R. Gachechiladze

ON A CONSEQUENCE OF GENERALIZED SIGNORINI'S PROBLEM

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In the present work, for the hemitropic elastic medium we consider the unilateral problem which is, in fact, a particular case of generalized Signorini's problem (see, e.g., [2], [12], [15]). The problem under consideration expresses in more detail mechanical meaning of Signorini's problem which, as is known, describes a contact between a part of the boundary of an elastic medium and a rigid body without friction ([2], [12], [3]. [6]). Note that the elastic body is subjected to the action of external forces, ordinary and superficial, the other portion of the surface is fixed. Here we present mathematical interpretation of that process under some restrictions ([7], Ch. 1).

Let a hemitropic elastic body occupy a bounded region Ω of the three- dimensional space $\mathbb{R}^3$ with the sufficiently smooth boundary $\partial\Omega = \Gamma$ which is divided into three mutually nonintersecting parts Γ_C, Γ_D and Γ_T, such that $\overline{\Gamma}_C \cap \overline{\Gamma}_D = \varnothing$ and $\Gamma = \Gamma_C \cup \Gamma_D \cup \overline{\Gamma}_T$. Assume that above Ω there is an absolutely rigid body (called an obstacle) with the boundary S which is expressed by the equality (see Fig. 1)

$$x_3 = \psi(x_1, x_2).$$

Assume that the elastic body is under the action of the body force F, the mass moment G; the surface force Ψ acts on Γ_T, and the contact between the elastic body and the obstacle takes place only on a part of the boundary Γ_C. We seek for a displacement $u(x) = (u_1(x), u_2(x), u_3(x))$ and rotation $\omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$, $x \in \Omega$.
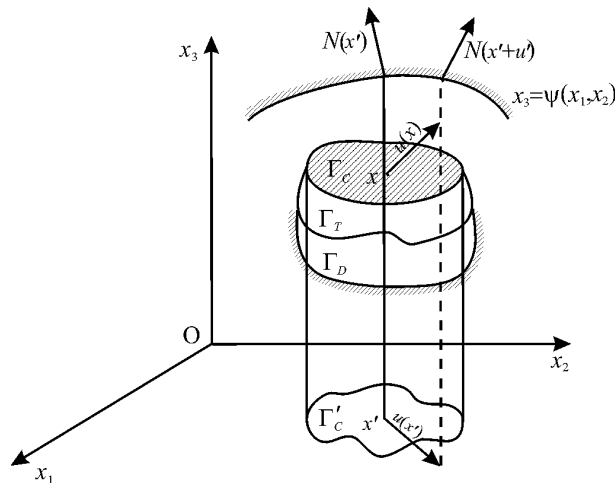

Fig. 1

Γ'_C is the orthogonal projection of the surface Γ_C onto the plane $x_1 0 x_2$, $x' = (x_1, x_2)$, $u'(x) = (u_1(x), u_2(x))$, $x = (x', x_3)$ and $u = (u', u_3)$. Since upon deformation some part Γ_C of the surface Γ remains always below S, it is clear that

$$x_3 + u_3(x) \leq \psi(x' + u'). \tag{1}$$

Let ψ be a twice continuously differentiable concave function. Then after linearization, assuming that the body Ω admits "small displacements", the conditions (1) take the form

$$u(x) \cdot N(x') \leq \varphi(x), \quad x \in \Gamma_C, \tag{2}$$

where $\varphi(x) = (\psi(x') - x_3) \big/ \sqrt{1 + |(\operatorname{grad} \psi)(x')|^2}$, and

$$N(x') = (N_1(x'), N_2(x'), N_3(x')) = \left(-\frac{\partial \psi}{\partial x_1}, -\frac{\partial \psi}{\partial x_2}, 1 \right) \Big/ \sqrt{1 + \left(\frac{\partial \psi}{\partial x_1} \right)^2 + \left(\frac{\partial \psi}{\partial x_2} \right)^2}$$

is the exterior normal of the rigid obstacle at the point $(x', \psi(x') \in S$.

The conditions (2) differ from those of the classical statement of Signorini's problem in which instead of N there appears the exterior normal n of the surface Γ_C. The passage from the normal N to n can be realized under additional restrictions; for example, we assume that the body Ω admits "small displacements", and the surfaces Γ_C and S are sufficiently close to consider them parallel (in most cases this is impossible).

Thus to derive the conditions (2), we need less restrictions than in the general case, and they describe the mechanical meaning of the problem more precisely than the standard Signorini's conditions.

The equilibrium equation of the hemitropic elastic body has the form ([11])

$$L(\partial)U(x) + \mathcal{F}(x) = 0, \quad x \in \Omega, \tag{3}$$

where

$$L(\partial) = \left\| \begin{matrix} L^{(1)}(\partial) & \vdots & L^{(2)}(\partial) \\ \dots\dots\dots\dots\dots\dots \\ L^{(3)}(\partial) & \vdots & L^{(4)}(\partial) \end{matrix} \right\|_{6 \times 6}$$

is the matrix differential operator corresponding to the statical state of the medium, $U(x) = (u(x), \omega(x))$, $u(x) = (u_1(x), u_2(x), u_3(x))$ is the displacement vector, $\omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$ is the rotation vector, and $\mathcal{F}(x) = (F(x), G(x))$.

Introduce the matrix differential stress operator

$$T(x, \partial, n) = \left\| \begin{matrix} T^{(1)}(x, \partial, n) & \vdots & T^{(2)}(x, \partial, n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ T^{(3)}(x, \partial, n) & \vdots & T^{(4)}(x, \partial, n) \end{matrix} \right\|_{6 \times 6},$$

where $n(x)$ is the exterior normal to Ω at the point $x \in \Gamma$. $L^{(j)}(\partial)$ and $T^{(j)}(x, \partial, n)$ $(j = 1, 2, 3, 4)$ are matrix differential operators of dimension 3×3.

$\tau U(x) = T^{(1)}(x, \partial, n)u(x) + T^{(2)}(x, \partial, n)\omega(x)$ is the vector of power stress;

$\mu U(x) = T^{(3)}(x, \partial, n)u(x) + T^{(4)}(x, \partial, n)\omega(x)$ is the vector of moment stress.

In the sequel, by $H^s(\Omega)$ and $H^s(\Gamma)$ ($s \in \mathbb{R}$) we will denote the real Sobolev–Slobodetski's spaces (see (13], [16]). Below, we will deal with weak solutions of the equation (3) for $\mathcal{F} \in (L_2(\Omega))^6$, i.e. with the functions $U \in (H^1(\Omega))^6$ for which the integral identity

$$\int_\Omega E(U, \Phi)\, dx = \int_\Omega \mathcal{F} \cdot \Phi\, dx, \quad \forall \Phi \in (C_0^\infty(\Omega))^6$$

is fulfilled; here $E(U, V)$ is the bilinear form corresponding to the operator $L(\partial)$. It should be noted that if $U \in (H^1(\Omega))^6$ and $L(\partial)U \in (L_2(\Omega))^6$, then using Green's formula, we

can determine $r_\Gamma T(x, \partial, n)U(x)$ as a functional of the class $(H^{-\frac{1}{2}}(\Gamma))^6$ by the relation

$$\langle r_\Gamma T(x, \partial, n)U(x), \Phi(x)\rangle_\Gamma = \int\limits_\Omega E(U, V)\, dx + \int\limits_\Omega L(\partial)U \cdot V\, dx,$$

$$\forall \Phi \in (H^{\frac{1}{2}}(\Gamma))^6 \ \text{ and } \ \forall V \in (H^1(\Gamma))^6, \ \ r_\Gamma V = \Phi;$$

here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the dual relation between the spaces $(H^{-\frac{1}{2}}(\Gamma))^6$ and $(H^{\frac{1}{2}}(\Gamma))^6$.

We consider the following

Problem. Find a vector function $U \in (H^1(\Omega))^6$, which is a weak solution of the equation (3), satisfying the conditions

$$r_{\Gamma_D} U(x) = 0, \quad r_{\Gamma_T} T(x, \partial, n)U(x) = \Psi(x); \tag{4}$$

$$r_{\Gamma_C} u(x) \cdot N(x', \psi(x')) \leq \varphi(x), \quad r_{\Gamma_C} \tau U(x) \cdot N(x', \psi(x')) \leq 0; \tag{5}$$

$$\langle r_{\Gamma_C} \tau U(x) \cdot N(x', \psi(x')), r_{\Gamma_C} u(x) \cdot N(x', \psi(x')) - \varphi(x)\rangle_{\Gamma_C} = 0; \tag{6}$$

$$r_{\Gamma_C} \tau U(x) - \left[r_{\Gamma_C} \tau U(x) \cdot N(x', \psi(x')) \right] N(x', \psi(x')) = 0, \quad r_{\Gamma_C} \mu U(x) = 0, \tag{7}$$

where $\mathcal{F} = (F, G) \in (L_2(\Omega))^6$, $\Psi \in (L_2(\Gamma_N))^6$, and φ is the function defined above.

For the sake of brevity, instead of $u(x) \cdot N(x', \psi(x'))$, $\tau U(x) \cdot N(x', \psi(x'))$ and $\tau U(x) - [\tau U(x) \cdot N(x', \psi(x'))]N(x', \psi(x'))$ we will write, respectively, $u_{N(x')}$, $[\tau U]_{N(x')}$ and $[\tau U]_{s(x')}$ (tangential constituent of the power stress).

The corresponding variational inequality has the form: find a vector function $U = (u, \omega) \in \mathcal{K}$ such that $\forall V \in \mathcal{K}$ the inequality

$$\int\limits_\Omega E(U, V - U)\, dx \geq \int\limits_\Omega \mathcal{F} \cdot (V - U)\, dx + \int\limits_{\Gamma_T} \Psi \cdot (V - U)\, d\Gamma \tag{8}$$

holds, where the convex closed set $\mathcal{K}$ is given by the formula

$$\mathcal{K} = \left\{ U = (u, \omega) \in (H^1(\Omega))^6 : \ r_{\Gamma_C} u_{N(x')}(x) \leq \varphi; \ r_{\Gamma_D} U = 0 \right\}.$$

We can prove that the variational inequality (8) is equivalent to the physical problem (3)–(7). To reduce (8) to a boundary variational inequality, we reduce the problem (3)–(7) by means of an auxiliary problem (which will be formulated below) to the homogeneous ($\mathcal{F} = 0$) problem.

Let $U_0 = (u_0, \omega_0) \in (H^1(\Omega))^6$ be a weak solution of the equation (3), $r_{\Gamma_D} U_0 = 0$ and $r_{\Gamma \setminus \Gamma_D} T(x, \partial, n)U_0(x) = 0$.

As is known, there exists a unique solution of the above-mentioned problem. Therefore the vector function $U^* = U - U_0$ will, instead of the condition (3), satisfy the condition (instead of U^* we write again U)

$$L(\partial)U = 0 \tag{9}$$

and the first inequality (5) will be fulfilled by replacing φ by $\varphi_0 :\equiv \varphi - r_{\Gamma_C}(o_0)_{N(x')}$. All the rest conditions of the problem remain unchanged.

Introduce the Steklov–Poincaré operator defined by the relation

$$\mathcal{A}f = \left\{ T(x, \partial, n)\widetilde{G}f(x) \right\}_\Gamma^+, \quad \forall f \in (H^{\frac{1}{2}}(\Gamma))^6,$$

where

$$\widetilde{G}f(x) = \int\limits_\Gamma \chi(x - y)\mathcal{H}^{-1}f(y)\, d_y\Gamma$$

is representable as a single layer potential (for the properties of functions representable in the form of the potential, see [8], [1], [10]), χ is the fundamental solution of the equation

(9), $\mathcal{H}$ is the direct value on the boundary Γ of the single layer potential, $\mathcal{H}^{-1}$ is the operator inverse to $\mathcal{H}$, and

$$\left\{ T(x,\partial,n)\widetilde{G}f(x) \right\}_{\Gamma}^{+} = \lim_{\Omega \ni z \to x \in \Gamma} T(z,\partial_z,n(x))\widetilde{G}f(z).$$

We prove that the operator $\mathcal{A}$ has the following properties ([5], [9]):

1) $\mathcal{A} : (H^{\frac{1}{2}}(\Gamma))^6 \to (H^{-\frac{1}{2}}(\Gamma))^6; \quad \langle \mathcal{A}f, f \rangle_\Gamma \geq 0, \ \forall f \in (H^{\frac{1}{2}}(\Gamma))^6;$

2) $\langle \mathcal{A}f, g \rangle_\Gamma = \langle \mathcal{A}g, f \rangle_\Gamma, \ \forall f, g \in (H^{(\frac{1}{2}}(\Gamma))^6;$

3) $\langle \mathcal{A}f, f \rangle_\Gamma \geq c\|Pf\|_{\frac{1}{2},\Gamma}^2$, where $I \setminus P$ is the operator of orthogonal projection of the space $(H^{\frac{1}{2}}(\Gamma))^6$ onto the kernel of the equation $\langle \mathcal{A}f, f \rangle_\Gamma = 0$.

Consider now the convex closed set

$$\mathcal{K}_1 = \left\{ h \in (H^{\frac{1}{2}}(\Gamma))^6 : h = (\xi,\eta), \ r_{\Gamma_D} h = 0, \ r_{\Gamma_C} \xi_{N(x')}(x) \leq \varphi_0(x) \right\}$$

and the boundary variational inequality: find $h_0 = (\xi_0, \eta_0) \in \mathcal{K}_1$ such that $\forall h \in \mathcal{K}_1$ the inequality

$$\langle \mathcal{A}h_0, h - h_0 \rangle_\Gamma \geq \langle \Psi, r_{\Gamma_T}(h - h_0) \rangle_{\Gamma_T} \tag{10}$$

is fulfilled.

Again, we can prove that the boundary variational inequality (10) is equivalent to the physical problem $(9), (4), (5)_{\varphi_0}, (6), (7)$ $((5)_{\varphi_0}$ is, in fact, the condition (5) with the function φ_0) in the following sense: if $h_0 \in \mathcal{K}_1$ is a solution of the inequality (10), then $\widetilde{G}h_0 \in (H^1(\Omega))^6$ is a solution of the physical problem $(9), (4), (5)_{\varphi_0}, (6), (7)$, and vice versa, if $U \in (H^1(\Omega))^6$ is a solution of that problem, then $h_0 = r_\Gamma U$ is a solution of the variational inequality (10).

In its turn, the problem of solvability of the variational inequality (10) is reduced to that of minimization on the set $\mathcal{K}_1$ of the energy functional

$$\Phi(h) = \frac{1}{2}\langle \mathcal{A}h, h \rangle - \int_{\Gamma_T} \Psi \cdot r_{\Gamma_T} h \, d\Gamma, \quad \forall h \in \mathcal{K}_1.$$

The functional Φ on the set $\mathcal{K}_1$ is strictly convex, and by virtue of the properties (1)–(3) the operator $\mathcal{A}$ is coercive (i.e., $\Phi(h) \to +\infty$, if $\|h\|_{\frac{1}{2},\Gamma} \to \infty$, $h \in \mathcal{K}_1$). Therefore from the general theory of variational inequalities ([2], [14], [4]) we can conclude that the functional Φ has on the set $\mathcal{K}_1$ a unique minimizing element, which, in its turn, owing to the equivalence is a solution of (10), and hence a solution of the physical problem.

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Author's address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: rgach@rmi.acnet.ge