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ON THE CAUCHY SINGULAR PROBLEM FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper, we consider the system of functional differential equations

$$x'(t) = f(x)(t) \tag{1}$$

with the initial condition

$$x(a) = 0, \tag{2}$$

where $f : C([a,b];\mathbb{R}^n) \to L_{loc}(]a,b];\mathbb{R}^n)$ is a continuous operator such that for any $\rho > 0$ the function

$$f^*_\rho(t) = \sup\Big\{\big\|f(x)(t)\big\| :\ x \in C([a,b];\mathbb{R}^n),\ \|x\|_C \le \rho\Big\}$$

satisfies the condition

$$f^*_\rho \in L_{loc}(]a,b];\mathbb{R}).$$

In regular case where $f^*_\rho \in L([a,b];\mathbb{R})$ for any $\rho > 0$, problem (1), (2) is investigated in detail (see, e.g., [1]–[4], [9], [10] and the references therein).

We are, in the main, interested in the case when f^*_ρ is nonintegrable on $[a,b]$, having singularity at $t = a$. In this case, the problem (1), (2) is singular one.

The methods of investigation of the singular problem (1), (2) in the case, when f is either the Nemytskii or evolution operator, have been developed in [5]–[7], [11]–[15]. Below, the problem (1), (2) will be investigated without an assumption that f is evolutionary.

We will use the following notation.

$\mathbb{R}^n$ is the space of vectors $x = (x_1,\dots,x_n)$ with real components $x_1,\dots,x_n$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

$x\cdot y$ is the scalar product of vectors x and $y \in \mathbb{R}^n$.

If $x = (x_1,\dots,x_n)$, then

$$\operatorname{sgn}(x) = \big(\operatorname{sgn}(x_1),\dots,\operatorname{sgn}(x_n)\big).$$

If $x \in \mathbb{R}$, then

$$[x]_+ = \frac{|x| + x}{2}\,.$$

$C([a,b];\mathbb{R}^n)$ is the Banach space of n-dimensional continuous vector functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\big\{\|x(t)\| :\ a \le t \le b\big\}.$$

If $x \in C([a,b];\mathbb{R}^n)$, then

$$\|x\|_{[a,t]} = \max\big\{\|x(s)\| :\ a \le s \le t\big\}.$$

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$L_{loc}(]a,b];\mathbb{R}^n)$ is the topological space of vector functions $x :]a,b] \to \mathbb{R}^n$, whose components are Lebesgue integrable on $[a+\varepsilon, b]$ for every $\varepsilon \in]0, b-a[$. A sequence $x_k \in L_{loc}(]a,b];\mathbb{R}^n)$ $(k = 1, 2, \dots)$ is said to be convergent to $x \in L_{loc}(]a,b];\mathbb{R}^n)$ if

$$\lim_{k\to+\infty} \int_{a+\varepsilon}^{b} \big\|x_k(t) - x(t)\big\|\,dt = 0$$

for every $\varepsilon \in]0, b-a[$.

Theorem 1. *Let there exist continuous nondecreasing functions* $\ell_0 : [a,b] \to [0,+\infty[$, $\ell : [a,b] \to [0,+\infty[$, $\varphi : [0,+\infty[\to [0,+\infty[$ *and an integrable function* $p : [a,b] \to [0,+\infty[$ *such that*

$$\ell_0(0) = 0, \ \ \ell(0) = 0, \ \ \varphi(\rho) > 0 \ \text{ for } \ \rho > 0,$$

and for an arbitrary $x \in C([a,b];\mathbb{R}^n)$ *in the interval* $[a,b]$ *the inequality*

$$\int_a^t \big[f(x)(s)\cdot \operatorname{sgn}(x(s))\big]_+\,ds \le \ell_0(t) + \ell(t)\|x\|_C + \int_a^t p(s)\varphi\big(\|x\|_{[a,s]}\big)\,ds$$

holds. If, moreover,

$$\liminf_{\rho\to+\infty} \phi\big(\ell_0(b) + \ell(b)\rho, \rho\big) > \int_a^b p(t)\,dt, \tag{3}$$

where

$$\phi(\rho_0, \rho) = \int_{\rho_0}^{\rho} \frac{ds}{\varphi(s)} \ \text{ for } \ \rho \ge \rho_0 \ge 0, \tag{4}$$

then the problem (1), (2) *has at least one solution.*

Theorem 2. *Let there exist continuous nondecreasing functions* $\ell : [a,b] \to [0,+\infty[$, $\varphi : [0,+\infty[\to [0,+\infty[$ *and an integrable function* $p : [0,+\infty[\to [0,+\infty[$ *such that* $\varphi(\rho) > 0$ *for* $\rho > 0$, *and*

$$\phi\big(\ell(b)\rho, \rho\big) > 0 \ \text{ for } \ \rho > 0, \tag{5}$$

and for any x *and* $y \in C([a,b];\mathbb{R}^n)$ *in the interval* $[a,b]$ *the inequality*

$$\int_a^t \Big[\big(f(x)(s) - f(y)(s)\big)\operatorname{sgn}\big(x(s) - y(s)\big)\Big]_+\,ds \le$$

$$\le \ell(t)\|x-y\|_C + \int_a^t p(s)\varphi\big(\|x-y\|_{[a,s]}\big)\,ds$$

holds. Then the problem (1), (2) *has no more than one solution.*

Theorem 3. *Let the conditions of Theorem* 2 *be fulfilled and*

$$\int_a^b \big\|f(0)(t)\big\|\,dt < +\infty. \tag{6}$$

If, moreover, $\ell(a) = 0$ *and the function* ϕ *satisfies the inequality* (3), *where*

$$\ell_0(t) = \big\|f(0)(t)\big\|,$$

then the problem (1), (2) *has one and only one solution.*

For $\varphi(s) \equiv s$, from Theorems 1, 2 and 3 follows

Corollary 1. *Let there exist continuous nondecreasing functions* $\ell_0 : [a,b] \to [0,+\infty[\,$, $\ell : [a,b] \to [0,+\infty[$ *and an integrable function* $p : [a,b] \to [0,+\infty[$ *such that* $\ell_0(a) = \ell(a) = 0$,

$$\ell(b)\exp\Big(\int_a^b p(s)\,ds\Big) < 1, \tag{7}$$

and for an arbitrary $x \in C([a,b];\mathbb{R}^n)$ *in the interval* $[a,b]$ *the inequality*

$$\int_a^t \big[f(x)(s)\cdot \operatorname{sgn}(x(s))\big]_+\,ds \le \ell_0(t) + \ell(t)\|x\|_C + \int_a^t p(s)\|x\|_{[a,s]}\,ds \tag{8}$$

holds. Then the problem (1), (2) *has at least one solution.*

Corollary 2. *Let there exist a continuous nondecreasing function* $\ell : [a,b] \to [0,+\infty[$ *and an integrable function* $p : [a,b] \to [0,+\infty[$ *such that for arbitrary* x *and* $y \in C([a,b];\mathbb{R}^n)$ *in the interval* $[a,b]$ *the inequality*

$$\int_a^t \Big[\big(f(x)(s) - f(y)(s)\big)\operatorname{sgn}\big(x(s)-y(s)\big)\Big]_+\,ds \le$$

$$\le \ell(t)\|x-y\|_C + \int_a^t p(s)\|x-y\|_{[a,s]}\,ds$$

holds. If, moreover, the functions ℓ *and* p *satisfy the condition* (7), *then the problem* (1), (2) *has no more than one solution.*

Corollary 3. *Let the conditions of Corollary* 2 *be fulfilled. If, moreover,* $\ell(a) = 0$ *and* f *satisfies the condition* (6), *then the problem* (1), (2) *has one and only one solution.*

Example. Let $n = 1$, $a = 0$, $b = 1$, $\lambda_0 \ge 0$, $\lambda > 0$, $\mu \ge 1$ and

$$f(x)(t) = -\lambda_0 \exp\Big(\frac{1}{t} + x^2(t)\Big)x(t) +$$
$$+ \lambda\big(1+|x(1)|\big)^{1+\mu} t \exp\big((1+|x(1)|)^\mu (t-1)\big).$$

Then

$$\int_0^t \big[f(x)(s)\cdot\operatorname{sgn}(x(s))\big]_+\,ds \le \lambda\big(1+|x(1)|\big)^{1+\mu}\int_0^t s\exp\big((1+|x(1)|)^\mu(s-1)\big)\,ds =$$

$$= \lambda\big(1+|x(1)|\big)t - \lambda\big(1+|x(1)|\big)^{1-\mu}\Big(\exp\big((1+|x(1)|)^\mu(t-1)\big) - \exp\big(-(1+|x(1)|)^\mu\big)\Big). \tag{9}$$

Consequently, the condition (8) holds, where

$$\ell_0(t) \equiv \ell(t) \equiv \lambda t, \quad p(t) \equiv 0.$$

If $\lambda < 1$, then inequality (7) is fulfilled, and according to Corollary 1, the problem (1), (2) has at least one solution. Suppose now that $\lambda_0 = 0$, $\lambda \ge 1$ and the problem (1), (2) has a solution x. Then by virtue of (9), we have $x(1) > 0$ and

$$x(1) = \lambda(1+x(1)) - \lambda(1+x(1))^{1-\mu}\big(1 - \exp\big(-(1+x(1))^\mu\big)\big) > x(1).$$

Hence if $\lambda_0 = 0$ and $\lambda \ge 1$, then the problem (1), (2) has no solution.

The above constructed example shows that in Theorem 1 (in Corollary 1) the condition (3) (the condition (7)) is unimprovable and it cannot be replaced by the condition

$$\liminf_{\rho\to+\infty} \phi\big(\ell_0(b) + \ell(b)\rho, \rho\big) \ge \int_a^b p(t)\,dt \quad \bigg(\ell(b)\exp\Big(\int_a^b p(s)\,ds\Big) \le 1\bigg).$$

An important particular case of (1) is the following integro-differential equation

$$\frac{dx(t)}{dt} = \int_a^b g\big(t, s, x(s), x(t)\big)\, d_s\sigma(t, s), \tag{10}$$

where $\sigma :]a, b] \times [a, b] \to [0, 1]$ is a measurable in the first and nondecreasing in the second argument function, and $g :]a, b] \times [a, b] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a vector function whose components are measurable in the first and continuous in the last $2n + 1$ arguments. Moreover,

$$\int_a^b g^*_\rho(\cdot, s)\, d_s\sigma(\cdot, s) \in L_{loc}(]a, b]),$$

where

$$g^*_\rho(t, s) = \max\Big\{\big\|g(t, s, x, y)\big\| : \ \|x\| \le \rho, \ \|y\| \le \rho\Big\}.$$

The corollaries of Theorems 1–3 presented below deal with the cases when the vector function g on the set $]a, b] \times [a, b] \times \mathbb{R}^{2n}$ satisfies either the condition

$$g(t, s, x, y) \cdot \operatorname{sgn}(y) \le h_0(t, s) + h_1(t, s)\|x\| + h_2(t, s)\varphi(\|y\|), \tag{11}$$

or the condition

$$\big(g(t, s, x, y) - g(t, s, \overline{x}, \overline{y})\big) \cdot \operatorname{sgn}(y - \overline{y}) \le h_1(t, s)\|x - \overline{x}\| + h_2(t, s)\varphi(\|y - \overline{y}\|), \tag{12}$$

where $h_i :]a, b] \times [a, b] \to [0, +\infty[$ $(i = 0, 1, 2)$ are measurable in the first and continuous in the second argument functions such that

$$\int_a^b d\tau \int_a^b h_i(\tau, s)\, d_s\sigma(s, \tau) < +\infty \ \ (i = 0, 1, 2),$$

and $\varphi : [0, +\infty[\to [0, +\infty[$ is a continuous, nondecreasing function such that

$$\varphi(\rho) > 0 \ \text{ for } \ \rho > 0.$$

Suppose

$$\ell_0(t) = \int_a^t d\tau \int_a^b h_0(\tau, s)\, d_s\sigma(s, \tau), \quad \ell(t) = \int_a^t d\tau \int_a^b h_1(\tau, s)\, d_s\sigma(s, \tau),$$

$$p(t) = \int_a^b h_3(t, s)\, d_s\sigma(s, t).$$

Moreover, under ϕ we mean the function, given by the equality (4).

Theorems 1–3 result in the following propositions.

Corollary 4. *If the conditions* (11) *and* (3) *are fulfilled, then the problem* (10), (2) *has at least one solution.*

Corollary 5. *If the conditions* (12) *and* (5) *are fulfilled, then the problem* (10), (2) *has no more than one solution.*

Corollary 6. *Let the conditions* (12), (5) *and*

$$\int_a^b dt \int_a^b \big\|g(t, s, 0, 0)\big\|\, d_s\sigma(s, t) < +\infty$$

be fulfilled. If, moreover, the inequality (3) *holds, where*

$$\ell_0(t) = \int_a^b d\tau \int_a^b \big\|g(t, s, 0, 0)\big\|\, d_s\sigma(s, \tau),$$

then the problem (10), (2) *has one and only one solution.*

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