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THE COEFFICIENT OF REDUCIBILITY OF LINEAR DIFFERENTIAL SYSTEMS

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We consider linear systems of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1_A}$$

with piecewise continuous bounded coefficients ($\|A(t)\| \leq a$ for $t \geq 0$) and perturbed systems (1_{A+Q}) with piecewise continuous on the non-negative half-line $[0, +\infty)$ perturbations Q satisfying either the condition

$$\|Q(t)\| \leq C_Q e^{-\sigma t}, \quad \sigma \geq 0, \quad t \geq 0, \tag{2}$$

or the more general condition

$$\|Q(t)\| \leq C_Q^{\varepsilon} e^{(\varepsilon-\sigma)t}, \quad \sigma \geq 0, \quad \forall \varepsilon > 0, \quad t \geq 0, \tag{3_1}$$

which is equivalent to the inequality

$$\lambda[Q] \equiv \varlimsup_{t\to+\infty} t^{-1} \ln \|Q(t)\| \leq -\sigma \leq 0. \tag{3_2}$$

If $\sigma = 0$, then we additionally suppose that

$$Q(t) \to 0 \ \text{ as } \ t \to +\infty. \tag{4}$$

A great number of papers (it seems impossible to compile the complete bibliography of them) are dedicated to the investigation of the classic notion of Lyapunov's reducibility (see [1, p. 43]) of linear systems. Here we are interested in properties of *the coefficient of reducibility* $r_2(A)$ and *the exponent of reducibility* $r_3(A)$ of (1_A) with respect to perturbations (2) and (3_1)–(3_2), respectively.

Definition 1 (see [2]). The infimum of the set $R_2(A)$ (the set $R_3(A)$) of all values of $\sigma > 0$ such that perturbed system (1_{A+Q}) with *any* perturbation Q satisfying condition (2) (conditions (3_1)–(3_2)) is reducible to the initial system (1_A) is called *the coefficient of reducibility* $r_2(A)$ (*the exponent of reducibility* $r_3(A)$) of (1_A).

To further investigate the properties of $r_2(A)$ and $r_3(A)$, we will use the following definition which is equivalent to Definition 1.

Definition 2. The number $r_2(A) > 0$ (the number $r_3(A) > 0$) is called the coefficient (the exponent) of reducibility of (1_A) if for any $0 < \sigma_1 < r_2(A) < \sigma_2$ ($0 < \sigma_1 < r_3(A) < \sigma_2$): 1) there exists a perturbation Q_1 satisfying (2) ((3_1)–(3_2)) with $\sigma = \sigma_1$ such that (1_A) and (1_{A+Q_1}) are not reducible to each other; 2) (1_{A+Q}) is reducible to (1_A) for any perturbation Q satisfying (2) ((3_1)–(3_2)) with $\sigma = \sigma_2$. We say that (1_A) has the zero coefficient $r_2(A) = 0$ (the zero exponent $r_3(A) = 0$) of reducibility if (1_{A+Q}) is reducible to (1_A) for any perturbation Q satisfying (2) ((3_1)–(3_2)) with any fixed $\sigma > 0$.

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Now we show that the coefficient and the exponent of reducibility of (1_A) are well-defined.

Let $R_2(A)$ be the set of all $\sigma > 0$ such that perturbed system (1_{A+Q}) is reducible to initial system (1_A) for any perturbation Q satisfying (2). Show that $r_2(A) = i_2(A) \equiv \inf R_2(A) \in [0, 2a]$, where the inclusion holds owing to Theorem 1 from [2]. Since any perturbation Q satisfying (2) with $\sigma = \alpha_2 > 0$ satisfies (2) with $\sigma = \alpha_1 \in (0, \alpha_2)$, we see that the set $R_2(A)$ can be represented as $R_2(A) = |i_2(A), +\infty)$. If now the equality $i_2(A) = 0$ holds, then the necessary condition of the definition of $r_2(A) = 0$ is fulfilled. In the case $i_2(A) > 0$ for all $\sigma_2 > i_2(A)$ Property 2) of the definition of $r_2(A)$ is also fulfilled. Property 1) of this definition is also fulfilled for any $\sigma_1 \in (0, i_2(A))$, otherwise we get $i_2(A) \le \sigma_1 < i_2(A)$ for some $\sigma_1 \in (0, i_2(A))$, which is impossible. Therefore the required equality $r_2(A) = i_2(A)$ is proved. In the same manner we can show that the reducibility exponent $r_3(A) \in [0, 2a]$ exists for any system (1_A), $\|A(t)\| \le a$ for $t \ge 0$.

Theorem 1. *The coefficient of reducibility $r_2(A)$ and the exponent of reducibility $r_3(A)$ are equal for any linear system (1_A).*

Proof. Suppose, contrary to our claim, that $r_2(A) \ne r_3(A)$. If $0 \le r_2(A) < r_3(A)$, then by definition, 1) there exists a perturbation Q satisfying (3_2), $\lambda[Q] < -\sigma_1 \equiv -(r_2+r_3)/2$, $\sigma_1 < r_3(A)$, so that (1_A) and (1_{A+Q}) are not reducible; 2) this perturbation Q satisfies the inequality $\|Q(t)\| \le C_1 \exp(-\sigma_1 t)$, $t \ge 0$, thus Q satisfies (2) with $\sigma = \sigma_1 > r_2(A) \ge 0$, and it follows (the second property of the definition above) that (1_{A+Q}) is reducible to (1_A). This contradiction implies the inequality $r_2(A) \ge r_3(A)$.

Similarly, one can show that the inequality $r_2(A) > r_3(A) \ge 0$ is also impossible. The theorem is proved. □

Now we can define the coefficient of reducibility $r(A)$ of (1_A) as the common value of the reducibility coefficient and the reducibility exponent:

$$r(A) = r_2(A) = r_3(A).$$

Let $\omega_0(A) \le \Omega_0(A)$ be the general (singular) lower and upper exponents (see [3, pp. 109–111]) of (1_A). The following result is proved in [4].

Theorem 2. *If a piecewise continuous matrix Q satisfies* (4) *and*

$$\left\| \int_t^{+\infty} Q(\tau)\, d\tau \right\| \le C_Q e^{-\sigma t}, \quad t \ge 0, \tag{5}$$

with some $\sigma > \Omega_0(A) - \omega_0(A)$, then systems (1_A) and (1_{A+Q}) can be reduced to each other by Lyapunov's transformation, i.e., are asymptotically equivalent.

Since the lower and upper general exponents $\omega_0(A)$ and $\Omega_0(A)$ of system (1_A), defined in terms of its Cauchy matrix $X_A(t,\tau)$ by the formulae [3, p. 117]

$$\omega_0(A) = \lim_{T\to+\infty} \frac{1}{T} \inf_{k\ge0} \ln \|X_A(kT, kT+T)\|^{-1},$$

$$\Omega_0(A) = \lim_{T\to+\infty} \frac{1}{T} \sup_{k\ge0} \ln \|X_A(kT+T, kT)\|,$$

admit the estimates $\omega_0(A) \ge -a$ and $\Omega_0(A) \le a$, we see that Theorem 1 implies the following assertion.

Corollary. *If condition* (5) *is satisfied for some $\sigma > 2a$, then systems (1_A) and (1_{A+Q}) are asymptotically equivalent.*

Therefore, the reducibility coefficient $r(A)$ of (1_A) belongs to the segment $[0, 2a]$. Moreover, the following assertion (see [4]) establishes the existence of systems (1_A) such that $r(A) = 2a$.

Theorem 3. *For each $a > 0$, there exist a system (1_A) with the piecewise continuous coefficient matrix A, $\|A(t)\| \le a$ for $t \ge 0$, and a piecewise continuous perturbation Q satisfying the condition*

$$\|Q(t)\| \le C_Q e^{-2at}, \quad t \ge 0, \tag{6}$$

such that the initial and perturbed linear systems (1_A) and (1_{A+Q}) are not asymptotically equivalent.

To prove this theorem, it suffices to consider the two-dimensional system (1_A) with the diagonal matrix $A(t) = \operatorname{diag}[-a(t), a(t)]$, where

$$a(t) = (-1)^i a, \ \ t \in [t_{2k+i}, t_{2k+i+1}), \ \ i = 0, 1,$$

and

$$t_0 = 0, \ \ t_{k+1} = t_k + e^{4at_k}, \ \ k \ge 0, \ \ \{t_k\} \uparrow +\infty.$$

It is easy to verify that $\omega_0(A) = -a$, $\Omega_0(A) = a$ for this system. We take the second-order lower triangular matrix with the entries

$$q_{ij}(t) = 0, \ \ i \le j, \ \ q_{21}(t) = q(t) = e^{-2at}, \ \ t \ge 0,$$

as the perturbation matrix $Q(t)$ satisfying (6).

Theorem 3 gives the structure of the set $R_2(A) = (2a, +\infty)$ for system (1_A) constructed above and, in view of the evident inclusion $R_3(A) \subset R_2(A)$ and the equality $r_3(A) = r_2(A)$, it also gives the structure of the set $R_3(A) = (2a, +\infty)$.

However, in the general case, the sets R_2 and R_3 do not coincide with each other. This fact is established by the following theorem.

Theorem 4. *For each $a > 0$, there exists a system (1_A) with the piecewise continuous coefficient matrix A, $\|A(t)\| \le a$ for $t \ge 0$, and with the reducibility coefficient $r(A) = 2a$ such that system (1_{A+Q}) with any piecewise continuous perturbation Q satisfying the condition*

$$\|Q(t)\| \le C_Q e^{-r(A)t} \ \text{for} \ t \ge 0 \tag{7}$$

is reducible to (1_A) and is not reducible to (1_A) for some perturbation Q satisfying (3_1)–(3_2) with $\sigma = r(A)$.

To construct the required system, we define two sequences: the sequence (a_m) of numbers $a_m = a(1 - 1/m)$, $a_0 = 0$, $m \in \mathbb{N}$, and the time sequence (t_m), t_m, $t_1 = 1$, $t_0 = 0$, satisfying the condition

$$\varepsilon_m \equiv t_m/t_{m+1} \le e^{-2}(1+m)^{-1}, \ \ m \in \mathbb{N}. \tag{8}$$

From (8) it follows that the length of each next half-interval $[t_m, t_{m+1})$ is greater than the previous one $[t_{m-1}, t_m)$, $m \in \mathbb{N}$, and $t_m \to +\infty$ as $m \to +\infty$.

Using these sequences, we define the entries of the diagonal matrix $A(t) = \operatorname{diag}[a_1(t), a_2(t)]$:

$$a_2(t) = -a_1(t) = (-1)^m a_m, \ \ t \in [t_m, t_{m+1}), \ \ m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}.$$

It is evident that $\sup\limits_{t \ge 0} \|A(t)\| = a$ and the coefficient of reducibility $r(A)$ of this system is equal to $2a$. Furthermore, system (1_{A+Q}) is asymptotically equivalent to system (1_A) for any perturbation Q satisfying (7). To prove the second part of the theorem, that is, to construct system (1_{A+Q}) which is not asymptotically equivalent to (1_A), it suffices to take the second-order matrix Q with the entries

$$q_{ij}(t) = 0, \ \ i \le j, \quad q_{21}(t) = \exp[-2at + p(t)], \ \ t \ge 0,$$

where $p(t) = 0$ for $t \in [0, 1)$, $p(t) = 4at/m$ for $t \in [t_m, t_{m+1})$, $m \in \mathbb{N}$. One can verify that Q satisfies (3_1)–(3_2) with $\sigma = r(A)$.

Thus, for the piecewise continuous perturbations (2), the reducibility coefficient of linear systems has the following property of two kinds: there exist a system (1_A) and a perturbation Q satisfying (2) with $\sigma = r(A)$ such that the perturbed system (1_{A+Q})

and the initial system (1_A) are not reducible (Theorem 3), as well as there exist systems (1_A) such that perturbed system (1_{A+Q}) with any perturbation Q satisfying the same condition (2) with $\sigma = r(A)$ is reducible to (1_A) (Theorem 4).

At the same time, Theorem 4 shows the inherent difference between the properties of the reducibility coefficient with respect to perturbations (2) and with respect to more general perturbations (3_1)–(3_2). Namely, there exist systems (1_A) such that perturbed system (1_{A+Q}): 1) for any perturbation Q satisfying (2) with $\sigma = r(A)$ is reducible to (1_A); 2) for some perturbation Q satisfying (3_1)–(3_2) with the same $\sigma = r(A)$ is no longer reducible to (1_A).

The following assertion gives the general integral test of reducibility of system (1_{A+Q}) to system (1_A).

Theorem 5. *If Q satisfies the condition*

$$\overline{\lim_{t\to+\infty}} \int\limits_t^{+\infty} \big\|X_A(t,\tau)Q(\tau)X_A(\tau,t)\big\|\,d\tau < 1,$$

where $X_A(t,\tau)$ is the Cauchy matrix of system (1_A), then the system (1_{A+Q}) is reducible to (1_A).

In conclusion, we note that the value of the norm of the coefficient matrix of the linear system and the value of its reducibility coefficient are independent.

Theorem 6. *For any numbers $2a \geq r \geq 0$ there exists a system (1_A) with the piecewise continuous coefficient matrix A such that $r(A) = r$ and $\|A(t)\| \leq a$ for $t \geq 0$.*

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