(\xi) + \widetilde{Q}(y_n)(\xi)\big]\,d\xi \quad \text{for } t \in [a,b].$$

Then, according to Lemmas 2.3 and 2.4 and the continuity of χ, $\widetilde{Q}$ and $\widetilde{h}$, we get the condition (2.23). Therefore, the operator Ω is a continuous operator transforming the set $C([a,b];R_-)$ into its relatively compact subset. According to Schauder's fixed point theorem, there exists $u \in C([a,b];R_-)$ such that $u = \Omega(u)$, i.e., $u \in \widetilde{C}([a,b];R_-)$ and, in view of (2.28),

$$u'(t) = \chi(\|u\|_C)\big[H(u,u)(t) + Q(u)(t)\big] \quad \text{for a.e. } t \in [a,b],$$

$$u(a) - \lambda u(b) = \chi(\|u\|_C)h(u).$$

Now, according to the assumptions of the lemma and (2.27), we get that u admits the estimate (2.2) and, consequently, u is a nonpositive solution to the problem (0.1), (0.2). $\square$

The following assertion can be proved in analogous way.

Lemma 2.6. *Let $H \in W_{ab}^{+}(\lambda;+)$ and let there exist a number $\rho > 0$ such that for every $\delta \in]0,1[$ an arbitrary function $u \in \widetilde{C}([a,b];R_+)$ satisfying* (2.24) *admits the estimate* (2.2)*. If, moreover,*

$$\begin{aligned} Q(v)(t) \geq 0 \ \ &\text{for a.e. } t \in [a,b], \ \ v \in C([a,b];R_+), \\ h(v) \geq 0 \ \ &\text{for } v \in C([a,b];R_+), \end{aligned}$$

then the problem (0.1), (0.2) *has at least one nonnegative solution.*

Lemma 2.7. *Let $H \in V_{ab}^{+}(\lambda;\geq)$, $c \in R_+$, and*

$$|Q(v)(t)| \leq q(t,\|v\|_C) \ \ \text{for a.e. } t \in [a,b], \ \ v \in C([a,b];R_-), \tag{2.35}$$

$$|h(v)| \leq c \ \ \text{for } v \in C([a,b];R_-). \tag{2.36}$$

Then there exists a number $\rho > 0$ such that an arbitrary function $u \in \widetilde{C}([a,b];R_-)$ satisfying (2.24) *with some $\delta \in]0,1[$ admits the estimate* (2.2)*.*

Proof. Assume on the contrary that for every $n \in N$ there exist $\delta_n \in]0,1[$ and $u_n \in \widetilde{C}([a,b];R_-)$ such that

$$u_n'(t) = \delta_n\big[H(u_n,u_n)(t) + Q(u_n)(t)\big] \ \ \text{for a.e. } t \in [a,b], \tag{2.37}$$

$$u_n(a) - \lambda u_n(b) = \delta_n h(u_n), \tag{2.38}$$

and (2.6) is fulfilled. Define the functions v_n by (2.7). Then, obviously, (2.8) is satisfied, and, since $H \in \mathcal{H}_{ab}$, using (2.37) and (2.38) we have

$$v_n'(t) = \delta_n\Big[H(v_n,v_n)(t) + \frac{1}{\|u_n\|_C}\,Q(u_n)(t)\Big] \quad \text{for a.e. } t \in [a,b], \ \ n \in N, \tag{2.39}$$

$$v_n(a) - \lambda v_n(b) = \frac{\delta_n}{\|u_n\|_C}\,h(u_n) \ \ \text{for } n \in N. \tag{2.40}$$

Furthermore, by virtue of (2.8), (2.35), and the assumptions $\delta_n \in]0,1[$ for $n \in N$ and $H \in \mathcal{H}_{ab}$, we have

$$\begin{aligned} |v_n(t) - v_n(s)| \leq \Big|\int_s^t \big[H(0,-1)(\xi) - H(-1,0)(\xi)\big]\,d\xi\Big| + \\ + \Big|\frac{1}{\|u_n\|_C}\int_s^t q(\xi,\|u_n\|_C)\,d\xi\Big| \ \ \text{for } s,\, t \in [a,b], \ \ n \in N. \end{aligned}$$

According to the Lebesgue theorem (see, e.g., [39, Theorem 3, p. 170]), since q satisfies (1.1), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_s^t \big[H(0,-1)(\xi) - H(-1,0)(\xi)\big]\, d\xi \right| \leq \frac{\varepsilon}{2} \ \text{ for } \ s,\, t \in [a,b], \ \ |s-t| \leq \delta,$$

$$\left| \frac{1}{\|u_n\|_C} \int_s^t q(\xi, \|u_n\|_C)\, d\xi \right| \leq \frac{\varepsilon}{2} \ \text{ for } \ s,\, t \in [a,b], \ \ |s-t| \leq \delta, \ \ n \in N.$$

Consequently,

$$|v_n(t) - v_n(s)| \leq \varepsilon \ \text{ for } \ s,\, t \in [a,b], \ \ |s-t| \leq \delta, \ \ n \in N. \tag{2.41}$$

Thus (2.8) and (2.41) guarantee that the sequence of functions $\{v_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous. Therefore, without loss of generality we can assume that there exist $\delta_0 \in [0,1]$ and $v_0 \in C([a,b]; R_-)$ such that (2.11) and (2.12) hold.

On the other hand, with regard to (1.1), (2.6), (2.11), (2.12), (2.35), (2.36) and the assumptions $\delta_n \in]0,1[$ for $n \in N$ and $H \in \mathcal{H}_{ab}$, we have

$$\lim_{n\to+\infty} \delta_n \int_a^t H(v_n, v_n)(\xi)\, d\xi = \delta_0 \int_a^t H(v_0, v_0)(\xi)\, d\xi \ \text{ uniformly on } \ [a,b], \tag{2.42}$$

$$\lim_{n\to+\infty} \delta_n \int_a^t \frac{1}{\|u_n\|_C}\, Q(u_n)(\xi)\, d\xi = 0 \ \text{ uniformly on } \ [a,b], \tag{2.43}$$

and

$$\lim_{n\to+\infty} \frac{\delta_n}{\|u_n\|_C}\, h(u_n) = 0. \tag{2.44}$$

Integration of (2.39) from a to t yields

$$v_n(t) = v_n(a) + \delta_n \int_a^t H(v_n, v_n)(\xi)\, d\xi + {} + \delta_n \int_a^t \frac{1}{\|u_n\|_C}\, Q(u_n)(\xi)\, d\xi \ \text{ for } \ t \in [a,b], \ \ n \in N,$$

and, consequently, as $n \to +\infty$, with regard to (2.12), (2.40), (2.42)–(2.44), we have

$$v_0(t) = v_0(a) + \delta_0 \int_a^t H(v_0, v_0)(\xi)\, d\xi \ \text{ for } \ t \in [a,b], \ \ v_0(a) - \lambda v_0(b) = 0. \tag{2.45}$$

Moreover, according to (2.8) and (2.12), we get (2.17). However, in view of (2.45) and the fact that $\delta_0 \in [0,1]$, we have $v_0 \in \widetilde{C}([a,b]; R_-)$ and

$$v_0'(t) \geq H(v_0, 0)(t) \ \text{ for a.e. } \ t \in [a,b], \quad v_0(a) - \lambda v_0(b) = 0.$$

Since $H \in V_{ab}^{+}(\lambda; \geq)$, we have $v_0(t) \geq 0$ for $t \in [a,b]$. Consequently, $v_0 \equiv 0$, which contradicts (2.17). $\square$

In analogous way one can prove the following

Lemma 2.8. *Let $H \in V_{ab}^{+}(\lambda; \leq)$, $c \in R_+$, and*

$$\begin{gathered}|Q(v)(t)| \leq q(t, \|v\|_C) \ \ \text{for a.e. } \ t \in [a,b], \ \ v \in C([a,b]; R_+), \\ |h(v)| \leq c \ \ \text{for } \ v \in C([a,b]; R_+).\end{gathered}$$

Then there exists a number $\rho > 0$ such that an arbitrary function $u \in \widetilde{C}([a,b]; R_+)$ satisfying (2.24) *with some $\delta \in]0,1[$ admits the estimate* (2.2).

Lemma 2.9. *Let $\delta \in [0,1]$, $\overline{H} \in \mathcal{H}_{ab}$, and $\overline{H}(0,\cdot\,)$ be an a-Volterra operator. Let, moreover, $w \in \widetilde{C}([a,b]; R)$ satisfy*

$$w'(t) \geq \delta \overline{H}(0,w)(t) \ \ \text{for a.e. } \ t \in [a,b], \tag{2.46}$$

$$w(a) - \lambda w(b) \geq 0, \tag{2.47}$$

and

$$\min\big\{w(t) : \ t \in [a,b]\big\} < 0. \tag{2.48}$$

Then there exist $t_* \in]a,b]$ and $t^* \in [a, t_*[$ such that

$$w(t_*) = \min\big\{w(t) : \ t \in [a,b]\big\}, \ \ w(t^*) = \max\big\{w(t) : \ t \in [a,t_*]\big\} > 0. \tag{2.49}$$

Proof. Put

$$\begin{gathered}m = -\min\big\{w(t); t \in [a,b]\big\}, \\ A = \big\{t \in [a,b]; w(t) = -m\big\}, \quad t_* = \sup A.\end{gathered}$$

Obviously, $m > 0$ and

$$w(t_*) = -m. \tag{2.50}$$

In view of (2.47) it is clear that

$$a \notin A. \tag{2.51}$$

We will show that

$$\max\big\{w(t) : \ t \in [a,t_*]\big\} > 0.$$

Assume on the contrary that

$$w(t) \leq 0 \ \ \text{for } \ t \in [a,t_*]. \tag{2.52}$$

Since $\overline{H}(0,\cdot\,)$ is an a-Volterra operator, integration of (2.46) from a to t_*, on account of (2.52), results in

$$w(t_*) - w(a) \geq \delta \int_a^{t_*} \overline{H}(0,w)(s)\, ds \geq 0.$$

The last inequality, in view of (2.50), yields $a \in A$, which contradicts (2.51). $\square$

3. Proofs

Theorem 1.1 follows from Lemmas 2.5 and 2.7. Theorem 1.2 follows from Lemmas 2.6 and 2.8.

Proof of Theorem 1.3. We will assume that $H \in W^+_{ab}(\lambda; -)$. The case $H \in W^+_{ab}(\lambda; +)$ can be proved analogously.

To prove the existence of a solution, it is sufficient to show that the problem

$$u'(t) = \delta H(u, u)(t), \quad u(a) - \lambda u(b) = 0 \tag{3.1}$$

has only the trivial solution for every $\delta \in [0, 1]$ (see Corollary 1.4 in [34]). Let u be a solution to (3.1). Since $H \in W^+_{ab}(\lambda; -)$, according to Lemma 2.2 there exists a solution α to the problem

$$\alpha'(t) = \delta H(-[u]_-, \alpha)(t), \quad \alpha(a) - \lambda\alpha(b) = 0. \tag{3.2}$$

Furthermore, since

$$H(-[u]_-, 0)(t) \leq 0 \ \text{ for a.e. } \ t \in [a, b]$$

and

$$\alpha'(t) - \delta H(-[u]_-, \alpha)(t) \leq u'(t) - \delta H(-[u]_-, u)(t) \ \text{ for a.e. } \ t \in [a, b],$$

in view of $H \in W^+_{ab}(\lambda; -)$ we have

$$\alpha(t) \leq 0, \quad \alpha(t) \leq u(t) \ \text{ for } \ t \in [a, b]. \tag{3.3}$$

Consequently,

$$\alpha(t) \leq -[u(t)]_- \ \text{ for } \ t \in [a, b]. \tag{3.4}$$

Now, using (3.3) and (3.4) in (3.2) we get

$$\alpha'(t) \geq \delta H(\alpha, \alpha)(t) \geq H(\alpha, 0)(t) \ \text{ for a.e. } \ t \in [a, b], \tag{3.5}$$

which together with $H \in V^+_{ab}(\lambda; \geq)$ implies

$$\alpha(t) \geq 0 \ \text{ for } \ t \in [a, b]. \tag{3.6}$$

Now (3.3) and (3.6) result in

$$u(t) \geq 0 \ \text{ for } \ t \in [a, b]. \tag{3.7}$$

However, (3.7) implies

$$u'(t) \leq H(u, 0)(t) \ \text{ for a.e. } \ t \in [a, b],$$

whence, according to $H \in V^+_{ab}(\lambda; \leq)$, we obtain

$$u(t) \leq 0 \ \text{ for } \ t \in [a, b]. \tag{3.8}$$

Consequently, (3.7) and (3.8) yield $u \equiv 0$.

If, in addition, (1.2) is fulfilled, then the existence of a nonpositive solution to (0.1), (0.2) follows from Theorem 1.1.

Therefore, let moreover (1.3) be fulfilled. Consider the auxiliary problem

$$u'(t) = H(u, u)(t) + Q(|u|)(t), \quad u(a) - \lambda u(b) = h(|u|). \tag{3.9}$$

According to the first part of the theorem, there exists a solution u to (3.9). Moreover, in view of (1.3) we have

$$u'(t) \geq H(u,u)(t) \geq H(-[u]_-,u)(t) \text{ for a.e. } t \in [a,b], \tag{3.10}$$

$$u(a) - \lambda u(b) \geq 0. \tag{3.11}$$

Let α be a solution to

$$\alpha'(t) = H(-[u]_-,\alpha)(t), \quad \alpha(a) - \lambda\alpha(b) = 0, \tag{3.12}$$

which is guaranteed by Lemma 2.2 and the inclusion $H \in W_{ab}^+(\lambda; -)$. Then in view of (3.10) and (3.11) we have

$$\alpha'(t) - H(-[u]_-,\alpha)(t) \leq u'(t) - H(-[u]_-,u)(t) \text{ for a.e. } t \in [a,b],$$

and, consequently, (3.3) and (3.4). Now, using (3.3) and (3.4) in (3.12), we get

$$\alpha'(t) \geq H(\alpha,\alpha)(t) \geq H(\alpha,0)(t) \text{ for a.e. } t \in [a,b],$$

which together with $H \in V_{ab}^+(\lambda; \geq)$ implies (3.6). Now (3.3) and (3.6) result in (3.7). Therefore, u is also a nonnegative solution to (0.1), (0.2). □

Proof of Proposition 1.1. First suppose that $H \in V_{ab}^+(\lambda; \geq)$. If u is a solution to the problem (1.4), then $-u$ is a solution to (0.5). According to the assumption $H \in V_{ab}^+(\lambda; \geq)$, we have $-u(t) \geq 0$ for $t \in [a,b]$, i.e., $u(t) \leq 0$ for $t \in [a,b]$.

Now suppose that the problem (1.4) has no nontrivial nonnegative solution. Let u be a solution to the problem (0.5). Put

$$v(t) = u(t) - \frac{u(a) - \lambda u(b)}{1-\lambda} \text{ for } t \in [a,b]. \tag{3.13}$$

Then

$$[v(t)]_-' \leq -H(-[v]_-,0)(t) \text{ for a.e. } t \in [a,b], \quad [v(a)]_- = \lambda[v(b)]_- .$$

Hence, $[v]_-$ is a nonnegative solution to the problem (1.4). Thus $[v]_- \equiv 0$, i.e., in view of the second inequality in (0.5) and (3.13), $u(t) \geq 0$ for $t \in [a,b]$. □

Proof of Theorem 1.4. First suppose that there exists $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities (1.5) and (1.6). Let u be a solution to the problem (0.5). We will show that

$$u(t) \geq 0 \text{ for } t \in [a,b]. \tag{3.14}$$

Assume on the contrary that (3.14) is not valid. Then there exists $t_0 \in [a,b]$ such that

$$u(t_0) < 0. \tag{3.15}$$

Put

$$\alpha = \max\Big\{ -\frac{u(t)}{\gamma(t)}; t \in [a,b]\Big\}.$$

Then, in view of (3.15), we have $\alpha > 0$ and

$$\alpha\gamma(t) + u(t) \geq 0 \text{ for } t \in [a,b]. \tag{3.16}$$

Moreover, there exists $t_* \in [a,b]$ such that

$$\alpha\gamma(t_*) + u(t_*) = 0. \tag{3.17}$$

By virtue of $H \in \mathcal{H}_{ab}$, (0.5), (1.5) and (3.16), we get

$$\alpha\gamma'(t) + u'(t) \geq -H(-\alpha\gamma, 0)(t) + H(u,0)(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b].$$

From the last inequality, (3.16) and (3.17), we obtain

$$\alpha\gamma(a) + u(a) = 0. \tag{3.18}$$

However, in view of (0.5), (1.6), (3.16), (3.18) and $\alpha > 0$, we have

$$0 \leq \lambda\alpha\gamma(b) + \lambda u(b) < \alpha\gamma(a) + u(a) = 0,$$

a contradiction.

Now assume that $H \in V_{ab}^{+}(\lambda; \geq)$. Obviously, the operator $H(\cdot\,,0)$ belongs to the set $W_{ab}^{+}(\lambda; -)$. Therefore, according to Theorem 1.1 there exists a nonpositive solution u to the problem

$$u'(t) = H(u,0)(t), \quad u(a) - \lambda u(b) = -1. \tag{3.19}$$

Moreover, by virtue of $H \in \mathcal{H}_{ab}$, from (3.19) it follows that $u'(t) \leq 0$ for almost every $t \in [a,b]$, $u(a) < 0$. Consequently,

$$u(t) < 0 \ \text{ for } \ t \in [a,b].$$

Put

$$\gamma(t) = -u(t) \ \text{ for } \ t \in [a,b].$$

Then $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ and it satisfies the inequalities (1.5) and (1.6). $\square$

Proof of Corollary 1.1. (a) It is not difficult to verify that the function

$$\gamma(t) = \exp\bigg(-\int_a^t H(-1,0)(s)\,ds \bigg) \ \text{ for } \ t \in [a,b]$$

satisfies the inequalities (1.5) and (1.6). Consequently, the assumptions of Theorem 1.4 are fulfilled.

(b) Define the operators $\varphi_i : C([a,b];R) \to C([a,b];R)$, $i \in N \cup \{0\}$, as follows:

$$\varphi_0(v)(t) \overset{\text{def}}{=} v(t), \quad \varphi_i(v)(t) \overset{\text{def}}{=} -\frac{\lambda}{1-\lambda} \int_a^b H(-\varphi_{i-1}(v),0)(s)\,ds -$$
$$- \int_a^t H(-\varphi_{i-1}(v),0)(s)\,ds \ \text{ for } \ t \in [a,b], \ \ i \in N. \tag{3.20}$$

According to Proposition 1.1, it is sufficient to show that the problem (1.4) has no nontrivial nonnegative solution. Let $u \in C([a,b];R_+)$ satisfy (1.4). Then

$$u(t) \leq \varphi_1(u)(t) \ \text{ for } \ t \in [a,b], \tag{3.21}$$

and since φ_i, $i \in N$, are nondecreasing operators, from (3.21), on account of (3.20), we get

$$u(t) \le \varphi_i(u)(t) \quad \text{for} \quad t \in [a,b], \quad i \in N, \tag{3.22}$$

and

$$u(t) \le \varphi_k(1)(t)\|u\|_C = \rho_k(t)\|u\|_C \quad \text{for} \quad t \in [a,b]. \tag{3.23}$$

Put

$$v(t) = \begin{cases} \dfrac{u(t)}{\rho_k(t)} & \text{if} \quad \rho_k(t) \neq 0 \\ 0 & \text{if} \quad \rho_k(t) = 0 \end{cases} \qquad \text{for} \quad t \in [a,b], \tag{3.24}$$

$$\beta = \sup\big\{v(t) : \ t \in [a,b]\big\}. \tag{3.25}$$

By virtue of (3.23)–(3.25), we have $\beta < +\infty$ and

$$u(t) \le \beta\rho_k(t) \quad \text{for} \quad t \in [a,b]. \tag{3.26}$$

Furthermore, in view of (1.7), (3.22), and (3.26),

$$u(t) \le \varphi_{m-k}(u)(t) \le \beta\varphi_{m-k}(\rho_k)(t) = \beta\rho_m(t) \le \alpha\beta\rho_k(t) \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (3.24) and (3.25), we get

$$\beta \le \alpha\beta.$$

Now, since $\alpha \in]0,1[$, we have $\beta = 0$, and consequently $u \equiv 0$.

(c) According to (1.8), there exists $\varepsilon > 0$ such that

$$\varepsilon\gamma_0 \exp\bigg(-\int_a^b H(-1,0)(\xi)\,d\xi\bigg) + \\ + \gamma_0 \int_a^b \widehat{H}(1,0)(s) \exp\bigg(-\int_s^b H(-1,0)(\xi)\,d\xi\bigg)\,ds \le 1, \tag{3.27}$$

where

$$\gamma_0 = \frac{1}{1 - \lambda \exp\big(-\int_a^b H(-1,0)(\xi)\,d\xi\big)}\,.$$

Put

$$\gamma(t) = \gamma_0 \exp\bigg(-\int_a^t H(-1,0)(\xi)\,d\xi\bigg) \times \\ \times \bigg[\varepsilon + \int_a^t \widehat{H}(1,0)(s) \exp\bigg(\int_a^s H(-1,0)(\xi)\,d\xi\bigg)\,ds + \\ + \lambda \int_t^b \widehat{H}(1,0)(s) \exp\bigg(-\int_s^b H(-1,0)(\xi)\,d\xi\bigg)\,ds\bigg] \quad \text{for} \quad t \in [a,b].$$

Obviously, $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ and it is a solution to the problem

$$\gamma'(t) = -H(-1,0)(t)\gamma(t) + \widehat{H}(1,0)(t), \quad \gamma(a) - \lambda\gamma(b) = \varepsilon. \tag{3.28}$$

Since $H, \widehat{H} \in \mathcal{H}_{ab}$ and $\gamma(t) > 0$ for $t \in [a,b]$, from (3.27) and (3.28) we have $\gamma'(t) \geq 0$ for almost every $t \in [a,b]$, $\gamma(t) \leq 1$ for $t \in [a,b]$, and, consequently,

$$\gamma'(t) \geq -H(-1,0)(t)\gamma(t) + \widehat{H}(\gamma,0)(t), \quad \gamma(a) - \lambda\gamma(b) > 0.$$

By virtue of Theorem 1.4 we find

$$\widetilde{H} \in V_{ab}^{+}(\lambda;\geq), \tag{3.29}$$

where

$$\widetilde{H}(v,w)(t) \stackrel{\text{def}}{=} -H(-1,0)(t)v(t) - \widehat{H}(-v,0)(t) \ \text{ for } \ t \in [a,b]. \tag{3.30}$$

According to Proposition 1.1 it is sufficient to show that the problem (1.4) has no nontrivial nonnegative solution. Let $u \in C([a,b];R_+)$ satisfy (1.4). Put

$$v(t) = \vartheta(u)(t) \ \text{ for } \ t \in [a,b], \tag{3.31}$$

where ϑ is defined by (1.10). Obviously, $v'(t) = -H(-u,0)(t) \geq u'(t)$ for almost every $t \in [a,b]$ and

$$0 \leq u(t) \leq v(t) \ \text{ for } \ t \in [a,b], \quad v(a) - \lambda v(b) = 0. \tag{3.32}$$

On the other hand, in view of $H \in \mathcal{H}_{ab}$, (1.9) and (3.30)–(3.32), we get

$$\begin{aligned}
v'(t) &= -H(-u,0)(t) \leq \\
&\leq -H(-1,0)(t)v(t) - H(-v,0)(t) + H(-1,0)(t)v(t) = \\
&= -H(-1,0)(t)v(t) - H(-\vartheta(u),0)(t) + H(-1,0)(t)\vartheta(u)(t) \leq \\
&\leq -H(-1,0)(t)v(t) + \widehat{H}(u,0)(t) \leq \\
&\leq -H(-1,0)(t)v(t) + \widehat{H}(v,0)(t) = -\widetilde{H}(-v,0)(t) \ \text{ for a.e. } \ t \in [a,b].
\end{aligned}$$

Now by (3.29), (3.32), and Proposition 1.1 we obtain $u \equiv 0$. $\square$

Proof of Theorem 1.5. Let $\delta \in [0,1]$, $y \in C([a,b];R_-)$, and let $u, v \in \widetilde{C}([a,b];R)$ satisfy (0.7), (0.8). Put $w(t) = u(t) - v(t)$ for $t \in [a,b]$. Then, in view of (1.11), the inequalities (2.46) and (2.47) are fulfilled. We will show that

$$w(t) \geq 0 \ \text{ for } \ t \in [a,b]. \tag{3.33}$$

Assume on the contrary that (2.48) is fulfilled. According to Lemma 2.9, there exist $t_* \in]a,b]$ and $t^* \in [a,t_*[$ such that (2.49) is valid. It is clear that there exists $t_0 \in]t^*,t_*[$ such that

$$w(t_0) = 0. \tag{3.34}$$

Put

$$\kappa = \max\left\{\frac{w(t)}{\gamma(t)} \, : \, t \in [a,t_0]\right\}.$$

Obviously, $\kappa > 0$ and there exists $t_1 \in [a,t_0[$ such that

$$\kappa\gamma(t_1) - w(t_1) = 0. \tag{3.35}$$

It is also evident that

$$\kappa\gamma(t) - w(t) \geq 0 \;\text{ for }\; t \in [a, t_0].$$

Due to (1.12), (1.13), (2.46), $\delta \in [0, 1]$, $\kappa > 0$, and the fact that $\overline{H}(0, \cdot\,)$ is an a-Volterra operator, we have

$$\kappa\gamma'(t) - w'(t) \leq \delta\Big[\overline{H}(0, \kappa\gamma)(t) - \overline{H}(0, w)(t)\Big] \leq 0 \;\text{ for a.e. }\; t \in [a, t_0].$$

Hence, in view of (3.35), we get

$$\kappa\gamma(t) - w(t) \leq 0 \;\text{ for }\; [t_1, t_0],$$

whence, together with (1.12) and (3.34), we find $0 < \kappa\gamma(t_0) \leq 0$, a contradiction. □

Proof of Theorem 1.6. (a) Let $\delta \in [0, 1]$, $y \in C([a, b]; R_-)$, and let $u, v \in \widetilde{C}([a, b]; R)$ satisfy (0.7), (0.8). Put $w(t) = u(t) - v(t)$ for $t \in [a, b]$. Then, in view of (1.11), the inequalities (2.46) and (2.47) are fulfilled. We will show that (3.33) holds. Put

$$M = \max\big\{w(t) :\; t \in [a, b]\big\}, \quad m = -\min\big\{w(t) :\; t \in [a, b]\big\} \tag{3.36}$$

and choose $t_M, t_m \in [a, b]$ such that

$$w(t_M) = M, \quad w(t_m) = -m. \tag{3.37}$$

Assume that (3.33) is not valid. Then $m > 0$, and either $M \geq 0$ or $M < 0$.

Suppose that $M < 0$, i.e.,

$$w(t) < 0 \;\text{ for }\; t \in [a, b]. \tag{3.38}$$

Then integration of (2.46) from a to b yields

$$w(b) - w(a) \geq \delta \int_a^b \overline{H}(0, w)(s)\, ds \geq 0,$$

which, on account of (3.38), contradicts (2.47).

Therefore, $M \geq 0$, $m > 0$, and either

$$t_M < t_m \tag{3.39}$$

or

$$t_m < t_M. \tag{3.40}$$

First suppose that (3.39) is fulfilled. Integration of (2.46) from t_M to t_m, in view of (3.36), (3.37) and the assumption $\delta \in [0, 1]$, results in

$$M + m \leq -\delta \int_{t_M}^{t_m} \overline{H}(0, w)(s)\, ds \leq M \int_a^b \big|\overline{H}(0, 1)(s)\big|\, ds.$$

Hence, according to (1.14) we get $m \leq 0$, a contradiction.

Now suppose that (3.40) holds. Integration of (2.46) from a to t_m and from t_M to b, respectively, on account of (3.36), (3.37) and the assumption $\delta \in [0,1]$, yields

$$w(a) + m \le -\delta \int_a^{t_m} \overline{H}(0,w)(s)\,ds \le M \int_a^{t_m} \big|\overline{H}(0,1)(s)\big|\,ds, \tag{3.41}$$

$$M - w(b) \le -\delta \int_{t_M}^{b} \overline{H}(0,w)(s)\,ds \le M \int_{t_M}^{b} \big|\overline{H}(0,1)(s)\big|\,ds. \tag{3.42}$$

Multiplying (3.42) by λ, in view of the assumption $M \ge 0$ we get

$$\lambda M - \lambda w(b) \le M \int_{t_M}^{b} \big|\overline{H}(0,1)(s)\big|\,ds.$$

Summing the last inequality and (3.41), taking into account (2.47) and the assumption $M \ge 0$, we obtain

$$\lambda M + m \le M \int_a^b \big|\overline{H}(0,1)(s)\big|\,ds.$$

Hence, according to (1.14) and the assumption $m > 0$, we get the contradiction $\lambda M < \lambda M$. Therefore, (3.33) is valid.

(b) Let $\delta \in [0,1]$, $y \in C([a,b];R_-)$, and let u, $v \in \widetilde{C}([a,b];R)$ satisfy (0.7), (0.8). Put $w(t) = u(t) - v(t)$ for $t \in [a,b]$. Then, in view of (1.11), the inequalities (2.46) and (2.47) are fulfilled. We will show that (3.33) is satisfied.

Assume on the contrary that the inequality (2.48) holds. According to Lemma 2.9, there exist $t_* \in]a,b]$ and $t^* \in [a,t_*[$ such that (2.49) is valid. Integration of (2.46) from t^* to t_* yields

$$w(t^*) - w(t_*) \le -\int_{t^*}^{t_*} \overline{H}(0,w)(s)\,ds.$$

Hence, in view of (2.48), (2.49) and the assumption that $\overline{H}(0,\cdot\,)$ is an a-Volterra operator, we find

$$w(t^*) < w(t^*) + |w(t_*)| \le w(t^*) \int_a^b |\overline{H}(0,1)(s)|\,ds.$$

The last inequality, on account of (1.15), implies $w(t^*) < w(t^*)$, a contradiction. □

Proofs of Proposition 1.2, Theorems 1.7–1.9, and Corollary 1.3 are similar to the proofs of Proposition 1.1, Theorems 1.4–1.6, and Corollary 1.1, respectively and therefore they will be omitted.

Proof of Theorem 1.10. Put

$$\gamma(t) \stackrel{\text{def}}{=} \frac{\lambda}{1-\lambda}(b-a)+(t-a)+\varepsilon \quad \text{for} \quad t\in[a,b].$$

Then $\gamma(a)-\lambda\gamma(b)=(1-\lambda)\varepsilon>0$, and, in view of (1.18), we have

$$\begin{aligned}\gamma'(t) &\geq p(t)\Big(\frac{\lambda}{1-\lambda}(b-a)+(\tau_1(t)-a)+\varepsilon\Big)=p(t)\gamma(\tau_1(t))=\\ &=-p(t)\max\big\{-\gamma(s):\ \tau_1(t)\leq s\leq\tau_2(t)\big\} \quad \text{for a.e.} \quad t\in[a,b] \quad \text{if} \quad i=1\end{aligned}$$

and

$$\begin{aligned}\gamma'(t) &\geq p(t)\Big(\frac{\lambda}{1-\lambda}(b-a)+(\tau_2(t)-a)+\varepsilon\Big)=p(t)\gamma(\tau_2(t))=\\ &=p(t)\max\big\{\gamma(s):\ \tau_1(t)\leq s\leq\tau_2(t)\big\} \quad \text{for a.e.} \quad t\in[a,b] \quad \text{if} \quad i=2.\end{aligned}$$

Consequently, the assumptions of Theorems 1.4 and 1.7, respectively, are fulfilled. □

Proof of Theorem 1.11. Let us note that according to (1.19) and (1.20) there exist $x_0\in\,]0,\omega[$ and $\varepsilon\in\,]0,1-\lambda e^{x_0\|p\|_L}[$ such that

$$\int\limits_t^{\tau_i(t)} p(s)\,ds \leq \frac{1}{x_0}\ln\Big(x_0+\frac{x_0(1-\lambda e^{x_0\|p\|_L}-\varepsilon)}{e^{x_0\|p\|_L}-1+\varepsilon}\Big) \quad \text{for a.e.} \quad t\in[a,b].$$

Hence we get

$$\exp\left(x_0\int\limits_{\tau_i(t)}^{t} p(s)\,ds\right) \geq \frac{1}{x_0}-\frac{1-\lambda e^{x_0\|p\|_L}-\varepsilon}{(1-\lambda)x_0e^{x_0\|p\|_L}} \quad \text{for a.e.} \quad t\in[a,b]. \tag{3.43}$$

Put

$$\gamma(t)=\exp\left(x_0\int\limits_a^t p(s)\,ds\right)-\frac{1-\lambda e^{x_0\|p\|_L}-\varepsilon}{1-\lambda} \quad \text{for} \quad t\in[a,b].$$

Then $\gamma\in\widetilde{C}([a,b];]0,+\infty[)$, $\gamma(a)-\lambda\gamma(b)>0$, and, in view of (3.43), we have

$$\begin{aligned}\gamma'(t)=x_0p(t)\exp\left(x_0\int\limits_a^{\tau_i(t)} p(s)\,ds\right)\exp\left(x_0\int\limits_{\tau_i(t)}^{t} p(s)\,ds\right)\geq\\ \geq p(t)\Bigg(\exp\left(x_0\int\limits_a^{\tau_i(t)} p(s)\,ds\right)-\\ -\exp\left(x_0\int\limits_a^{\tau_i(t)} p(s)\,ds\right)\frac{1-\lambda e^{x_0\|p\|_L}-\varepsilon}{(1-\lambda)e^{x_0\|p\|_L}}\Bigg)\geq\\ \geq p(t)\gamma(\tau_i(t)) \quad \text{for a.e.} \quad t\in[a,b].\end{aligned}$$

However, since γ is a nondecreasing function, the last inequality means

$$\gamma'(t) \geq -p(t) \max\big\{ -\gamma(s) : \ \tau_1(t) \leq s \leq \tau_2(t)\big\} \quad \text{for a.e. } t \in [a,b] \ \text{ if } \ i=1 \tag{3.44}$$

and

$$\gamma'(t) \geq p(t) \max\big\{\gamma(s) : \ \tau_1(t) \leq s \leq \tau_2(t)\big\} \quad \text{for a.e. } t \in [a,b] \ \text{ if } \ i=2. \tag{3.45}$$

Consequently, the assumptions of Theorems 1.4 and 1.7 are fulfilled, respectively. □

Proof of Corollary 1.5. The validity of the corollary follows immediately from Theorem 1.11 for $x = \frac{1}{\|p\|_L} \ln \frac{1}{\lambda}$. □

Proof of Corollary 1.6. If $p \equiv 0$, then the validity of the corollary follows from Theorem 1.10. If $p \not\equiv 0$, then the corollary follows from Theorem 1.11, because

$$\eta > \sup\Big\{\frac{1}{x}\ln x : \ x>0\Big\} = \frac{1}{e}. \qquad \square$$

Proof of Theorem 1.12. (a) Put

$$\gamma(t) = \exp\bigg(\int_a^t p(s)\,ds\bigg) \ \text{ for } \ t \in [a,b].$$

Then $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ and, in view of (1.21), $\gamma(a) - \lambda\gamma(b) > 0$. Furthermore,

$$\gamma'(t) = p(t)\exp\bigg(\int_a^t p(s)\,ds\bigg) \geq p(t)\gamma(\tau_i(t)) \ \text{ for a.e. } \ t \in [a,b].$$

However, the last inequality yields (3.44) and (3.45), respectively. Therefore, the assumptions of Theorem 1.4, respectively Theorem 1.7, are fulfilled.

(b) The assertion follows from Corollary 1.1 (b), respectively Corollary 1.3 (b), for $m=2$ and $k=1$.

(c) Put

$$\widehat{H}(v,w)(t) \overset{\text{def}}{=} p(t)\sigma_i(t) \int_t^{\tau_i(t)} p(s) \max\big\{v(\xi) : \ \tau_1(s) \leq \xi \leq \tau_2(s)\big\}\,ds \quad \text{for a.e. } \ t \in [a,b].$$

Then the assertion follows from Corollary 1.1 (c), respectively Corollary 1.3 (c). □

Proof of Theorem 1.13. It immediately follows from Corollaries 1.2 and 1.4. □

Proof of Theorem 1.14. Let us note that if $g \in L([a,b];R_+)$, then

$$g(t)\max\big\{u(s):\ \mu_1(t)\le s\le\mu_2(t)\big\}-g(t)\max\big\{v(s):\ \mu_1(t)\le s\le\mu_2(t)\big\}\le \\ \le g(t)\max\big\{u(s)-v(s):\ \mu_1(t)\le s\le\mu_2(t)\big\}\ \text{ for a.e. } t\in[a,b]. \tag{3.46}$$

Therefore we can put

$$\overline{H}(v,w)(t)\overset{\text{def}}{=}-g(t)\max\{w(s):\ \mu_1(t)\le s\le\mu_2(t)\}\ \text{ for a.e. } t\in[a,b]. \tag{3.47}$$

Then the assertion (a) follows from Theorem 1.6 (b) and Remark 1.2. The assertion (b) follows from Theorems 1.5 and 1.8 with $\gamma(t)=b-t$ for $t\in[a,b]$.

(c) Let $u,\ v\in\widetilde{C}([a,b];R)$ and $\delta\in[0,1]$ be such that (0.7) and (0.8) are satisfied with $y\in C([a,b];R)$ and H defined by (0.4). We will show that (0.9) is fulfilled. Put $w(t)=u(t)-v(t)$ for $t\in[a,b]$. Then, according to (3.46), we have

$$w'(t)\ge-\delta g(t)\max\big\{w(s):\ \mu_1(t)\le s\le\mu_2(t)\big\}\ \text{ for a.e. } t\in[a,b], \\ w(a)-\lambda w(b)\ge 0.$$

We will show that $w(t)\ge 0$ for $t\in[a,b]$. Obviously, there exists $q_0\in L([a,b];R_+)$ such that

$$w'(t)=-\delta g(t)\max\big\{w(s):\mu_1(t)\le s\le\mu_2(t)\big\}+q_0(t)\ \text{ for a.e. } t\in[a,b]. \tag{3.48}$$

From (3.48) we get

$$w'(t)=-\delta g(t)w(t)-\delta g(t)\max\big\{w(s):\ \mu_1(t)\le s\le\mu_2(t)\big\}+ \\ +\delta g(t)w(t)+q_0(t)\ \text{ for a.e. } t\in[a,b]. \tag{3.49}$$

On the other hand, integration of (3.48) from a to t yields

$$w(t)=w(a)-\delta\int_a^t g(s)\max\big\{w(\xi):\ \mu_1(s)\le\xi\le\mu_2(s)\big\}\,ds+ \\ +\int_a^t q_0(s)\,ds\ \text{ for } t\in[a,b]. \tag{3.50}$$

Using (3.50) in (3.49), we obtain

$$w'(t)=-\delta g(t)w(t)- \\ -\delta g(t)\max\bigg\{-\delta\int_a^s g(\xi)\max\{w(\eta):\ \mu_1(\xi)\le\eta\le\mu_2(\xi)\}\,d\xi+ \\ +\int_a^s q_0(\xi)\,d\xi:\ \mu_1(t)\le s\le\mu_2(t)\bigg\}- \\ -\delta g(t)\int_a^t \delta g(s)\max\big\{w(\xi):\ \mu_1(s)\le\xi\le\mu_2(s)\big\}\,ds+$$

$$+\,\delta g(t)\int\limits_a^t q_0(s)\,ds + q_0(t) \ \text{ for a.e. } \ t\in[a,b].$$

Hence we get

$$w'(t) = -\delta g(t)w(t) -$$
$$-\delta g(t)\max\Big\{\delta\int\limits_s^t g(\xi)\max\big\{w(\eta):\ \mu_1(\xi)\le\eta\le\mu_2(\xi)\big\}\,d\xi -$$
$$-\int\limits_s^t q_0(\xi)\,d\xi:\ \mu_1(t)\le s\le\mu_2(t)\Big\} + q_0(t) \ \text{ for a.e. } \ t\in[a,b]. \tag{3.51}$$

Now, since $g(t)(\mu_2(t)-t)\le 0$ and $q_0(t)\ge 0$ for almost every $t\in[a,b]$, the equality (3.51) results in

$$w'(t)\ge -\delta g(t)w(t) - \delta^2 g(t)\times$$
$$\times\max\Big\{\int\limits_s^t g(\xi)\max\big\{w(\eta):\ \mu_1(\xi)\le\eta\le\mu_2(\xi)\big\}\,d\xi:\ \mu_1(t)\le s\le\mu_2(t)\Big\}$$
$$\text{for a.e. } \ t\in[a,b].$$

Put

$$x(t) = w(t)\exp\left(\delta\int\limits_a^t g(s)\,ds\right) \ \text{ for } \ t\in[a,b]. \tag{3.52}$$

Then, in view of $w(a)\ge\lambda w(b)$, we have $x(a)\ge\lambda_0 x(b)$, where $\lambda_0 = \lambda e^{-\delta\|g\|_L}\in[0,1[$. Furthermore,

$$x'(t)\ge -\delta^2 g(t)\max\Big\{\int\limits_s^t g(\xi)\times$$
$$\times\max\Big\{x(\eta)\exp\Big(\int\limits_\eta^t\delta g(\nu)\,d\nu\Big):\mu_1(\xi)\le\eta\le\mu_2(\xi)\Big\}\,d\xi:\mu_1(t)\le s\le\mu_2(t)\Big\} \tag{3.53}$$
$$\text{for a.e. } \ t\in[a,b].$$

Define

$$\overline{H}(v,z)(t) \overset{\text{def}}{=} -\delta^2 g(t)\max\Big\{\int\limits_s^t g(\xi)\times$$
$$\times\max\Big\{z(\eta)\exp\Big(\int\limits_\eta^t\delta g(\nu)\,d\nu\Big):\mu_1(\xi)\le\eta\le\mu_2(\xi)\Big\}\,d\xi:\mu_1(t)\le s\le\mu_2(t)\Big\}$$
$$\text{for a.e. } \ t\in[a,b].$$

Then $\overline{H} \in \mathcal{H}_{ab}$ and $\overline{H}(0,\cdot\,)$ is an a-Volterra operator. Suppose that the function x assumes negative values. Then, according to Lemma 2.9, there exist $t_* \in\,]a,b]$ and $t^* \in [a,t_*[$ such that

$$x(t_*) = -m < 0, \quad x(t^*) = M > 0, \tag{3.54}$$

where

$$m = -\min\big\{x(t) :\ t \in [a,b]\big\}, \quad M = \max\big\{x(t) :\ t \in [a,t_*]\big\}.$$

Now integration of (3.53) from t^* to t_*, on account of (3.54), results in

$$\begin{aligned} -m - M &\geq M \int_{t^*}^{t_*} \overline{H}(0,1)(s)\, ds \geq \\ &\geq -M \int_a^b g(s) \int_{\mu_1(s)}^{s} g(\xi) \exp\Bigg(\int_{\mu_1(\xi)}^{s} g(\eta)\, d\eta \Bigg)\, d\xi\, ds. \end{aligned}$$

However hence, with respect to (1.22) and (3.54), we get a contradiction $M < M$. Therefore, $x(t) \geq 0$ for $t \in [a,b]$, and consequently, on account of (3.52), we have $w(t) \geq 0$ for $t \in [a,b]$.

(d) According to (1.23) there exist $x_0 > 0$ and $\varepsilon \in\,]0,1[$ such that

$$\int_{\mu_1(t)}^{t} g(s)\, ds \leq \frac{1}{x_0} \ln\Bigg(x_0 + \frac{x_0(1-\varepsilon)}{e^{x_0\|g\|_L} - 1 + \varepsilon} \Bigg) \ \text{ for a.e. } \ t \in [a,b].$$

Hence we get

$$\exp\Bigg(x_0 \int_t^{\mu_1(t)} g(s)\, ds \Bigg) \geq \frac{1}{x_0} - \frac{1-\varepsilon}{x_0 e^{x_0\|g\|_L}} \ \text{ for a.e. } \ t \in [a,b]. \tag{3.55}$$

Put

$$\gamma(t) = \exp\Bigg(x_0 \int_t^b g(s)\, ds \Bigg) - 1 + \varepsilon \ \text{ for } \ t \in [a,b].$$

Then $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ and, in view of (3.55), we have

$$\begin{aligned} \gamma'(t) &= -x_0 g(t) \exp\Bigg(x_0 \int_t^{\mu_1(t)} g(s)\, ds \Bigg) \exp\Bigg(x_0 \int_{\mu_1(t)}^{b} g(s)\, ds \Bigg) \leq \\ &\leq -g(t) \Bigg[\exp\Bigg(x_0 \int_{\mu_1(t)}^{b} g(s)\, ds \Bigg) - \exp\Bigg(x_0 \int_{\mu_1(t)}^{b} g(s)\, ds \Bigg) \frac{1-\varepsilon}{e^{x_0\|g\|_L}} \Bigg] \leq \\ &\leq -g(t)\gamma(\mu_1(t)) \ \text{ for a.e. } \ t \in [a,b]. \end{aligned}$$

However, since γ is a nonincreasing function, the last inequality means

$$\gamma'(t) \leq -g(t) \max\big\{\gamma(s) :\ \mu_1(t) \leq s \leq \mu_2(t)\big\} \ \text{ for a.e. } \ t \in [a,b].$$

Now if we define $\overline{H}$ by (3.47), then, according to (3.46), the assertion follows from Theorems 1.5 and 1.8. $\Box$

Proof of Corollary 1.7. If $g \equiv 0$, then the validity of the corollary follows from Theorem 1.14 (a). If $g \not\equiv 0$, then the corollary follows from Theorem 1.14 (d), because

$$\vartheta > \sup\left\{\frac{1}{x}\ln x:\ x > 0\right\} = \frac{1}{e}\,. \qquad \Box$$

Proof of Theorem 1.15. Define $\overline{H}$ by (3.47). Then, in view of (3.46), the assertion follows immediately from Theorem 1.6 (a) and Remark 1.2. $\Box$

4. Examples

The following example shows that the solvability of the problem (0.1), (0.2) does not mean, in general, the unique solvability of this problem, even in the case where $Q \equiv 0$ and $h \equiv const$.

Example 4.1. Let $t_0 \in]a,b[$ and consider the boundary value problem

$$\begin{gathered} u'(t) = p(t)\Big[\max\big\{u(s):\ \tau_1(t) \le s \le \tau_2(t)\big\} - u(\tau_1(t))\Big], \\ u(a) - \lambda u(b) = -c, \end{gathered} \tag{4.1}$$

where

$$\tau_1(t) = \begin{cases} a & \text{for a.e. } t \in [a,t_0] \\ t_0 & \text{for a.e. } t \in]t_0,b] \end{cases}, \quad \tau_2(t) = \begin{cases} t_0 & \text{for a.e. } t \in [a,t_0] \\ b & \text{for a.e. } t \in]t_0,b] \end{cases},$$

$p \in L([a,b];R_+)$, $c > 0$, $\lambda \in [0,e^{-2}[$, and

$$\int_a^{t_0} p(s)\,ds = 1, \quad \int_{t_0}^{b} p(s)\,ds = 1. \tag{4.2}$$

Define

$$H(u,v)(t) \stackrel{\text{def}}{=} p(t)\Big[\max\big\{u(s):\ \tau_1(t) \le s \le \tau_2(t)\big\} - v(\tau_1(t))\Big] \quad \text{for a.e. } t \in [a,b].$$

Then, according to Theorem 1.12 (a), we have $H \in V_{ab}^{+}(\lambda;\ge)$. Moreover, it can be easily shown that $H \in W_{ab}^{+}(\lambda;-)$. Indeed, let u, $v \in \widetilde{C}([a,b];R)$ satisfy (0.7) and (0.8) with some $\delta \in [0,1]$ and $y \in C([a,b];R_-)$. Then, if we put $w(t) = u(t) - v(t)$ for $t \in [a,b]$, the inequalities (0.7), (0.8) are equivalent to

$$w'(t) \ge -\delta p(t) w(\tau_1(t)) \ \text{ for a.e. } t \in [a,b], \quad w(a) - \lambda w(b) \ge 0. \tag{4.3}$$

Integration of (4.3) from a to $t \in [a, t_0]$ and from t_0 to $t \in [t_0, b]$ yields respectively

$$w(t) \geq w(a)\left(1 - \delta \int_a^t p(s)\,ds\right) \quad \text{for } t \in [a, t_0], \tag{4.4}$$

$$w(t) \geq w(t_0)\left(1 - \delta \int_{t_0}^t p(s)\,ds\right) \quad \text{for } t \in [t_0, b]. \tag{4.5}$$

Furthermore, from (4.4) and (4.5), with regard to (4.2), we get

$$w(t_0) \geq w(a)(1-\delta), \quad w(b) \geq w(t_0)(1-\delta), \tag{4.6}$$

whence we have

$$w(b) \geq w(a)(1-\delta)^2. \tag{4.7}$$

Assume that $w(a) < 0$. Then, in view of the second inequality in (4.3) and (4.7), we find

$$0 > w(a) \geq \lambda w(b) > w(b) \geq w(a)(1-\delta)^2 \geq w(a),$$

a contradiction. Therefore $w(a) \geq 0$, and, consequently, in view of (4.6) we have also $w(t_0) \geq 0$. Now from (4.4) and (4.5) we get $w(t) \geq 0$ for $t \in [a, b]$, which means $u(t) \geq v(t)$ for $t \in [a, b]$.

Thus, according to Theorem 1.1, the problem (4.1) has at least one nonpositive solution.

On the other hand, since $c > 0$, we can choose $d > 0$ such that $c + \lambda d > d$, and for every $k \in [d, c + \lambda d]$ let us define

$$u_k(t) = \begin{cases} -(c+\lambda d)\displaystyle\int_t^{t_0} p(s)\,ds - k\int_a^t p(s)\,ds & \text{for } t \in [a, t_0] \\ -k\displaystyle\int_t^b p(s)\,ds - d\int_{t_0}^t p(s)\,ds & \text{for } t \in]t_0, b] \end{cases}.$$

Now it can be easily verified that for every $k \in [d, c + \lambda d]$ the function u_k is a nonpositive solution to (4.1). Thus, the problem (4.1) has infinitely many nonpositive solutions.

The following example shows that the requirement on the operators Q and h to transform nonpositive functions into the set of nonpositive functions and nonpositive numbers, respectively, introduced in Theorem 1.1, is essential and it cannot be omitted. In other words, the inclusion $H \in V_{ab}^{+}(\lambda; \geq) \cap W_{ab}^{+}(\lambda; -)$ does not guarantee, in general, the solvability of the problem (0.1), (0.2) for arbitrary sublinear Q and h. The same is true for the inclusion $H \in V_{ab}^{+}(\lambda; \leq) \cap W_{ab}^{+}(\lambda; +)$.

Example 4.2. On the segment $[a, b]$ consider the problem

$$u'(t) = p(t) \max\big\{u(s):\ a \leq s \leq b\big\} + q_0(t), \quad u(a) = c. \tag{4.8}$$

Here $p \in L([a,b];R_+)$, $q_0 \in L([a,b];R)$, $c \in R$, and

$$\int_a^b p(s)\,ds \geq 1.$$

Put

$$H(u,v)(t) \overset{\text{def}}{=} p(t) \max\big\{u(s):\ a \leq s \leq b\big\}.$$

Obviously, if $u(a) \geq 0$, then also $\max\{u(s) : a \leq s \leq b\} \geq 0$, and thus every function $u \in \widetilde{C}([a,b];R)$ satisfying (0.5) (with $\lambda = 0$) admits

$$u'(t) \geq H(u,0)(t) = p(t) \max\big\{u(s):\ a \leq s \leq b\big\} \geq 0 \text{ for a.e. } t \in [a,b].$$

Consequently $H \in V_{ab}^{+}(0;\geq)$. On the other hand, the function

$$u(t) = 1 - \int_t^b p(s)\,ds \text{ for } t \in [a,b]$$

satisfies (0.6) (with $\lambda = 0$) but it assumes positive values. Therefore, $H \notin V_{ab}^{+}(0;\leq)$. Moreover, it can be easily verified that $H \in W_{ab}^{+}(0;-)$ and $H \in W_{ab}^{+}(0;+)$. According to Theorem 1.1, the problem (4.8) has at least one nonpositive solution whenever $q_0(t) \leq 0$ for almost every $t \in [a,b]$ and $c \leq 0$.

On the other hand, it is not difficult to show that the problem (4.8) is not solvable if $q_0(t) \geq 0$ for almost every $t \in [a,b]$ and $c \geq 0$, $c + \|q_0\|_L > 0$. Indeed, assume on the contrary that there exists $u \in \widetilde{C}([a,b];R)$ satisfying (4.8). Then from (4.8) we get $u'(t) \geq 0$ for almost every $t \in [a,b]$. Consequently, the maximum value of the function u is reached at the point b and $u(b) \geq u(a) = c \geq 0$. Having this in mind, by integration of (4.8) from a to b we obtain

$$u(b) = c + u(b) \int_a^b p(s)\,ds + \int_a^b q_0(s)\,ds,$$

whence we get

$$0 \geq u(b)\bigg(1 - \int_a^b p(s)\,ds\bigg) = c + \int_a^b q_0(s)\,ds > 0,$$

a contradiction.

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