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ON THE WELL-POSEDNESS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $-\infty < a < b < +\infty$, $I = [a,b]$, n be a natural number, and let $f : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $h : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be continuous operators. Consider the boundary value problem

$$\frac{dx(t)}{dt} = f(x)(t), \tag{1}$$

$$h(x) = 0, \tag{2}$$

by a solution of which we mean an absolutely continuous vector function $x : I \to \mathbb{R}^n$ satisfying both the system (1.1) almost everywhere on I and the condition (1.2).

The well-posedness of this problem is more or less satisfactorily investigated only in the cases when f is either the linear, or the Nemytski operator (see, e.g., [1]–[9] and the references therein). In a general case to which we propose the present paper, the well-posedness of the problem (1), (2) remains still little studied.

In what follows, the following notation will be used.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$\mathbb{R}^n$ is the space of n-dimensional vectors $x = (x_i)_{i=1}^n$ with components $x_i \in \mathbb{R}$ ($i = 1, \dots, n$) and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$C(I;\mathbb{R}^n)$ is the space of continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\big\{\|x(t)\| : \ t \in I\big\};$$

$L(I;\mathbb{R}^n)$ is the space of vector functions $x : I \to \mathbb{R}^n$ with Lebesgue integrable components and the norm

$$\|x\|_C = \int_a^b \|x(t)\|\, dt;$$

$L(I;\mathbb{R}_+) = \big\{x \in L(I;\mathbb{R}) : \ x(t) \geq 0 \text{ for } t \in I\big\}$;

$M(I \times \mathbb{R}_+;\mathbb{R}_+)$ is the set of nondecreasing in the second argument functions $\omega : I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(\cdot, \rho) \in L(I;\mathbb{R}_+)$ for $\rho \in \mathbb{R}_+$ and $\omega(t, 0) = 0$ for $t \in I$.

If $x^0 \in C(I;\mathbb{R}^n)$, $\rho \in]0, +\infty[$, $\eta^* \in L(I;\mathbb{R}_+)$ and $\eta : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$, then we put

$$\mathcal{U}(x^0;\rho) = \big\{x \in C(I;\mathbb{R}^n) : \ \|x - x^0\| < \rho\big\}$$

and denote by $\mathcal{U}_{\eta,\eta^*}(x^0;\rho)$ the set of absolutely continuous vector functions $x \in \mathcal{U}(x^0;\rho)$ such that

$$\big\|x'(t) - \eta(x^0)(t)\big\| \leq \eta^*(t) \ \text{ for almost all } \ t \in I.$$

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Along with (1), (2) we will consider the perturbed problem

$$\frac{dx(t)}{dt} = f(x)(t) + \eta(x)(t), \tag{3}$$

$$h(x) + \gamma(x) = 0, \tag{4}$$

where $\eta : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\gamma : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ are continuous operators.

Let x^0 be a solution of the problem (1), (2), and let ρ be a positive constant. Introduce the following definitions.

Definition 1. The problem (1), (2) is said to be $(x^0;\rho)$**-well-posed** if for any $\varepsilon \in]0,\rho[$, $\rho^* \in]0,+\infty[$, $\eta^* \in L(I;\mathbb{R}_+)$ and $\omega \in M(I\times\mathbb{R}_+;\mathbb{R}_+)$ there exists $\delta > 0$ such that no matter how are the continuous operators $\eta : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\gamma : C(I;\mathbb{R}^n) \to \mathbb{R}^n$, satisfying the conditions

$$\big\|\eta(x)(t) - \eta(y)(t)\big\| \le \omega\big(t, \|x-y\|_C\big), \quad \|\gamma(x)\| \le \rho \ \text{ for } \ t \in I, \ \ x \text{ and } y \in \mathcal{U}(x^0;\rho),$$

$$\bigg\|\int_a^t \eta(x)(s)\,ds\bigg\| \le \delta, \quad \|\gamma(x)\| < \delta \ \text{ for } \ t \in I, \ \ x \in \mathcal{U}_{\eta,\eta^*}(x^0;\rho),$$

the perturbed problem (3), (4) has at least one solution contained in the ball $\mathcal{U}(x^0;\rho)$, and each of such solutions belongs also to the ball $\mathcal{U}(x^0;\varepsilon)$.

Definition 2. The problem (1), (2) is said to be **well-posed** if it is $(x^0;\rho)$-well-posed for an arbitrary $\rho > 0$.

Definition 3. The pair (p,ℓ) of continuous operators $p : C(I;\mathbb{R}^n)\times C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell : C(I;\mathbb{R}^n)\times C(I;\mathbb{R}^n) \to \mathbb{R}^n$ is said to be **consistent** if:

(i) for any $x \in C(I;\mathbb{R}^n)$, the operators $p(x,\cdot) : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell(x,\cdot) : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ are linear;

(ii) there exist an integrable in the first argument and nondecreasing in the second argument function $\alpha : I\times\mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing function $\alpha_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for arbitrary x and $y \in C(I;\mathbb{R}^n)$ and for almost all $t \in I$ the inequalities

$$\big\|p(x,y)(t)\big\| \le \alpha\big(t,\|x\|_C\big)\|y\|_C, \quad \|\ell(x,y)\| \le \alpha_0(\|x\|_C)\|y\|_C$$

are fulfilled;

(iii) there exists a positive constant β such that for any $x \in C(I;\mathbb{R}^n)$, $q \in L(I;\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$, an arbitrary solution y of the boundary value problem

$$\frac{dy(t)}{dt} = p(x,y)(t) + q(t), \quad \ell(x,y) = c_0$$

admits the estimate

$$\|y\|_C \le \beta\big(\|c_0\| + \|q\|_L\big).$$

Definition 4. A solution x^0 of the problem (1), (2) is said to be **strongly isolated in radius** ρ_0, if there exist a consistent pair (p,ℓ) of continuous operators $p : C(I;\mathbb{R}^n)\times C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell : C(I;\mathbb{R}^n)\times C(I;\mathbb{R}^n) \to \mathbb{R}^n$ and continuous operators $q : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $c_0 : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ such that

$$\sup\big\{\|q(x)(\cdot)\| : \ x \in C(I;\mathbb{R}^n)\big\} \in L(I;\mathbb{R}_+), \quad \sup\big\{\|c_0(x)\| : \ x \in C(I;\mathbb{R}^n)\big\} < +\infty, \tag{5}$$

$$f(x)(t) = p(x,x)(t) + q(x)(t), \quad h(x) = \ell(x,x) - c_0(x) \ \text{ for } \ x \in \mathcal{U}(x^0;\rho),$$

and the boundary value problem

$$\frac{dx(t)}{dt} = p(x,x)(t) + q(x)(t), \quad \ell(x,x) = c_0(x) \tag{6}$$

has no solution, different from x^0.

Theorem 1. *If the problem* (1), (2) *has a solution* x^0 *which is strongly isolated in radius* $\rho > 0$*, then this problem is* $(x^0;\rho)$*-well-posed.*

Corollary 1. *Let there exist a solution x^0 of the problem* (1), (2)*, constants $\rho_0 > 0$, $\alpha_0 > 0$, a function $\alpha \in L(I;\mathbb{R}_+)$ and continuous operators $p : \mathcal{U}(x^0;\rho_0) \times C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell : \mathcal{U}(x^0;\rho_0) \times C(I;\mathbb{R}^n) \to \mathbb{R}^n$ such that for arbitrary $x \in \mathcal{U}(x^0;\rho_0)$, $y \in C(I;\mathbb{R}^n)$ and for almost all $t \in I$ the conditions*

$$\big\|p(x,y)(t)\big\| \leq \alpha(t)\|y\|_C, \quad \|\ell(x,y)\| \leq \alpha_0\|y\|_C,$$
$$f(x)(t) - f(x^0)(t) = p(x, x - x^0)(t), \quad h(x) - h(x^0) = \ell(x, x - x^0)$$

are fulfilled. Let, moreover, for an arbitrary $x \in \mathcal{U}(x^0;\rho)$ the operators $p(x,\cdot) : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$ and $\ell(x,\cdot) : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be linear and the homogeneous problem

$$\frac{dy(t)}{dt} = p(x^0, y)(t), \quad \ell(x^0, y) = 0$$

have only a trivial solution. Then for sufficiently small $\rho > 0$ the problem (1), (2) *is $(x^0;\rho)$-well-posed.*

Corollary 2. *Let $p : C(I;\mathbb{R}^n) \times C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$, $q : C(I;\mathbb{R}^n) \to L(I;\mathbb{R}^n)$, $\ell : C(I;\mathbb{R}^n) \times C(I;\mathbb{R}^n) \to \mathbb{R}^n$ and $c_0 : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ be continuous operators such that the pair (p,ℓ) is consistent and the conditions* (5) *are fulfilled. Then the unique solvability of the problem* (6) *guarantees its well-posedness.*

For an arbitrary natural number k, we consider now the boundary value problem

$$\frac{dx(t)}{dt} = f(x)(t) + \eta_k\big(t, \zeta(x)(t)\big), \tag{7_k}$$
$$h(x) + \gamma_k(x) = 0, \tag{8_k}$$

where $\eta_k : I \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector function satisfying the local Carathéodory conditions, while $\zeta : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^m)$ and $\gamma_k : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ are continuous operators, and ζ and m are independent of k.

By $X_k(x^0;\rho)$ we denote the set of solutions of the problem (7_k), (8_k) contained in the ball $\mathcal{U}(x^0;\rho)$.

Theorem 2. *Let the problem* (1), (2) *have a solution x^0 which is strongly isolated in radius $\rho > 0$, and let there exist $\rho_0 > 0$, $\omega \in M(I \times \mathbb{R}_+;\mathbb{R}_+)$ and a continuous function $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_0(0) = 0$,*

$$\|\zeta(x)\|_C \leq \rho_0, \quad \big\|\zeta(x) - \zeta(\overline{x})\big\|_C \leq \omega_0\big(\|x - \overline{x}\|_C\big),$$
$$\big\|\gamma_k(x) - \gamma_k(\overline{x})\big\| \leq \omega_0\big(\|x - \overline{x}\|_C\big) \ \text{ for } x \text{ and } \overline{x} \in \mathcal{U}(x^0;\rho)$$

and

$$\big\|\eta_k(t,z) - \eta_k(t,\overline{z})\big\| \leq \omega\big(t, \|z - \overline{z}\|\big) \ \text{ for } t \in I, \ \|z\| \leq \rho_0, \ \|\overline{z}_0\| \leq \rho_0.$$

Let, moreover,

$$\lim_{k \to +\infty} \gamma_k(x) = 0 \ \text{ for } x \in \mathcal{U}(x^0;\rho),$$
$$\sup\bigg\{\bigg\|\int_a^t \eta_k(s,z)\,ds\bigg\| : \ t \in I, \ z \in \mathbb{R}^m, \ \|z\| \leq \rho_0\bigg\} \to 0 \ \text{ as } k \to +\infty.$$

Then there exists a natural number k_0 such that $X_k(x^0;\rho) \neq \varnothing$ for $k \geq k_0$ and

$$\sup\big\{\|x - x^0\| : \ x \in X_k(x^0;\rho)\big\} \to 0 \ \text{ as } k \to +\infty.$$

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