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NON-NOETHER SYMMETRIES IN HAMILTONIAN DYNAMICAL SYSTEMS

Abstract. We discuss geometric properties of non-Noether symmetries and their possible applications in integrable Hamiltonian systems. The correspondence between non-Noether symmetries and conservation laws is revisited. It is shown that in regular Hamiltonian systems such symmetries canonically lead to Lax pairs on the algebra of linear operators on the cotangent bundle over the phase space. Relationship between non-Noether symmetries and other widespread geometric methods of generating conservation laws such as bi-Hamiltonian formalism, bidifferential calculi and Frölicher–Nijenhuis geometry is considered. It is proved that the integrals of motion associated with a continuous non-Noether symmetry are in involution whenever the generator of the symmetry satisfies a certain Yang–Baxter type equation. Action of one-parameter group of symmetry on the algebra of integrals of motion is studied and involutivity of group orbits is discussed. Hidden non-Noether symmetries of the Toda chain, Korteweg–de Vries equation, Benney system, nonlinear water wave equations and Broer–Kaup system are revealed and discussed.

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Key words and phrases. Non-Noether symmetry, Conservation law, bi-Hamiltonian system, Bidifferential calculus, Lax pair, Frölicher–Nijenhuis operator, Korteweg–de Vries equation, Broer–Kaup system, Benney system, Toda chain.

რეზიუმე. შესწავლილია არანოეთერისეული სიმეტრიების გეომეტრიული თვისებები და მათი გამოყენების არეალი ჰამილტონურ დინამიკურ სისტემებში. შემოღებულია იანგ-ბაქსტერის განტოლების მსგავსი ინვოლუციურობის კრიტერიუმი. დამტკიცებულია, რომ თუ სიმეტრიის გენერატორი აკმაყოფილებს ამ კრიტერიუმს, მაშინ სიმეტრიასთან დაკავშირებული შენახვადი სიდიდეები ინვოლუციაში არიან. დამტკიცებულია, რომ გარკვეულ პირობებში არანოეთერისეული სიმეტრიის საშუალებით შესაძლებელია ბიჰამილტონური სისტემის აგება და პირიქით, დადგენილია, თუ რა შემთხვევაში უკავშირდება მოცემული ბიჰამილტონური სისტემა სიმეტრიას. არანოეთერისეული სიმეტრიის და პუასონური ბივექტორული ველის საშუალებით აგებულია დე რამის კომპლექსის გარკვეული დეფორმაცია. ნაჩვენებია, რომ სათანადო პირობებში არანოეთერისეული სიმეტრიის საშუალებით შესაძლებელია ფაზური სივრცის მხებ ფიბრაციაზე ინვარიანტული ფროლიხერ-ნეიენჰეისის ოპერატორის აგება. ნაპოვნია ტოდას მოდელის, კორტევეგ-დე ფრიზის განტოლების, ბენის სისტემის, წყლის არაწრფივი ტალღების განტოლებების და ბროერ-კოპის სისტემის არანოეთერისეული სიმეტრიები და განხილულია მათი თვისებები.

1. Introduction

Symmetries play essential role in dynamical systems, because they usually simplify analysis of evolution equations and often provide quite elegant solution of problems that otherwise would be difficult to handle. In Lagrangian and Hamiltonian dynamical systems special role is played by Noether symmetries — an important class of symmetries that leave action invariant and have some exceptional features. In particular, Noether symmetries deserved special attention due to the celebrated Noether's theorem that established a correspondence between symmetries that leave the action functional invariant, and conservation laws of Euler–Lagrange equations. This correspondence can be extended to Hamiltonian systems where it becomes more tight and evident than in Lagrangian case and gives rise to a Lie algebra homomorphism between the Lie algebra of Noether symmetries and the algebra of conservation laws (that form Lie algebra under Poisson bracket).

The role of symmetries that are not of Noether type has been suppressed for quite a long time. However, after some publications of Hojman, Harleston, Lutzky and others (see [16], [36], [39], [40], [49]–[57]) it became clear that non-Noether symmetries also can play important role in Lagrangian and Hamiltonian dynamics. In particular, according to Lutzky [51], in Lagrangian dynamics there is a definite correspondence between non-Noether symmetries and conservation laws. Moreover, unlike the noetherian case, each generator of a non-Noether symmetry may produce whole family of conservation laws (maximal number of conservation laws that can be associated with the non-Noether symmetry via Lutzky's theorem is equal to the dimension of configuration space of the Lagrangian system). This fact makes non-Noether symmetries especially valuable in infinite dimensional dynamical systems, where potentially one can recover infinite sequence of conservation laws knowing single generator of a non-Noether symmetry.

The existence of correspondence between non-Noether symmetries and conserved quantities raised many questions concerning relationship among this type of symmetries and other geometric structures emerging in the theory of integrable models. In particular one could notice suspicious similarity between the method of constructing conservation laws from a generator of a non-Noether symmetry and the way conserved quantities are produced in either Lax theory, bi-Hamiltonian formalism, bicomplex approach or Lenard scheme. It also raised the natural question whether the set of conservation laws associated with a non-Noether symmetry is involutive or not, and since it appeared that in general it may not be involutive, there emerged the need of involutivity criteria similar to Yang–Baxter equation used in Lax theory or compatibility condition in bi-Hamiltonian formalism and bicomplex approach. It was also unclear how to construct conservation laws in case of infinite dimensional dynamical systems where volume forms used

in Lutzky's construction are no longer well-defined. Some of these questions were addressed in [11]–[14], while in the present review we would like to summarize all these issues and to provide some examples of integrable models that possess non-Noether symmetries.

The review is organized as follows. In the first section we briefly recall some aspects of geometric formulation of Hamiltonian dynamics. Further, in the second section, a correspondence between non-Noether symmetries and integrals of motion in regular Hamiltonian systems is discussed. Lutzky's theorem is reformulated in terms of bivector fields and an alternative derivation of conserved quantities suitable for computations in infinite dimensional Hamiltonian dynamical systems is suggested. Non-Noether symmetries of two and three particle Toda chains are used to illustrate the general theory. In the subsequent section geometric formulation of Hojman's theorem [36] is revisited and examples are provided. Section 4 reveals a correspondence between non-Noether symmetries and Lax pairs. It is shown that a non-Noether symmetry canonically gives rise to a Lax pair of certain type. The Lax pair is explicitly constructed in terms of the Poisson bivector field and the generator of symmetry. Examples of Toda chains are discussed. Next section deals with integrability issues. An analogue of the Yang–Baxter equation that, being satisfied by a generator of symmetry ensures involutivity of the set of conservation laws produced by this symmetry, is introduced. The relationship between non-Noether symmetries and bi-Hamiltonian systems is considered in Section 6. It is proved that under certain conditions a non-Noether symmetry endows the phase space of a regular Hamiltonian system with a bi-Hamiltonian structure. We also discuss conditions under which the non-Noether symmetry can be "recovered" from the bi-Hamiltonian structure. The theory is illustrated by examples of Toda chains. Next section is devoted to bicomplexes and their relationship with non-Noether symmetries. Special kind of deformation of De Rham complex induced by a symmetry is constructed in terms of Poisson bivector field and the generator of the symmetry. Examples of two and three particle Toda chain are discussed. Section 8 deals with Frölicher–Nijenhuis recursion operators. It is shown that under certain conditions a non-Noether symmetry gives rise to an invariant Frölicher–Nijenhuis operator on the tangent bundle over the phase space. The last section of theoretical part contains some remarks on action of one-parameter group of symmetry on algebra of integrals of motion. Special attention is devoted to involutivity of the group orbits.

Subsequent sections of the present review provide examples of integrable models that possess interesting non-Noether symmetries. In particular, Section 10 reveals a non-Noether symmetry of the n-particle Toda chain. Bi-Hamiltonian structure, conservation laws, bicomplex, Lax pair and Frölicher–Nijenhuis recursion operator of Toda hierarchy are constructed using this symmetry. Further we focus on infinite dimensional integrable Hamiltonian systems emerging in mathematical physics. In Section 11 the case

of Korteweg–de Vries equation is discussed. A symmetry of this equation is identified and used in construction of infinite sequence of conservation laws and bi-Hamiltonian structure of KdV hierarchy. Next section is devoted to non-Noether symmetries of integrable systems of nonlinear water wave equations, such as dispersive water wave system, Broer–Kaup system and dispersionless long wave system. Last section focuses on Benney system and its non-Noether symmetry, which appears to be local, gives rise to infinite sequence of conserved densities of Benney hierarchy and endows it with a bi-Hamiltonian structure.

2. Regular Hamiltonian Systems

The basic concept in geometric formulation of Hamiltonian dynamics is the notion of symplectic manifold. Such a manifold plays the role of the phase space of the dynamical system and therefore many properties of the dynamical system can be quite effectively investigated in the framework of symplectic geometry. Before we consider symmetries of Hamiltonian dynamical systems, let us briefly recall some basic notions from symplectic geometry.

The symplectic manifold is a pair (M, ω) where M is a smooth even dimensional manifold and ω is a closed,

$$d\omega = 0, \tag{1}$$

and nondegenerate 2-form on M. Being nondegenerate means that the contraction of an arbitrary non-zero vector field with ω does not vanish:

$$i_X\omega = 0 \Leftrightarrow X = 0 \tag{2}$$

(here i_X denotes contraction of the vector field X with a differential form). Otherwise one can say that ω is nondegenerate if its n-th outer power does not vanish ($\omega^n \neq 0$) anywhere on M. In Hamiltonian dynamics M is usually the phase space of a classical dynamical system with finite number of degrees of freedom and the symplectic form ω is a basic object that defines the Poisson bracket structure, algebra of Hamiltonian vector fields and the form of Hamilton's equations.

The symplectic form ω naturally defines an isomorphism between vector fields and differential 1-forms on M (in other words, the tangent bundle TM of the symplectic manifold can be quite naturally identified with the cotangent bundle T^*M). The isomorphic map Φ_ω from TM into T^*M is obtained by taking contraction of the vector field with ω

$$\Phi_\omega : X \to -i_X\omega \tag{3}$$

(the minus sign is the matter of convention). This isomorphism gives rise to natural classification of vector fields. Namely, a vector field X_h is said to be Hamiltonian if its image is an exact 1-form or in other words if it satisfies Hamilton's equation

$$i_{X_h}\omega + dh = 0 \tag{4}$$

for some function h on M. Similarly, a vector field X is called locally Hamiltonian if it's image is a closed 1-form

$$i_X\omega + u = 0, \qquad du = 0.$$

One of the nice features of locally Hamiltonian vector fields, known as Liouville's theorem, is that these vector fields preserve the symplectic form ω. In other words, Lie derivative of the symplectic form ω along arbitrary locally Hamiltonian vector field vanishes

$$L_X\omega = 0 \Leftrightarrow i_X\omega + du = 0, \qquad du = 0.$$

Indeed, using Cartan's formula that expresses Lie derivative in terms of contraction and exterior derivative

$$L_X = i_X d + d i_X$$

one gets

$$L_X\omega = i_X d\omega + d i_X\omega = d i_X\omega$$

(since $d\omega = 0$) but according to the definition of locally Hamiltonian vector field

$$d i_X\omega = -du = 0.$$

So locally Hamiltonian vector fields preserve ω and vise versa, if a vector field preserves the symplectic form ω then it is locally Hamiltonian.

Clearly, Hamiltonian vector fields constitute a subset of locally Hamiltonian ones since every exact 1-form is also closed. Moreover, one can notice that Hamiltonian vector fields form an ideal in the algebra of locally Hamiltonian vector fields. This fact can be observed as follows. First of all for arbitrary couple of locally Hamiltonian vector fields X, Y we have $L_X\omega = L_Y\omega = 0$ and

$$L_X L_Y\omega - L_Y L_X\omega = L_{[X,Y]}\omega = 0,$$

so locally Hamiltonian vector fields form a Lie algebra (the corresponding Lie bracket is ordinary commutator of vector fields). Further it is clear that for arbitrary Hamiltonian vector field X_h and locally Hamiltonian one Z one has

$$L_Z\omega = 0$$

and

$$i_{X_h}\omega + dh = 0$$

that implies

$$\begin{aligned} L_Z(i_{X_h}\omega + dh) &= L_{[Z,X_h]}\omega + i_{X_h}L_Z\omega + dL_Z h = \\ &= L_{[Z,X_h]}\omega + dL_Z h = 0. \end{aligned}$$

Thus the commutator $[Z, X_h]$ is a Hamiltonian vector field $X_{L_Z h}$, or in other words Hamiltonian vector fields form an ideal in the algebra of locally Hamiltonian vector fields.

The isomorphism Φ_ω can be extended to higher order vector fields and differential forms by linearity and multiplicativity. Namely,

$$\Phi_\omega(X \wedge Y) = \Phi_\omega(X) \wedge \Phi_\omega(Y).$$

Since Φ_ω is an isomorphism, the symplectic form ω has a unique counter image W known as the Poisson bivector field. The property $d\omega = 0$ together with non degeneracy implies that the bivector field W is also nondegenerate ($W^n \neq 0$) and satisfies the condition

$$[W, W] = 0 \tag{5}$$

where the bracket $[\,,\,]$ known as Schouten bracket or supercommutator, is actually the graded extension of ordinary commutator of vector fields to the case of multivector fields, and can be defined by linearity and derivation property

$$\begin{aligned}[C_1 \wedge C_2 \wedge \cdots \wedge C_n, S_1 \wedge S_2 \wedge \cdots \wedge S_n] = \\ = (-1)^{p+q}[C_p, S_q] \wedge C_1 \wedge C_2 \wedge \cdots \wedge \hat{C}_p \wedge \cdots \wedge C_n \\ \wedge S_1 \wedge S_2 \wedge \cdots \wedge \hat{S}_q \wedge \cdots \wedge S_n\end{aligned}$$

where the over hat denotes omission of the corresponding vector field. In terms of the bivector field W, Liouville's theorem mentioned above can be rewritten as follows

$$[W(u), W] = 0 \Leftrightarrow du = 0 \tag{6}$$

for each 1-form u. It follows from the graded Jacoby identity satisfied by the Schouten bracket and property $[W, W] = 0$ satisfied by the Poisson bivector field.

Being the counter image of a symplectic form, W gives rise to the map Φ_W transforming differential 1-forms into vector fields, which is inverted to the map Φ_ω and is defined by

$$\Phi_W : u \to W(u); \qquad \Phi_W \Phi_\omega = id.$$

Further we will often use these maps.

In Hamiltonian dynamical systems the Poisson bivector field is a geometric object that underlies the definition of the Poisson bracket — a kind of Lie bracket on the algebra of smooth real functions on phase space. In terms of a bivector field W, the Poisson bracket is defined by

$$\{f, g\} = W(df \wedge dg). \tag{7}$$

The condition $[W, W] = 0$ satisfied by the bivector field ensures that for every triple (f, g, h) of smooth functions on the phase space the Jacobi identity

$$\{f\{g, h\}\} + \{h\{f, g\}\} + \{g\{h, f\}\} = 0 \tag{8}$$

is satisfied. Interesting property of the Poisson bracket is that the map from the algebra of real smooth functions on the phase space into the algebra of

Hamiltonian vector fields defined by the Poisson bivector field

$$f \to X_f = W(df)$$

appears to be a homomorphism of Lie algebras. In other words, the commutator of two vector fields associated with two arbitrary functions reproduces the vector field associated with the Poisson bracket of these functions

$$[X_f, X_g] = X_{\{f,g\}}. \tag{9}$$

This property is a consequence of the Liouville theorem and the definition of the Poisson bracket. Further we also need another useful property of Hamiltonian vector fields and the Poisson bracket

$$\{f, g\} = W(df \wedge dg) = \omega(X_f \wedge X_g) = L_{X_f} g = -L_{X_g} g. \tag{10}$$

It also follows from the Liouville theorem and the definition of Hamiltonian vector fields and Poisson brackets.

To define dynamics on M one has to specify time evolution of observables (smooth functions on M). In Hamiltonian dynamical systems time evolution is governed by Hamilton's equation

$$\frac{d}{dt} f = \{h, f\}, \tag{11}$$

where h is some fixed smooth function on the phase space called Hamiltonian. In local coordinate frame z_k, the bivector field W has the form

$$W = W_{bc} \frac{\partial}{\partial z_b} \wedge \frac{\partial}{\partial z_c}$$

and Hamilton's equation rewritten in terms of local coordinates takes the form

$$\dot{z}_b = W_{bc} \frac{\partial h}{\partial z_c}.$$

Note that the functions W_{ab} are not arbitrary: to ensure the validity of the condition $[W, W] = 0$ condition they should fulfill the restriction

$$\sum_{a=1}^{n} \left[W_{ab} \frac{\partial W_{cd}}{\partial z_a} + W_{ac} \frac{\partial W_{bd}}{\partial z_a} + W_{ad} \frac{\partial W_{bc}}{\partial z_a} \right] = 0$$

and at the same time the determinant of the matrix formed by the functions W_{ab} should not vanish to ensure that the Poisson bivector field W is nondegenerate.

3. Non-Noether Symmetries

Now let us focus on symmetries of Hamilton's equation (11). Generally speaking, symmetries play very important role in Hamiltonian dynamics due to different reasons. They not only give rise to conservation laws but also often provide very effective solutions to problems that otherwise would be difficult to solve. Here we consider the special class of symmetries of Hamilton's equation called non-Noether symmetries. Such symmetries appear to

be closely related to many geometric concepts used in Hamiltonian dynamics including bi-Hamiltonian structures, Frölicher–Nijenhuis operators, Lax pairs and bicomplexes.

Before we proceed let us recall that each vector field E on the phase space generates a one-parameter continuous group of transformations $g_a = e^{aL_E}$ (here L denotes Lie derivative) that acts on the observables as follows

$$g_a(f) = e^{aL_E}(f) = f + aL_E f + \frac{1}{2}(aL_E)^2 f + \cdots . \tag{12}$$

Such a group of transformations is called symmetry of Hamilton's equation (11) if it commutes with the time evolution operator

$$\frac{d}{dt} g_a(f) = g_a(\frac{d}{dt} f). \tag{13}$$

In terms of the vector fields this condition means that the generator E of the group g_a commutes with the vector field $W(h) = \{h, \}$, i.e.

$$[E, W(h)] = 0. \tag{14}$$

However we would like to consider a more general case where E is a time dependent vector field on the phase space. In this case (14) should be replaced with

$$\frac{\partial}{\partial t} E = [E, W(h)]. \tag{15}$$

Further one should distinguish between groups of symmetry transformations generated by Hamiltonian, locally Hamiltonian and non-Hamiltonian vector fields. First kind of symmetries are known as Noether symmetries and are widely used in Hamiltonian dynamics due to their tight connection with conservation laws. The second group of symmetries is rarely used, while the third group of symmetries that further will be referred as non-Noether symmetries seems to play important role in integrability issues due to their remarkable relationship with bi-Hamiltonian structures and Frölicher–Nijenhuis operators. Thus if in addition to (14) the vector field E does not preserve Poisson bivector field $[E, W] \neq 0$, then g_a is called non-Noether symmetry.

Now let us focus on non-Noether symmetries. We would like to show that the presence of such a symmetry essentially enriches the geometry of the phase space and under certain conditions can ensure integrability of the dynamical system. Before we proceed let us recall that a non-Noether symmetry leads to a number of integrals of motion. More precisely the relationship between non-Noether symmetries and the conservation laws is described by the following theorem. This theorem was proposed by Lutzky in [51]. Here it is reformulated in terms of Poisson bivector field.

Theorem 1. *Let (M, h) be a regular Hamiltonian system on the $2n$-dimensional Poisson manifold M. Then, if the vector field E generates a*

non-Noether symmetry, the functions

$$Y^{(k)} = \frac{\hat{W}^k \wedge W^{n-k}}{W^n}, \qquad k = 1, 2, \ldots, n, \tag{16}$$

where $\hat{W} = [E, W]$, *are integrals of motion.*

Proof. By definition

$$\hat{W}^k \wedge W^{n-k} = Y^{(k)} W^n$$

(the definition is correct since the space of $2n$-degree multivector fields on $2n$-degree manifold is one dimensional). Let us take time derivative of this expression along the vector field $W(h)$,

$$\frac{d}{dt}\hat{W}^k \wedge W^{n-k} = \left(\frac{d}{dt}Y^{(k)}\right) W^n + Y^{(k)}[W(h), W^n],$$

or

$$k\left(\frac{d}{dt}\hat{W}\right) \wedge \hat{W}^{k-1} \wedge W^{n-k} + (n-k)[W(h), W] \wedge \hat{W}^k \wedge W^{n-k-1} =$$
$$= \left(\frac{d}{dt}Y^{(k)}\right) W^n + nY^{(k)}[W(h), W] \wedge W^{n-1}. \tag{17}$$

But according to the Liouville theorem the Hamiltonian vector field preserves W i.e.

$$\frac{d}{dt} W = [W(h), W] = 0.$$

Hence, by taking into account that

$$\frac{d}{dt} E = \frac{\partial}{\partial t} E + [W(h), E] = 0,$$

we get

$$\frac{d}{dt}\hat{W} = \frac{d}{dt}[E, W] = \left[\frac{d}{dt}E, W\right] + [E[W(h), W]] = 0,$$

and as a result (17) yields

$$\frac{d}{dt} Y^{(k)} W^n = 0.$$

But since the dynamical system is regular ($W^n \neq 0$), we obtain that the functions $Y^{(k)}$ are integrals of motion. □

Remark 3.1. Instead of conserved quantities $Y^{(1)} \ldots Y^{(n)}$, the solutions $c_1 \ldots c_n$ of the secular equation

$$(\hat{W} - cW)^n = 0 \tag{18}$$

can be associated with the generator of symmetry. By expanding the expression (18) it is easy to verify that the conservation laws $Y^{(k)}$ can be

expressed in terms of the integrals of motion $c_1 \dots c_n$ in the following way

$$Y^{(k)} = \frac{(n-k)!k!}{n!} \sum_{i_1 > i_2 > \dots > i_k} c_{i_1} c_{i_2} \dots c_{i_k}. \tag{19}$$

Note also that the conservation laws $Y^{(k)}$ can be also defined by means of the symplectic form ω using the following formula

$$Y^{(k)} = \frac{(L_E \omega)^k \wedge \omega^{n-k}}{\omega^n}, \qquad k = 1, 2, \dots, n, \tag{20}$$

while the conservation laws $c_1 \dots c_n$ can be derived from the secular equation

$$(L_E \omega - c\omega)^n = 0. \tag{21}$$

However, all these expressions fail in case of infinite dimensional Hamiltonian systems where the volume form

$$\Omega = \omega^n$$

does not exist since $n = \infty$. But fortunately in this case one can define conservation laws using the alternative formula

$$C^{(k)} = i_{W^k} (L_E \omega)^k \tag{22}$$

as far as it involves only finite degree differential forms $(L_E \omega)^k$ and well-defined multivector fields W^k. Note that in finite dimensional case the sequence of conservation laws $C^{(k)}$ is related to families of conservation laws $Y^{(k)}$ and c_k in the following way

$$C^{(k)} = \sum_{i_1 > i_2 > \dots > i_k} c_{i_1} c_{i_2} \dots c_{i_k} = \frac{n!}{(n-k)!k!} Y^{(k)}. \tag{23}$$

Note also that by taking Lie derivative of known conservation along the generator of symmetry E one can construct new conservation laws

$$\frac{d}{dt} Y = L_{X_h} Y = 0 \Rightarrow \frac{d}{dt} L_E Y = L_{X_h} L_E Y = L_E L_{X_h} Y = 0$$

since $[E, X_h] = 0$.

Remark 3.2. Besides continuous non-Noether symmetries generated by non-Hamiltonian vector fields one may encounter discrete non-Noether symmetries — noncannonical transformations that doesn't necessarily form a group but commute with the evolution operator

$$\frac{d}{dt} g(f) = g\left(\frac{d}{dt} f\right).$$

Such symmetries give rise to the same conservation laws

$$Y^{(k)} = \frac{g(W)^k \wedge W^{n-k}}{W^n}, \qquad k = 1, 2, \dots, n. \tag{24}$$

Example. Let M be R^4 with the coordinates z_1, z_2, z_3, z_4 and the Poisson bivector field

$$W = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} + \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_4} \tag{25}$$

and let's take

$$h = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + e^{z_3 - z_4}. \tag{26}$$

This is the so-called two particle non periodic Toda model. One can check that the vector field

$$E = \sum_{a=1}^{4} E_a \frac{\partial}{\partial z_a}$$

with the components

$$\begin{aligned} E_1 &= \frac{1}{2} z_1^2 - e^{z_3 - z_4} - \frac{t}{2}(z_1 + z_2) e^{z_3 - z_4} \\ E_2 &= \frac{1}{2} z_2^2 + 2 e^{z_3 - z_4} + \frac{t}{2}(z_1 + z_2) e^{z_3 - z_4} \\ E_3 &= 2 z_1 + \frac{1}{2} z_2 + \frac{t}{2}(z_1^2 + e^{z_3 - z_4}) \\ E_4 &= z_2 - \frac{1}{2} z_1 + \frac{t}{2}(z_2^2 + e^{z_3 - z_4}) \end{aligned} \tag{27}$$

satisfies the condition (15) and as a result generates a symmetry of the dynamical system. The symmetry appears to be non-Noether with the Schouten bracket $[E, W]$ equal to

$$\begin{aligned} \hat{W} = [E, W] = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_4} + \\ + e^{z_3 - z_4} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4}. \end{aligned} \tag{28}$$

Calculation of volume vector fields $\hat{W}^k \wedge W^{n-k}$ gives rise to

$$\begin{aligned} W \wedge W &= -2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4}, \\ \hat{W} \wedge W &= -(z_1 + z_2) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4}, \\ \hat{W} \wedge \hat{W} &= -2(z_1 z_2 - e^{z_3 - z_4}) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4}, \end{aligned}$$

and the conservation laws associated with this symmetry are just

$$\begin{aligned} Y^{(1)} &= \frac{\hat{W} \wedge W}{W \wedge W} = \frac{1}{2}(z_1 + z_2), \\ Y^{(2)} &= \frac{\hat{W} \wedge \hat{W}}{W \wedge W} = z_1 z_2 - e^{z_3 - z_4}. \end{aligned} \tag{29}$$

It is remarkable that the same symmetry is also present in higher dimensions. For example, in case where M is R^6 with the coordinates

$$z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6.$$

The Poisson bivector equal to

$$W = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_5} + \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_6} \tag{30}$$

and the following Hamiltonian

$$h = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2 + e^{z_4 - z_5} + e^{z_5 - z_6}, \tag{31}$$

we still can construct a symmetry similar to (27). More precisely the vector field

$$E = \sum_{a=1}^{6} E_a \frac{\partial}{\partial z_a}$$

with the components specified as follows

$$\begin{aligned}
E_1 &= \frac{1}{2}z_1^2 - 2e^{z_4 - z_5} - \frac{t}{2}(z_1 + z_2)e^{z_4 - z_5}\,, \\
E_2 &= \frac{1}{2}z_2^2 + 3e^{z_4 - z_5} - e^{z_5 - z_6} + \frac{t}{2}(z_1 + z_2)e^{z_4 - z_5}\,, \\
E_3 &= \frac{1}{2}z_3^2 + 2e^{z_5 - z_6} + \frac{t}{2}(z_2 + z_3)e^{z_5 - z_6}\,, \\
E_4 &= 3z_1 + \frac{1}{2}z_2 + \frac{1}{2}z_3 + \frac{t}{2}(z_1^2 + e^{z_4 - z_5})\,, \\
E_5 &= 2z_2 - \frac{1}{2}z_1 + \frac{1}{2}z_3 + \frac{t}{2}(z_2^2 + e^{z_4 - z_5} + e^{z_5 - z_6})\,, \\
E_6 &= z_3 - \frac{1}{2}z_1 - \frac{1}{2}z_2 + \frac{t}{2}(z_3^2 + e^{z_5 - z_6})
\end{aligned} \tag{32}$$

satisfies the condition (15) and generates a non-Noether symmetry of the dynamical system (three particle non periodic Toda chain). Calculation of the Schouten bracket $[E, W]$ gives rise to the expression

$$\begin{aligned}
\hat{W} = [E, W] = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_4} + z_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_5} + z_3 \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_6} + \\
+ e^{z_4 - z_5} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + e^{z_5 - z_6} \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} + \\
+ \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_4} \wedge \frac{\partial}{\partial z_5} + \frac{\partial}{\partial z_5} \wedge \frac{\partial}{\partial z_6}.
\end{aligned} \tag{33}$$

Volume multivector fields $\hat{W}^k \wedge W^{n-k}$ can be calculated in the manner similar to the R^4 case and give rise to the well known conservation laws of

three particle Toda chain.

$$
\begin{aligned}
Y^{(1)} &= \frac{1}{6}(z_1+z_2+z_3) = \frac{\hat{W}\wedge W\wedge W}{W\wedge W\wedge W}\,,\\
Y^{(2)} &= \frac{1}{3}(z_1z_2+z_1z_3+z_2z_3-e^{z_4-z_5}-e^{z_5-z_6}) = \frac{\hat{W}\wedge\hat{W}\wedge W}{W\wedge W\wedge W}\,,\\
Y^{(3)} &= z_1z_2z_3-z_3e^{z_4-z_5}-z_1e^{z_5-z_6} = \frac{\hat{W}\wedge\hat{W}\wedge\hat{W}}{W\wedge W\wedge W}.
\end{aligned}
\tag{34}
$$

4. Non-Liouville Symmetries

Besides Hamiltonian dynamical systems that admit invariant symplectic form ω, there are dynamical systems that either are not Hamiltonian or admit Hamiltonian realization but the explicit form of symplectic structure ω is unknown or too complex. However, usually such a dynamical system possesses an invariant volume form Ω which like the symplectic form can be effectively used in construction of conservation laws. Note that the volume form for a given manifold is an arbitrary differential form of maximal degree (equal to the dimension of the manifold). In case of regular Hamiltonian systems, n-th outer power of the symplectic form ω naturally gives rise to the invariant volume form known as Liouville form

$$\Omega = \omega^n,$$

and sometimes it is easier to work with Ω than with the symplectic form itself. In the generic Liouville dynamical system time evolution is governed by the equations of motion

$$\frac{d}{dt}f = X(f), \tag{35}$$

where X is some smooth vector field that preserves the Liouville volume form Ω

$$\frac{d}{dt}\Omega = L_X\Omega = 0.$$

A symmetry of the equations of motion still can be defined by the condition

$$\frac{d}{dt}g_a(f) = g_a(\frac{d}{dt}f)$$

which in terms of vector fields implies that the generator of symmetry E should commute with time evolution operator X

$$[E, X] = 0.$$

Throughout this chapter a symmetry will be called non-Liouville if it is not a conformal symmetry of Ω, or in other words if

$$L_E\Omega \neq c\Omega$$

for any constant c. Such symmetries may be considered as analogues of non-Noether symmetries defined in Hamiltonian systems and similarly to the Hamiltonian case one can try to construct conservation laws by means of the

generator of symmetry E and the invariant differential form Ω. Namely, we have the following theorem, which is a reformulation of Hojman's theorem in terms of the Liouville volume form.

Theorem 2. *Let (M, X, Ω) be a Liouville dynamical system on a smooth manifold M. Then, if the vector field E generates a non-Liouville symmetry, the function*

$$J = \frac{L_E \Omega}{\Omega} \tag{36}$$

is a conservation law.

Proof. By definition

$$L_E \Omega = J\Omega$$

and J is not constant (again the definition is correct since the space of volume forms is one dimensional). By taking Lie derivative of this expression along the vector field X that defines time evolution, we get

$$\begin{aligned} L_X L_E \Omega &= L_{[X,E]}\Omega + L_E L_X \Omega = \\ &= L_X(J\Omega) = (L_X J)\Omega + J L_X \Omega \end{aligned}$$

but since the Liouville volume form is invariant, $L_X \Omega = 0$, and the vector field E is the generator of a symmetry satisfying the commutation relation $[E, X] = 0$, we obtain

$$(L_X J)\Omega = 0$$

or

$$\frac{d}{dt} J = L_X J = 0.$$

□

Remark 4.1. In fact the theorem is valid for a larger class of symmetries. Namely, one can consider symmetries with time dependent generators. Note, however, that in this case the condition $[E, X] = 0$ should be replaced by

$$\frac{\partial}{\partial t} E = [E, X].$$

Note also that by calculating Lie derivative of the conservation law J along the generator of the symmetry E, one can recover additional conservation laws

$$J^{(m)} = (L_E)^m \Omega.$$

Example. Let us consider a symmetry of the three particle non periodic Toda chain. This dynamical system with equations of motion defined by the vector field

$$X = -e^{z_4 - z_5} \frac{\partial}{\partial z_1} + (e^{z_4 - z_5} - e^{z_5 - z_6}) \frac{\partial}{\partial z_2} + e^{z_5 - z_6} \frac{\partial}{\partial z_3} +$$

$$+z_1\frac{\partial}{\partial z_4}+z_2\frac{\partial}{\partial z_5}+z_3\frac{\partial}{\partial z_6}$$

possesses the invariant volume form

$$\Omega = dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge dz_5\wedge dz_6.$$

One can check that Ω is really an invariant volume form, i.e. Lie derivative of Ω along X vanishes

$$\frac{d}{dt}\Omega = L_X\Omega = \left[\frac{\partial X_1}{\partial z_1}+\frac{\partial X_2}{\partial z_2}+\frac{\partial X_3}{\partial z_3}+\frac{\partial X_4}{\partial z_4}+\frac{\partial X_5}{\partial z_5}+\frac{\partial X_6}{\partial z_6}\right]\Omega = 0.$$

The symmetry (32) is clearly non-Liouville one as far as

$$L_E\Omega = \left[\frac{\partial E_1}{\partial z_1}+\frac{\partial E_2}{\partial z_2}+\frac{\partial E_3}{\partial z_3}+\frac{\partial E_4}{\partial z_4}+\frac{\partial E_5}{\partial z_5}+\frac{\partial E_6}{\partial z_6}\right]\Omega =$$
$$= (z_1+z_2+z_3)dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge dz_5\wedge dz_6 = (z_1+z_2+z_3)\Omega$$

and the main conservation law associated with this symmetry via Theorem 2 is total momentum

$$J = \frac{L_E\Omega}{\Omega} = z_1+z_2+z_3.$$

Other conservation laws can be recovered by taking Lie derivative of J along the generator of symmetry E, in particular

$$J^{(1)} = L_E J = \frac{1}{2}z_1^2+\frac{1}{2}z_2^2+\frac{1}{2}z_3^2+e^{z_4-z_5}+e^{z_5-z_6}$$
$$J^{(2)} = L_E J^{(1)} = \frac{1}{2}(z_1^3+z_2^3+z_3^3)+$$
$$+\frac{3}{2}(z_1+z_2)e^{z_4-z_5}+\frac{3}{2}(z_2+z_3)e^{z_5-z_6}.$$

5. Lax Pairs

The presence of a non-Noether symmetry not only leads to a sequence of conservation laws, but also endows the phase space with a number of interesting geometric structures and it appears that such a symmetry is related to many important concepts used in the theory of dynamical systems. One of the such concepts is Lax pair that plays quite important role in construction of completely integrable models. Let us recall that the Lax pair of a Hamiltonian system on a Poisson manifold M is a pair (L,P) of smooth functions on M with values in some Lie algebra g such that the time evolution of L is given by the adjoint action

$$\frac{d}{dt}L = [L,P] = -ad_P L, \tag{37}$$

where $[\,,\,]$ is Lie bracket on g. It is well known that each Lax pair leads to a number of conservation laws. When g is some matrix Lie algebra, the conservation laws are just traces of powers of L

$$I^{(k)} = \frac{1}{2}Tr(L^k) \tag{38}$$

since the trace is invariant under coadjoint action

$$\frac{d}{dt}I^{(k)} = \frac{1}{2}\frac{d}{dt}Tr(L^k) = \frac{1}{2}Tr\left(\frac{d}{dt}L^k\right) = \frac{k}{2}Tr\left(L^{k-1}\frac{d}{dt}L\right) =$$
$$= \frac{k}{2}Tr(L^{k-1}[L,P]) = \frac{1}{2}Tr([L^k,P]) = 0.$$

It is remarkable that each generator of a non-Noether symmetry canonically leads to a Lax pair of a certain type. Such Lax pairs have definite geometric origin, their Lax matrices are formed by coefficients of invariant tangent valued 1-form on the phase space. In the local coordinates z_a, where the bivector field W, the symplectic form ω and the generator of the symmetry E have the following form

$$W = \sum_{ab} W_{ab}\frac{\partial}{\partial z_a}\wedge\frac{\partial}{\partial z_b}, \quad \omega = \sum_{ab}\omega_{ab}dz_a\wedge dz_b, \quad E = \sum_a E_a\frac{\partial}{\partial z_a},$$

the corresponding Lax pair can be calculated explicitly. Namely, we have the following theorem (see also [55]–[56]):

Theorem 3. *Let (M,h) be a regular Hamiltonian system on an $2n$-dimensional Poisson manifold M. Then, if the vector field E on M generates a non-Noether symmetry, the following $2n\times 2n$ matrix valued functions on M*

$$\begin{aligned} L_{ab} &= \sum_{dc}\omega_{ad}\left[E_c\frac{\partial W_{db}}{\partial z_c} - W_{bc}\frac{\partial E_d}{\partial z_c} + W_{dc}\frac{\partial E_b}{\partial z_c}\right], \\ P_{ab} &= \sum_c\left[\frac{\partial W_{bc}}{\partial z_a}\frac{\partial h}{\partial z_c} + W_{bc}\frac{\partial^2 h}{\partial z_a z_c}\right] \end{aligned} \tag{39}$$

form the Lax pair (37) *of the dynamical system* (M,h).

Proof. Let us consider the following operator on the space of 1-forms

$$\overline{R}_E(u) = \Phi_\omega([E,\Phi_W(u)]) - L_E u \tag{40}$$

(here Φ_W and Φ_ω are maps induced by the Poisson bivector field and the symplectic form). It is remarkable that $\overline{R}_E$ appears to be an invariant linear operator. First of all let us show that $\overline{R}_E$ is really linear, or in other words, that for arbitrary 1-forms u and v and function f the operator $\overline{R}_E$ has the following properties

$$\overline{R}_E(u+v) = \overline{R}_E(u) + \overline{R}_E(v)$$

and

$$\overline{R}_E(fu) = f\overline{R}_E(u).$$

The first property is an obvious consequence of linearity of the Schouten bracket, Lie derivative and the maps Φ_W, Φ_ω. The second property can be checked directly

$$\begin{aligned} \overline{R}_E(fu) &= \Phi_\omega([E,\Phi_W(fu)]) - L_E(fu) = \\ &= \Phi_\omega([E,f\Phi_W(u)]) - (L_E f)u - fL_E u = \end{aligned}$$

$$= \Phi_\omega((L_E f)\Phi_W(u)) + \Phi_\omega(f[E, \Phi_W(u)]) - (L_E f)u - fL_E u =$$
$$= L_E f \Phi_\omega \Phi_W(u) + f\Phi_\omega([E, \Phi_W(u)]) - (L_E f)u - fL_E u =$$
$$= f(\Phi_\omega([E, \Phi_W(u)]) - L_E u) = f\overline{R}_E(u)$$

as far as $\Phi_\omega \Phi_W(u) = u$. Now let us check that $\overline{R}_E$ is an invariant operator:

$$\frac{d}{dt}\overline{R}_E = L_{X_h}\overline{R}_E = L_{X_h}(\Phi_\omega L_E \Phi_W - L_E) =$$
$$= \Phi_\omega L_{[X_h,E]}\Phi_W - L_{[X_h,E]} = 0$$

because, being a Hamiltonian vector field, X_h commutes with the maps Φ_W, Φ_ω (this is a consequence of the Liouville theorem) and commutes with E as far as E generates the symmetry $[X_h, E] = 0$. In terms of the local coordinates $\overline{R}_E$ has the following form

$$\overline{R}_E = \sum_{ab} L_{ab} dz_a \otimes \frac{\partial}{\partial z_b}$$

and the invariance condition

$$\frac{d}{dt}\overline{R}_E = L_{W(h)}\overline{R}_E = 0$$

yields

$$\frac{d}{dt}\overline{R}_E = \frac{d}{dt}\sum_{ab} L_{ab} dz_a \otimes \frac{\partial}{\partial z_b} =$$
$$= \sum_{ab}\left[\frac{d}{dt}L_{ab}\right] dz_a \otimes \frac{\partial}{\partial z_b} + \sum_{ab} L_{ab}(L_{W(h)}dz_a) \otimes \frac{\partial}{\partial z_b} +$$
$$+ \sum_{ab} L_{ab} dz_a \otimes \left[L_{W(h)}\frac{\partial}{\partial z_b}\right] = \sum_{ab}\left[\frac{d}{dt}L_{ab}\right] dz_a \otimes \frac{\partial}{\partial z_b} +$$
$$+ \sum_{abcd} L_{ab}\frac{\partial W_{ad}}{\partial z_c}\frac{\partial h}{\partial z_d} dz_c \otimes \frac{\partial}{\partial z_b} +$$
$$+ \sum_{abcd} L_{ab} W_{ad}\frac{\partial^2 h}{\partial z_c \partial z_d} dz_c \otimes \frac{\partial}{\partial z_b} +$$
$$+ \sum_{abcd} L_{ab}\frac{\partial W_{cd}}{\partial z_b}\frac{\partial h}{\partial z_d} dz_a \otimes \frac{\partial}{\partial z_c} + \sum_{abcd} L_{ab} W_{cd}\frac{\partial^2 h}{\partial z_b \partial z_d} dz_a \otimes \frac{\partial}{\partial z_c} =$$
$$= \sum_{ab}\left[\frac{d}{dt}L_{ab} + \sum_c (P_{ac}L_{cb} - L_{ac}P_{cb})\right] dz_a \otimes \frac{\partial}{\partial z_b} = 0,$$

or in matrix notation

$$\frac{d}{dt}L = [L, P].$$

So we have proved that a non-Noether symmetry canonically yields a Lax pair on the algebra of linear operators on cotangent bundle over the phase space. □

Remark 5.1. The conservation laws (38) associated with the Lax pair (39) can be expressed in terms of the integrals of motion c_i in quite simple way:

$$I^{(k)} = \frac{1}{2} Tr(L^k) = \sum_i c_i^k. \tag{41}$$

This correspondence follows from the equation (18) and the definition of the operator $\overline{R}_E$ (40). One can also write down a recursion relation that determines the conservation laws $I^{(k)}$ in terms of the conservation laws $C^{(k)}$

$$I^{(m)} + (-1)^m m C^{(m)} + \sum_{k=1}^{m-1} (-1)^k I^{(m-k)} C^{(k)} = 0. \tag{42}$$

Example. Let us calculate the Lax matrix of two particle Toda chain associated with the non-Noether symmetry (27). Using (39) it is easy to check that Lax matrix has eight nonzero elements

$$L = \begin{pmatrix} z_1 & 0 & 0 & -e^{z_3 - z_4} \\ 0 & z_2 & e^{z_3 - z_4} & 0 \\ 0 & 1 & z_1 & 0 \\ -1 & 0 & 0 & z_2 \end{pmatrix}, \tag{43}$$

while the matrix P involved in Lax pair

$$\frac{d}{dt} L = [L, P]$$

has the following form

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -e^{z_3 - z_4} & e^{z_3 - z_4} & 0 & 0 \\ e^{z_3 - z_4} & -e^{z_3 - z_4} & 0 & 0 \end{pmatrix}. \tag{44}$$

The conservation laws associated with this Lax pair are total momentum and energy of the two particle Toda chain:

$$\begin{aligned} I^{(1)} &= \frac{1}{2} Tr(L) = z_1 + z_2 \\ I^{(2)} &= \frac{1}{2} Tr(L^2) = z_1^2 + z_2^2 + 2e^{z_3 - z_4}. \end{aligned} \tag{45}$$

Similarly one can construct the Lax matrix of three particle Toda chain. It has 16 nonzero elements

$$L = \begin{pmatrix} z_1 & 0 & 0 & 0 & -e^{z_4 - z_5} & 0 \\ 0 & z_2 & 0 & e^{z_4 - z_5} & 0 & -e^{z_5 - z_6} \\ 0 & 0 & z_3 & 0 & e^{z_5 - z_6} & 0 \\ 0 & -1 & -1 & z_1 & 0 & 0 \\ 1 & 0 & -1 & 0 & z_2 & 0 \\ 1 & 1 & 0 & 0 & 0 & z_3 \end{pmatrix} \tag{46}$$

with the matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -e^{z_4-z_5} & e^{z_4-z_5} & 0 & 0 & 0 & 0 \\ e^{z_4-z_5} & -e^{z_4-z_5}-e^{z_5-z_6} & e^{z_5-z_6} & 0 & 0 & 0 \\ 0 & e^{z_5-z_6} & -e^{z_5-z_6} & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

Corresponding conservation laws reproduce total momentum, energy and second Hamiltonian involved in bi-Hamiltonian realization of the Toda chain

$$\begin{aligned} I^{(1)} &= \frac{1}{2}Tr(L) = z_1 + z_2 \\ I^{(2)} &= \frac{1}{2}Tr(L^2) = z_1^2 + z_2^2 + z_3^2 + 2e^{z_4-z_5} + 2e^{z_5-z_6} \\ I^{(3)} &= \frac{1}{2}Tr(L^3) = z_1^3 + z_2^3 + z_3^3 + \\ &\quad + 3(z_1+z_2)e^{z_4-z_5} + 3(z_2+z_3)e^{z_5-z_6}. \end{aligned} \tag{48}$$

6. Involutivity of Conservation Laws

Now let us focus on the integrability issues. We know that n integrals of motion are associated with each generator of a non-Noether symmetry. At the same time we know that, according to the Liouville–Arnold theorem, a regular Hamiltonian system (M, h) on a $2n$-dimensional symplectic manifold M is completely integrable (can be solved completely) if it admits n functionally independent integrals of motion in involution. One can understand functional independence of a set of conservation laws $c_1, c_2, \ldots, c_n$ as linear independence of either differentials of conservation laws $dc_1, dc_2, \ldots, dc_n$ or corresponding Hamiltonian vector fields $X_{c_1}, X_{c_2}, \ldots, X_{c_n}$. Strictly speaking, we can say that conservation laws $c_1, c_2, \ldots, c_n$ are functionally independent if Lesbegue measure of the set of points of the phase space M, where the differentials $dc_1, dc_2, \ldots, dc_n$ become linearly dependent is zero. Involutivity of conservation laws means that all possible Poisson brackets of these conservation laws vanish pairwise

$$\{c_i, c_j\} = 0, \qquad i, j = 1, \ldots, n.$$

In terms of vector fields, the existence of involutive family of n functionally independent conservation laws $c_1, c_2, \ldots, c_n$ implies that the corresponding Hamiltonian vector fields $X_{c_1}, X_{c_2}, \ldots, X_{c_n}$ span the Lagrangian subspace (isotropic subspace of dimension n) of tangent space (at each point of M). Indeed, due to the property (10)

$$\{c_i, c_j\} = \omega(X_{c_i}, X_{c_j}) = 0,$$

thus the space spanned by $X_{c_1}, X_{c_2}, \ldots, X_{c_n}$ is isotropic. The dimension of this space is n so it is Lagrangian. Note also that the distribution

$X_{c_1}, X_{c_2}, \ldots, X_{c_n}$ is integrable since due to (9)

$$[X_{c_i}, X_{c_j}] = X_{\{c_i,c_j\}} = 0,$$

and according to the Frobenius theorem there exists a submanifold of M such that the distribution $X_{c_1}, X_{c_2}, \ldots, X_{c_n}$ spans the tangent space of this submanifold. Thus for the phase space geometry the existence of complete involutive set of integrals of motion implies the existence of an invariant Lagrangian submanifold.

Now let us look at the conservation laws $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}$ associated with a generator of a non-Noether symmetry. Generally speaking, these conservation laws might appear to be neither functionally independent nor involutive. However, it is reasonable to ask the question – what condition should be satisfied by the generator of a non-Noether symmetry to ensure the involutivity ($\{Y^{(k)}, Y^{(m)}\} = 0$) of conserved quantities? In Lax theory the situation is very similar — each Lax matrix leads to a set of conservation laws but in general this set is not involutive. However in Lax theory there is certain condition known as Classical Yang–Baxter Equation (CYBE) that being satisfied by the Lax matrix ensures that conservation laws are in involution. Since involutivity of conservation laws is closely related to integrability, it is essential to have some analogue of CYBE for the generator of a non-Noether symmetry. To address this issue, we would like to propose the following theorem.

Theorem 4. *If the vector field E on a $2n$-dimensional Poisson manifold M satisfies the condition*

$$[[E[E,W]]W] = 0 \tag{49}$$

and the bivector field W has maximal rank ($W^n \neq 0$), then the functions (16) *are in involution*

$$\{Y^{(k)}, Y^{(m)}\} = 0\,.$$

Proof. First of all let us note that the identity (5) satisfied by the Poisson bivector field W is responsible for the Liouville theorem

$$[W,W] = 0 \qquad \Leftrightarrow \qquad L_{W(f)}W = [W(f),W] = 0 \tag{50}$$

that follows from the graded Jacoby identity satisfied by the Schouten bracket. By taking Lie derivative of the expression (5) we obtain another useful identity

$$L_E[W,W] = [E[W,W]] = [[E,W]W] + [W[E,W]] = 2[\hat{W},W] = 0.$$

This identity gives rise to the following relation

$$[\hat{W},W] = 0 \qquad \Leftrightarrow \qquad [\hat{W}(f),W] = -[\hat{W},W(f)], \tag{51}$$

and finally the condition (49) ensures the third identity

$$[\hat{W},\hat{W}] = 0$$

yielding the Liouville theorem for $\hat{W}$:

$$[\hat{W},\hat{W}]=0 \qquad \Leftrightarrow \qquad [\hat{W}(f),\hat{W}]=0. \tag{52}$$

Indeed,

$$\begin{aligned}[\hat{W},\hat{W}]=[[E,W]\hat{W}]=[[\hat{W},E]W]=\\ =-[[E,\hat{W}]W]=-[[E[E,W]]W]=0.\end{aligned}$$

Now let us consider two different solutions $c_i \neq c_j$ of the equation (18). By taking Lie derivative of the equation

$$(\hat{W}-c_iW)^n=0$$

along the vector fields $W(c_j)$ and $\hat{W}(c_j)$ and using the Liouville theorem for the bivectors W and $\hat{W}$ we obtain the following relations

$$(\hat{W}-c_iW)^{n-1}(L_{W(c_j)}\hat{W}-\{c_j,c_i\}W)=0, \tag{53}$$

and

$$(\hat{W}-c_iW)^{n-1}(c_iL_{\hat{W}(c_j)}W+\{c_j,c_i\}_*W)=0, \tag{54}$$

where

$$\{c_i,c_j\}_*=\hat{W}(dc_i\wedge dc_j)$$

is the Poisson bracket calculated by means of the bivector field $\hat{W}$. Now multiplying (53) by c_i, subtracting (54) and using the identity (51) gives rise to

$$(\{c_i,c_j\}_*-c_i\{c_i,c_j\})(\hat{W}-c_iW)^{n-1}W=0. \tag{55}$$

Thus, either

$$\{c_i,c_j\}_*-c_i\{c_i,c_j\}=0 \tag{56}$$

or the volume field $(\hat{W}-c_iW)^{n-1}W$ vanishes. In the second case we can repeat the procedure (53)–(55) for the volume field $(\hat{W}-c_iW)^{n-1}W$ yielding after n iterations $W^n=0$, which according to our assumption (that the dynamical system is regular) is not true. As a result, we arrive at (56) and by the simple interchange of indices $i\leftrightarrow j$ we get

$$\{c_i,c_j\}_*-c_j\{c_i,c_j\}=0. \tag{57}$$

Finally by comparing (56) and (57) we obtain that the functions c_i are in involution with respect to both Poisson structures (since $c_i\neq c_j$)

$$\{c_i,c_j\}_*=\{c_i,c_j\}=0,$$

and according to (19) the same is true for the integrals of motion $Y^{(k)}$. □

Remark 6.1. Theorem 4 is useful in multidimensional dynamical systems where involutivity of conservation laws can not be checked directly.

7. Bi-Hamiltonian Systems

Further we will focus on non-Noether symmetries that satisfy the condition (49). Besides yielding involutive families of conservation laws, such symmetries appear to be related to many known geometric structures such as bi-Hamiltonian systems [53] and Frölicher–Nijenhuis operators (torsionless tangent valued differential 1-forms). The relationship between non-Noether symmetries and bi-Hamiltonian structures was already implicitly outlined in the proof of Theorem 4. Now let us pay more attention to this issue.

Originally bi-Hamiltonian structures were introduced by F. Magri in analisys of integrable infinite dimensional Hamiltonian systems such as Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) hierarchies, Nonlinear Schrödinger equation and Harry Dym equation. Since that time bi-Hamiltonian formalism is effectively used in construction of involutive families of conservation laws in integrable models.

A generic bi-Hamiltonian structure on $2n$ dimensional manifold consists of two Poisson bivector fields W and $\hat{W}$ satisfying certain compatibility condition $[\hat{W}, W] = 0$. If, in addition, one of these bivector fields is nondegenerate ($W^n \neq 0$), then the bi-Hamiltonian system is called regular. Further we will discuss only regular bi-Hamiltonian systems. Note that each Poisson bivector field by definition satisfies condition (5). So we actually impose four restrictions on the bivector fields W and $\hat{W}$:

$$[W, W] = [\hat{W}, W] = [\hat{W}, \hat{W}] = 0 \tag{58}$$

and

$$W^n \neq 0. \tag{59}$$

During the proof of Theorem 4 we already showed that the bivector fields W and $\hat{W} = [E, W]$ satisfy the conditions (58) (see (50)–(52)). Thus we can formulate the following statement.

Theorem 5. *Let (M, h) be a regular Hamiltonian system on a $2n$-dimensional manifold M endowed with a regular Poisson bivector field W. If a vector field E on M generates a non-Noether symmetry and satisfies the condition*

$$[[E[E, W]]W] = 0,$$

then the following bivector fields on M

$$W, \hat{W} = [E, W]$$

form an invariant bi-Hamiltonian system $([W, W] = [\hat{W}, W] = [\hat{W}, \hat{W}] = 0)$.

Proof. See proof of Theorem 4. □

Example. One can check that the non-Noether symmetry (27) satisfies the condition (49) while the bivector fields

$$W = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} + \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_4}$$

and

$$\hat{W} = [E, W] = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_4} + \\ + e^{z_3 - z_4} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4}$$

form a bi-Hamiltonian system $[W, W] = [W, \hat{W}] = [\hat{W}, \hat{W}] = 0$. Similarly, one can recover the bi-Hamiltonian system of three particle Toda chain associated with the symmetry (32). It is formed by the bivector fields

$$W = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_5} + \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_6}$$

and

$$\hat{W} = [E, W] = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_4} + z_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_5} + z_3 \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_6} + \\ + e^{z_4 - z_5} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + e^{z_5 - z_6} \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} + \\ + \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_4} \wedge \frac{\partial}{\partial z_5} + \frac{\partial}{\partial z_5} \wedge \frac{\partial}{\partial z_6}.$$

In terms of differential forms, a bi-Hamiltonian structure is formed by a couple of closed differential 2-forms: a symplectic form ω (such that $d\omega = 0$ and $\omega^n \neq 0$) and $\omega^* = L_E \omega$ (clearly $d\omega^* = dL_E \omega = L_E d\omega = 0$). It is important that by taking Lie derivative of Hamilton's equation

$$i_{X_h} \omega + dh = 0$$

along the generator E of the symmetry

$$L_E(i_{X_h} \omega + dh) = i_{[E, X_h]} \omega + i_{X_h} L_E \omega + L_E dh = i_{X_h} \omega^* + dL_E h = 0,$$

one obtains another Hamilton's equation

$$i_{X_h} \omega^* + dh^* = 0,$$

where $h^* = L_E h$. This is actually a second Hamiltonian realization of the equations of motion and thus under certain conditions the existence of a non-Noether symmetry gives rise to an additional presymplectic structure ω^* and an additional Hamiltonian realization of the dynamical system. In many integrable models admitting bi-Hamiltonian realization (including Toda chain, Korteweg–de Vries hierarchy, Nonlinear Schrödinger equation, Broer–Kaup system and Benney system), non-Noether symmetries that are responsible for existence of bi-Hamiltonian structures have been found and motivated further investigation of relationship between symmetries and bi-Hamiltonian structures. Namely, it seems to be interesting to know whether

in the general case the existence of a bi-Hamiltonian structure is related to a non-Noether symmetry. Let us consider more general case and suppose that we have a couple of differential 2-forms ω and ω^* such that

$$d\omega = d\omega^* = 0, \qquad \omega^n \neq 0,$$

$$i_{X_h}\omega + dh = 0$$

and

$$i_{X_h}\omega^* + dh^* = 0.$$

The question is whether there exists a vector field E (the generator of a non-Noether symmetry) such that $[E, X_h] = 0$ and $\omega^* = L_E\omega$.

The answer depends on ω^*. Namely, if ω^* is the exact form (there exists 1-form θ^* such that $\omega^* = d\theta^*$); then one can argue that such a vector field exists and thus any exact bi-Hamiltonian structure is related to a hidden non-Noether symmetry. To outline the proof of this statement, let us introduce the vector field E^* defined by

$$i_{E^*}\omega = \theta^*$$

(such a vector field always exist because ω is a nondegenerate 2-form). By construction

$$L_{E^*}\omega = \omega^*.$$

Indeed,

$$L_{E^*}\omega = di_{E^*}\omega + i_{E^*}d\omega = d\theta^* = \omega^*,$$

and

$$i_{[E^*,X_h]}\omega = L_{E^*}(i_{X_h}\omega) - i_{X_h}L_{E^*}\omega = -d(E^*(h) - h^*) = -dh'.$$

In other words, $[X_h, E^*]$ is a Hamiltonian vector field

$$[X_h, E] = X_{h'}.$$

One can also construct the locally Hamiltonian vector field X_g that satisfies the same commutation relation. Namely, let us define the function (in the general case this can be done only locally)

$$g(z) = \int_0^t h' dt$$

where integration along a solution of the Hamilton's equation, with the fixed origin and the end point in $z(t) = z$, is assumed. And then it is easy to verify that the locally Hamiltonian vector field associated with $g(z)$ by construction satisfies the same commutation relations as E^* (namely $[X_h, X_g] = X_{h'}$). Using E^* and $X_{h'}$, one can construct the generator of the non-Noether symmetry — the non-Hamiltonian vector field $E = E^* - X_g$ commuting with X_h and satisfying

$$L_E\omega = L_{E^*}\omega - L_{X_g}\omega = L_{E^*}\omega = \omega^*$$

(thanks to Liouville's theorem $L_{X_g}\omega = 0$). So in case of the regular Hamiltonian system every exact bi-Hamiltonian structure is naturally associated with some (non-Noether) symmetry of the space of solutions. In the case where bi-Hamiltonian structure is not exact (ω^* is closed but not exact), due to

$$\omega^* = L_E\omega = di_E\omega + i_E d\omega = di_E\omega$$

it is clear that such a bi-Hamiltonian system is not related to a symmetry. However, in all known cases bi-Hamiltonian structures seem to be exact.

8. Bidifferential Calculi

Another important concept that is often used in the theory of dynamical systems and may be related to the non-Noether symmetry is bidifferential calculus (bicomplex approach). Recently A. Dimakis and F. Müller-Hoissen applied bidifferential calculi to a wide range of integrable models including KdV hierarchy, KP equation, self-dual Yang-Mills equation, Sine-Gordon equation, Toda models, non-linear Schrödinger and Liouville equations. It turns out that these models can be effectively described and analyzed using the bidifferential calculi [17], [24]. Here we would like to show that each generator of non-Noether symmetry satisfying the condition $[[E[E,W]]W] = 0$ gives rise to certain bidifferential calculus.

Before we proceed let us specify what kind of bidifferential calculi we plan to consider. Under a bidifferential calculus we mean a graded algebra of differential forms over the phase space

$$\Omega = \bigcup_{k=0}^{\infty} \Omega^{(k)} \tag{60}$$

($\Omega^{(k)}$ denotes the space of k-degree differential forms) equipped with a couple of differential operators

$$d, \bar{d} : \Omega^{(k)} \to \Omega^{(k+1)} \tag{61}$$

satisfying the conditions $d^2 = \bar{d}^2 = d\bar{d} + \bar{d}d = 0$ (see [24]). In other words, we have two De Rham complexes M, Ω, d and $M, \Omega, \bar{d}$ on the algebra of differential forms over the phase space. And these complexes satisfy certain compatibility condition — their differentials anticommute with each other $d\bar{d} + \bar{d}d = 0$. Now let us focus on non-Noether symmetries. It is interesting that if the generator of a non-Noether symmetry satisfies the equation $[[E[E,W]]W] = 0$, then we are able to construct an invariant bidifferential calculus of a certain type. This construction is summarized in the following theorem:

Theorem 6. *Let (M,h) be a regular Hamiltonian system on a Poisson manifold M. Then, if a vector field E on M generates a non-Noether symmetry and satisfies the equation*

$$[[E[E,W]]W] = 0,$$

the differential operators

$$du = \Phi_\omega([W, \Phi_W(u)]), \tag{62}$$

$$\bar{d}u = \Phi_\omega([[E, W]\Phi_W(u)]) \tag{63}$$

form an invariant bidifferential calculus ($d^2 = \bar{d}^2 = d\bar{d} + \bar{d}d = 0$) *over the graded algebra of differential forms on* M.

Proof. First of all we have to show that d and $\bar{d}$ are really differential operators , i.e., they are linear maps from $\Omega^{(k)}$ into $\Omega^{(k+1)}$, satisfy the derivation property and are nilpotent ($d^2 = \bar{d}^2 = 0$). Linearity is obvious and follows from the linearity of the Schouten bracket $[,]$ and the maps Φ_W, Φ_ω. Then, if u is a k-degree form, Φ_W maps it on a k-degree multivector field and the Schouten brackets $[W, \Phi_W(u)]$ and $[[E, W]\Phi_W(u)]$ result the $k+1$-degree multivector fields that are mapped on $k+1$-degree differential forms by Φ_ω. So, d and $\bar{d}$ are linear maps from $\Omega^{(k)}$ into $\Omega^{(k+1)}$. The derivation property follows from the same feature of the Schouten bracket $[,]$ and linearity of the maps Φ_W and Φ_ω. Now we have to prove the nilpotency of d and $\bar{d}$. Let us consider d^2u:

$$d^2u = \Phi_\omega([W, \Phi_W(\Phi_\omega([W, \Phi_W(u)]))]) = \Phi_\omega([W[W, \Phi_W(u)]]) = 0$$

as a result of the property (50) and the Jacoby identity for the bracket $[,]$. In the same manner

$$\bar{d}^2u = \Phi_\omega([[W, E][[W, E]\Phi_W(u)]]) = 0$$

according to the property (52) of $[W, E] = \hat{W}$ and the Jacoby identity. Thus we have proved that d and $\bar{d}$ are differential operators (in fact d is an ordinary exterior differential and the expression (62) is its well-known representation in terms of Poisson bivector field). It remains to show that the compatibility condition $d\bar{d} + \bar{d}d = 0$ is fulfilled. Using the definitions of $d, \bar{d}$ and the Jacoby identity, we get

$$(d\bar{d} + \bar{d}d)(u) = \Phi_\omega([[[W, E]W]\Phi_W(u)]) = 0$$

as far as (51) is satisfied. So, d and $\bar{d}$ form a bidifferential calculus over the graded algebra of differential forms. It is also clear that the bidifferential calculus $d, \bar{d}$ is invariant, since both d and $\bar{d}$ commute with time evolution operator $W(h) = \{h, \}$. □

Remark 8.1. Conservation laws that are associated with the bidifferential calculus (62) (63) and form the Lenard scheme (see [24])

$$(k+1)\bar{d}I^{(k)} = kdI^{(k+1)}$$

coincide with the sequence of integrals of motion (41). Proof of this correspondence lays outside the scope of the present memoir, but can be done in the manner similar to [17].

Example. The symmetry (27) endows R^4 with the bicomplex structure $d, \bar{d}$, where d is the ordinary exterior derivative while $\bar{d}$ is defined by

$$\begin{aligned} \bar{d}z_1 &= z_1 dz_1 - e^{z_3 - z_4} dz_4 \,, \\ \bar{d}z_2 &= z_2 dz_2 + e^{z_3 - z_4} dz_3 \,, \\ \bar{d}z_3 &= z_1 dz_3 + dz_2 \,, \\ \bar{d}z_4 &= z_2 dz_4 - dz_1 \end{aligned} \tag{64}$$

and is extended to whole De Rham complex by linearity, derivation property and the compatibility property $d\bar{d} + \bar{d}d = 0$. By direct calculations one can verify that the calculus constructed in this way is consistent and satisfies $\bar{d}^2 = 0$. To illustrate the technique, let us explicitly check that $\bar{d}^2 z_1 = 0$.

Indeed,

$$\begin{aligned} \bar{d}^2 z_1 = \bar{d}\bar{d}z_1 &= \bar{d}(z_1 dz_1 - e^{z_3 - z_4} dz_4) = \\ &= \bar{d}z_1 \wedge dz_1 + z_1 \bar{d}dz_1 - e^{z_3 - z_4} \bar{d}z_3 \wedge dz_4 + \\ &\quad + e^{z_3 - z_4} \bar{d}z_4 \wedge dz_4 - e^{z_3 - z_4} \bar{d}dz_4 = \\ &= \bar{d}z_1 \wedge dz_1 - z_1 d\bar{d}z_1 - e^{z_3 - z_4} \bar{d}z_3 \wedge dz_4 + \\ &\quad + e^{z_3 - z_4} \bar{d}z_4 \wedge dz_4 + e^{z_3 - z_4} d\bar{d}z_4 = 0 \end{aligned}$$

because of the properties

$$\begin{aligned} \bar{d}z_1 \wedge dz_1 &= e^{z_3 - z_4} dz_1 \wedge dz_4, \\ -z_1 d\bar{d}z_1 &= z_1 e^{z_3 - z_4} dz_3 \wedge dz_4, \\ -e^{z_3 - z_4} \bar{d}z_3 \wedge dz_4 &= -z_1 e^{z_3 - z_4} dz_1 \wedge dz_4 - e^{z_3 - z_4} dz_2 \wedge dz_4, \\ e^{z_3 - z_4} \bar{d}z_4 \wedge dz_4 &= e^{z_3 - z_4} dz_2 \wedge dz_4 \end{aligned}$$

and

$$e^{z_3 - z_4} d\bar{d}z_4 = -e^{z_3 - z_4} dz_1 \wedge dz_4.$$

Similarly one can show that

$$\bar{d}^2 z_2 = \bar{d}^2 z_3 = \bar{d}^2 z_4 = 0$$

and thus $\bar{d}$ is a nilpotent operator: $\bar{d}^2 = 0$. Note also that the conservation laws

$$\begin{aligned} I^{(1)} &= z_1 + z_2 \\ I^{(2)} &= z_1^2 + z_2^2 + 2e^{z_3 - z_4} \end{aligned}$$

form the simplest Lenard scheme

$$2\bar{d}I^{(1)} = dI^{(2)}.$$

Similarly one can construct the bidifferential calculus associated with the non-Noether symmetry (32) of three particle Toda chain. In this case $\bar{d}$ can

be defined by

$$\begin{aligned}
\bar{d}z_1 &= z_1 dz_1 - e^{z_4 - z_5} dz_5 \,, \\
\bar{d}z_2 &= z_2 dz_2 + e^{z_4 - z_5} dz_4 - e^{z_5 - z_6} dz_6 \,, \\
\bar{d}z_3 &= z_3 dz_3 + e^{z_5 - z_6} dz_5 \,, \\
\bar{d}z_4 &= z_1 dz_4 - dz_2 - dz_3 \,, \\
\bar{d}z_5 &= z_2 dz_5 + dz_1 - dz_3 \,, \\
\bar{d}z_6 &= z_3 dz_6 + dz_1 + dz_2,
\end{aligned} \tag{65}$$

and as in the case of two particle Toda it can be extended to the whole De Rham complex by linearity, derivation property and the compatibility property $d\bar{d} + \bar{d}d = 0$. One can check that the conservation laws of Toda chain

$$\begin{aligned}
I^{(1)} &= z_1 + z_2 \\
I^{(2)} &= z_1^2 + z_2^2 + z_3^2 + 2e^{z_4 - z_5} + 2e^{z_5 - z_6} \\
I^{(3)} &= z_1^3 + z_2^3 + z_3^3 + 3(z_1 + z_2)e^{z_4 - z_5} + 3(z_2 + z_3)e^{z_5 - z_6}
\end{aligned}$$

form the Lenard scheme

$$\begin{aligned}
2\bar{d}I^{(1)} &= dI^{(2)} \\
3\bar{d}I^{(2)} &= 2dI^{(3)}.
\end{aligned}$$

9. Frölicher–Nijenhuis Geometry

Finally we would like to reveal some features of the operator $\overline{R}_E$ (40) and to show how the Frölicher–Nijenhuis geometry arises in a Hamiltonian system that possesses certain non-Noether symmetry. From the geometric properties of the tangent valued forms we know that the traces of powers of a linear operator F on the tangent bundle are in involution whenever its Frölicher–Nijenhuis torsion $T(F)$ vanishes, i.e. whenever for arbitrary vector fields X, Y the condition

$$T(F)(X,Y) = [FX, FY] - F([FX, Y] + [X, FY] - F[X, Y]) = 0$$

is satisfied. Torsionless forms are also called Frölicher–Nijenhuis operators and are widely used in the theory of integrable models, where they play the role of recursion operators and are used in construction of involutive families of conservation laws. We would like to show that each generator of non-Noether symmetry satisfying the equation $[[E[E, W]]W] = 0$ canonically leads to an invariant Frölicher–Nijenhuis operator on the tangent bundle over the phase space. This operator can be expressed in terms of the generator of symmetry and the isomorphism defined by the Poisson bivector field. Strictly speaking, we have the following theorem.

Theorem 7. *Let (M, h) be a regular Hamiltonian system on the Poisson manifold M. If the vector field E on M generates a non-Noether symmetry*

and satisfies the equation

$$[[E[E,W]]W] = 0,$$

then the linear operator defined for every vector field X by the equation

$$R_E(X) = \Phi_W(L_E\Phi_\omega(X)) - [E,X] \tag{66}$$

is an invariant Frölicher–Nijenhuis operator on M.

Proof. Invariance of R_E follows from the invariance of $\overline{R}_E$ defined by (40) (note that for an arbitrary 1-form vector field u and a vector field X the contraction $i_X u$ has the property $i_{R_E X}u = i_X \overline{R}_E u$, so R_E is actually transposed to $\overline{R}_E$). It remains to show that the condition (49) ensures vanishing of the Frölicher–Nijenhuis torsion $T(R_E)$ of R_E, i.e. for arbitrary vector fields X, Y we must get

$$\begin{aligned} T(R_E)(X,Y) = [R_E(X), R_E(Y)] - R_E([R_E(X),Y]+ \\ + [X, R_E(Y)] - R_E([X,Y])) = 0. \end{aligned} \tag{67}$$

First let us introduce the following auxiliary 2-forms

$$\omega = \Phi_\omega(W), \qquad \omega^* = \overline{R}_E\omega, \qquad \omega^{**} = \overline{R}_E\omega^*. \tag{68}$$

Using the realization (62) of the differential d and the property (5) yields

$$d\omega = \Phi_\omega([W,W]) = 0.$$

Similarly, using the property (51) we obtain

$$d\omega^* = d\Phi_\omega([E,W]) - dL_E\omega = \Phi_\omega([[E,W]W]) - L_E d\omega = 0.$$

And finally, taking into account that $\omega^* = 2\Phi_\omega([E,W])$ and using the condition (49), we get

$$d\omega^{**} = 2\Phi_\omega([[E[E,W]]W]) - 2dL_E\omega^* = -2L_E d\omega^* = 0.$$

So the differential forms $\omega, \omega^*, \omega^{**}$ are closed

$$d\omega = d\omega^* = d\omega^{**} = 0. \tag{69}$$

Now let us consider the contraction of $T(R_E)$ and ω.

$$\begin{aligned} i_{T(R_E)(X,Y)}\omega &= i_{[R_E X, R_E Y]}\omega - i_{[R_E X, Y]}\omega^* - i_{[X, R_E Y]}\omega^* + i_{[X,Y]}\omega^{**} = \\ &= L_{R_E X} i_Y \omega^* - i_{R_E Y} L_X \omega^* - L_{R_E X} i_Y \omega^* + \\ &\quad + i_Y L_{R_E X}\omega^* - L_X i_{R_E Y}\omega^* + i_{R_E Y} L_X \omega^* + i_{[X,Y]}\omega^{**} = \\ &= i_Y L_X \omega^{**} - L_X i_Y \omega^{**} + i_{[X,Y]}\omega^{**} = 0, \end{aligned} \tag{70}$$

where we have used (68) (69), the property

$$L_X i_Y \omega = i_Y L_X \omega + i_{[X,Y]}\omega$$

of the Lie derivative and the relations of the following type

$$\begin{aligned} L_{R_E X}\omega &= d i_{R_E X}\omega + i_{R_E X} d\omega = d i_X \omega^* = \\ &= L_X \omega^* - i_X d\omega^* = L_X \omega^*. \end{aligned}$$

So we have proved that for arbitrary vector fields X, Y the contraction of $T(R_E)(X,Y)$ and ω vanishes. But since the bivector W is non-degenerate ($W^n \neq 0$), its counter image

$$\omega = \Phi_\omega(W)$$

is also non-degenerate and the vanishing of the contraction (70) implies that the torsion $T(R_E)$ itself is zero. So we get

$$\begin{aligned} T(R_E)(X,Y) = [R_E(X), R_E(Y)] - R_E([R_E(X), Y] + \\ + [X, R_E(Y)] - R_E([X,Y])) = 0. \end{aligned} \qquad \square$$

Example. Note that the operator R_E associated with the non-Noether symmetry (27) reproduces the well-known Frölicher–Nijenhuis operator

$$\begin{aligned} R_E = z_1 dz_1 \otimes \frac{\partial}{\partial z_1} - dz_1 \otimes \frac{\partial}{\partial z_4} + z_2 dz_2 \otimes \frac{\partial}{\partial z_2} + dz_2 \otimes \frac{\partial}{\partial z_3} + \\ + z_1 dz_3 \otimes \frac{\partial}{\partial z_3} + e^{z_3 - z_4} dz_3 \otimes \frac{\partial}{\partial z_2} + z_2 dz_4 \otimes \frac{\partial}{\partial z_4} - e^{z_3 - z_4} dz_4 \otimes \frac{\partial}{\partial z_1} \end{aligned}$$

(compare with [30]). Note that the operator $\overline{R}_E$ plays the role of recursion operator for the conservation laws

$$\begin{aligned} I^{(1)} &= z_1 + z_2 \\ I^{(2)} &= z_1^2 + z_2^2 + 2e^{z_3 - z_4}. \end{aligned}$$

Indeed, one can check that

$$2\overline{R}_E(dI^{(1)}) = dI^{(2)}.$$

Similarly using the non-Noether symmetry (32), one can construct the recursion operator of three particle Toda chain

$$\begin{aligned} R_E = z_1 dz_1 \otimes \frac{\partial}{\partial z_1} - e^{z_4 - z_5} dz_5 \otimes \frac{\partial}{\partial z_1} + \\ + z_2 dz_2 \otimes \frac{\partial}{\partial z_2} + e^{z_4 - z_5} dz_4 \otimes \frac{\partial}{\partial z_2} - \\ - e^{z_5 - z_6} dz_6 \otimes \frac{\partial}{\partial z_2} + z_3 dz_3 \otimes \frac{\partial}{\partial z_3} + e^{z_5 - z_6} dz_5 \otimes \frac{\partial}{\partial z_3} + \\ + z_1 dz_4 \otimes \frac{\partial}{\partial z_4} - dz_2 \otimes \frac{\partial}{\partial z_4} - dz_3 \otimes \frac{\partial}{\partial z_4} + \\ + z_2 dz_5 \otimes \frac{\partial}{\partial z_5} + dz_1 \otimes \frac{\partial}{\partial z_5} - dz_3 \otimes \frac{\partial}{\partial z_5} + \\ + z_3 dz_6 \otimes \frac{\partial}{\partial z_6} + dz_1 \otimes \frac{\partial}{\partial z_6} + dz_2 \otimes \frac{\partial}{\partial z_6}, \end{aligned}$$

and as in case of two particle Toda chain, the operator $\overline{R}_E$ appears to be the recursion operator for the conservation laws

$$\begin{aligned} I^{(1)} &= z_1 + z_2, \\ I^{(2)} &= z_1^2 + z_2^2 + z_3^2 + 2e^{z_4 - z_5} + 2e^{z_5 - z_6}, \\ I^{(3)} &= z_1^3 + z_2^3 + z_3^3 + +3(z_1 + z_2)e^{z_4 - z_5} + 3(z_2 + z_3)e^{z_5 - z_6} \end{aligned}$$

and fulfills the following recursion condition

$$dI^{(3)} = 3\overline{R}_E(dI^{(2)}) = 6(\overline{R}_E)^2(dI^{(1)}).$$

10. One-Parameter Families of Conservation Laws

The one-parameter group of transformations g_a defined by (12) naturally acts on the algebra of integrals of motion. Namely, for each conservation law

$$\frac{d}{dt}J = 0$$

one can define a one-parameter family of conserved quantities $J(a)$ by applying the group of transformations g_a to J

$$J(a) = g_a(J) = e^{aL_E}J = J + aL_E J + \frac{1}{2}(aL_E)^2 J + \cdots .$$

The property (13) ensures that $J(a)$ is conserved for arbitrary values of the parameter a

$$\frac{d}{dt}J(a) = \frac{d}{dt}g_a(J) = g_a\left(\frac{d}{dt}J\right) = 0$$

and thus each conservation law gives rise to whole family of conserved quantities that form an orbit of the group of transformations g_a.

Such an orbit $J(a)$ is called involutive if the conservation laws that form it are in involution

$$\{J(a), J(b)\} = 0$$

(for arbitrary values of the parameters a, b). On a $2n$-dimensional symplectic manifold each involutive family that contains n functionally independent integrals of motion naturally gives rise to an integrable system (due to Liouville–Arnold theorem). So in order to identify those orbits that may be related to integrable models it is important to know how the involutivity of the family of conserved quantities $J(a)$ is related to properties of the initial conserved quantity $J(0) = J$ and the nature of the generator E of the group $g_a = e^{aL_E}$. In other words, we would like to know what condition must be satisfied by the generator of symmetry E and the integral of motion J to ensure that $\{J(a), J(b)\} = 0$. To address this issue and to describe the class of vector fields that possess nontrivial involutive orbits, we would like to propose the following theorem

Theorem 8. *Let M be a Poisson manifold endowed with a 1-form s such that*

$$[W[W(s),W](s)] = c_0[W(s)[W(s),W]] \qquad (c_0 \neq -1). \tag{71}$$

Then each function J satisfying the property

$$W(L_{W(s)}dJ) = c_1[W(s),W](dJ) \qquad (c_1 \neq 0) \tag{72}$$

($c_{0,1}$ are some constants) gives rise to an involutive set of functions

$$J^{(m)} = (L_{W(s)})^m J, \qquad \{J^{(m)}, J^{(k)}\} = 0. \tag{73}$$

Proof. First let us introduce a linear operator R on bundle of multivector fields and define it for arbitrary multivector field V by the condition

$$R(V) = \frac{1}{2}([W(s),V] - \Phi_W(L_{W(s)}\Phi_\omega(V))). \tag{74}$$

The proof of linearity of this operator is identical to the proof given for (40), so we will skip it. Further it is clear that

$$R(W) = [W(s),W] \tag{75}$$

and

$$\begin{aligned} R^2(W) = R([W(s),W]) &= \frac{1}{2}([W(s)[W(s),W]] - \Phi_W((L_{W(s)})^2\omega)) = \\ &= \frac{1+c_0}{2}[W(s)[W(s),W]], \end{aligned} \tag{76}$$

where we have used the property

$$\begin{aligned} \Phi_W((L_{W(s)})^2\omega) &= \Phi_W(L_{W(s)}L_{W(s)}\omega) = \\ &= \Phi_W(i_{W(s)}dL_{W(s)}\omega) + \Phi_W(di_{W(s)}L_{W(s)}\omega) = \\ &= [W, \Phi_W(i_{W(s)}L_{W(s)}\omega)] = [W[W(s),W](s)] = \\ &= c_0[W(s)[W(s),W]]. \end{aligned}$$

At the same time, by taking Lie derivative of (75) along the vector field $W(s)$ one gets

$$[W[W(s),W](s)] = (L_{W(s)}R + R^2)(W). \tag{77}$$

Comparing (76) and (77) yields

$$(1+c_0)(L_{W(s)}R + R^2) = 2R^2,$$

and thus

$$L_{W(s)}R = \frac{1-c_0}{1+c_0}R^2. \tag{78}$$

Further let us rewrite the condition (72) as follows

$$W(L_{W(s)}dJ) = c_1 R(W)(dJ). \tag{79}$$

Due to linearity of the operator R this condition can be extended to

$$R^m(W)(L_{W(s)}dJ) = c_1 R^{m+1}(W)(dJ). \tag{80}$$

Now assuming that the following condition is true

$$W((L_{W(s)})^m dJ) = c_m R^m(W)(dJ), \tag{81}$$

let us take its Lie derivative along the vector field $W(s)$. We get

$$\begin{aligned} R(W)((L_{W(s)})^m dJ) + W((L_{W(s)})^{m+1} dJ) = \\ = mc_m \frac{1-c_0}{1+c_0} R^{m+1}(W)(dJ) + c_m R^{m+1}(W)(dJ), \end{aligned} \tag{82}$$

where we have used the properties (75) and (78). Note also that (81) together with linearity of the operator R implies that

$$R^k W((L_{W(s)})^m dJ) = c_m R^{k+m}(W)(dJ), \tag{83}$$

and thus (82) reduces to

$$W((L_{W(s)})^{m+1} dJ) = c_{m+1} R^{m+1}(W)(dJ), \tag{84}$$

where c_{m+1} is defined by

$$c_{m+1} = mc_n \frac{1-c_0}{1+c_0}\,.$$

So we have proved that if the the assumption (81) is valid for m, then it is also valid for $m+1$. We also know that for $m=1$ it matches (79) and thus by induction we have proved that the condition (81) is valid for arbitrary m while c_m can be determined by

$$c_m = c_0(m-1)! \left[\frac{1-c_0}{1+c_0}\right]^{m-1}.$$

Now using (81) and (83) it is easy to show that the functions $(L_{W(s)})^m J$ are in involution. Indeed,

$$\begin{aligned} \{(L_{W(s)})^m J, (L_{W(s)})^k J\} = W(d(L_{W(s)})^m J \wedge d(L_{W(s)})^k J) = \\ = W((L_{W(s)})^m dJ \wedge (L_{W(s)})^k dJ) = c_m c_k W(dJ \wedge dJ) = 0. \end{aligned}$$

So we have proved the functions (73) are in involution.

Further we will use this theorem to prove involutivity of a family of conservation laws constructed using a non-Noether symmetry of Toda chain. □

11. Toda Model

To illustrate features of non-Noether symmetries we often refer to two and three particle non-periodic Toda systems. However, it turns out that non-Noether symmetries are present in generic n-particle non-periodic Toda chains as well. Moreover, they preserve basic features of the symmetries (27), (32). In case of n-particle Toda model the symmetry yields n functionally independent conservation laws in involution, gives rise to bi-Hamiltonian structure of Toda hierarchy, reproduces the Lax pair of Toda system, endows the phase space with a Frölicher–Nijenhuis operator and leads to an

invariant bidifferential calculus on the algebra of differential forms over the phase space of Toda system.

First of all let us remind that Toda model is a $2n$-dimensional Hamiltonian system that describes the motion of n particles on the line governed by the exponential interaction. Equations of motion of the non periodic n-particle Toda model are

$$\begin{aligned} &\frac{d}{dt}q_i = p_i, \\ &\frac{d}{dt}p_i = \epsilon(i-1)e^{q_{i-1}-q_i} - \epsilon(n-i)e^{q_i-q_{i+1}} \end{aligned} \tag{85}$$

($\epsilon(k) = -\epsilon(-k) = 1$ for any natural k and $\epsilon(0) = 0$) and can be rewritten in the Hamiltonian form (11) with the canonical Poisson bracket defined by

$$W = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

and Hamiltonian equal to

$$h = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i-q_{i+1}}.$$

Note that in two and three particle case we have used slightly different notation

$$z_i = p_i \qquad z_{n+i} = q_i \qquad i = 1, 2, (3); \quad n = 2(3)$$

for the local coordinates. The group of transformations g_a generated by the vector field E will be a symmetry of Toda chain if for each p_i, q_i satisfying the Toda equations (85) $g_a(p_i), g_a(q_i)$ also satisfy it. Substituting the infinitesimal transformations

$$\begin{aligned} g_a(p_i) &= p_i + aE(p_i) + O(a^2), \\ g_a(p_i) &= q_i + aE(q_i) + O(a^2) \end{aligned}$$

into (85) and grouping first order terms gives rise to the conditions

$$\begin{aligned} &\frac{d}{dt}E(q_i) = E(p_i)\,, \\ &\frac{d}{dt}E(p_i) = \epsilon(i-1)e^{q_{i-1}-q_i}(E(q_{i-1}) - E(q_i)) - \\ &\qquad\qquad - \epsilon(n-i)e^{q_i-q_{i+1}}(E(q_i) - E(q_{i+1})). \end{aligned} \tag{86}$$

One can verify that the vector field defined by

$$
\begin{aligned}
E(p_i) &= \frac{1}{2}p_i^2 + \epsilon(i-1)(n-i+2)e^{q_{i-1}-q_i} - \epsilon(n-i)(n-i)e^{q_i-q_{i+1}} + \\
&+ \frac{t}{2}(\epsilon(i-1)(p_{i-1}+p_i)e^{q_{i-1}-q_i} - \\
&- \epsilon(n-i)(p_i+p_{i+1})e^{q_i-q_{i+1}}), \\
E(q_i) &= (n-i+1)p_i - \frac{1}{2}\sum_{k=1}^{i-1} p_k + \frac{1}{2}\sum_{k=i+1}^{n} p_k + \\
&+ \frac{t}{2}(p_i^2 + \epsilon(i-1)e^{q_{i-1}-q_i} + \epsilon(n-i)e^{q_i-q_{i+1}})
\end{aligned}
\tag{87}
$$

satisfies (15) and generates a symmetry of Toda chain. It appears that this symmetry is non-Noether since it does not preserve the Poisson bracket structure $[E,W] \neq 0$ and additionally one can check that Yang–Baxter equation $[[E[E,W]]W] = 0$ is satisfied. This symmetry may play important role in analysis of Toda model. First let us note that calculating $L_E W$ leads to the following Poisson bivector field

$$
\begin{aligned}
\hat{W} = [E,W] &= \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i=1}^{n-1} e^{q_i-q_{i+1}} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{i+1}} + \\
&+ \sum_{j>i} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_j},
\end{aligned}
$$

and W and $L_E W$ together give rise to a bi-Hamiltonian structure of Toda model (compare with [30]). Thus the bi-Hamiltonian realization of Toda chain can be considered as manifestation of hidden symmetry. The conservation laws (22) associated with the symmetry reproduce the well-known set of conservation laws of Toda chain.

$$
\begin{aligned}
I^{(1)} &= C^{(1)} = \sum_{i=1}^{n} p_i, \\
I^{(2)} &= (C^{(1)})^2 - 2C^{(2)} = \sum_{i=1}^{n} p_i^2 + 2\sum_{i=1}^{n-1} e^{q_i-q_{i+1}}, \\
I^{(3)} &= C^{(1)})^3 - 3C^{(1)}C^{(2)} + 3C^{(3)} = \\
&= \sum_{i=1}^{n} p_i^3 + 3\sum_{i=1}^{n-1}(p_i+p_{i+1})e^{q_i-q_{i+1}}, \\
I^{(4)} &= C^{(1)})^4 - 4(C^{(1)})^2C^{(2)} + 2(C^{(2)})^2 + 4C^{(1)}C^{(3)} - 4C^{(4)} = \\
&= \sum_{i=1}^{n} p_i^4 + 4\sum_{i=1}^{n-1}(p_i^2 + 2p_ip_{i+1} + p_{i+1}^2)e^{q_i-q_{i+1}} +
\end{aligned}
\tag{88}
$$

$$+ 2\sum_{i=1}^{n-1} e^{2(q_i - q_{i+1})} + 4\sum_{i=1}^{n-2} e^{q_i - q_{i+2}},$$

$$I^{(m)} = (-1)^{m+1} m C^{(m)} + \sum_{k=1}^{m-1} (-1)^{k+1} I^{(m-k)} C^{(k)}.$$

The condition $[[E[E,W]]W] = 0$ satisfied by the generator of symmetry E ensures that the conservation laws are in involution, i.e. $\{C^{(k)}, C^{(m)}\} = 0$. Thus the conservation laws as well as the bi-Hamiltonian structure of the non-periodic Toda chain appear to be associated with a non-Noether symmetry.

Using the formula (39) one can calculate the Lax pair associated with the symmetry (87). The Lax matrix calculated in this way has the following non-zero entries (note that in case $n = 2$ and $n = 3$ this formula yields the matrices (44)-(47))

$$\begin{aligned} &L_{k,k} = L_{n+k,n+k} = p_k, \\ &L_{n+k,k+1} = -L_{n+k+1,k} = \epsilon(n-k)e^{q_k - q_{k+1}}, \\ &L_{k,n+m} = \epsilon(m-k), \\ &m, k = 1, 2, \ldots, n, \end{aligned} \tag{89}$$

while the non-zero entries of matrix P involved in the Lax pair are

$$\begin{aligned} &P_{k,n+k} = 1, \\ &P_{n+k,k} = -\epsilon(k-1)e^{q_{k-1} - q_k} - \epsilon(n-k)e^{q_k - q_{k+1}}, \\ &P_{n+k,k+1} = \epsilon(n-k)e^{q_k - q_{k+1}}, \\ &P_{n+k,k-1} = \epsilon(k-1)e^{q_{k-1} - q_k}, \\ &k = 1, 2, \ldots, n. \end{aligned} \tag{90}$$

This Lax pair constructed from the generator of non-Noether symmetry exactly reproduces the known Lax pair of Toda chain.

Like two and three particle Toda chain, n-particle Toda model also admits an invariant bidifferential calculus on the algebra of differential forms over the phase space. This bidifferential calculus can be constructed using a non-Noether symmetry (see (63)). It consists of two differential operators $d, \bar{d}$, where d is the ordinary exterior derivative while $\bar{d}$ can be defined by

$$\begin{aligned} &\bar{d}q_i = p_i dq_i + \sum_{j>i} dp_j - \sum_{i>j} dp_j, \\ &\bar{d}p_i = p_i dp_i - e^{q_i - q_{i+1}} dq_{i+1} + e^{q_{i-1} - q_i} dq_i \end{aligned} \tag{91}$$

and is extended to the whole De Rham complex by linearity, derivation property and compatibility property $d\bar{d} + \bar{d}d = 0$. By direct calculations one can verify that calculus constructed in this way is consistent and satisfies property $\bar{d}^2 = 0$. One can also check that the conservation laws (88) form

the Lenard scheme

$$(k+1)\bar{d}I^{(k)} = kdI^{(k+1)}.$$

Further let us focus on the Frölicher–Nijenhuis geometry. Using the formula (66) one can construct an invariant Frölicher–Nijenhuis operator out of the generator of non-Noether symmetry of Toda chain. The operator constructed in this way has the form

$$\overline{R}_E = \sum_{i=1}^{n} p_i \left[dp_i \otimes \frac{\partial}{\partial q_i} + dq_i \otimes \frac{\partial}{\partial p_i} \right] - \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} dq_{i+1} \otimes \frac{\partial}{\partial p_i} + \\ + \sum_{i=1}^{n-1} e^{q_{i-1} - q_i} dq_i \otimes \frac{\partial}{\partial p_i} - \sum_{j>i} \left[dp_i \otimes \frac{\partial}{\partial q_j} - dp_j \otimes \frac{\partial}{\partial q_i} \right]. \quad (92)$$

One can check that the Frölicher–Nijenhuis torsion of this operator vanishes and it plays the role of recursion operator for n-particle Toda chain in sense that conservation laws $I^{(k)}$ satisfy the recursion relation

$$(k+1)R_E(dI^{(k)}) = kdI^{(k+1)}. \quad (93)$$

Thus a non-Noether symmetry of Toda chain not only leads to n functionally independent conservation laws in involution, but also essentially enriches the phase space geometry by endowing it with an invariant Frölicher–Nijenhuis operator, a bi-Hamiltonian system, a bicomplex structure and a Lax pair.

Finally, in order to outline possible applications of Theorem 8 let us study the action of the non-Noether symmetry (87) on conserved quantities of Toda chain. The vector field E defined by (87) generates the one-parameter group of transformations (12) that maps an arbitrary conserved quantity J to

$$J(a) = J + aJ^{(1)} + \frac{a^2}{2!}J^{(2)} + \frac{a^3}{3!}J^{(3)} + \cdots,$$

where

$$J^{(m)} = (L_E)^m J.$$

In particular, let us focus on the family of conserved quantities obtained by the action of $g_a = e^{aL_E}$ on total momenta of Toda chain

$$J = \sum_{i=1}^{n} p_i. \quad (94)$$

By direct calculations one can check that the family $J(a)$ that forms an orbit of non-Noether symmetry generated by E reproduces the entire involutive

family of integrals of motion (88). Namely

$$
\begin{aligned}
J^{(1)} &= L_E J = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \\
J^{(2)} &= L_E J^{(1)} = (L_E)^2 J = \frac{1}{2}\sum_{i=1}^{n} p_i^3 + \frac{3}{2}\sum_{i=1}^{n-1}(p_i + p_{i+1}) e^{q_i - q_{i+1}} \\
J^{(3)} &= L_E J^{(2)} = (L_E)^3 J = \frac{3}{4}\sum_{i=1}^{n} p_i^4 + \\
&\quad + 3\sum_{i=1}^{n-1}(p_i^2 + 2p_i p_{i+1} + p_{i+1}^2) e^{q_i - q_{i+1}} + \\
&\quad + \frac{3}{2}\sum_{i=1}^{n-1} e^{2(q_i - q_{i+1})} + 3\sum_{i=1}^{n-2} e^{q_i - q_{i+2}} \\
J^{(m)} &= L_E J^{(m-1)} = (L_E)^m J.
\end{aligned}
\tag{95}
$$

Involutivity of this set of conservation laws can be verified using Theorem 8. In particular one can notice that the differential 1-form s defined by

$$E = W(s)$$

(where E is the generator of the non-Noether symmetry (87)) satisfies the condition

$$[W[W(s), W](s)] = 3[W(s)[W(s), W]]$$

while the conservation law J defined by (94) has the property

$$W(L_{W(s)} dJ) = -[W(s), W](dJ)$$

and thus according to Theorem 8 the conservation laws (95) are in involution.

12. Korteweg–de Vries Equation

Toda model provides a good example of a finite dimensional integrable Hamiltonian system that possesses a non-Noether symmetry. However, there are many infinite dimensional integrable Hamiltonian systems and in this case in order to ensure integrability one should construct an infinite number of conservation laws. Fortunately in several integrable models this task can be effectively simplified by identifying an appropriate non-Noether symmetry. First let us consider a well-known infinite dimensional integrable Hamiltonian system – the Korteweg–de Vries equation (KdV). The KdV equation has the following form

$$u_t + u_{xxx} + u u_x = 0 \tag{96}$$

(here u is smooth function of $(t,x) \in R^2$). The generators of symmetries of KdV should satisfy condition

$$E(u)_t + E(u)_{xxx} + u_x E(u) + uE(u)_x = 0 \tag{97}$$

which is obtained by substituting the infinitesimal transformation $u \to u + aE(u) + O(a^2)$ into the KdV equation and grouping first order terms.

Later we will focus on the symmetry generated by the following vector field

$$\begin{aligned} E(u) = 2u_{xx} + \frac{2}{3}u^2 + \frac{1}{6}u_x v + \frac{x}{2}(u_{xxx} + uu_x) - \\ - \frac{t}{4}(6u_{xxxxx} + 20u_x u_{xx} + 10uu_{xxx} + 5u^2 u_x) \end{aligned} \tag{98}$$

(here v is defined by $v_x = u$).

If u is subjected to zero boundary conditions $u(t,-\infty) = u(t,+\infty) = 0$, then the KdV equation can be rewritten in the Hamiltonian form

$$u_t = \{h, u\} \tag{99}$$

with the Poisson bivector field equal to

$$W = \int_{-\infty}^{+\infty} dx \frac{\delta}{\delta u} \wedge \left[\frac{\delta}{\delta u}\right]_x \tag{100}$$

and the Hamiltonian defined by

$$h = \int_{-\infty}^{+\infty} (u_x^2 - \frac{u^3}{3}) dx. \tag{101}$$

By taking the Lie derivative of the symplectic form along the generator of the symmetry, one gets the second Poisson bivector

$$[E, W] = W = \int_{-\infty}^{+\infty} dx \left[\left[\frac{\delta}{\delta u}\right]_{xx} \wedge \left[\frac{\delta}{\delta u}\right]_x + \frac{2}{3}u\frac{\delta}{\delta u} \wedge \left[\frac{\delta}{\delta u}\right]_x\right] \tag{102}$$

involved in the bi-Hamiltonian structure of the KdV hierarchy and proposed by Magri [58].

Now let us show how non-Noether symmetry can be used to construct conservation laws of the KdV hierarchy. By integrating KdV it is easy to show that

$$J^{(0)} = \int_{-\infty}^{+\infty} u dx$$

is a conserved quantity. At the same time Lie derivative of any conserved quantity along the generator of symmetry is conserved as well, while taking

the Lie derivative of $J^{(0)}$ along E gives rise to infinite sequence of conservation laws $J^{(m)} = (L_E)^m J^{(0)}$ that reproduce the well-known conservation laws of the KdV equation

$$\begin{aligned}
J^{(0)} &= \int_{-\infty}^{+\infty} u dx, \\
J^{(1)} &= L_E J^{(0)} = \frac{1}{4} \int_{-\infty}^{+\infty} u^2 dx, \\
J^{(2)} &= (L_E)^2 J^{(0)} = \frac{5}{8} \int_{-\infty}^{+\infty} (\frac{u^3}{3} - u_x^2) dx, \\
J^{(3)} &= (L_E)^3 J^{(0)} = \frac{35}{16} \int_{-\infty}^{+\infty} (\frac{5}{36} u^4 - \frac{5}{3} u u_x^2 + u_{xx}^2) dx, \\
J^{(m)} &= (L_E)^m J^{(0)}.
\end{aligned} \tag{103}$$

Thus the conservation laws and bi-Hamiltonian structures of the KdV hierarchy are related to the non-Noether symmetry of the KdV equation.

13. Nonlinear Water Wave Equations

Among nonlinear partial differential equations that describe propagation of waves in shallow water there are many remarkable integrable systems. We have already discussed the case of the KdV equation that possess non-Noether symmetries leading to an infinite sequence of conservation laws and a bi-Hamiltonian realization of this equation. Now let us consider other important water wave systems. It is reasonable to start with the dispersive water wave system, since many other models can be obtained from it by reduction. Evolution of dispersive water wave system is governed by the following set of equations

$$\begin{aligned}
u_t &= u_x w + u w_x, \\
v_t &= u u_x - v_{xx} + 2 v_x w + 2 v w_x, \\
w_t &= w_{xx} - 2 v_x + 2 w w_x.
\end{aligned} \tag{104}$$

Each symmetry of this system must satisfy the linear equation

$$\begin{aligned}
E(u)_t &= (wE(u))_x + (uE(w))_x, \\
E(v)_t &= (uE(u))_x - E(v)_{xx} + 2(wE(v))_x + 2(vE(w))_x, \\
E(w)_t &= E(w)_{xx} - 2E(v)_x + 2(wE(w))_x
\end{aligned}$$

obtained by substituting the infinitesimal transformations

$$\begin{aligned} u &\to u + aE(u) + O(a^2), \\ v &\to v + aE(v) + O(a^2), \\ w &\to w + aE(w) + O(a^2) \end{aligned}$$

into the equations of motion (104) and grouping first order (in a) terms. One of the solutions of this equation yields the following symmetry of the dispersive water wave system

$$\begin{aligned} E(u) &= uw + x(uw)_x + 2t(uw^2 - 2uv + uw_x)_x, \\ E(v) &= \frac{3}{2}u^2 + 4vw - 3v_x + x(uu_x + 2(vw)_x - v_{xx}) + \\ &\quad + 2t(u^2w - uu_x - 3v^2 + 3vw^2 - 3v_xw + v_{xx})_x, \\ E(w) &= w^2 + 2w_x - 4v + x(2ww_x + w_{xx} - 2v_x) - \\ &\quad - 2t(u^2 + 6vw - w^3 - 3ww_x - w_{xx})_x, \end{aligned} \tag{105}$$

and it is remarkable that this symmetry is local in the sense that $E(u)$ at the point x depends only on u and its derivatives evaluated at the same point (this is not the case in KdV where the symmetry is non-local due to presence of the non-local field v defined by $v_x = u$).

Before we proceed let us note that the dispersive water wave system is actually an infinite dimensional Hamiltonian dynamical system. Assuming that the fields u, v and w are subjected to zero boundary conditions

$$u(\pm\infty) = v(\pm\infty) = w(\pm\infty) = 0,$$

it is easy to verify that the equations (104) can be represented in the Hamiltonian form

$$\begin{aligned} u_t &= \{h, u\}, \\ v_t &= \{h, v\}, \\ w_t &= \{h, w\} \end{aligned}$$

with the Hamiltonian equal to

$$h = -\frac{1}{4}\int_{-\infty}^{+\infty} (u^2w + 2vw^2 - 2v_xw - 2v^2)dx \tag{106}$$

and the Poisson bracket defined by the following Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\frac{\delta}{\delta u} \wedge \left[\frac{\delta}{\delta u}\right]_x + \frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta w}\right]_x\right] dx. \tag{107}$$

Now using our symmetry that appears to be non-Noether, one can calculate the second Poisson bivector field involved in the bi-Hamiltonian realization of dispersive water wave system

$$\hat{W} = [E, W] =$$

$$= -2 \int_{-\infty}^{+\infty} \Big[u \frac{\delta}{\delta v} \wedge \Big[\frac{\delta}{\delta u}\Big]_x + v \frac{\delta}{\delta v} \wedge \Big[\frac{\delta}{\delta v}\Big]_x + \Big[\frac{\delta}{\delta v}\Big]_x \wedge \Big[\frac{\delta}{\delta w}\Big]_x +$$

$$+ w \frac{\delta}{\delta v} \wedge \Big[\frac{\delta}{\delta w}\Big]_x + \Big[\frac{\delta}{\delta w}\Big]_x \wedge \frac{\delta}{\delta w}\Big] dx. \tag{108}$$

Note that $\hat{W}$ give rise to the second Hamiltonian realization of the model

$$u_t = \{h^*, u\}_*,$$
$$v_t = \{h^*, v\}_*,$$
$$w_t = \{h^*, w\}_*,$$

where

$$h^* = -\frac{1}{4} \int_{-\infty}^{+\infty} (u^2 + 2vw) dx$$

and $\{,\}_*$ is the Poisson bracket defined by the bivector field $\hat{W}$.

Now let us pay attention to conservation laws. By integrating the third equation of the dispersive water wave system (104) it is easy to show that

$$J^{(0)} = \int_{-\infty}^{+\infty} w dx$$

is a conservation law. Using the non-Noether symmetry one can construct other conservation laws by taking Lie derivative of $J^{(0)}$ along the generator of symmetry and in this way the entire infinite sequence of conservation laws of dispersive water wave system can be reproduced

$$J^{(0)} = \int_{-\infty}^{+\infty} w dx,$$

$$J^{(1)} = L_E J^{(0)} = -2 \int_{-\infty}^{+\infty} v dx,$$

$$J^{(2)} = L_E J^{(1)} = (L_E)^2 J^{(0)} = -2 \int_{-\infty}^{+\infty} (u^2 + 2vw) dx, \tag{109}$$

$$J^{(3)} = L_E J^{(2)} = (L_E)^3 J^{(0)} = -6 \int_{-\infty}^{+\infty} (u^2 w + 2vw^2 - 2v_x w - 2v^2) dx,$$

$$J^{(4)} = L_E J^{(3)} = (L_E)^4 J^{(0)} =$$

$$= -24 \int_{-\infty}^{+\infty} (u^2 w^2 + u^2 w_x - 2u^2 v - 6v^2 w + 2vw^3 - 3v_x w^2 - 2v_x w_x) dx,$$

$$J^{(n)} = L_E J^{(n-1)} = (L_E)^n J^{(0)}.$$

Thus the conservation laws and the bi-Hamiltonian structure of the dispersive water wave system can be constructed by means of non-Noether symmetry.

Note that the symmetry (105) can be used in many other partial differential equations that can be obtained by reduction from the dispersive water wave system. In particular, one can use it in the dispersionless water wave system, the Broer–Kaup system, the dispersionless long wave system, Burger's equation etc. In case of the dispersionless water waves system

$$\begin{aligned} u_t &= u_x w + u w_x, \\ v_t &= u u_x + 2 v_x w + 2 v w_x, \\ w_t &= -2 v_x + 2 w w_x \end{aligned} \tag{110}$$

the symmetry (105) is reduced to

$$\begin{aligned} E(u) &= uw + x(uw)_x + 2t(uw^2 - 2uv)_x, \\ E(v) &= \frac{3}{2}u^2 + 4vw + x(uu_x + 2(vw)_x) + 2t(u^2 w - 3v^2 + 3vw^2)_x, \\ E(w) &= w^2 - 4v + x(2ww_x - 2v_x) - 2t(u^2 + 6vw - w^3)_x, \end{aligned} \tag{111}$$

and the corresponding conservation laws (109) reduce to

$$\begin{aligned} J^{(0)} &= \int_{-\infty}^{+\infty} w dx, \\ J^{(1)} &= L_E J^{(0)} = -2 \int_{-\infty}^{+\infty} v dx, \\ J^{(2)} &= L_E J^{(1)} = (L_E)^2 J^{(0)} = -2 \int_{-\infty}^{+\infty} (u^2 + 2vw) dx, \\ J^{(3)} &= L_E J^{(2)} = (L_E)^3 J^{(0)} = -6 \int_{-\infty}^{+\infty} (u^2 w + 2vw^2 - 2v^2) dx, \\ J^{(4)} &= L_E J^{(3)} = (L_E)^4 J^{(0)} = \\ &= -24 \int_{-\infty}^{+\infty} (u^2 w^2 - 2u^2 v - 6v^2 w + 2vw^3) dx, \\ J^{(n)} &= L_E J^{(n-1)} = (L_E)^n J^{(0)}. \end{aligned} \tag{112}$$

Another important integrable model that can be obtained from the dispersive water wave system is the Broer–Kaup system

$$\begin{aligned} v_t &= \frac{1}{2}v_{xx} + v_x w + v w_x, \\ w_t &= -\frac{1}{2}w_{xx} + v_x + w w_x. \end{aligned} \tag{113}$$

One can check that the symmetry (105) of the dispersive water wave system, after reduction, reproduces the non-Noether symmetry of Broer–Kaup model

$$\begin{aligned} E(v) &= 4vw + 3v_x + x(2(vw)_x + v_{xx}) + \\ &\quad + t(3v^2 + 3vw^2 + 3v_x w + v_{xx})_x, \\ E(w) &= w^2 - 2w_x + 4v + x(2ww_x - w_{xx} + 2v_x) + \\ &\quad + t(6vw + w^3 - 3ww_x + w_{xx})_x \end{aligned} \tag{114}$$

and gives rise to the infinite sequence of conservation laws of the Broer–Kaup hierarchy

$$\begin{aligned} J^{(0)} &= \int_{-\infty}^{+\infty} w dx, \\ J^{(1)} &= L_E J^{(0)} = 2\int_{-\infty}^{+\infty} v dx, \\ J^{(2)} &= L_E J^{(1)} = (L_E)^2 J^{(0)} = 4\int_{-\infty}^{+\infty} vw dx, \\ J^{(3)} &= L_E J^{(2)} = (L_E)^3 J^{(0)} = 12\int_{-\infty}^{+\infty} (vw^2 + v_x w + v^2) dx, \\ J^{(4)} &= L_E J^{(3)} = (L_E)^4 J^{(0)} = 24\int_{-\infty}^{+\infty} (6v^2 w + 2vw^3 + 3v_x w^2 - 2v_x w_x) dx, \\ J^{(n)} &= L_E J^{(n-1)} = (L_E)^n J^{(0)}. \end{aligned} \tag{115}$$

And exactly like in the dispersive water wave system one can rewrite the equations of motion (113) in the Hamiltonian form

$$\begin{aligned} v_t &= \{h, v\}, \\ w_t &= \{h, w\}, \end{aligned}$$

where the Hamiltonian is

$$h = \frac{1}{2}\int_{-\infty}^{+\infty} (vw^2 + v_x w + v^2) dx$$

while the Poisson bracket is defined by the Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \left[\frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta w} \right]_x \right] dx. \tag{116}$$

And again, using the symmetry (114) one can recover the second Poisson bivector field involved in the bi-Hamiltonian realization of the Broer–Kaup system by taking the Lie derivative of (116)

$$\hat{W} = [E, W] = -2 \int_{-\infty}^{+\infty} \left[v \frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta v} \right]_x - \left[\frac{\delta}{\delta v} \right]_x \wedge \left[\frac{\delta}{\delta w} \right]_x + \right.$$
$$\left. + w \frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta w} \right]_x + \frac{\delta}{\delta w} \wedge \left[\frac{\delta}{\delta w} \right]_x \right] dx. \tag{117}$$

This bivector field gives rise to the second Hamiltonian realization of the Broer–Kaup system

$$\begin{aligned} v_t &= \{h^*, v\}_*, \\ w_t &= \{h^*, w\}_* \end{aligned}$$

with

$$h^* = -\frac{1}{4} \int_{-\infty}^{+\infty} vw dx.$$

So the non-Noether symmetry of the Broer–Kaup system yields an infinite sequence of conservation laws of Broer–Kaup hierarchy and endows it with bi-Hamiltonian structure.

By suppressing dispersive terms in the Broer–Kaup system one reduces it to a more simple integrable model — the dispersionless long wave system

$$\begin{aligned} v_t &= v_x w + v w_x, \\ w_t &= v_x + w w_x. \end{aligned} \tag{118}$$

In this case the symmetry (105) reduces to the more simple non-Noether symmetry

$$\begin{aligned} E(v) &= 4vw + 2x(vw)_x + 3t(v^2 + vw^2)_x, \\ E(w) &= w^2 + 4v + 2x(ww_x + v_x) + t(6vw + w^3)_x, \end{aligned} \tag{119}$$

while the conservation laws of the Broer–Kaup hierarchy reduce to the sequence of conservation laws of the dispersionless long wave system

$$
\begin{aligned}
J^{(0)} &= \int\limits_{-\infty}^{+\infty} w dx, \\
J^{(1)} &= L_E J^{(0)} = 2 \int\limits_{-\infty}^{+\infty} v dx, \\
J^{(2)} &= L_E J^{(1)} = (L_E)^2 J^{(0)} = 4 \int\limits_{-\infty}^{+\infty} vw dx, \\
J^{(3)} &= L_E J^{(2)} = (L_E)^3 J^{(0)} = 12 \int\limits_{-\infty}^{+\infty} (vw^2 + v^2) dx, \\
J^{(4)} &= L_E J^{(3)} = (L_E)^4 J^{(0)} = 48 \int\limits_{-\infty}^{+\infty} (3v^2 w + vw^3) dx, \\
J^{(n)} &= L_E J^{(n-1)} = (L_E)^n J^{(0)}.
\end{aligned}
\tag{120}
$$

At the same time the bi-Hamitonian structure of Broer–Kaup hierarchy after reduction gives rise to the bi-Hamiltonian structure of the dispersionless long wave system

$$
\begin{aligned}
W &= \int\limits_{-\infty}^{+\infty} \left[\frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta w} \right]_x \right] dx, \\
\hat{W} &= [E, W] = \\
&= -2 \int\limits_{-\infty}^{+\infty} \left[v \frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta v} \right]_x + w \frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta w} \right]_x + \frac{\delta}{\delta w} \wedge \left[\frac{\delta}{\delta w} \right]_x \right] dx.
\end{aligned}
\tag{121}
$$

Among other reductions of the dispersive water wave system one should probably mention Burger's equation

$$
w_t = w_{xx} + w w_x. \tag{122}
$$

However, Hamiltonian realization of this equation is unknown (for instance, the Poisson bivector field of the dispersive water wave system (107) vanishes during reduction).

14. Benney System

Now let us consider another integrable system of nonlinear partial differential equations — the Benney system. Time evolution of this dynamical

system is governed by the equations of motion

$$\begin{aligned} u_t &= vv_x + 2(uw)_x, \\ v_t &= 2u_x + (vw)_x, \\ w_t &= 2v_x + 2ww_x. \end{aligned} \tag{123}$$

To determine symmetries of the system one has to look for solutions of the linear equation

$$\begin{aligned} E(u)_t &= (vE(v))_x + 2(uE(w))_x + 2(wE(u))_x, \\ E(v)_t &= 2E(u)_x + (vE(w))_x + (wE(v))_x, \\ E(w)_t &= 2E(v)_x + 2(wE(w))_x \end{aligned} \tag{124}$$

obtained by substituting the infinitesimal transformations

$$\begin{aligned} u &\to u + aE(u) + O(a^2), \\ v &\to v + aE(v) + O(a^2), \\ w &\to w + aE(w) + O(a^2) \end{aligned}$$

into the equations (123) and grouping first order terms. In particular, one can check that the vector field E defined by

$$\begin{aligned} E(u) &= 5uw + 2v^2 + x(2(uw)_x + vv_x) + 2t(4uv + v^2w + 3uw^2)_x, \\ E(v) &= vw + 6u + x((vw)_x + 2u_x) + 2t(4uw + 3v^2 + vw^2)_x, \\ E(w) &= w^2 + 4v + 2x(ww_x + v_x) + 2t(w^3 + 4vw + 4u)_x \end{aligned} \tag{125}$$

satisfies the equation (124) and therefore generates a symmetry of the Benney system. The fact that this symmetry is local simplifies further calculations.

At the same time, it is well-known that under the zero boundary conditions

$$u(\pm\infty) = v(\pm\infty) = w(\pm\infty) = 0$$

the Benney equations can be rewritten in the Hamiltonian form

$$\begin{aligned} u_t &= \{h, u\}, \\ v_t &= \{h, v\}, \\ w_t &= \{h, w\} \end{aligned}$$

with the Hamiltonian

$$h = -\frac{1}{2}\int_{-\infty}^{+\infty} (2uw^2 + 4uv + v^2w)dx \tag{126}$$

and the Poisson bracket defined by the following Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\frac{\delta}{\delta v} \wedge \left[\frac{\delta}{\delta v}\right]_x + \frac{\delta}{\delta u} \wedge \left[\frac{\delta}{\delta w}\right]_x\right] dx. \tag{127}$$

Using the symmetry (125) that in fact is a non-Noether one, we can reproduce the second Poisson bivector field involved in the bi-Hamiltonian structure of the Benney hierarchy (by taking the Lie derivative of W along E)

$$\begin{aligned}\hat{W} &= [E, W] = \\ &= -3 \int_{-\infty}^{+\infty} \Big[u \frac{\delta}{\delta u} \wedge \Big[\frac{\delta}{\delta u}\Big]_x + v \frac{\delta}{\delta u} \wedge \Big[\frac{\delta}{\delta v}\Big]_x + \\ &\quad + w \frac{\delta}{\delta u} \wedge \Big[\frac{\delta}{\delta w}\Big]_x + 2 \frac{\delta}{\delta v} \wedge \Big[\frac{\delta}{\delta w}\Big]_x \Big] dx. \end{aligned} \tag{128}$$

The Poisson bracket defined by the bivector field $\hat{W}$ gives rise to the second Hamiltonian realization of the Benney system

$$\begin{aligned} u_t &= \{h^*, u\}_*, \\ v_t &= \{h^*, v\}_*, \\ w_t &= \{h^*, w\}_* \end{aligned}$$

with the new Hamiltonian

$$h^* = \frac{1}{6} \int_{-\infty}^{+\infty} (v^2 + 2uw) dx.$$

Thus the symmetry (125) is closely related to the bi-Hamiltonian realization of Benney hierarchy.

The same symmetry yields an infinite sequence of conservation laws of Benney system. Namely, one can construct a sequence of integrals of motion by applying the non-Noether symmetry (125) to

$$J^{(0)} = \int_{-\infty}^{+\infty} w dx$$

(the fact that $J^{(0)}$ is conserved can be verified by integrating the third equation of the Benney system). The sequence looks like

$$\begin{aligned} J^{(0)} &= \int_{-\infty}^{+\infty} w dx, \\ J^{(1)} &= L_E J^{(0)} = 2 \int_{-\infty}^{+\infty} v dx, \\ J^{(2)} &= L_E J^{(1)} = (L_E)^2 J^{(0)} = 8 \int_{-\infty}^{+\infty} u dx, \end{aligned}$$

$$J^{(3)} = L_E J^{(2)} = (L_E)^3 J^{(0)} = 12 \int\limits_{-\infty}^{+\infty} (v^2 + 2uw)dx, \tag{129}$$

$$J^{(4)} = L_E J^{(3)} = (L_E)^4 J^{(0)} = 48 \int\limits_{-\infty}^{+\infty} (2uw^2 + 4uv + v^2 w)dx,$$

$$J^{(5)} = L_E J^{(4)} = (L_E)^5 J^{(0)} =$$

$$= 240 \int\limits_{-\infty}^{+\infty} (4u^2 + 8uvw + 2uw^3 + 2v^3 + v^2 w^2)dx,$$

$$J^{(n)} = L_E J^{(n-1)} = (L_E)^n J^{(0)}.$$

So the conservation laws and the bi-Hamiltonian structure of the Benney hierarchy are closely related to its symmetry, which can play an important role in analysis of the Benney system and other models that can be obtained from it by reduction.

15. Conclusions

The fact that many important integrable models, such as the Korteweg–de Vries equation, the Broer–Kaup system, the Benney system and the Toda chain, possess non-Noether symmetries that can be effectively used in analysis of these models, inclines us to think that non-Noether symmetries can play an essential role in the theory of integrable systems and properties of this class of symmetries should be investigated further. The present review indicates that in many cases non-Noether symmetries lead to maximal involutive families of functionally independent conserved quantities and in this way ensure integrability of the dynamical system. To determine the involutivity of conservation laws in cases when it can not be checked by direct computations (for instance, one can not check directly the involutivity in many generic n-dimensional models like the Toda chain and infinite dimensional models like the KdV hierarchy) we propose an analogue of the Yang–Baxter equation, that being satisfied by the generator of symmetry, ensures the involutivity of the family of conserved quantities associated with this symmetry.

Another important feature of non-Noether symmetries is their relationship with several essential geometric concepts emerging in the theory of integrable systems, such as Frölicher–Nijenhuis operators, Lax pairs, bi-Hamiltonian structures and bicomplexes. On the one hand, this relationship enlarges the possible scope of applications of non-Noether symmetries in Hamiltonian dynamics and, on the other hand, it indicates that the existence of invariant Frölicher–Nijenhuis operators, bi-Hamiltonian structures and bicomplexes in many cases can be considered as manifestation of hidden symmetries of the dynamical system.

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