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CRITERIA OF CORRECTNESS OF LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

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Let $P \in L([a,b];\mathbb{R}^{n\times n})$, $p \in L([a,b];\mathbb{R}^n)$, $Q_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,m)$, $q_j \in \mathbb{R}^n$ $(j=1,\dots,m)$, $a=\tau_0<\tau_1<\dots<\tau_m\le\tau_{m+1}=b$, $c_0\in\mathbb{R}^n$, and $\ell : \mathrm{BVC}([a,b];\tau_1,\dots,\tau_m;\mathbb{R}^n) \to \mathbb{R}^n$ be a linear bounded operator such that the impulsive system

$$\frac{dx}{dt} = P(t)x + p(t), \tag{1}$$

$$x(\tau_j+) - x(\tau_j-) = Q_j x(\tau_j) + q_j \quad (j=1,\dots,m) \tag{2}$$

has a unique solution x_0 satisfying the boundary condition $\ell(x) = c_0$.

Consider sequences of matrix- and vector-functions $P_k \in L([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $p_k \in L([a,b];\mathbb{R}^n)$ $(k=1,2,\dots)$, sequences of constant matrices $Q_{kj} \in \mathbb{R}^{n\times n}$ $(j=1,\dots,m;\ k=1,2,\dots)$ and constant vectors $q_{kj} \in \mathbb{R}^n$ $(j=1,\dots,m;\ k=1,2,\dots)$ and $c_{0k} \in \mathbb{R}^n$ $(k=1,2,\dots)$ and a sequence of linear bounded operators $\ell_k : \mathrm{BVC}([a,b];\tau_1,\dots,\tau_m;\mathbb{R}^n) \to \mathbb{R}^n$ $(k=1,2,\dots)$.

In this paper necessary and sufficient conditions as well as effective sufficient conditions are established for a sequence of boundary value problems

$$\frac{dx}{dt} = P_k(t)x + p_k(t), \tag{3}$$

$$x(\tau_j+) - x(\tau_j-) = Q_{kj} x(\tau_j) + q_{kj} \quad (j=1,\dots,m), \tag{4}$$

$$\ell_k(x) = c_{0k} \tag{5}$$

$(k=1,2,\dots)$ to have a unique solution x_k for sufficiently large k and

$$\lim_{k\to\infty} x_k(t) = x_0(t) \tag{6}$$

uniformly on $[a,b]$.

Analogous questions are investigated e.g. in [1], [2], [5], [6] (see the references therein, too) for systems of ordinary differential equations and in [3], [4] for systems of generalized ordinary differential equations.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty,\infty[$. $\mathbb{R}^{n\times l}$ is the space of all real $n\times l$-matrices $X=(x_{ij})_{i,j=1}^{n,l}$ with the norm $\|X\| = \max\limits_{j=1,\dots,l} \sum\limits_{i=1}^{n} |x_{ij}|$. $O_{n\times l}$ is the zero $n\times l$-matrix.

$\det(X)$ is the determinant of a matrix $X \in \mathbb{R}^{n\times n}$. I_n is the identity $n\times n$-matrix. δ_{ij} is the Kronecker symbol, i.e. $\delta_{ii}=1$ and $\delta_{ij}=0$ for $i\ne j$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$.

$\mathrm{BVC}([a,b];\tau_1,\dots,\tau_m;\mathbb{R}^{n\times l})$ is the Banach space of all continuous on the intervals $[a,\tau_1]$, $]\tau_k,\tau_{k+1}]$ $(k=1,\dots,m)$ matrix-functions of bounded variation $X:[a,b]\to\mathbb{R}^{n\times l}$ with the norm $\|X\|_S = \sup\big\{\|X(t)\| :\ t\in[a,b]\big\}$.

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$L([a,b];\mathbb{R}^{n\times l})$ is the set of all measurable and Lebesgue integrable on $[a,b]$ matrix-functions.

$C([a,b];\mathbb{R}^{n\times l})$ is the set of all continuous on $[a,b]$ matrix-functions.

$\widetilde{C}([a,b];\mathbb{R}^{n\times l})$ is the set of all absolutely continuous on $[a,b]$ matrix-functions.

$\widetilde{C}([a,b]\setminus\{\tau_j\}_{j=1}^m;\mathbb{R}^{n\times l})$ is the set of all matrix-functions restrictions of which on every closed interval $[c,d]$ from $[a,b]\setminus\{\tau_j\}_{j=1}^m$ belong to $\widetilde{C}([c,d];\mathbb{R}^{n\times l})$.

On the set $C([a,b];\mathbb{R}^{n\times l})\times\underbrace{\mathbb{R}^{n\times l}\times\cdots\times\mathbb{R}^{n\times l}}_{m}\times L([a,b];\mathbb{R}^{l\times k})$ we introduce the operator

$$\mathcal{B}_0(\Phi,G_1,\dots,G_m,X)(t)\equiv\int_a^t\Phi(s)X(s)\,ds+\sum_{j=0,\ \tau_j\in[a,t[}^{m}G_j\int_{t_j}^{t}X(s)\,ds,$$

where $G_0=O_{n\times n}$.

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x\in\widetilde{C}([a,b]\setminus\{\tau_j\}_{j=1}^m;\mathbb{R}^{n\times l})\cap\mathrm{BVC}([a,b];\tau_1,\dots,\tau_m;\mathbb{R}^n)$ satisfying the system (1) for a.e. $t\in[a,b]$ and the equality (2) for every $j\in\{1,\dots,n\}$.

We assume everywhere that $\det(I_n+Q_j)\neq 0$ $(j=1,\dots,m)$.

Note that this condition guarantees the unique solvability of the system (1), (2) under the Cauchy condition $x(t_0)=c_0$.

Definition 1. We say that a sequence $(P_k,p_k,\{Q_{kj}\}_{j=1}^m,\{q_{kj}\}_{j=1}^m,\ell_k)$ $(k=1,2,\dots)$ belongs to the set $S(P,p,\{Q_j\}_{j=1}^m,\{q_j\}_{j=1}^m,\ell)$ if for every $c_0\in\mathbb{R}^n$ and $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying the condition $\lim\limits_{k\to\infty}c_k=c_0$ the problem (3)–(5) has a unique solution x_k for any sufficiently large k and the condition (6) holds uniformly on $[a,b]$.

Theorem 1. *Let*

$$\lim_{k\to\infty}\ell_k(y)=\ell(y)\quad\textit{for}\ \ y\in\mathrm{BVC}([a,b];\tau_1,\dots,\tau_m;\mathbb{R}^n).\tag{7}$$

Then

$$\big((P_k,p_k,\{Q_{kj}\}_{j=1}^m,\{q_{kj}\}_{j=1}^m,\ell_k)\big)_{k=1}^{\infty}\in S(P,p,\{Q_j\}_{j=1}^m,\{q_j\}_{j=1}^m,\ell)\tag{8}$$

if and only if there exist sequences of matrix-functions Φ, $\Phi_k\in\widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *and constant matrices* G_j, $G_{kj}\in\mathbb{R}^{n\times n}$, $G_0=G_{k0}=O_{n\times n}$ $(j=0,\dots,m;$ $k=1,2,\dots)$ *such that*

$$\lim_{k\to\infty}\sup\sum_{j=0}^{m}\int_{\tau_j}^{\tau_{j+1}}\Big\|\Phi_k'(t)+\Big(\Phi_k(t)+\sum_{i=0}^{j}Q_{kj}\Big)P_k(t)\Big\|\,dt<\infty,\tag{9}$$

$$\inf\Big\{\Big|\det\Big(\Phi(t)+\sum_{i=0}^{j}G_i\Big)\Big|:\ t\in]\tau_j,\tau_{j+1}]\Big\}>0\ \ (j=0,\dots,m),\tag{10}$$

$$\lim_{k\to\infty}G_{kj}=G_j\ \ (j=1,\dots,m),\tag{11}$$

$$\lim_{k\to\infty}Q_{kj}=Q_j,\quad \lim_{k\to\infty}q_{kj}=q_j\ \ (j=1,\dots,m),\tag{12}$$

and the conditions

$$\lim_{k\to\infty}\Phi_k(t)=\Phi(t),\tag{13}$$

$$\lim_{k\to\infty}\mathcal{B}_0(\Phi_k,G_{k1},\dots,G_{km},P_k)(t)=\mathcal{B}_0(\Phi,G_1,\dots,G_m,P)(t),\tag{14}$$

$$\lim_{k\to\infty}\mathcal{B}_0(\Phi_k,G_{k1},\dots,G_{km},p_k)(t)=\mathcal{B}_0(\Phi,G_1,\dots,G_m,p)(t)\tag{15}$$

are fulfilled uniformly on $[a,b]$.

Remark 1. The conditions (14) and (15) are fulfilled uniformly on $[a,b]$ if and only if the conditions

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\Big(\Phi_k(s)+\sum_{i=0}^{j}G_{ki}\Big)P_k(s)\,ds=\int_{\tau_j}^{t}\Big(\Phi(s)+\sum_{i=0}^{j}G_i\Big)P(s)\,ds,$$

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\Big(\Phi_k(s)+\sum_{i=0}^{j}G_{ki}\Big)p_k(s)\,ds=\int_{\tau_j}^{t}\Big(\Phi(s)+\sum_{i=0}^{j}G_i\Big)p(s)\,ds,$$

respectively, are fulfilled uniformly on $[\tau_j,\tau_{j+1}]$ for every $j\in\{0,\dots,m\}$.

Corollary 1. *Let the conditions* (7) *and* (12) *hold. Let, moreover, there exist matrix-functions* Φ, $\Phi_k\in\widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *such that the conditions* (9) *and*

$$\inf\Big\{\big|\det\big(\Phi(t)+(1-\delta_{0j})jI_n\big)\big|:\ t\in]\tau_j,\tau_{j+1}]\Big\}>0\ \ (j=0,\dots,m)$$

hold and the conditions (13),

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\big(\Phi_k(s)+(1-\delta_{0j})jI_n\big)P_k(s)\,ds=\int_{\tau_j}^{t}\big(\Phi(s)+(1-\delta_{0j})jI_n\big)P(s)\,ds$$

and

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\big(\Phi_k(s)+(1-\delta_{0j})jI_n\big)p_k(s)\,ds=\int_{\tau_j}^{t}\big(\Phi(s)+(1-\delta_{0j})jI_n\big)p(s)\,ds$$

be fulfilled uniformly on $[\tau_j,\tau_{j+1}]$ *for every* $j\in\{0,\dots,m\}$. *Then the condition* (8) *holds.*

Corollary 2. *Let the conditions* (7) *and* (12) *hold. Let, moreover, there exist matrix-functions* Φ, $\Phi_k\in\widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *such that*

$$\lim_{k\to\infty}\sup\int_a^b\big\|\Phi_k'(t)+\Phi_k(t)P_k(t)\big\|\,dt<\infty,\quad \inf\Big\{\big|\det(\Phi(t))\big|:\ t\in[a,b]\Big\}>0$$

and the conditions (13) *and*

$$\lim_{k\to\infty}\int_a^t\Phi_k(s)P_k(s)\,ds=\int_a^t\Phi(s)P(s)\,ds,\quad \lim_{k\to\infty}\int_a^t\Phi_k(s)p_k(s)\,ds=\int_a^t\Phi(s)p(s)\,ds$$

are fulfilled uniformly on $[a,b]$. *Then the condition* (8) *holds.*

Corollary 3. *Let the conditions* (7), (11) *and* (12) *hold. Let, moreover, there exist constant matrices* G_j, $G_{kj}\in\mathbb{R}^{n\times n}$, $G_0=G_{k0}=O_{n\times n}$ $(j=0,\dots,m;\ k=1,2,\dots)$ *such that*

$$\lim_{k\to\infty}\sup\sum_{j=0}^{m}\int_{\tau_j}^{\tau_{j+1}}\Big\|\Big(I_n+\sum_{i=0}^{j}Q_{ki}\Big)P_k(t)\Big\|\,dt<\infty,\tag{16}$$

$$\det\Big(I_n+\sum_{i=1}^{j}G_i\Big)\neq 0\ \ (j=1,\dots,m)$$

and the conditions

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_{ki}\Big)P_k(s)\,ds=\int_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_i\Big)P(s)\,ds,$$

$$\lim_{k\to\infty}\int_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_{ki}\Big)p_k(s)\,ds=\int_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_i\Big)p(s)\,ds$$

are fulfilled uniformly on $[\tau_j,\tau_{j+1}]$ *for every* $j\in\{0,\dots,m\}$. *Then the condition* (8) *holds.*

Corollary 4. *Let the conditions* (7)*,* (12) *and* (16) *hold and the conditions*

$$\lim_{k\to\infty}\int_a^t P_k(s)\,ds=\int_a^t P(s)\,ds,\quad \lim_{k\to\infty}\int_a^t p_k(s)\,ds=\int_a^t p(s)\,ds \tag{17}$$

be fulfilled uniformly on $[a,b]$*. Then the condition* (8) *holds.*

Corollary 5. *Let the conditions* (7)*,* (12)*, and* (16) *hold and the condition* (17) *be fulfilled uniformly on* $[a,b]$*. Then the condition* (8) *holds.*

Remark 2. In Theorem 1 and Corollaries 1–5 we can assume without loss of generality that $\Phi(t)\equiv I_n$ and $G_j=O_{n\times n}$ $(j=1,\dots,m)$ everywhere they appear. So that the condition (10) in Theorem 1 as well as the analogous conditions in the corollaries are valid automatically.

These results follow from analogous results for a system of so-called generalized ordinary differential equations contained in [4] because the system (1), (2) is its particular case.

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