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CONTACT AND BOUNDARY-CONTACT PROBLEMS OF STATICS OF ANISOTROPIC INHOMOGENEOUS ELASTIC BODY

Abstract. The paper presents the proofs of the existence and uniqueness of solutions of the contact and boundary-contact problems of inhomogeneous anisotropic elastic body in the two-dimensional case. The potential method and the theory of Fredholm integral equations is used. These problems for isotropic elastic body have been solved earlier by D.I. Sherman [1], who used for their solution the method of general solutions due to Kolosov–Muskhelishvili, complex potentials and also the methods of the theory of a complex variable. The boundary conditions of the above-mentioned problems will be written in natural way. In his work D.I. Sherman instead of a stress vector takes its integral. First we consider the contact problem after which the boundary-contact problems are treated comparatively elementarily.

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რეზიუმე. ნაშრომში მოყვანილია არაერთგვაროვანი ანიზოტროპული დრეკადი სხეულისათვის საკონტაქტო და სასაზღვრო-საკონტაქტო ამოცანების ამონახსნების არსებობისა და ერთადერთობის დამტკიცებები ორგანზომილებიან შემთვევაში. გამოყენებულია პოტენციალთა მეთოდი და ფრედჰოლმის ინტეგრალურ განტოლებათა თეორია. იზოტროპული დრეკადი სხეულებისათვის ადრე ეს ამოცანები დ.ი. შერმანის მიერ იყო ამოხსნილი, რომელმაც გამოიყენა კოლოსკოვ-მუსხელიშვილის ზოგადი ამოხსნების მეთოდი, კომპლექსური პოტენციალები და კომპლექსური ცვლადის ფუნქციათა თეორიის მეთოდები. ზემოხსენებული ამოცანების სასაზღვრო პირობები ჩვენს მიერ ბუნებრივად იქნება ჩაწერილი. დ.ი. შერმანი ძაბვის ვექტორის ნაცვლად მის ინტეგრალს იღებს.

პირველად ჩვენ განვიხილავთ საკონტაქტო ამოცანას, რის შემდეგაც სასაზღვრო-საკონტაქტო ამოცანების განხილვა შედარებით ელემენტარულად ხდება.

1. The Basic Contact Problem

Let an infinite plane be divided into two parts by a closed curve S_1. One (infinite) part occupying the domain $D_0$is characterized by Hook's constants $A_{11}^{(0)}$, $A_{12}^{(0)}$, $A_{13}^{(0)}$, $A_{22}^{(0)}$, $A_{23}^{(0)}$, $A_{33}^{(0)}$, while the other (finite) part occupying the domain D_1 is characterized by Hook's constants $A_{11}^{(1)}$, $A_{12}^{(1)}$, $A_{13}^{(1)}$, $A_{22}^{(1)}$, $A_{23}^{(1)}$, $A_{33}^{(1)}$. These constants in the domains D_0 and D_1 are independent of each other.

We introduce the following notation:

$$C^{(j)}(\partial_x)u^{(j)} = \begin{bmatrix} C_{11}^{(j)}(\partial_x), & C_{12}^{(j)}(\partial_x) \\ C_{21}^{(j)}(\partial_x), & C_{22}^{(j)}(\partial_x) \end{bmatrix} u^{(j)}, \tag{1.1}$$

where $u^{(1)}$ is the (real) vector of displacement in the domain D_j $(j = 0, 1)$, and

$$\left.\begin{aligned}
C_{11}^{(j)}(\partial x)u^{(j)} &= A_{11}^{(j)}\frac{\partial^2 u_1^{(j)}}{\partial x_1^2} + 2A_{12}^{(j)}\frac{\partial^2 u_1^{(j)}}{\partial x_1 \partial x_2} + A_{33}^{(j)}\frac{\partial^2 u_1^{(j)}}{\partial x_2^2}\,, \\
C_{12}^{(j)}(\partial x)u^{(j)} &= A_{13}^{(j)}\frac{\partial^2 u_2^{(j)}}{\partial x_1^2} + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 u_2^{(j)}}{\partial x_1 \partial x_2} + A_{23}^{(j)}\frac{\partial^2 u_2^{(j)}}{\partial x_2^2}\,, \\
C_{21}^{(j)}(\partial x)u^{(j)} &= A_{13}^{(j)}\frac{\partial^2 u_1^{(j)}}{\partial x_1^2} + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 u_1^{(j)}}{\partial x_1 \partial x_2} + A_{23}^{(j)}\frac{\partial^2 u_1^{(j)}}{\partial x_2^2}\,, \\
C_{22}(\partial x)u^{(j)} &= A_{33}^{(j)}\frac{\partial^2 u_2^{(j)}}{\partial x_1^2} + 2A_{23}^{(j)}\frac{\partial^2 u_2^{(j)}}{\partial x_1 \partial x_2} + A_{22}^{(j)}\frac{\partial^2 u_2^{(j)}}{\partial x_2^2}\,,
\end{aligned}\right\} \tag{1.2}$$

where x_1 and x_2 are the coordinates of the point x which is either in D_0 or in D_1.

Let now

$$T^{(j)}u^{(1)} = ((T^{(1)}u^{(1)})_1, (T^{(1)}u^{(1)})_2), \tag{1.3}$$

where

$$\left.\begin{aligned}
(T^{(j)}u^{(j)})_1 &= (A_{11}^{(j)}\varepsilon_x^{(j)} + A_{12}^{(j)}\varepsilon_y^{(j)} + A_{13}^{(j)}\varepsilon_{xy}^{(j)})n_1 + \\
&\quad + (A_{13}^{(j)}\varepsilon_x^{(j)} + A_{23}^{(j)}\varepsilon_y^{(j)} + A_{33}^{(j)}\varepsilon_{xy}^{(j)})n_2, \\
(T^{(j)}u^{(j)})_2 &= (A_{13}^{(j)}\varepsilon_x^{(j)} + A_{22}^{(j)}\varepsilon_y^{(j)} + A_{33}^{(j)}\varepsilon_{xy}^{(j)})n_1 + \\
&\quad + (A_{12}^{(j)}\varepsilon_x^{(j)} + A_{22}^{(j)}\varepsilon_y^{(j)} + A_{23}^{(j)}\varepsilon_{xy}^{(j)})n_2, \\
\varepsilon_x^{(j)} = \frac{\partial u_1^{(j)}}{\partial x_1}, \quad \varepsilon_y^{(j)} &= \frac{\partial u_2^{(j)}}{\partial x_2}, \quad \varepsilon_{xy}^{(j)} = \frac{\partial u_1^{(j)}}{\partial x_2} + \frac{\partial u_2^{(j)}}{\partial x_1}\,.
\end{aligned}\right\} \tag{1.4}$$

We define the regular vector as follows [2]. The vector $u^{(j)}$ is said to be regular in the domain D_j if it has continuous second order derivatives in that domain, and $u^{(j)}$ itself and its first order derivatives are continuous vectors up to the boundary S_1.

Now we formulate the basic contact problem: find regular vectors $u^{(0)}(\mathrm{x})$ and $u^{(1)}(x)$ in the domains D_0 and D_1 which satisfy the equation $C^{(j)}u^{(j)}=0$

and the contact conditions

$$\left.\begin{aligned}(u^{(1)}(t))^{+}-(u^{(0)}(t))^{-}&=f(t),\\(T^{(1)}u^{(1)})^{+}-(T^{(0)}u^{(0)})^{-}&=F(t),\ \ t\in S_1,\end{aligned}\right\}\tag{1.5}$$

where $f(t)\in C^{1,\alpha}(S_1)$, $f'(t)$ and $F(t)\in C^{0,\alpha}(S_1)$, $\alpha>0$ [3]. The symbols $(u^{(1)})^{+}$ and $(u^{(0)})^{-}$ denote the boundary values $u^{(1)}$ and $u^{(0)}$ for $t\in S_1$ from D_1 and D_0, respectively. $(T^{(1)}u^{(1)})^{+}$ and $(T^{(0)}u^{(0)})^{-}$ for $t\in S_1$ denote limiting values of the expressions $T^{(1)}u^{(1)}$ and $T^{(0)}u^{(0)}$ at the point t from D_1 and D_0, respectively. It should be noted that the normals are viewed in the positive direction from D_1 to D_0. Moreover, we require that $S_1\in C^{2,\alpha}$, $\alpha>0$, i.e. S_1 is of Hölder continuous curvature.

The solutions $u^{(1)}$ and $u^{(0)}$ are sought in the form

$$\left.\begin{aligned}u^{(1)}(x)=\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(1)}\Gamma^{(1)})'X^{(1)}+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-\\-\Gamma^{(1)}z^{(1)}h\Big\}dS,\quad x\in D_1,\\u^{(0)}(x)=\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-\\-\Gamma^{(0)}z^{(0)}h\Big\}dS,\quad x\in D_0,\end{aligned}\right\}\tag{1.6}$$

where g and h are unknown real vectors,

$$\begin{aligned}(T^{(1)}\Gamma^{(1)})'&=\sum_{k=1}^{2}\begin{bmatrix}N_k^{(j)}, & M_k^{(j)}\\L_k^{(j)}, & R_k^{(j)}\end{bmatrix}\frac{\partial\ln\sigma_{kj}}{\partial S(y)},\\\frac{\partial\Gamma^{(j)}}{\partial S(y)}&=\sum_{k=1}^{2}\begin{bmatrix}A_k^{(j)}, & B_k^{(j)}\\B_k^{(j)}, & C_k^{(j)}\end{bmatrix}\frac{\partial\ln\sigma_{kj}}{\partial S(y)},\\\Gamma^{(j)}&=\sum_{k=1}^{2}\begin{bmatrix}A_k^{(j)}, & B_k^{(j)}\\B_k^{(j)}, & C_k^{(j)}\end{bmatrix}\ln\sigma_{kj},\end{aligned}\tag{1.7}$$

$\sigma_{kj}=(x_1-y_1)+\alpha_{kj}(x_2-y_2)$, where y_1 and y_2 are the coordinates of the point $y\in S_1$, and $\alpha_{kj}=a_{kj}+ib_{kj}$, $b_{kj}>0$, is the root of the characteristic equation [2]

$$a_{11}^{(j)}\alpha_{kj}^4-2a_{13}^{(j)}\alpha_{kj}^3+(2a_{12}^{(j)}+a_{33}^{(j)})\alpha_{kj}^2-2a_{23}^{(j)}\alpha_{kj}+a_{22}^{(j)}=0,\ \ k=1,2,\ \ j=0,1,$$

where the elastic constants $a_{11}^{(j)}$, $a_{13}^{(j)}$, $2a_{12}^{(j)}+a_{33}^{(j)}$, $a_{23}^{(j)}$, $a_{22}^{(j)}$ are expressed uniquely through Hook's coefficients [2], $X^{(j)}$, $Y^{(j)}$, $Z^{(j)}$ are constant real matrices which will be defined below.

In the sequel, we will take into account that

$$\frac{\partial \ln \sigma_{kj}}{\partial S(y)} = \frac{\partial \ln r}{\partial S(y)} + i\frac{\partial \Theta}{\partial S(y)} + \frac{\partial}{\partial (y)} \ln\left(1 + \lambda_{kj}\frac{\overline{\sigma}}{\sigma}\right),$$
$$\ln \sigma_{kj} = \ln r + i\Theta + \ln\left(1 + \lambda_{kj}\frac{\overline{\sigma}}{\sigma}\right) + \ln(1 - i\alpha_{kj})$$
$$\frac{\partial^2 \ln \sigma_{kj}}{\partial S(x)\partial S(y)} = \frac{\partial^2(\ln r + i\Theta)}{\partial S(x)\partial S(y)} + \frac{\partial^2}{\partial S(x)\partial S(y)} \ln\left(1 + \lambda_{kj}\frac{\overline{\sigma}}{\sigma}\right),$$
$$\sigma = x_1 - y_1 + i(x_2 - y_2), \quad \overline{\sigma} = x_1 - y_1 - i(x_2 - y_2),$$

where $\frac{\partial}{\partial S(x)} = n_1(x)\frac{\partial}{\partial x_2} - n_2(x)\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial S(y)} = n_1(y)\frac{\partial}{\partial y_2} - n_2\frac{\partial}{\partial y_1}$ are tangential derivatives at the points x and y, $n(y) = (n_1(y), n_2(y))$ is the unit normal vector at the point $u \in S_1$, directed from D_1 to D_0,

$$\lambda_{kj} = \frac{1 + i\alpha_{kj}}{1 - i\alpha_{kj}}, \quad |\lambda_{kj}| < 1,$$
$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad \Theta = \operatorname{arctg}\frac{y_2 - x_2}{y_1 - x_1}.$$

The potentials having the terms with the kernels $\ln(1+\lambda_{kj}\frac{\overline{\sigma}}{\sigma})$, $\frac{\partial}{\partial S(y)}\ln(1+\lambda_{kj}\frac{\overline{\sigma}}{\sigma})$ and $\frac{\partial^2}{\partial S(x)\partial S(y)}\ln(1+\lambda_{kj}\frac{\overline{\sigma}}{\sigma})$ that continuously cross the boundary S_0 can temporary be neglected since they do not affect the boundary values. Taking into account the above reasoning, we can write [2]

$$\begin{aligned}
(T^{(j)}\Gamma^{(j)})' &= \sum_{k=1}^{2}\begin{bmatrix} N_k^j, & M_k^{(j)} \\ C_k^{(j)}, & R_k^{(j)} \end{bmatrix} \frac{\partial \ln \sigma_{kj}}{\partial S(y)} = \\
&= \left(E + i\omega_j \begin{bmatrix} -A_j, & C_j \\ -B_j, & A_j \end{bmatrix}\right)\Big(\frac{\partial \ln r}{\partial S(y)} + i\frac{\partial \Theta}{\partial S(y)}\Big), \\
\frac{\partial \Gamma^{(j)}}{\partial S(y)} &= \sum_{k=1}^{2}\begin{bmatrix} A_k^{(j)}, & B_k^{(j)} \\ B_k^{(j)}, & C_k^{(j)} \end{bmatrix} \frac{\partial \ln \sigma_{kj}}{\partial S(y)} = \\
&= \frac{im_j}{B_jC_j - A_j^2}\begin{bmatrix} C_j, & A_j \\ A_j, & B_j \end{bmatrix}\Big(\frac{\partial \ln r}{\partial S(y)} + i\frac{\partial \Theta}{\partial S(y)}\Big), \\
m_j &= a_{11}^{(j)}[1 - \omega_j^2(B_jC_j - A_j^2)], \\
\Gamma^{(j)} &= \sum_{k=1}^{2}\begin{bmatrix} A_k^{(j)}, & B_k^{(j)} \\ B_k^{(j)}, & C_k^{(j)} \end{bmatrix} \ln \sigma_{kj} = \\
&= \frac{im_j}{B_jC_j - A_j^2}\begin{bmatrix} C_j, & A_j \\ A_j, & B_j \end{bmatrix}[(\ln r + i\Theta) + \ln(1 - i\alpha_{kj})],
\end{aligned} \tag{1.8}$$

where E is the unit matrix.

Bearing in mind the above formulas and passing to the limit in (1.6) as $x \to t \in S_1$, after simple calculations we obtain

$$\left.\begin{aligned}
&(U^{(1)}(t))^+ = X^{(1)}g + \frac{1}{\pi}\int\limits_{S_1}\bigg\{\frac{\partial\Theta}{\partial S(y)}X^{(1)}+\\
&+\bigg(\omega_1\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix}X^{(1)} + \frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}Y^{(1)}\bigg)\frac{\partial\ln r}{\partial S(y)}g-\\
&-\frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}z^{(1)}\ln rh\bigg\}dS,\\
&(U^{(0)}(t))^- = -X^{(0)}g + \frac{1}{\pi}\int\limits_{S_1}\bigg\{\frac{\partial\Theta}{\partial S(y)}X^{(0)}+\\
&+\bigg(\omega_0\begin{bmatrix}-A_0, & C_0\\ -B_0, & A_0\end{bmatrix}X^{(0)} + \frac{m_0}{B_0C_0-A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}Y^{(0)}\bigg)\frac{\partial\ln r}{\partial S(y)}g-\\
&-\frac{m_0}{B_0C_0-A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}z^{(0)}\ln rh\bigg\}dS\,.
\end{aligned}\right\}\tag{1.9}$$

Now with regard for (1.3) and (1.4), from (1.6) we have

$$\begin{aligned}
T^{(1)}u^{(1)} &= \frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[T^{(1)}(T^{(1)}\Gamma^{(1)})'X^{(1)}+\\
&\quad+\frac{\partial T^{(1)}\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g - T^{(1)}\Gamma^{(1)}z^{(1)}h\Big\}dS, \quad x\in D_1,\\
T^{(0)}u^{(0)} &= \frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[T^{(0)}(T^{(0)}\Gamma^{(0)})'X^{(0)}+\\
&\quad+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g - T^{(0)}\Gamma^{(0)}z^{(0)}h\Big\}dS, \quad x\in D_1,
\end{aligned}\tag{1.10}$$

where

$$\begin{aligned}
T^{(j)}(T^{(j)}\Gamma^{(j)})' &= \frac{2}{a_{11}^{(1)}}\sum_{k=1}^{2}\begin{bmatrix}\alpha_k^2, & \alpha_{kj}\\ -\alpha_{kj}, & -1\end{bmatrix}d_{kj} = \frac{-i}{a_{11}^{(j)}}\begin{bmatrix}B_j, & -A_j\\ -A_j, & C_j\end{bmatrix},\\
T^{(j)}\Gamma^{(j)} &= E + i\omega_j\begin{bmatrix}-A, & -B_j\\ C_j, & A_j\end{bmatrix}.
\end{aligned}$$

Taking into account the last formulas and substituting in (1.10), after the passage to the limit as $x \to t \in S_1$ we obtain

$$
\begin{aligned}
&(T^{(1)}u^{(1)})^+ = Y^{(1)}\frac{\partial g}{\partial S(t)} + z^{(1)}h + \frac{1}{\pi}\int\limits_{S_1}\bigg\{\frac{\partial^2\Theta}{\partial S(t)\partial S(y)}X^{(1)}+\\
&+\bigg[\begin{bmatrix} B_1, & -A_1\\ -A_1, & C_1\end{bmatrix}\frac{X^{(1)}}{a_{11}^{(1)}} - \omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix}Y^{(1)}\bigg]\frac{\partial\ln r}{\partial S(t)}\frac{\partial g}{\partial S(y)}-\\
&\qquad -\bigg(\frac{\partial\Theta}{\partial S(t)} + \omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix}\frac{\partial\ln r}{\partial S(t)}\bigg)z^{(1)}h\bigg\}dS,\\
&(T^{(0)}u^{(0)})^- = -Y^{(0)}\frac{\partial g}{\partial S(t)} - z^{(0)}h + \frac{1}{\pi}\int\limits_{S}\bigg\{\frac{\partial^2\Theta}{\partial S(t)\partial S(y)}X^{(0)}+\\
&+\bigg[\begin{bmatrix} B_0, & -A_0\\ -A_0, & C_0\end{bmatrix}\frac{X^{(0)}}{a_{11}^{(0)}} - \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}Y^{(0)}\bigg]\frac{\partial\ln r}{\partial S(x)}\frac{\partial g}{\partial S(y)}-\\
&\qquad -\bigg(\frac{\partial\Theta}{\partial S(x)} + \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}\frac{\partial\ln r}{\partial S(t)}z^{(0)}h\bigg)\bigg\}dS.
\end{aligned}
\tag{1.11}
$$

Taking now into consideration the contact conditions (1.5), we easily see that to obtain Fredholm integral equations it is sufficient that the unknown constant real matrices satisfy the following conditions:

$$
\begin{aligned}
&X^{(1)} + X^{(0)} = E, \quad Y^{(1)} + Y^{(0)} = 0, \quad z^{(1)} + z^{(0)} = E,\\
&\omega_1\begin{bmatrix} -A_1, & C_1\\ -B_1, & A_1\end{bmatrix}X^{(1)} - \omega_0\begin{bmatrix} -A_0, & C_0\\ -B_0, & A_0\end{bmatrix}X^{(0)}+\\
&\quad + \bigg[\frac{m_1}{B_1C_1 - A_1^2}\begin{bmatrix} C_1, & A_1\\ A_1, & B_1\end{bmatrix} + \frac{m_0}{B_0C_0 - A_0^2}\begin{bmatrix} C_0, & A_0\\ A_0, & B_0\end{bmatrix}\bigg]Y^{(1)} = 0,\\
&\begin{bmatrix} B_1, & -A_1\\ A_1, & C_1\end{bmatrix}\frac{X^{(1)}}{a_{11}^{(1)}} - \begin{bmatrix} B_0, & -A_0\\ -A_0, & C_0\end{bmatrix}\frac{X^{(0)}}{a_{11}^{(0)}}+\\
&\quad + \bigg[\omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix} + \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}\bigg]Y^{(1)} = 0,\\
&\omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix}z^{(1)} - \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}z^{(0)} = 0.
\end{aligned}
$$

Thus for the determination of $X^{(j)}$, $Y^{(j)}$, $z^{(j)}$, $j = 0, 1$, we have obtained six equations. First of all, we find $z^{(1)}$ and $z^{(0)}$. The third and the sixth equations yield

$$
\begin{aligned}
&\bigg(\omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix} + \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}\bigg)z^{(1)} = \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix},\\
&\bigg(\omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix} + \omega_0\begin{bmatrix} -A_0, & -B_0\\ C_0, & A_0\end{bmatrix}\bigg)z^{(0)} = \omega_1\begin{bmatrix} -A_1, & -B_1\\ C_1, & A_1\end{bmatrix},
\end{aligned}
$$

whence

$$z^{(1)}=\frac{1}{\Delta_1}\left\{\omega_1\begin{bmatrix}A_1, & B_1\\ -C_1, & A_1\end{bmatrix}+\omega_0\begin{bmatrix}A_0, & B_0\\ -C_0, & -A_0\end{bmatrix}\right\}\omega_0\begin{bmatrix}-A_0, & -B_0\\ C_0, & A_0\end{bmatrix}=$$
$$=\frac{1}{\Delta_1}\left\{\begin{bmatrix}B_1C_0-A_0A_1, & B_1A_0-A_1B_0\\ C_1A_0-A_1C_0, & B_0C_1-A_0A_1\end{bmatrix}\omega_1\omega_0+\right.$$
$$\left.+\,\omega_0^2(B_0C_0-A_0^2)E\right\},$$
$$z^{(0)}=\frac{1}{\Delta_1}\left\{\omega_1\begin{bmatrix}A_{11}, & B_1\\ -C_1, & -A_1\end{bmatrix}+\omega_0\begin{bmatrix}A_0, & B_0\\ -C_0, & -A_0\end{bmatrix}\right\}\omega_1\begin{bmatrix}-A_1, & -B_1\\ C_1, & A_1\end{bmatrix}= \tag{1.12}$$
$$=\frac{1}{\Delta_1}\left\{\omega_1^2(B_1C_1-A_1^2)E+\right.$$
$$\left.+\begin{bmatrix}B_0C_1-A_0A_1, & B_0A_1-A_0B_1\\ C_0A_1-A_0C_1, & B_1C_0-A_0A_1\end{bmatrix}\omega_0\omega_1\right\},$$

where

$$\Delta_1=\omega_0^2(B_0C_0-A_0^2)+\omega_1^2(B_1C_1-A_1^2)+(B_1C_0+B_0C_1-2A_0A_1)\omega_0\omega_1. \tag{1.13}$$

Let us now prove that $\Delta_1>0$. To this end we notice [2] that

$$B_0C_0-A_0^2=\frac{1}{b_{10}b_{20}[(a_{10}-a_{20})^2+(b_{10}+b_{20})^2]},$$
$$B_1C_1-A_1^2=\frac{1}{b_{11}b_{21}[(a_{11}-a_{21})^2+(b_{11}+b_{21})^2]},$$
$$B_1C_0+B_0C_1-2A_0A_1=(B_0C_0-A_0^2)(B_1C_1-A_1^2)\{b_{10}b_{11}[(a_{20}-a_{21})^2+$$
$$+b_{20}^2+b_{21}^2]+b_{10}b_{21}[(a_{20}-a_{11})^2+b_{20}^2+b_{11}^2]+$$
$$+b_{20}b_{11}[(a_{10-a_{11}})^2+b_{10}^2+b_{21}^2]+b_{20}b_{21}[(a_{10}-a_{11})^2+b_{20}^2+b_{21}]\}.$$

Since $b_{klj}>0$, $k=1,2$, $j=0,1$, we immediately get $\Delta_1>0$.

To solve the fourth and the fifth equations, we multiply the left-hand side of the fifth equation by the matrix $\begin{bmatrix}0, & 1\\ -1, & 0\end{bmatrix}$; obviously $\det\begin{bmatrix}0, & 1\\ -1, & 0\end{bmatrix}=1$, and we have

$$\omega_1\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix}X^{(1)}-\omega_0\begin{bmatrix}-A_0, & C_0\\ -B_0, & A_0\end{bmatrix}X^{(0)}+$$
$$+\left\{\frac{m_1}{B_1C_1-a_1^2}\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}+\right.$$
$$\left.+\frac{m_0}{B_0C_0-A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}\right\}Y^{(1)}=0,$$
$$\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix}\frac{X^{(1)}}{a_{11}^{(1)}}-\begin{bmatrix}-A_0, & C_0\\ -B_0, & A_0\end{bmatrix}\frac{X^{(0)}}{a_1^{(0)}}+$$
$$+\left\{\omega_1\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}+\omega_0\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}\right\}Y^{(1)}=0,$$

which directly leads to

$$\left(\frac{\omega_1}{\omega_{11}^{(0)}} - \frac{\omega_0}{\omega_{11}^{(1)}}\right)\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix} X^{(1)} + \left\{\frac{1}{B_0C_0 - A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix} + \right.$$
$$\left. + \left(\frac{m_1}{(B_1C_1 - A_1^2)a_{11}^{(0)}} + \omega_0\omega_1\right)\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}\right\} Y^{(1)} = 0,$$
$$\left(\frac{\omega_1}{a_{11}^{(0)}} - \frac{\omega_0}{a_{11}^{(1)}}\right)\begin{bmatrix}-A_0, & C_0\\ -B_0 & A_0\end{bmatrix} X^{(0)} + \left\{\frac{1}{B_1C_1 - A_1^2}\begin{bmatrix}C_1, & A_1\\ A_1 & B_1\end{bmatrix} + \right.$$
$$\left. + \left(\frac{m_0}{a_{11}^{(1)}(B_0C_0 - A_0^2)} + \omega_0\omega_1\right)\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}\right\} Y^{(1)} = 0.$$

Thus we have

$$\begin{gathered}\left(\frac{\omega_1}{a_{11}^{(0)}} - \frac{\omega_0}{a_{11}^{(1)}}\right)(B_1C_1 - A_1^2)X^{(1)} + \\ + \left\{\frac{1}{B_0C_0 - A_0^2}\begin{bmatrix}A_1C_0 - C_1A_0, & A_1A_0 - B_0C_1\\ B_1C_0 - A_0A_1, & B_1A_0 - A_1B_0\end{bmatrix} + \right. \\ \left. + \left(\frac{m_1}{a_{11}^{(0)}(B_1C_1 - A_1^2)} + \omega_0\omega_1\right)\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}(B_1C_1 - A_1^2)\right\} Y^{(1)} = 0, \\ \left(\frac{\omega_1}{a_{11}^{(0)}} - \frac{\omega_0}{a_{11}^{(1)}}\right)(B_0C_0 - A_0^2)X^{(0)} + \\ + \left\{\frac{1}{B_1C_1 - A_1^2}\begin{bmatrix}-A_1C_0 + C_1A_0, & A_0A_1 - B_1C_0\\ B_0C_1 - A_0A_1, & B_0A_1 - A_0B_1\end{bmatrix} + \right. \\ \left. + \left(\frac{m_0}{a_{11}^{(0)}(B_0C_0 - A0^2)} + \omega_0\omega_1\right)\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}(B_0C_0 - A_0^2)\right\} Y^{(1)} = 0.\end{gathered} \tag{1.14}$$

Dividing the former by $(B_1C_1 - A_1^2)$ and the latter by $(B_0C_0 - A_0^2)$, with regard for $X^{(1)} + X^{(0)} = E$ we obtain

$$\begin{gathered}\left(\frac{\omega_1}{a_{11}^{(0)}} - \frac{\omega_0}{a_{11}^{(1)}}\right)E + \\ + \left\{\frac{1}{(B_1C_1 - A_1^2)(B_0C_0 - A_0^2)}\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}(B_1C_0 + B_0C_1 - 2A_1A_1) + \right. \\ \left. + \left(\frac{m_1}{a_{11}^{(0)}(B_1C_1 - A_1^2)} + \frac{m_0}{\omega_{11}^{(1)}(B_1C_1 - A_1^2)} + 2\omega_0\omega_1\right)\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}\right\} Y^{(1)} = 0,\end{gathered}$$

and hence

$$\begin{gathered}Y^{(1)} = \left(-\frac{\omega_1}{a_{11}^{(0)}} + \frac{\omega_0}{a_{11}^{(1)}}\right)\frac{1}{\Delta_2}\begin{bmatrix}0, & 1\\ -1, & 0\end{bmatrix}, \\ Y^{(1)} = \left(\frac{\omega_1}{a_{11}^{(0)}} - \frac{\omega_0}{a_{11}^{(1)}}\right)\frac{1}{\Delta_2}\begin{bmatrix}0, & 1\\ -1, & 0\end{bmatrix},\end{gathered} \tag{1.15}$$

where

$$\Delta_2 = \frac{1}{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)}\Big(B_1C_0 + B_0C_1 - 2A_1A_0 + \\ + \frac{m_1}{a_{11}^{(0)}(B_1C_1 - A_1^2)} + \frac{m_0}{a_{11}^{(1)}(B_1C_1 - A_1^2)} + 2\omega_0\omega_1\Big). \tag{1.16}$$

Taking into account the above reasoning and the fact that $m_1 > 0$, $m_0 > 0$, $a_{11}^{(1)} > 0$ and $a_{11}^{(0)} > 0$ [2], we can easily notice that $\Delta_2 > 0$.

Thus we have

$$\left.\begin{aligned} X^{(1)} = & -\frac{1}{\Delta_2}\bigg\{\frac{1}{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)} \times \\ & \times \begin{bmatrix} A_1C_0 - C_1A_0, & A_1A_0 - B_0C_1 \\ B_1C_0 - A_0A_1, & B_1A_0 - B_0A_1 \end{bmatrix} \frac{a_{11}^{(0)}a_{11}^{(1)}}{\omega_1 a_{11}^{(1)} - \omega_0 a_{11}^{(0)}} + \\ & + \Big(\frac{m_1}{a_{11}^{(0)}(B_1C_1 - A_1^2)} + \omega_0\omega_1\Big)\begin{bmatrix} 0, & -1 \\ 1, & 0 \end{bmatrix}\bigg\}, \\ X^{(0)} = & -\frac{1}{\Delta_2}\bigg\{\frac{1}{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)} \times \\ & \times \begin{bmatrix} A_0C_1 - C_0A_1, & A_0A_1 - B_0C_1 \\ B_0C_1 - A_0A_1, & B_0A_1 - A_0B_1 \end{bmatrix} \frac{a_{11}^{(0)}a_{11}^{(1)}}{\omega_1 a_{11}^{(1)} - \omega_0 a_{11}^{(0)}} + \\ & + \Big(\frac{m_0}{a_{11}^{(1)}(B_1C_1 - A_1^2)} + \omega_0\omega_1\Big)\begin{bmatrix} 0, & -1 \\ 1, & 0 \end{bmatrix}\bigg\}, \end{aligned}\right\} \tag{1.17}$$

where $a_1 a_{11}^{(1)} - \omega_0 a_{11}^{(0)} \neq 0$. If $\omega_1 a_{11}^{(1)} - \omega_0 a_0$, then from (1.15) we arrive at $Y^{(1)} = Y^{(1)} = 0$, and instead of six we obtain four equations which can be solved in an easier way. Indeed, in this case the fourth and the fifth equations transform into one equation which has the form

$$\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B, & A_1 \end{bmatrix} X^{(1)} - \omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix} X^{(0)} = 0,$$

and

$$X^{(1)} + X^{(0)} = E.$$

Therefore we have:

$$\bigg\{\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix} + \omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix}\bigg\} X^{(0)} = \omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix},$$

$$\bigg\{\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix} + \omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix}\bigg\} X^{(1)} = \omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix}.$$

Hence when $\omega_1 a_{11}^{(1)} = \omega_0 a_{11}^{(0)}$, we have

$$
\begin{aligned}
X^{(1)} &= \frac{1}{\Delta_3}\Big\{\omega_1\begin{bmatrix}A_1, & -C_1\\ B_1, & -A_1\end{bmatrix}+\omega_0\begin{bmatrix}A_0, & -C_0\\ B_0, & -A_0\end{bmatrix}\Big\}\omega_1\begin{bmatrix}-A_0, & C_0\\ -B_0, & A_0\end{bmatrix}=\\
&= \frac{1}{\Delta_3}\Big\{\omega_1\omega_0\begin{bmatrix}B_0C_1-A_0A_1, & A_1C_0-C_1A_0\\ A_1B_0-B_1A_0, & B_1C_0-A_0A_1\end{bmatrix}+\\
&\quad +\omega_0^2(B_0C_0-A_0^2)E\Big\},\\
X^{(0)} &= \frac{1}{\Delta_3}\Big\{\omega_1\begin{bmatrix}A_1, & -C_1\\ B_1, & -A_1\end{bmatrix}+\omega_0\begin{bmatrix}A_0, & -C_0\\ B_0, & -A_0\end{bmatrix}\Big\}\omega_1\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix}=\\
&= \frac{1}{\Delta_3}\Big\{\omega_0\omega_1\begin{bmatrix}B_1C_0-A_0A_1, & -A_0C_1+C_0A_1\\ -A_0B_1+B_0A_1, & B_1C_0-A_0A_1\end{bmatrix}+\\
&\quad +\omega_0^2(B_0C_0-A_0^2)E\Big\},
\end{aligned}
\tag{1.18}
$$

where $\Delta_3=\Delta_1=\omega_0^2(B_0C_0-A_0^2)+\omega_1^2(B_1C_1-A_1^2)+\omega_0\omega_1(B_1C_0+B_0C_1-2A_0A_1)>0$.

Thus we have defined $X^{(0)}$, $X^{(1)}$, $Y^{(1)}$, $Y^{(0)}$, $z^{(0)}$, $z^{(1)}$ in the general case. In what follows, we will use just these constants which have the form (1.12), (1.15), (1.17) and (1.18).

It should be noted that in case $x=t\in S_1$ we have the identity

$$\int\limits_{S_1}\frac{\partial^2}{\partial S(t)\partial S(y)}g\,dS=-\int\limits_{S}\frac{\partial\Theta}{\partial S(t)}\frac{dg}{\gamma S(y)}dS.$$

Here $S_1\in C^{2,\alpha}$, $\alpha>0$. We have used the above identity in (1.11).

Taking into account the obtained values $X^{(1)}$, $x^{(0)}$, $y^{(1)}$, $y^{(0)}$, $Z^{(0)}$, $Z^{(1)}$, we can write (1.11) in somewhat different manner. Moreover, in (1.11) we will restore the terms neglected earlier.

Thus we obtain

$$
\begin{aligned}
&(U^{(1)}(t))^+(u^{(0)})^- = g+\frac{1}{\pi}\int\limits_{S_1}\bigg\{\frac{\partial\Theta}{\partial S(y)}(X^{(1)}-X^{(0)})\times\\
&\times\operatorname{Im}\sum_{k=1}^{2}\left(\begin{bmatrix}N_k^{(1)}, & M_k^{(1)}\\ L_k^{(1)}, & R_k^{(1)}\end{bmatrix}X^{(1)}+\frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}A_k^{(1)}, & B_k^{(1)}\\ B_k^{(1)}, & C_k^{(1)}\end{bmatrix}Y^{(1)}\right)\times\\
&\times\frac{\partial}{\partial S(y)}\ln\left(1+\lambda_{k1}\frac{\overline{\sigma}}{\sigma}\right)g-\left[\frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}z^{(1)}\ln r+\right.\\
&\left.+\operatorname{Im}\sum_{k=1}^{2}\frac{m_1}{(B_1C_1-A_1^2)}\ln\left(1+\lambda_{k1}\frac{\overline{\sigma}}{\sigma}\right)z^{(1)}\right]h\,dS-\\
&-\operatorname{Im}\bigg\{\sum_{k=1}^{2}\begin{bmatrix}N_k^{(0)}, & M_k^{(0)}\\ L_k^{(0)}, & R_k^{(0)}\end{bmatrix}X^{(0)}+
\end{aligned}
$$

$$+\frac{m_0}{B_0C_0-A_0^2}\begin{bmatrix}A_k^{(0)}, & B_k^{(0)}\\ B_k^{(0)}, & C_k^{(0)}\end{bmatrix}Y^{(0)}\Bigg)\frac{\partial}{\partial S(y)}\ln\left(1+\lambda_{k0}\frac{\overline{\sigma}}{\sigma}\right)\Bigg\}dS,$$

$$(T^{(1)}u^{(1)}(t))^+-(T^{(0)}u^{(0)})^-=h+\frac{1}{\pi}\int\limits_{S_1}\Bigg\{\frac{\partial^2\Theta}{\partial S(x)\partial S(y)}(X^{(1)}-X^{(0)})+$$

$$+\operatorname{Im}\sum_{k=1}^{2}\Bigg(\begin{bmatrix}\alpha_{k1}^2, & -\alpha_{k1}\\ -\alpha_{k1}, & 1\end{bmatrix}d_{k1}X^{(1)}+ \tag{1.19}$$

$$+\frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}N_k^{(1)}, & L_k^{(1)}\\ M_k^{(1)}, & R_k^{(1)}\end{bmatrix}Y^{(1)}\Bigg)\frac{\partial^2}{\partial S(y)\partial S(y)}\ln\left(1+\lambda_{k1}\frac{\overline{\sigma}}{\sigma}\right)-$$

$$-\sum_{k=1}^{2}\Bigg(\begin{bmatrix}\alpha_{k0}^2, & -\alpha_{k0}\\ -\alpha_{k0}, & 1\end{bmatrix}d_{k0}X^{(0)}+\frac{m_0}{B_0C_0-A_0^2}\begin{bmatrix}N_k^{(0)}, & L_k^{(0)}\\ M_k^{(0)}, & R_k^{(0)}\end{bmatrix}Y^{(0)}\Bigg)\times$$

$$\times\frac{\partial^2}{\partial S(t)\partial S(y)}\ln\left(1+\lambda_{k0}\frac{\overline{\sigma}}{\sigma}\right)-\Bigg(\Bigg[\frac{m_0}{B_0C_0-A_0^2}\begin{vmatrix}-A_0, & -B_0\\ C_0 & A_0\end{vmatrix}+$$

$$+\frac{\partial\Theta}{\partial S(t)}z^{(0)}hdS+\operatorname{Im}\sum_{k=1}^{2}\Bigg\{\begin{bmatrix}N_k^{(1)}, & L_k^{(1)}\\ M_k^{(1)}, & R_k^{(1)}\end{bmatrix}\ln\left(1+\lambda_{k1}\frac{\overline{\sigma}}{\sigma}\right)z^{(1)}-$$

$$\times\begin{bmatrix}N_k^{(0)}, & Lk^{(0)}\\ M_k^{(0)}, & R_k^{(0)}\end{bmatrix}\ln\left(1+\lambda_{k0}\frac{\overline{\sigma}}{\sigma}\right)z^{(0)}\Bigg\}+$$

$$+\left(\frac{m_1}{B_1C_1-A_1^2}\begin{bmatrix}-A_0, & -B_0\\ C_0, & A_0\end{bmatrix}+\frac{\partial\Theta}{\partial S(t)}\right)z^{(0)}h\Bigg]\Bigg)dS.$$

Hence

$$\int\limits_{S_1}(T^{(1)}u^{(1)}(t))^+dS-\int\limits_{S_1}(T^{(0)}u^0(t))^-dS=-2z_0\int\limits_{S}hdS.$$

Since $\int_{S_1}(T^{(1)}u^{(1)}(t))^+=0$, $\int_{S_1}(T^{(0)}u^{(0)}(t))^-=-2z_0\int hdS$ and $\det z_0\neq 0$, we obtain

$$\int\limits_{S_1}hds=0. \tag{1.20}$$

This condition ensures that $u^{(1)}(x)$ and $u^{(0)}(x)$ are equal to zero at infinity.

Now we prove the following

Theorem. *The homogeneous basic contact problem has only zero solution.*

Proof. Using the well-known formulas [2]

$$N^{(1)}u^{(1)}=\frac{1}{m_1}\begin{bmatrix}B_1, & -A_1\\ -A_1, & C_1\end{bmatrix}\frac{\partial V^{(1)}}{\partial S(x)},\quad N^{(0)}u^{(0)}=\frac{1}{m_0}\begin{bmatrix}B_0, & -A_0\\ -A_0, & C_0\end{bmatrix}\frac{\partial V^{(0)}}{\partial S(x)},$$

we obtain $N^{(1)}u^{(1)}=0$ and $N^{(0)}u^{(0)}=0$. But since $u^{(0)}$ is equal to zero at infinity, therefore

$$u^{(0)}(x)=0,\quad x\in D_0.$$

Just analogously, $u^{(1)}=C_1$, $x\in D_1$, where C_1 is a constant. But since $u^{(1)}$ is defined to within a constant vector and if we assume that $u^{(1)}(x_1)=0$, where $x_1=(x_{11},x_{12})\in D_1$, then we obtain $u^{(1)}(x)=0$, $x\in D_1$.

It is known [2] that $u^{(1)}$, $V^{(1)}$, $u^{(0)}$, $V^{(0)}$ are conjugate vector-functions. Writing $V^{(1)}(x)$ and $V^{(0)}(x)$, we have

$$\begin{aligned}
V^{(1)}(x)=-\frac{1}{\pi}\int\limits_{S_1}\mathrm{Re}\Big\{\Big[(T^{(1)}\Gamma^{(1)})'X^{(1)}&+\\
+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-\Gamma^{(1)}z^{(1)}h\Big\}dS,&\quad x\in D_1,\\
V^{(0)}(x)=-\frac{1}{\pi}\int\limits_{S_1}\mathrm{Re}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}&+\\
+\frac{\partial\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-\Gamma^{(0)}z^{(0)}h\Big\}dS,&\quad x\in D_0.
\end{aligned}\tag{1.21}$$

Similarly,

$$\begin{aligned}
T^{(1)}V^{(1)}(x)=-\frac{1}{\pi}\int\limits_{S_1}\mathrm{Re}\Big\{\Big[T^{(1)}(T^{(1)}\Gamma^{(1)})'X^{(1)}&+\\
+\frac{\partial T^{(1)}\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-T^{(1)}\Gamma^{(1)}z^{(1)}h\Big\}dS&\\
T^{(0)}V^{(0)}(x)=-\frac{1}{\pi}\int\limits_{S_1}\mathrm{Re}\Big\{\Big[T^{(0)}(T^{(0)}\Gamma^{(0)})'X^{(0)}&+\\
+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(x)}Y^{(0)}\Big]g-T^{(0)}\Gamma^{(0)}z^{(0)}h\Big\}dS\,.&
\end{aligned}\tag{1.22}$$

The expressions (1.21) and (1.22) yield

$$(V^{(1)}(t))^+-(V^{(1)})^-=$$
$$=\left\{2\omega_1\begin{bmatrix}-A_1, & C_1\\ -B_1, & A_1\end{bmatrix}X^{(1)}+\frac{2m_1}{B_1C_1-A_1^2}\begin{bmatrix}C_1, & A_1\\ A_1, & B_1\end{bmatrix}M^{(1)}\right\}g$$
$$(V^{(0)}(t))^+-(V^{(0)})^-=$$
$$=\left\{2\omega_0\begin{bmatrix}-A_0, & C_0\\ -B_0, & A_0\end{bmatrix}X^{(0)}+\frac{2m_0}{B_0C_0-A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & A_0\end{bmatrix}M^{(0)}\right\}g.$$

Analogously,

$$(T^{(1)}V^{(1)})^+-(T^{(1)}V^{(1)})^-=\left[2\begin{bmatrix}B_1, & -A_1\\ -A_1, & C_1\end{bmatrix}\frac{X^{(1)}}{a_{11}^{(0)}}-\right.$$

$$-2\omega_1 \begin{bmatrix} -A_{11}, & -B_1 \\ C_1, & A_1 \end{bmatrix} Y^{(1)}\Bigg] \frac{\partial g}{\partial S(t)} + 2\omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} z^{(1)} h,$$

$$(T^{(0)}V^{(0)})^+ - (T^{(0)}V^{(0)})^- = \Bigg[2 \begin{bmatrix} B_0, & -A_0 \\ -A_0, & C_0 \end{bmatrix} \frac{X^{(0)}}{a_{11}^{(0)}} -$$

$$-2\omega_0 \begin{bmatrix} -A_0, & -B_0 \\ C_0, & A_0 \end{bmatrix} Y^{(0)}\Bigg] \frac{\partial g}{\partial S(t)} - 2\omega_0 \begin{bmatrix} -A_0, & -B_0 \\ C_0, & A_0 \end{bmatrix} z^{(0)} h.$$

Taking now into account the fact that $(V^1)^+ = (V^{(0)}(t))^- = (T^{(1)}V^{(1)})^+ = (T^{(0)}V^{(0)})^- = 0$ and the above formulas, we get

$$(V^{(1)}(t))^+ + (V^{(1)}(t))^- = 2\Bigg\{ \omega_0 \begin{bmatrix} -A_0, & -C_0 \\ -B_0, & A_0 \end{bmatrix} X^{(0)} -$$

$$-\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix} X^{(1)} - \Bigg[\frac{m_0}{B_0C_0 - A_0^2} \begin{bmatrix} C_0, & A_0 \\ A_0, & B_0 \end{bmatrix} +$$

$$+ \frac{m_1}{B_1C_1 - A_1^2} \begin{bmatrix} C_1, & A_1 \\ A_1, & B_1 \end{bmatrix} \Bigg] \Bigg\} M^{(0)} + 2\omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} z^{(1)} h = 0,$$

$$(T^{(0)}V^{(0)})^+ (T^{(0)}V^{(0)})^- = 2\Bigg\{ \begin{bmatrix} B_0, & -A_0 \\ -A_0, & C_0 \end{bmatrix} \frac{X^{(0)}}{a_{11}^{(0)}} -$$

$$- \begin{bmatrix} B_1, & -A_1 \\ -A_1, & C_1 \end{bmatrix} \frac{X^{(1)}}{a_{11}^{(1)}} + \Bigg[\omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} +$$

$$+ \omega_0 \begin{bmatrix} -A_0, & -B_0 \\ C_0, & A_0 \end{bmatrix} \Bigg] \Bigg\} M^{(0)} + 2\omega_0 \begin{bmatrix} -A_0, & -B_0 \\ C_0, & A_0 \end{bmatrix} z^{(0)} h = 0.$$

Thus we have obtained

$$(V^{(1)})^- = -(V^{(0)})^+, \quad (T^{(1)}V^{(1)})^- = -(T^{(0)}V^{(0)})^+.$$

According to Green's formulas [2]

$$\int\limits_{D_0} E(V^{(1)}, V^{(1)}) d\sigma = -\int\limits_{S_1} (V^{(1)})^- (T^{(1)}V^{(1)})^- dS,$$

$$\int\limits_{D_1} E(V^{(0)}, V^{(0)}) dS = +\int\limits_{S_1} (V^{(0)})^+ (T^{(0)}V^{(0)})^+ dS,$$

where $E(V^{(1)}, V^{(1)})$ is the doubled potential energy, we have

$$\int\limits_{D_0} E(V^{(1)}, V^{(1)}) d\sigma + \int\limits_{D_1} E(V^{(0)}, V^{(0)}) d\sigma = 0.$$

Therefore

$$V^{(1)} = A^{(1)} + \varepsilon^{(1)} \begin{pmatrix} -X_2 \\ X_1 \end{pmatrix}, \quad x \in D_1, \quad V^{(0)} = A^{(0)} + \varepsilon^{(0)} \begin{pmatrix} -X_2 \\ X_1 \end{pmatrix}, \quad x \in D_0.$$

Since $V^{(0)}(\infty) = 0$, therefore $A^{(0)} = 0$ and $\varepsilon^{(0)} = 0$, i.e. $(V^{(0)})^+ = 0$, and hence $(V^{(1)}(t))^- = 0$. With regard for the above formulas of discontinuity, we obtain

$$\left\{\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix} X^{(1)} + \frac{m_1}{B_1C_1 - A_1^2} \begin{bmatrix} C_1, & A_1 \\ A_1, & B_1 \end{bmatrix} M^{(1)} \right\} g = 0,$$

$$\left\{\omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix} X^{(0)} + \frac{m_0}{B_0C_0 - A_0^2} \begin{bmatrix} C_0, & A_0 \\ A_0, & B_0 \end{bmatrix} M^{(0)} \right\} g = 0,$$

and hence $g = 0$. Analogously we find that $h = 0$. Thus we have proved that the homogeneous contact problem has only the trivial solution. □

2. Solution of the First Boundary Value Problem of an Inhomogeneous Anisotropic Elastic Body.

Let an elastic anisotropic body with the constants $A_{11}^{(0)}$, $A_{12}^{(0)}$, $(A_{13})^{(0)}$, $A_{22}^{(0)}$, $A_{23}^{(0)}$, $A_{33}^{(0)}$ and the boundary S_0 wholly contain in itself a finite inclusion with properties in general different from those of the medium, characterized by the constants $A_{11}^{(1)}$, $A_{12}^{(1)}$, $(A_{13})^{(1)}$, $A_{22}^{(1)}$, $A_{23}^{(1)}$, $A_{33}^{(1)}$ and bounded by the curve S_1. Denote the domain occupied by the inclusion by D_1 and the remaining domain by D_0. The complement of $D_1 \cup S_1 \cup D_0 \cup S_0$ with respect to the whole plane we denote by D_2. The counter-clockwise direction on each of the contours is taken to be positive; the direction of the outer normal is also positive. A point belonging to the domain D_0 will be the origin of coordinates.

The first boundary-contact problem for such inhomogeneous elastic body is formulated as follows: find regular vectors $u^{(0)}(x)$ and $u^{(1)}(x)$ in the domains D_0 and D_1 that satisfy the equation $C^{(j)}u^{(j)} = 0$, the condition $(u^{(0)}(t))^+ = f_0(t)$, and the contact conditions on S_1,

$$\left.\begin{aligned} (u^{(1)}(x))^+ - (u^{(0)}(t))^- &= f(t), \\ (T^{(1)}u^{(1)})^+ - (T^{(0)}u^{(0)}(t))^- &= F(t), \end{aligned}\right\} \tag{2.1}$$

where the signs + and − denote the same as in Section 1. $f_0 \in C^{1,\alpha}(S_1)$ and $f'(t)$ and $F(t) \in C^{0,\alpha}(S_1)$, $\alpha > 0$ [3]. f_0, f and F are the given vectors.

A solution of the above-posed boundary-contact problem is sought in the form

$$\begin{aligned} u^{(0)}(x) &= \frac{1}{\pi}\int\limits_{S_0} \operatorname{Im}(N^{(0)}\Gamma^{(0)})'\mu(y)dS + \frac{1}{\pi}\int\limits_{S_1} \operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)} + \\ &\quad + \frac{\partial\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g - \Gamma^{(0)}z^{(0)}h\Big\}dS, \quad x \in D_0, \\ u^{(1)}(x) &= \frac{1}{\pi}\int\limits_{S_1} \operatorname{Im}\Big\{\Big[\int\limits_{S_1} \operatorname{Im}\Big\{\Big[(T^{(1)}\Gamma^{(1)})'X^{(1)} + \\ &\quad + \frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g - \Gamma^{(1)}z^{(1)}h\Big\}dS, \quad x \in D_1, \end{aligned} \tag{2.2}$$

where μ, g and h are unknown real vectors, and

$$(N^{(0)}\Gamma^{(0)})' = \sum_{k=1}^{2} \begin{bmatrix} E_k^{(0)}, & F_k^{(0)} \\ G_k^{(0)}, & H_k^{(0)} \end{bmatrix},$$

where the coefficients are defined in [2].

Taking into account the conditions of the first boundary-contact problem and the properties of the potentials contained in (2.2), for determination of the unknown vectors μ, g, h we obtain the following system of Fredholm integral equations

$$\begin{aligned}
&\mu(t) + \frac{1}{\pi}\int\limits_{S_0} \operatorname{Im}(N^{(0)}\Gamma^{(0)})'\mu(y)dS + \frac{1}{\pi}\int\limits_{S_1} \operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)} + \\
&\quad + \frac{\partial \Gamma^{(0)}}{\partial S(y)} Y^{(0)}\Big] g - \Gamma^{(0)} z^{(0)} h\Big\} dS = f_0(t), \quad t \in S_0, \\
&g + \frac{1}{\pi}\int\limits_{S_1} \Big\{ \frac{\partial \Theta}{\partial S(y)} X^{(1)} + \Big(\omega_1 \begin{bmatrix} -A_1, & C_1 \\ -B_1, & A_1 \end{bmatrix} X^{(1)} + \\
&\quad + \frac{m_1}{B_1C_1 - A_1^2} \begin{bmatrix} C_1, & A_1 \\ A_1, & B_1 \end{bmatrix} Y^{(1)}\Big) \frac{\partial \ln r}{\partial S(y)} g - \\
&\quad - \frac{m_1}{B_1C_1 - A_1^2} \begin{bmatrix} C_1, & A_1 \\ A_1, & B_1 \end{bmatrix} z^{(1)} h\Big\} dS - \frac{1}{\pi}\int\limits_{S_1} \Big\{ \frac{\partial \Theta}{\partial S(y)} X^{(0)} + \\
&\quad + \Big(\omega_0 \begin{bmatrix} -A_0, & C_0 \\ -B_0, & A_0 \end{bmatrix} X^{(0)} + \frac{m_0}{B_0C_0 - A_0^2} \begin{bmatrix} C, & A_0 \\ A_0, & B_0 \end{bmatrix} Y^{(0)} \frac{\partial \ln r}{\partial S(y)} g - \\
&\quad - \frac{m_0}{B_0C_0 - A_0^2} \begin{bmatrix} C_0, & A_0 \\ A_0, & B_0 \end{bmatrix} z^{(0)} h\Big\} dS - \\
&\quad - \frac{1}{\pi}\int\limits_{S_1} \operatorname{Im}(N^{(0)}\Gamma^{(0)})'\mu(y)dS = f(t), \\
&h + \frac{1}{\pi}\int\limits_{S_1} \Big\{ \frac{\partial^2 \Theta}{\partial S(t)\partial S(y)} X^{(1)} + \Big[\begin{bmatrix} B_1, & -A_1 \\ -A_1, & C_1 \end{bmatrix} \frac{X^{(1)}}{a_{11}^1} - \\
&\quad - \omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} Y^{(1)}\Big] \frac{\partial \ln r}{\partial S(t)} \frac{\partial g}{\partial S(y)} - \Big(\frac{\partial \Theta}{\partial S(t)} + \\
&\quad + \omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} \frac{\partial \ln r}{\partial S(t)}\Big) z^{(1)} h\Big\} dS - \frac{1}{\pi}\int\limits_{S_1} \Big\{ \frac{\partial^2 \Theta}{\partial S(t)\partial S(y)} X^{(0)} + \\
&\quad + \Big[\begin{bmatrix} B_0, & -A_0 \\ -A_0, & C_0 \end{bmatrix} \frac{X^{(0)}}{a_{11}^0} - \omega_1 \begin{bmatrix} -A_1, & -B_1 \\ C_1, & A_1 \end{bmatrix} Y^{(0)}\Big] \frac{\partial \ln r}{\partial S(t)} \frac{\partial g}{\partial S(y)} - \\
&\quad - \Big(\frac{\partial \Theta}{\partial S(t)} + \omega_0 \begin{bmatrix} -A_0, & -B_0 \\ C_0, & A_0 \end{bmatrix} \frac{\partial \ln r}{\partial S(t)}\Big) z^{(0)} h\Big\} dS - \\
&\quad - \frac{1}{\pi}\int\limits_{S_0} \operatorname{Im}(T^{(0)}N^{(0)}\Gamma^{(0)})'\mu(y)dS = F(t).
\end{aligned} \tag{2.3}$$

Let us now prove that the system (2.3) is uniquely solvable. Suppose that the homogeneous system corresponding to (2.3) has a solution, which we will denote by μ_0, g_0 and h_0. Using Green's formulas

$$\int_{D_0} E(u^{(0)}, u^{(0)})d\sigma = \int_{S_0} u^{(0)}T^{(0)}u^{(0)}dS - \int_{S_1} u^{(0)}T^{(0)}u^{(0)}dS,$$
$$\int_{D_1} E(u^{(1)}, u^{(1)})d\sigma = \int_{S_1} u^{(1)}T^{(1)}u^{(1)}dS$$

and taking into account the conditions of the boundary-contact problem, we obtain

$$\int_{D_0} E(u^{(0)}, u^{(0)})d\sigma + \int_{D_1} E(u^{(1)}, u^{(1)}d\sigma = 0,$$

whence $u(x)^{(0)}(x) = A^{(0)} + \varepsilon^{(0)}\binom{-x_2}{x_1}$, $u^{(1)}(x) = A^{(1)} + \varepsilon^{(1)}\binom{-x_2}{x_1}$. Since $u^{(0)}(x) = 0$ at infinity, we find that $u^{(0)}(x) = 0$ and $u^{(1)}(x) = 0$. Note that the operation $N^{(0)}$ from (2.2) passes continuously through the boundary S_0, i.e. $(N^{(0)}u^{(0)})^- = 0$. Using now Green's formulas, in the domain D_2 we get $u^{(0)} = 0$, $x \in D_2$. Thus we have obtained $(u^{(0)}(t))^+ - (u^{(0)}(t))^- = 2\mu_0(t)$.

Since $(u^{(0)}(t))^+ = (u^{(0)}(t))^- = 0$, therefore $\mu_0(t) = 0$, and from (2.2) it remains to prove that $g_0 = h_0 = 0$. But this is just how the matter stands, because (2.2) and (2.3) provide us with the Fredholm homogeneous integral equation for the contact problem.

Thus we have proved that the first boundary-contact problem has a solution, because the homogeneous boundary-contact problem has only the trivial solution. It should be noted that here we have used the condition $\int_{S_1} hdS = 0$, which follows from the second contact condition as far as

$$\int_{S_1} (T^{(0)}u^{(0)})^- dS = -2z_0 \int_{S_1} hdS = 0$$

and

$$\int_{S_1} (T^{(1)}u^{(1)})^+ dS = 0, \quad \int_{S_1} (T^{(0)}u^{(0)})^- dS = 0.$$

3. Solution of the Second Boundary-Contact Problem of an Inhomogeneous Anisotropic Elastic Body

The second boundary-contact problem is formulated analogously to the first one, the only difference being that the boundary-contact conditions are of the form

$$(T^{(0)}u^{(0)})^+ = F_0(t), \quad t \in S_0, \tag{3.1}$$

$$\left.\begin{aligned} (u^{(1)}(t))^+ - (u^{(0)})^- &= f(t), \\ (T^{(1)}u^{(1)})^+ - (T^{(0)}u^{(0)})^- &= F(t). \end{aligned}\right\} \tag{3.2}$$

All the definitions, agreements and formulas remain as before.

A solution of the second boundary-contact problem is sought in the form

$$\left.\begin{aligned} u^{(0)}(x)&=\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}M^{(0)}(x,y)\mu dS+\\ &+\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial\Gamma^{(0)}}{\partial S(y)}\,Y^{(0)}\Big]g-\\ &-\Gamma^{(0)}z^{(0)}h\Big\}dS,\quad x\in D_0,\\ u^{(1)}(x)&=\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(1)}\Gamma^{(1)})'X^{(1)}+\\ &+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]\,g-\Gamma^{(1)}h\Big\}dS,\quad x\in D_1, \end{aligned}\right\} \tag{3.3}$$

where

$$\begin{aligned} M(x,y)&=\sum_{k=1}^{2}\begin{bmatrix}A'_{k0}, & B'_{k0}\\ C'_{k0}, & D'_{k0}\end{bmatrix}\ln\Big(1-\frac{z_k^{(0)}}{\zeta_{k0}}\Big)=\\ &=\frac{ia_{11}}{B_0C_0-A_0^2}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}+\omega_0a_{11}^{(0)}\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}, \end{aligned} \tag{3.4}$$

and the remaining matrices involved in (3.3) are defined in Section 2. The unknown matrices μ, g and h will be defined below.

From (3.3), with regard for the boundary-contact conditions (3.1), (3.2) and properties of potentials contained in (3.3) and (3.4), for determination of μ, g and h we obtain the following system of Fredholm integro-differential equations of second kind:

$$\begin{aligned} &-\mu(t)+\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}\{T^{(0)}M^{(0)}(t,y)\mu dS\}+\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}\Big\{\Big[T^{(0)}(T^{(0)}\Gamma^{(0)})'X^{(0)}+\\ &\quad+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(y)}\,Y^{(0)}\Big]g-T^{(0)}\Gamma^{(0)}z^{(0)hdS}\Big\}=F_0(t),\quad t\in S_0,\\ &\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(1)}\Gamma^{(1)})'X^{(1)}+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-\Gamma^{(1)}h\Big\}dS-\\ &\quad-\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial\Gamma^{(0)}}{\partial S(y)}\Big]g-\Gamma^{(0)}z^{(0)}h\Big\}dS+\\ &\quad+X^{(1)}g+X^{(0)}g-\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}M^{0}(t,y)\mu dS=f(t),\\ &h+\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[T^{(1)}(T^{(1)}\Gamma^{(1)})'X^{(1)}+\frac{\partial T^{(1)}\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-T^{(1)}\Gamma^{(1)}h\Big\}dS- \end{aligned} \tag{3.5}$$

$$-\frac{1}{\pi}\int\limits_S\Big\{\operatorname{Im}\Big[T^{(0)}(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-T^{(0)}\Gamma^{(0)}z^{(0)}h\Big\}dS-$$

$$-\frac{1}{\pi}\int\limits_S\operatorname{Im}T^{(0)}M^{(0)}(t,\mu)dS=F(t),\quad t\in S_1.$$

Instead of (3.5) we consider the equations

$$\begin{aligned}
&-\mu+\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}\{T^{(0)}M^{(0)}(t,y)\}\mu dS+\\
&\quad+\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix}p_{k0}, & q_{k0}\\ r_{k0}, & S_{k0}\end{bmatrix}\frac{\partial\ln z_{k0}}{\partial S(t)}\int\limits_{S_0}gdS-\\
&\quad-\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{pmatrix}q_{k0}\\ S_{k0}\end{pmatrix}\frac{\partial}{\partial S(t)}\frac{1}{z_{k0}}M^{(0)}+\\
&\quad+\frac{1}{\pi}\int\limits_S\operatorname{Im}\Big\{\Big[T^{(0)}(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-T^{(0)\Gamma^{(0)}}h\Big\}dS\\
&\quad=F_0(t),\\
&g(t)+\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[\Big(T^{(1)}\Gamma^{(1)}\Big)'X^{(1)}+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-T^{(1)}\Gamma^{(1)}z^{(1)}h\Big\}dS-\\
&\quad-\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-\Gamma^{(0)}z^{(0)}h\Big\}dS-\\
&\quad-\frac{1}{\pi}\int\limits_{S_0}\operatorname{Im}M^{(0)}(t,y)\mu dS=f(t),\\
&h(t)+\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[\Big(T^{(1)}\Gamma^{(1)}\Big)'X^{(1)}+\frac{\partial\Gamma^{(1)}}{\partial S(y)}Y^{(1)}\Big]g-T^{(1)}\Gamma^{(1)}z^{(1)}h\Big\}dS-\\
&\quad-\frac{1}{\pi}\int\limits_{S_1}\operatorname{Im}\Big\{\Big[(T^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial T^{(0)}\Gamma^{(0)}}{\partial S(y)}Y^{(0)}\Big]g-\\
&\quad-T^{(0)}\Gamma^{(0)}z^{(0)}h\Big\}dS-\frac{1}{\pi}\int\limits_S\operatorname{Im}T^{(0)}M^{(0)}(t,y)\mu dS=F(t).
\end{aligned}\tag{3.6}$$

Let us now prove that the system (3.6) is always solvable. Towards this end we denote a solution of the homogeneous system (3.6) by μ_0, g_0 and h_0 and prove that it is equal to zero. From (3.6), integrating with respect to S_0, we obtain

$$\int\limits_{S_0}gdS=0,\quad M^{(0)}=0.\tag{3.7}$$

For $M^{(0)}$ we have the following expression:

$$M^{(0)} = \Big(\frac{\partial u_2^{(0)}}{\partial x_1} - \frac{\partial u_1^{(0)}}{\partial x_2}\Big)_{x_1=x_2=0}, \tag{3.8}$$

where $u^{(0)}(x)$ is defined by means of (3.3). In this case (3.6) coincides with (3.5), which satisfies the conditions

$$(T^{(0)}u^{(0)})^+ = 0,$$
$$(u^{(1)}(t))^+ - (u^{(1)})' = 0$$
$$(T^{(1)}u^{(1)})^+ - (T^{(0)}u^{(0)})^- = 0.$$

Using Green's formulas [2]

$$\begin{aligned} \int\limits_{D_1} E_0(u^{(0)},u^{(0)})d\sigma &= \int\limits_{S_0} u^{(0)}T^{(0)}u^{(0)}(t)dS - \int\limits_{S_1} u^{(0)}T^{(0)}u^{(0)}dS,\\ \int\limits_{D_1} E_1(u^{(1)},u^{(1)})d\sigma &= \int\limits_{S} u^{(1)}T^{(1)}u^{(1)}dS, \end{aligned} \tag{3.9}$$

since $(u^{(1)})^+ = (u^{(0)})^-$ and $(T^{(1)}u^{(1)})^+ = (T^0u^{(0)})^-$, we find from (3.9) that

$$\int\limits_{D_0} E_0(u^{(0)},u^{(0)})d\sigma + \int\limits_{D_1} E_1(u^{(1)},u^{(1)})dS = 0.$$

Thus we obtain

$$u^{(0)}(x) = A^{(0)} + \varepsilon^{(0)}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}, \quad x\in D_0,$$
$$u^{(1)}(x) = A^{(1)} + \varepsilon^{(1)}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}, \quad x\in D_1.$$

Taking into account the fact that according to (3.7) $u^{(0)}(x)=0$ in the domain D_0, we have $V^{(0)}(x)=0$ (this follows from the formula $N^{(0)}u^{(0)} = \frac{1}{m_0}\begin{bmatrix} B_0, & -A_0\\ -A_0, & C_0\end{bmatrix}\frac{\partial V^{(0)}(x)}{\partial S}$, where $u^{(0)}(x)$ and $V^{(0)}(x)$ are the conjugate vector-functions).

It is not difficult to see that

$$(T^{(0)}u^{(0)})^+ - (T^{(0)}V^{(0)})^- = 0,$$

but since $V^{(0)} = 0$, where C is a constant, we have $(T^{(0)}V^{(0)})^+ = (T^{(0)}V^{(0)})^- = 0$. Using the uniqueness theorem, for the domain D_2 we find that $V^{(0)}(x)=0$, $x\in D_2$. Now we can see [2] that

$$(L^{(0)}V^{(0)}(t))^+ - (L^{(0)}V^{(0)}(t))^- = \frac{2\begin{bmatrix}0, & -1\\ 1, & 0\end{bmatrix}\begin{bmatrix}C_0, & A_0\\ A_0, & B_0\end{bmatrix}}{\omega_0(B_0C_0-A_0^2)}\mu_0(t),$$

whence it follows that $\mu_0(t)=0$. In this case $u^{(0)}(x)$ and $u^{(1)}$ coincide with the potentials introduced in Section 1. As far as the contact problem has always a solution, the second boundary-value problem is always solvable. Thus we have proved that the second boundary-contact problem has always a solution if the resultant vector on each of the contours is equal to zero, and the principal moment of external stresses acting on the boundaries of the domain D_0 is equal to zero.

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