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N. Manjavidze

BOUNDARY VALUE PROBLEMS FOR ANALYTIC AND GENERALIZED ANALYTIC FUNCTIONS ON A CUT PLANE

Abstract. Boundary value problems with discontinuous coefficients for analytic and generalized analytic functions on a cut plane are considered with a special emphasis on conditions of normal solvability and index formulae.

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რეზიუმე. ნაშრომში განხილულია წყვეტილ-კოეფიციენტებიანი სასაზღვრო ამოცანები ანალიზური და განზოგადებული ანალიზური ფუნქციებისათვის ჭრილებიან სიბრტყეზე. განსაკუთრებული ყურადღება ექცევა ამოხსნადობის აუცილებელ და საკმარის პირობებს და ინდექსის ფორმულებს.

Introduction

The aim of the present work is to investigate certain boundary value problems of Riemann–Hilbert type on a plane cut along several regular arcs, find conditions of normal solvability, and obtain index formulae. Despite this topic is conceptually related to the well-developed classical theories of boundary value problems for analytic functions and singular integral equations [60], [22], [70], boundary value problems on a plane with curvilinear cuts remain insufficiently explored up to present time, which is directly indicated in such sources as [59] and [28]. In particular, this refers to Riemann–Hilbert problems on a plane with curvilinear cuts.

For example, in [28] it is explicitly stated that investigation of such problems is important for the theory of cracks and singular problems of the plane elasticity theory but there is a lack of general results applicable to problems of such type. One of the reasons for such situation is that, mathematically, boundary value problems on a plane with curvilinear cuts reduce to boundary value problems in multiply connected domains which present substantial difficulties and require using some deep and technically complicated technical tools such as conformal mappings [27] and function theory on Riemann surfaces [73]. In particular, the results available for boundary value problems in multiply connected domains as a rule do not lead to explicit solvability conditions (cf. [73]). So it does not seem possible to develop solvability theory for boundary value problems on a plane with curvilinear cuts by merely interpreting them as boundary value problems in multiply connected domains. Thus one needs to develop some specific direct methods to circumvent this difficulty.

The approach adopted in the present paper is based on reduction of boundary value problems on a plane with curvilinear cuts to analogous classical problems with discontinuous coefficients in a simply-connected domain. This method has its roots in the works of B. Khvedelidze and G. Manjavidze (see [33], [44], [45], [49]). Further important contributions belong to I. Simonenko [67], G. Litvinchuk (see [42] and references therein), I. Gohberg and N. Krupnik [26], V. Kokilashvili [37], [38], V. Paatashvili [63], R. Duduchava [17], [18], [43] and other authors (see references in [43] and [38]). Applications of those results in elasticity theory are discussed, e.g., in [5], [28].

For a simply connected domain, a powerful tool for investigating boundary values of such type is provided by Birkhoff (or Wiener-Hopf) factorization of matrix functions on closed contours [60], [70]. As is well-known, analogs of Birkhoff factorization can be developed for discontinuous matrix functions [33], [45], [43], [38], so we suggest some constructions which enable us to use those analogs for investigation of boundary value problems on a plane with curvilinear cuts. For brevity and convenience, instead of writing *plane cut along several smooth arcs* or *plane with curvilinear cuts* we write simply *cut plane* (cf. [47], [51], [52]). Let us add some historical and methodological remarks clarifying the concepts and approach used in the paper.

At present there are numerous papers on the theory of boundary value problems in the plane. In particular, the classical problems of linear conjugation, Riemann–Hilbert type problems, differential boundary value problems, one-dimensional singular integral equations, and their applications in mathematical physics were thoroughly studied in [60], [69], [70], [22], [58] to mention just a few fundamental sources.

Later on, there appeared many modifications and generalizations of those classical studies (see, e.g., [10], [24], [43], [33], [35]). We concentrate here on recent developments concerning problems on a cut plane and problems with displacement (shift) which appeared very useful in many applied problems (see, e.g., [44], [49], [58]). We basically follow the analytic approach developed by G.Manjavidze [44], [45], [46], [49], [2], [3] and at several places combine it with geometric methods from [35], [21].

Systematic research in the theory of linear conjugation problems with shift for analytic functions has been started after the appearance of the works [40], [41]. In the years to follow many papers concerned with various aspects of this theory were published (see, e.g., [33], [70], [44], [10], [73], [49], [67], [56], and references in [42]). Moreover, it turned out that certain aspects of the classical theory can be generalized to similar problems for hyperanalytic and generalized analytic functions [16], [24], [2]. Applications of the problems of linear conjugation with displacement (shift) in the theory of elasticity can be found in [4], [44], [68].

Much less was known about such problems on a cut plane. In the present paper we aim at showing that certain important results of the classical theory such as solvability conditions and index formulae can be generalized to boundary value problems on a cut plane [51], [52], [53]. We also indicate some applications of the main results which confirm that the main results are applicable to practically important problems. According to this aim, the structure of the paper is as follows.

We begin with introducing some classes of analytic functions in multiply connected domains and cut plane which are used throughout the text. In particular, we present basic concepts and facts about generalized analytic functions [11], [12] which are needed in the sequel.

Sections 2 and 3 contain the main results. Here we develop Fredholm theory for Riemann–Hilbert problems and linear conjugation problems on a cut plane. It appeared convenient and logically more consistent to begin with considering linear conjugation problems on a cut plane. Such problems are studied in Section 2. To this end, we begin with considering the problem of linear conjugation with piecewise continuous coefficient on a smooth contour and reduce it to factorization of a discontinuous matrix-function. We obtain solvability conditions and index formula for linear conjugation problems on a cut plane (Theorems 2.2 and 2.3) and relate them to analogous results for systems of singular integral equations.

Riemann–Hilbert problems on a cut plane are considered in Section 3. We first establish their relation to singular integral equations, which enables us to use the results obtained in the previous section. Finally, we obtain solvability conditions and index formula for Riemann–Hilbert problems on a cut plane.

In conclusion, as a sort of personal touch, the author wishes to dedicate this paper to the memory of her father Giorgi Manjavidze who introduced her to mathematical research and majority of the concepts and ideas used in this paper.

1. Preliminary Remarks

1.1. **Classical boundary value problems for analytic functions.** We briefly describe the main classical boundary value problems and establish the terminology to be used throughout the text. First of all, the word "plane" always means the plane $\mathbb{C}$ of complex variable canonically identified with the real plane $\mathbb{R}^2$ by setting $z = x + \imath y$. Under plane domain we understand a connected open subset of $\mathbb{C}$. The term "extended complex plane" $\overline{\mathbb{C}}$ means the same as "Riemann sphere", i.e. one adds to $\mathbb{C}$ an ideal point at infinity and introduces a topology by claiming that neighbourhoods of ∞ are complements to closed circles in $\mathbb{C}$.

The word "smooth" is used as an equivalent of "continuously differentiable". In most cases we explicitly indicate the classes of smoothness of the objects with which we deal at the moment. Analytic functions in a plane domain D are defined as those smooth functions which satisfy Cauchy–Riemann equations at every point of D. A mapping $f = (f_1, \dots, f_m) : D \to \mathbb{C}^m$ is called analytic if all of its components are analytic functions in D. Under a smooth arc (open curve) we mean the image of a smooth mapping of an interval into $\mathbb{C}$ while under a smooth contour (or closed curve) is meant the image of a smooth mapping of the circle S^1 into $\mathbb{C}$.

In many modern papers the word "contour" is substituted by the word "loop" borrowed from topology (cf. [35], [36]). This terminology is rather flexible and convenient and we will use it sometimes. Strictly speaking, the contour is just a particular case of the loop which is used to denote the loop in the plane. The term "loop" can be used with respect to an arbitrary topological space X (e.g., X can be a matrix group or a function space) and means exactly a subset $C \subset X$ which is the image of some continuous mapping $F : S^1 \to X$. This is especially convenient when one deals with functions or matrix functions on a contour. For example, one can say "matrix loop" instead of "matrix-function on the contour", which is obviously much more short and flexible. Correspondingly, one can speak of analytic, smooth, continuous (Hölder, L_p, etc.) loops.

An arc or a contour is called simple if it does not have self-intersections. The complement to a finite collection of nonintersecting simple smooth arcs is called plane with curvilinear cuts or simply cut plane. Notice that this set is always connected so this is an example of the plane domain. If the cuts are oriented, then one can obviously define left and right boundary values of functions defined on the cut plane.

A domain D is called simply connected if each smooth loop in it can be continuously contracted (homotoped) to a point without leaving D. If it is not the case, i.e. if there exist loops which cannot be contracted within D, then D is called multiply connected. One can define the (degree of) connectivity of D as the number of non-homotopic classes of loops in D. Thus simply connected domains have connectivity zero. If there is only one nontrivial homotopy class of loops, then the domain is called 1-connected. Such are, for example, any annulus, the complement to any point in $\mathbb{C}$ (the punctured plane), and the plane cut along a simple smooth arc. Notice that the punctured plane becomes connected if considered as the subset of the extended complex plane $\overline{\mathbb{C}}$. The same refers to the plane cut along one smooth arc. For this reason, boundary value problems in domains of the two latter types are easier to investigate than for an arbitrary cut plane.

We make now a few remarks concerning the terminology related with boundary value problems, which is really important because there exist various traditions in interpreting terms like linear conjugation problem or Riemann–Hilbert problem . First of all, by the *classical* boundary value problem of certain type we always mean the *linear* boundary value problem of the corresponding type.

Second and most important, we adopt a *broader meaning* of the key term *Riemann–Hilbert problem*. Namely, according to a modern geometric approach to boundary value problems in the plane (see, e.g., [36]) the general (nonlinear) Riemann–Hilbert problem for analytic functions in a plane domain is formulated as follows.

Riemann–Hilbert problem. Let D be a bounded plane domain with the smooth boundary Γ and let M be a real surface in $\Gamma \times \mathbb{C}^m$. The Riemann–Hilbert problem requires to find out if there exists a continuous mapping $f : \overline{D} \to \mathbb{C}^m$ which is analytic in D and satisfies

$$(z, f(z)) \in M, \quad \forall z \in \Gamma. \tag{1.1}$$

Usually one is also asked to describe the totality of such mappings called solutions to a given Riemann–Hilbert problem . For $z \in \Gamma$, put $M_z = \{w \in \mathbb{C}^m : (z, w) \in M\}$. We say that the above Riemann–Hilbert problem is linear if, for each $z \in \Gamma$, M_z is a real affine plane in $\mathbb{C}^m$. We are basically interested in linear Riemann–Hilbert problems but we need this general definition to describe some applications in Section 12.

As is well known (see, e.g., [36]), this formulation includes as particular cases both the classical *Riemann problem* (the problem of linear conjugation) and the *Hilbert problem* [22], [60]. This explains the name chosen for the above problem in this paper. In fact, practically everything what we will do in the sequel can be interpreted in terms of Riemann–Hilbert problems in the above sense.

Recall that the *Hilbert problem* consists in finding an analytic vector function X in D with prescribed real part g on the boundary Γ, i.e. satisfying

$$\operatorname{Re} X(z) = g(z), \quad \forall z \in \Gamma. \tag{1.2}$$

To formulate the Riemann problem (the linear conjugation problem), one assumes that the plane is decomposed into the inner domain D_+ and the outer domain D_- defined by Γ and one is given an $n \times n$ matrix-function G on Γ. Then the *Riemann problem* consists in finding two n-vector-functions $X_\pm$ which are analytic in $D_\pm$ respectively and satisfy on Γ the famous conjugation (transmission) condition

$$X_+(z) = G(z)X_-(z), \quad \forall z \in \Gamma. \tag{1.3}$$

Certain regularity of X_- at infinity is also supposed, for example, one may require that it is vanishing at ∞ or has finite order there [60], [22]. In this paper, for the Riemann problem we use another standard term, "linear conjugation problem", which seems more convenient in our context.

As is well known, each linear conjugation problem for n-vector-functions can be reduced to the Hilbert problem for $2n$-vector-functions so in principle it suffices to investigate only Hilbert problems or Riemann–Hilbert problems. However, linear conjugation problems have strong specifics and especially nice geometric interpretation so it is convenient to investigate them separately under their own name "linear conjugation problem" and we will do so in the sequel. As will be explained below, similar problems can be formulated not only for analytic functions but also for hyperanalytic functions and generalized analytic functions.

The famous *Birkhoff factorization theorem* [60] states that a sufficiently regular non-degenerate matrix function on $S^1 = \mathbb{T}$ can be represented in the form

$$G(t) = G_+(t)\operatorname{diag}(z^{\mathbf{k}})G_-(t), \tag{1.4}$$

where the matrix functions $G_\pm(t)$ are regular, non-degenerate and holomorphic in the domains $D_\pm$ respectively, $G_-(\infty)$ is the identity matrix, and

$$\operatorname{diag}(z^{\mathbf{k}}) = \operatorname{diag}(z^{k_1}, \dots, z^{k_n}), \quad k_1, \dots, k_n \in \mathbb{Z},$$

is a diagonal matrix function on $\mathbb{T}$ [60], [70].

The integer numbers k_i are called (left) partial indices [60], [70] of the matrix function $G(t)$. For a given matrix function $G(t)$, there can exist different factorizations of the form (1.4) but (left) partial indices are uniquely defined up to the order [70]. Analogously one can define a right Birkhoff factorization of $G(t)$ and right partial indices. We will only deal with the left factorizations because they are well-suited for investigation of linear conjugation problems of the form (1.3).

Partial indices exhibit quite non-trivial behaviour. The right partial indices need not be equal to the left ones. However, for sufficiently regular (rational, Hölder) matrix functions the sum of all left partial indices (*left total index*) is equal to the analogously defined *right total index*. Actually, both the left and right total indices are equal to the Fredholm index of the corresponding linear conjugation problem (1.3).

In fact, even for very regular (smooth, rational) matrix functions their collections of left and right partial indices are practically independent of each other (except the restriction that both total indices should be equal). For example, it was proved in [43] that for each two integer vectors $k, l \in \mathbb{Z}$ with $\sum k_i = \sum l_i$ there exists a non-degenerate rational matrix-function on the unit circle whose vectors of left and right partial indices are k and l respectively.

The problem of computing (left or right) partial indices of a concrete matrix function is far from trivial because in most cases they are not topological invariants and one has to take into account the analytic properties of a given matrix function. After several decades of gradual progress, this problem was eventually solved for several important classes of matrix functions [15], [43]. Recently these results were simplified and generalized in [1]. Thus the problem of computing partial indices nowadays can be considered as an algorithmically solvable one.

1.2. **Function classes.** We introduce now the basic function classes needed in the sequel. We use the terms and notation basically from [22], [60], [69], [70], [44], [8], [57], [43], [24], [72]. Let S be a subset of the plane of complex variable $z = x + iy$. The function $f = f(z)$ is said to satisfy $H(\mu)$-condition (i.e. the Hölder condition with the exponent μ) on S if f is defined on S and satisfies the inequality

$$|f(z_1) - f(z_2)| \leq A|z_1 - z_2|^{\mu}, \quad \forall z_1, z_2 \in S, \tag{H}$$

where A and μ are constants not depending on z_1, z_2 (with $A \geq 0$, $0 < \mu \leq 1$).

Denote by $H_\mu(S)$ the class of the functions satisfying the condition (H) (the constant A is not fixed). The union of the classes $H_\mu(S)$, $0 < \mu \leq 1$, is denoted by $H(S)$. It is evident that the functions of the class $H(S)$ are continuous; therefore sometimes the functions from this class will be referred to as Hölder-continuous.

If S is a domain, then denote by $C^m(S)[H^m_\mu(S)]$ the class of all functions satisfying the following conditions

$$f \in C(\bar{S})[f \in H(S)], \quad \frac{\partial^m f}{\partial x^{m-k} \partial y^k} \in C(S)[H_\mu(S)], \quad k = 0, \dots, m.$$

Consider, moreover, the class of functions $f(z) = f(x, y)$ defined and measurable in S and satisfying the condition

$$\iint\limits_S |f(z)|^p dxdy < \infty, \quad p \geq 1.$$

The class of all functions satisfying this condition is denoted by $L_p(\bar{S})$; by $L_p(\bar{S})$ we denote also the Banach space with the norm

$$\|f\|_{L_p} = \left(\iint\limits_{S} |f(z)|^p dxdy \right)^{1/p}.$$

Denote by $L_p(S)$ the class of all functions f for which the $p-$th power of the absolute value $|f|$ is summable (integrable) on every closed subset of the domain S. Let Γ be a simple rectifiable curve $z = z(s)$, where s is the arc abscissa, $0 \leq s \leq \ell$, and ℓ is the length of Γ.

A curve $\Gamma \in C^m$ if the derivatives of the function $z(s)$ with respect to s up to and including the order m are continuous on the segment $[0, \ell]$ (it is assumed that if Γ is closed, then $z^{(k)}(0) = z^{(k)}(\ell)$, $k = 1, \dots, m$); if in addition the derivative $z^{(m)} \in H_\mu([0, \ell])$, then $\Gamma \in H_\mu^m$. The curves of the class C^1 are called the smooth ones.

The curves consisting of a finite number of smooth curves are called the piecewise smooth ones. We write $D \in C^m[D \in H_\mu^m]$ if the boundary of the domain D consists of a finite number of simple closed curves of the class $C^m[H_\mu^m]$.

Let Γ be a simple curve, $c_1, c_2, \dots, c_r$ be points of Γ ordered according to the orientation of Γ. Denote by $C_0(\Gamma, c_1, \dots, c_r)$ the class of functions which are continuous on Γ except perhaps the points c_k where they may have discontinuities of the first kind; such functions are called the piecewise continuous functions.

A function $f(t)$ belongs to the class $H_0^\mu(\Gamma, c_1, \dots, c_r)$ if $f \in C_0(\Gamma, c_1, \dots, c_r)$ and f satisfies the $H(\mu)$-condition on each closed arc $c_k c_{k+1}$ provided the limits $f(c_k + 0)$ and $f(c_{k+1} - 0)$ are interpreted as the values of f at the points c_k and c_{k+1}, where $k = 1, \dots, r$ and $c_{r+1} = c_1$.

Denote by $C_0(\Gamma)$ [resp. $H_0(\Gamma)$] the union of the classes $C_0(\Gamma, c_1, \dots, c_r)$ [resp. $H_0^\mu(\Gamma, c_1, \dots, c_r)$], $0 < \mu \leq 1$. We say that $f(t) \in H^*(\Gamma)$ if the function $f(t)$ given on Γ admits the representation

$$f(t) = f_0(t) \prod_{k=1}^{r} |t - c_k|^{-\alpha}, \quad c_k \in \Gamma, \quad f_0(t) \in H_0(\Gamma), \quad \alpha < 1.$$

If $\prod\limits_{k=1}^{r} |t - c_k|^\varepsilon f(t) \in H(\Gamma)$ for arbitrarily small $\varepsilon > 0$, then we write $f(t) \in H_\varepsilon^*(\Gamma)$.

Let Γ be a rectifiable curve $t = t(s)$, $0 \leq s \leq \ell$, and $f(t)$ be a function defined on Γ. We say that $f(t)$ is measurable (respectively, summable) on Γ if the function $f(t(s))$ of the real variable s is measurable (respectively, summable) on the segment $[0, \ell]$; if $f(t)$ is summable, we define

$$\int\limits_{\Gamma} f(t)dt = \int\limits_{0}^{\ell} f(t(s))t'(s)ds.$$

Let $\rho(t) \geq 0$, $f(t)$ be measurable functions defined on Γ. We say that $f(t) \in L_p(\Gamma, \rho)$ if $\rho(t)|f(t)|^p$ $(p \geq 1)$ is a summable function on Γ; we write $L_p(\Gamma)$ instead of $L_p(\Gamma, 1)$. By $L_p(\Gamma, \rho)$ denote the corresponding Banach space with the norm

$$\|f\|_{L_p(\Gamma,\rho)} = \left(\int_\Gamma \rho(t)|f(t)^p|dt \right)^{1/p}.$$

The spaces $L_p(\Gamma, \rho)$ and $L_q(\Gamma, \rho^{1-q})$, are called conjugate spaces if $\frac{1}{p}+\frac{1}{q} = 1$, i.e. $q = p/(p-1)$.

As a rule, we assume that the weight function has the form

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\nu_k}, \quad t_k \in \Gamma, \quad -1 < \nu_k < p-1, \quad p > 1. \tag{W}$$

It is clear that in this case $L_p(\Gamma, \rho) \subset L_\lambda(\Gamma)$ for some $\lambda > 1$.

Let Γ be the union of simple smooth curves in the complex z-plane. Let $\phi(z)$ be a function defined and continuous in a neighborhood of Γ except perhaps at the points of Γ themselves. Let t be some point of Γ different from the end points and the points of self-intersection (if there are any). We say that the function $\phi(z)$ is continuously extendable in the point t from the left (respectively, from the right) if $\phi(z)$ tends to a definite limit $\phi^+(t)$ (respectively, $\phi^-(t)$) as z tends to t along any path remaining on the left (respectively, on the right) of Γ. If the mentioned limit exists when z tends to t on along some non-tangential path remaining on the left (respectively, on the right) from Γ, then we say that $\phi(z)$ has the angular boundary value $\phi^+(t)[\phi^-(t)]$.

A piecewise-holomorphic function ϕ is a holomorphic function in the plane cut along Γ (except perhaps at infinity) continuously extendable on Γ from both sides everywhere except perhaps the finite set of points c_k; near these points c_k the function $\phi(z)$ is supposed to satisfy the following estimate

$$|\phi(z)| \leq \frac{\text{const}}{|z - c_k|^\alpha}, \quad 0 \leq \alpha < 1.$$

At the point $z = \infty$ the function may have a pole.

The notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K. If K is some linear normed space with the norm $\|\cdot\|_K$, then $\|A\|_K = \max\limits_{\alpha,\beta} \|A_{\alpha\beta}\|_K$. Sometimes an $(n \times 1)$-matrix A is called a vector, and it is convenient to write it as the row $A = (A_1, \ldots, A_n)$.

Let D be a simply connected domain in the extended complex plane bounded by a rectifiable Jordan curve Γ. By definition the class $E_p(D)$, $p > 0$, is the set of all analytic functions in D for which there exists a sequence of domains D_k with rectifiable boundaries Γ_k satisfying the conditions:

$$\overline{D}_k \subset D, \quad D_k \subset D_{k+1}, \quad \underset{k}{\cup} D_k = D, \quad \sup \int_{\Gamma_k} |f(z)|^p |dz| < \infty.$$

Let Γ be a simple closed rectifiable curve bounding the finite domain D^+ and the infinite domain D^- (the domain D^+ remains on the left when passing along Γ in the positive direction); the Cauchy type integral

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)dt}{t-z}, \quad f(t) \in L_1(\Gamma), \tag{C}$$

has the angular boundary values $\phi^+(t)$ and $\phi^-(t)$ from D^+ and D^- (from both sides of Γ) almost everywhere on Γ.

Denote by $E_p^{\pm}(\Gamma)$, $p \geq 1$, ($E_p^{\pm}(\Gamma, \rho)$, where ρ is a function of the form (W)), the class of the functions $\phi(z)$ representable in the form

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)dt}{t-z} + P(z), \quad f \in L_p(\Gamma) \ (L_p(\Gamma, \rho)), \tag{E}$$

where Γ is a simple closed rectifiable curve and $P(z)$ is some polynomial.

Denote also by $E_{p,0}^{\pm}(\Gamma)$ ($E_{p,0}^{\pm}(\Gamma, \rho)$) the class of functions of the form (E) with $P(z) = 0$. By $E_{\infty}^{\pm}(\Gamma)$ ($E_{\infty,0}^{\pm}(\Gamma)$) we denote the intersection

$$\underset{p>1}{\cap} E_p^{\pm}(\Gamma) \ \ (\underset{p>1}{\cap} E_{p,0}^{\pm}(\Gamma)).$$

For the functions of the class $E_p^{\pm}(\Gamma)$ the following propositions are valid:

a) $E_p^{\pm}(\Gamma) \subset E_r^{\pm}(\Gamma)$, $p > r$.

b) If $\phi(z) \in E_1^{\pm}(\Gamma)$ and $\phi^+(t) = \phi^-(t)$ almost everywhere on Γ, then $\phi(z)$ is some polynomial.

c) If $\phi(z) \in E_p^{\pm}(\Gamma)$ $p > 1$, $\Gamma \in R$, then $\phi(z) \in E_p(D^+)$, $\phi(z) - P(z) \in E_p(D^-)$.

It is evident that if

$$\phi_1(z) \in E_p(D^+), \ \ \phi_2(z) \in E_p(D^-), \ \ p \geq 1,$$

then the function

$$\phi(z) = \begin{cases} \phi_1(z), & z \in D^+, \\ \phi_2(z), & z \in D^-, \end{cases}$$

belongs to $E_p^{\pm}(\Gamma)$.

d) Let $\phi_1(z) \in E_p^{\pm}(\Gamma, \rho)$, $\phi_2(z) \in E_q^{\pm}(\Gamma, \rho^{1-q})$. Then $\phi_1(z)\phi_2(z) \in E_1^{\pm}(\Gamma)$.

e) If $\phi(z) \in E_1^{\pm}(\Gamma)$, then $\phi(z) \in E_{1-\varepsilon}(D^+)$, $\phi(z) \in E_{1-\varepsilon}(D^-)$ for arbitrary small positive ε [44].

Let X and Y be Banach spaces and A be a linear bounded operator mapping X into Y. Recall that the operator A is said to be Fredholm if

a) the image of the operator A in Y is closed (i.e. the operator A is normally solvable in the sense of Hausdorff);

b) the null spaces (kernels) $N = \{x \in X,\ Ax = 0\}$ and $N^*\{f \in Y^* : A^*f = 0\}$ are finite dimensional subspaces (A^* is the conjugate operator, X^* and Y^* are the conjugate spaces).

The difference $\ell - \ell^*$, where ℓ and ℓ^* denote the dimensions of the subspaces N and N^*, respectively, is called the index $ind\, A$ of the Fredholm operator A.

1.3. **Generalized analytic functions.** In the theory of generalized analytic functions the following integral operators

$$(Tf)[z] = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)d\sigma_\zeta}{\zeta - z}, \quad (\Pi f)[z] = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)d\sigma_\zeta}{(\zeta - z)^2}$$

play an important role, where D is some domain in the z-plane, $z = x + iy$, and $f(\zeta)$ is a function of the class $L_p(\bar{D}), p \geq 1$. The main properties of the operators T, Π are the following.

The generalized derivatives satisfy

$$\partial_{\bar{z}} Tf = f, \quad \partial_z Tf = \Pi f.$$

If D is a bounded domain, then Tf is a linear completely continuous operator from the space $L_p(\bar{D})$, $p > 2$, into the space $H^\alpha(D)$, $\alpha = (p-2)/p$.

If the boundary Γ of D is the union of a finite number of piecewise-smooth contours, then the operator T is a linear bounded operator from $L_p(\bar{D}), 1 < p \leq 2$, into $L_j(\Gamma), 1 < j < p/(2-p)$.

Let $D \in H_\alpha^{m+1}, f(z) \in H_\alpha^m(D), 0 < \alpha < 1, m \geq 0$. Then $Tf \in H_\alpha^{m+1}(D), \partial_z Tf = \Pi f \in H_\alpha^m(D)$. Πf is a linear bounded operator in the spaces $H^\alpha(D)$ and $L_p(\bar{D}), p > 1$.

Let $q(z)$ be a measurable bounded function in the whole plane $\mathbb{C}, |q(z)| \leq q_0 < 1, q(z) = 0$ in a neighborhood of $z = \infty$, and let f be a solution of the equation

$$f - q\Pi f = q$$

belonging to the class $L_p(\mathbb{C}), p > 2$. Then the function

$$\omega(z) = z + Tf$$

is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\bar{z}}\omega - q(z)\partial_z\omega = 0.$$

These and other properties of the operators T and Π are formulated and proved in [46], [47].

A vector $w(z) = (w_1, \ldots, w_n)$ is called generalized analytic vector in the domain D if it is a solution of an elliptic system of the form

$$\partial_{\bar{z}} w - Q(z)\partial_z w + A(z)w + B(z)\bar{w} = 0, \tag{1.5}$$

where $A(z), B(z)$ are given square matrices of order n of the class $L_{p_0}(D)$, $p_0 > 2$, and $Q(z)$ is a matrix of the following special form: it is quasidiagonal and every block $Q^r = (q^r_{ik})$ is a lower (upper) triangular matrix satisfying the conditions

$$q^r_{11} = \cdots = q^r_{m_r,m_s} = q^r, \quad |q^r| \le q_0 < 1,$$
$$q^r_{ik} = q^r_{i+s,k+s} \quad (i+s \le n, k+s \le n).$$

Moreover, we suppose $Q(z) \in W^1_p(C), p > 2$, and $Q(z) = 0$ outside of some circle.

The equation

$$\partial_{\bar z} w - \partial_z (Q' w) - A'(z) w - \overline{B'(z) w} = 0 \tag{1.5'}$$

is called conjugate to the equation (1.5), the accent $'$ denotes transposition of matrix.

If $A(z) \equiv B(z) \equiv 0$, then the equations (1.5) and (1.5$'$) pass into

$$\partial_{\bar z} w - Q(z) w_z = 0, \tag{1.6}$$
$$\partial_{\bar z} w - \partial_z (Q' w) = 0. \tag{1.6'}$$

Solutions of the equation (1.2) are called Q-holomorplic vectors.

The equation (1.6) has a solution of the form

$$\zeta(z) = zI + T\omega, \tag{1.7}$$

where I is the unit matrix and $\omega(z)$ is a solution of the equation

$$\omega(z) + Q(z)\Pi\omega = Q(z)$$

belonging to $L_p(\mathbb{C}), p > 2$.

The solution (1.7) of the equation (1.6) is analogous to the fundamental homeomorphism of the Beltrami equation.

The matrix

$$V(t,z) = \partial_t \zeta(t)[\zeta(t) - \zeta(z)]^{-1} \tag{1.8}$$

is called the generalized Cauchy kernel for the equation (1.6) and the following assertions are true [13], [76]:

$$V(t,z) = \frac{1}{t-z}\left[I + Q(z)\frac{\bar t - \bar z}{t-z}\right]^{-1} + \frac{R_1(t,z)}{|t-z|^\alpha},$$
$$V(t,z) = \frac{1}{t-z}\left[I + Q(z)\frac{\bar t - \bar z}{t-z}\right]^{-1} + \frac{R_2(t,z)}{|t-z|^\alpha}, \quad \alpha \le 1,$$
$$R_1(t,z),\ R_2(t,z) \in H(\mathbb{C}\times\mathbb{C}), \quad R-1(z,z) = 0,$$
$$|V_{ik}(t,z)| \le \frac{\text{const}}{|t-z|}.$$

Next consider the generalized Cauchy-type integral defined by the matrix (1.8)

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma V(t,z) d_Q t \mu(t), \tag{1.9}$$

where Γ is a closed simple smooth curve, $\mu(t) \in L_1(\Gamma)$ and

$$d_Q t = I dt + Q(t) d\bar{t}.$$

If the density $\mu(t)$ in (1.9) is Hölder-continuous on Γ, the integral (1.9) is Hölder-continuous in $\bar{D}^+$ and $\bar{D}^-$ (D^+ and D^- are the domains bounded by Γ); the boundary values of Φ on Γ are given by

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\mu(t) + \frac{1}{2\pi i}\int_{\Gamma} V(\tau,t) d_Q\tau \mu(\tau). \tag{1.10}$$

If $\mu(t) \in L_p(\Gamma), p > 1$, then the formulas (1.10) are fulfilled almost everywhere on Γ, provided $\Phi^{\pm}$ are now understood as angular boundary values of the vector $\Phi(z)$. The analogues of the integral operators T and Π,

$$\begin{aligned} (\tilde{T}f)[z] &= -\frac{1}{\pi}\iint_D V(t,z) f(t) d\sigma_t, \\ (\tilde{\Pi}f)[z] &= -\frac{1}{\pi}\iint_D \partial_z V(t,z) f(t) d\sigma_t \end{aligned} \tag{1.11}$$

play an important role while studying generalized analytic vectors.

Let $Q \in H^{\alpha_0}(\mathbb{C})$. Then $(\tilde{T}f)$is a completely continuous operator from $L_p(\bar{D})$, $p > 2$, into $H^{\alpha}(D)$, $\alpha = \min\{\alpha_0, (p-2)/p\}$. Moreover, the operator $\tilde{\Pi}$ is a linear bounded operator from $L_p(\bar{D})$ into $L_p(\bar{D})$, and the relations

$$(\partial_{\bar{z}} - Q\partial_z)\tilde{T}f = f, \quad \partial_z \tilde{T}f = \widetilde{\Pi} f \tag{1.12}$$

are true.

Using Q-holomorphic vectors, generalized analytic vectors $w(z)$ can be represented as follows

$$\begin{aligned} w(z) = \Phi(z) + \iint_D \Gamma_1(z,t)\Phi(t) d\sigma_t + \\ + \iint_D \Gamma_2(z,t)\overline{\Phi(t)} d\sigma_t + \sum_{k=1}^{N} c_k w_k(z), \end{aligned} \tag{1.13}$$

where $\Phi(z)$ is a $Q-$ holomorphic vector, and $w_k(z)(k = 1,\dots,N)$ is a complete system of linearly independent solutions of the Fredholm equation

$$Kw \equiv w(t-z) - \frac{1}{\pi}\iint_D V(t,z)[A(t)w(t) + B(t)\overline{w(t)}] d\sigma_t = 0.$$

The $w_k(z)$ turn out to be continuous vectors in the whole plane vanishing at infinity, and the c_k's are arbitrary real constants; the kernels $\Gamma_1(z,t)$ and

$\Gamma_2(z,t)$, finally, satisfy the system of the integral equations

$$\begin{aligned}
&\Gamma_1(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\iint_D V(\tau,z)[A(\tau)\Gamma_1(\tau,t)+\\
&\qquad + B(\tau)\overline{\Gamma_2(\tau,t)}d\sigma_t = -\frac{1}{2}\sum_{k=1}^{N}\{v_k(z),\bar{v}_k(t)\},\\
&\Gamma_2(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\iint_D V(\tau,z)[A(\tau)\Gamma_2(\tau,t)+\\
&\qquad + B(\tau)\overline{\Gamma_1(\tau,t)}d\sigma_t = -\frac{1}{2}\sum_{k=1}^{N}\{v_k(z),\bar{v}_k(t)\},
\end{aligned} \tag{1.14}$$

where the $v_k(z) \in L_p(\bar{D})(k = 1, \ldots, N)$ form a system of linearly independent solutions of the Fredholm integral equation

$$v(z) + \frac{\overline{A'(z)}}{\pi}\iint_D \overline{V(z,t)}v(t)d\sigma_t + \frac{B'(z)}{\pi}\iint_D V'(z,t)\overline{v(t)}d\sigma_t = 0.$$

In the formulas (1.14) the curly bracket $\{v,w\}$ means the diagonal product of the vectors v and w: $\{v,w\}$ is the quadratic matrix of order n, whose elements $\{v,w\}_{ik}$ are defined by $\{v,w\}_{ik} = v_i w_k,\ \ i,k = 1,\ldots,n$.

Notice that in the formula (1.13) $\Phi(z)$ is not an arbitrary Q-holomorphic vector. It has to satisfy the conditions

$$\operatorname{Re}\iint_D \Phi(z)v_k(z)d\sigma_z = 0, \quad k = 1,\ldots,N. \tag{1.15}$$

Finally it should be mentioned that, generally speaking, the Liouville theorem is not true for solutions of (1.5). This explains the appearance of the constants c_k in the representation formula (1.13) and the fact that the condition (1.15) has to be satisfied (cf. [11], [44] and [33]).

2. Linear Conjugation Problems on a Cut Plane

2.1. **Factorization of matrix functions and solvability conditions.** Let Γ be a simple closed piecewise-smooth curve $\Gamma, a(t)$ and $b(t)$ be given $(n\times n)$ and $(n\times l)$ matrices respectively on Γ; $a(t)$ be a piecewise continuous matrix, $\inf|\det a(t)| > 0, b(t) \in L_p(\Gamma,\rho)$, $p > 1$, the weight function ρ have the form

$$\rho(t) = \prod_{k=1}^{r}|t-t_k|^{\nu_k}, \quad t_k \in \Gamma, \quad -1 < \nu_k < p-1. \tag{2.1}$$

The set $\{t_k\}$ contains all discontinuity points of the matrix $a(t)$, it may contain also other points of Γ. In many applications one needs to consider analogs of the Hilbert problem (1.2) or the linear conjugation problem (1.3) with discontinuous coefficient where one does **not** require that the boundary condition (1.2) or (1.3) is fulfilled *everywhere* [33], [49], [43].

In line with this, in this section by the boundary value problem of linear conjugation is understood the following problem: find an $(n \times l)$-matrix $\Phi(z) \in E_p^{\pm}(\Gamma, \rho)$ satisfying the boundary condition

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t) \tag{2.2}$$

almost everywhere on Γ.

It is convenient to introduce a version of Birkhoff factorization adjusted to the above problem.

Definition 2.1. A square matrix $\chi(z)$ of order n is said to be a *normal matrix* of the boundary value problem (2.2) (or for the matrix $a(t)$) if it satisfies the following conditions:

$$\chi(z) \in E_q^{\pm}(\Gamma, \rho), \quad \chi^{-1}(z) \in E_p^{\pm}(\Gamma, \rho^{1-q}), \quad q = \frac{p}{p-1},$$
$$\chi^+(t) = a(t)\chi^-(t)$$

almost everywhere on Γ.

A normal matrix $\chi(z)$ is called *canonical* if it has normal form at infinity, i.e. $\lim\limits_{z\to\infty} (z^{-\sigma} \det \chi(z))$ (σ is the sum of columns orders of $\chi(z)$) is finite and nonzero.

Since it is possible to consider various classes $E_p^{\pm}(\Gamma, \rho)$, we speak about the canonical (normal) matrices of the class $E_p^{\pm}(\Gamma, \rho)$. The matrix $a(t)$ is called *factorizable* in $E_p^{\pm}(\Gamma, \rho)$, if for $a(t)$ there exists a canonical matrix of the same class $E_p^{\pm}(\Gamma, \rho)$. It is easy to prove the following proposition.

Proposition 2.1 ([70]). *If $\chi_1(z)$ and $\chi_2(z)$ are normal* (*in particular canonical*) *matrices of the problem* (2.2) *of one and the same class, then $\chi_1(z) = \chi_2(z)P(z)$, where $P(z)$ is a polynomial matrix with the constant nonzero determinant.*

Consequently, the determinants of all normal (canonical) matrices of the given class of the boundary value problem (2.2) have the same orders at infinity. This enables one to define the *index* (or *total index*) of the problem (2.2) of class $E_p^{\pm}(\Gamma, \rho)$ (or the index of class $E_p^{\pm}(\Gamma, \rho)$ of the matrix $a(t)$) as the order at infinity of the determinant of any normal (canonical) matrix of the given class $E_p^{\pm}(\Gamma, \rho)$ taken with the opposite sign [60].

Having a normal matrix $\chi(z)$ of some class, we may obtain a canonical matrix by multiplying $\chi(z)$ from the right by the corresponding polynomial matrix with constant nonzero determinant. Let $\chi(z)$ be a canonical matrix (of the given class) for the matrix $a(t)$. Denote by $-\Upsilon_1, \dots, -\Upsilon_n$ the orders of the columns of $\chi(z)$ at infinity. The integers $\Upsilon_1, \dots, \Upsilon_n$ are called the *partial indices* of the matrix $a(t)$ or of the boundary value problem (2.2) (in the given class) [70]. The sum of the partial indices $\Upsilon_1 + \Upsilon_2 + \cdots + \Upsilon_n$ is equal to the index of $a(t)$ (or of the problem (2.2) in the given class).

Notice that if $\chi(z)$ is a canonical matrix of the class $E_p^{\pm}(\Gamma, \rho)$ of the matrix $a(t)$, then the matrix $[\chi'(z)]^{-1}$ will be a canonical matrix of the class

$E_p^{\pm}(\Gamma, \rho^{1-q})$ of the matrix $[a'(t)]^{-1}$. The following result is fundamental for the whole theory.

Theorem 2.1 ([49], [46]; cf. [15], [43], [21])**.** *Let $a(t)$ be a piecewise continuous nonsingular matrix with the points of discontinuity t_k $(k = 1, \dots, r,)$, let λ_{kj} $(k = 1, \dots, r, j = 1, \dots, n)$ be the roots of the characteristic equation* $\det(tI - a^{-1}(t_k - 0)a(t_k + 0)) = 0$ *and put* $\mu_{kj} = \arg \lambda_{kj}/2\pi$ *with* $0 \leq \arg \lambda_{kj} < 2\pi$.

If the inequalities

$$\frac{1+\nu_k}{p} \neq \mu_{kj} \tag{2.3}$$

are fulfilled, then there exists a canonical matrix of the problem (2.2) *of the class* $E_p^{\pm}(\Gamma, \rho)$ *and all solutions of the problem* (2.2) *in this class are given by the formula*

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int\limits_{\Gamma} \frac{[\chi^+(t)]^{-1} b(t) dt}{t - z} + \chi(z) P(z), \tag{2.4}$$

where $P(z)$ is an arbitrary polynomial $(n \times l)$-matrix. The index of the matrix $a(t)$ is calculated by the formula

$$\Upsilon = \frac{1}{2\pi} \left[\arg \frac{\det a(t)}{\prod\limits_{k=1}^{r} (t - z_0)^{\sigma_k}} \right]_{\Gamma}, \tag{2.5}$$

where $\sigma_k = \sum\limits_{j=1}^{r} \rho_{kj}$ *and*

$$\begin{aligned} 1 < \mathrm{Re}\rho_{kj} \leq 0 \quad &\text{if} \quad \mu_{kj} < \frac{1+\nu_k}{p}, \\ 0 \leq \mathrm{Re}\rho_{kj} < 1 \quad &\text{if} \quad \mu_{kj} > \frac{1+\nu_k}{p}, \end{aligned} \qquad \rho_{kj} = -\frac{1}{2\pi} \ln \lambda_{kj}.$$

Here $(z - z_0)^{\sigma_k}$ is a single-valued branch defined on the whole plane cut along the line l_k connecting the point $z_0 \in D^+$ with the point t_k and then with the point at infinity; the symbol $[\ ]_\Gamma$ denotes the increment of the expression contained in the brackets while passing Γ in the positive direction. Applying standard arguments of Fredholm theory, from Theorem 2.1 one derives the explicit solvability conditions.

Theorem 2.2. *If the conditions* (2.3) *are fulfilled, then for the problem* (2.2) *to be solvable in the class* $E_{p,0}^{\pm}(\Gamma, \rho)$ *it is necessary and sufficient the fulfillment of the conditions*

$$\int\limits_{\Gamma} b(t) \Psi^+(t) dt = 0, \tag{2.6}$$

where $\Psi(z)$ is an arbitrary solution of the adjoint homogeneous problem of class $E_{q,0}^{\pm}(\Gamma, \rho^{1-q})$.

Notice that the index of this problem is also available by the formula (2.4).

Remark 2.1. If $a(t) \in C(\Gamma)$, det $a(t) \neq 0$, then a canonical matrix $\chi(z)$ of any class $E_p^{\pm}(\Gamma, \rho)$ exists and has the following properties $\chi(z) \in E_p^{\pm}(\Gamma)$, $\chi^{-1}(z) \in E_p^{\pm}(\Gamma)$ for any p. If $a(t) \in H(\Gamma)$, then $\chi^{\pm}(t) \in H(\Gamma)$, $[\chi^{\pm}(t)]^{-1} \in H(\Gamma)$.

2.2. **Connection with singular integral equations.** We briefly describe now the well-known connection between linear conjugation problems and singular integral equations [60], [70]. Recall that the so-called *characteristic system of singular integral equations* is defined as the following system:

$$\sum_{\beta=1}^{n}[A_{\alpha\beta}(t_0)\varphi_\beta(t_o) + \frac{B_{\alpha\beta}(t_0)}{\pi i}\int\limits_\Gamma \frac{\varphi_\beta(t)dt}{t-t_0}] = f_\alpha(t_0), \quad \alpha = 1,\dots,n, \tag{2.7}$$

where $A_{\alpha\beta}$, $B_{\alpha\beta}$ are given piecewise-continuous functions on Γ, f_α are given functions on Γ of the class $L_p(\Gamma, \rho)$. We look for the solutions of the system (2.7) in $L_p(\Gamma, \rho)$.

Introducing the matrices and vectors

$$A = (A_{\alpha\beta}), \quad B = (B_{\alpha\beta}), \quad \varphi = (\varphi_1,\dots,\varphi_n), \quad f = (f_1,\dots,f_n),$$

we may rewrite (2.7) in the form

$$K^0\varphi \equiv A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int\limits_\Gamma \frac{\varphi(t)}{t-t_0}\,dt = f(t_0). \tag{2.8}$$

Let φ be a solution of the equation (2.7). Put

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(t)}{t-z}dt. \tag{2.9}$$

Then we have

$$\varphi(t) = \Phi^+(t) - \Phi^-(t), \quad \frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi(t)dt}{t-t_0} = \Phi^+(t_0) + \Phi^-(t_0). \tag{2.10}$$

If we substitute these values in the equation (2.8), we get

$$S(t)\Phi^+(t) = D(t)\Phi^-(t) + f(t), \tag{2.11}$$

where $S = A + B$, $D = A - B$.

Let

$$\inf|\det\ S(t)| > 0, \quad \inf|\det D(t)| > 0, \quad t \in \Gamma. \tag{2.12}$$

Then

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t), \tag{2.13}$$

where $a = S^{-1}D$, $b = S^{-1}f$.

Therefore the equation (2.8) is reduced to the boundary value problem (2.13): to every solution of (2.8) of the class $\text{Ł}_p(\Gamma, \rho)$ there corresponds a solution of the problem (2.13) of class $\text{Ł}_{p,0}^{\pm}(\Gamma, \rho)$ given by formula (2.9), and to every such solution of (2.13) there corresponds a solution of the equation (2.8) of the class $L_p(\Gamma,\rho)$ given by the formula (2.10). This connection between the equations and boundary value problems (2.2) gives us the possibility to establish the following result [49], [43].

Theorem 2.3. *Let the conditions* (2.12), (2.3) *be fulfilled. For the equation* (2.8) *to be solvable in the class* $L_p(\Gamma,\rho)$ *it is necessary and sufficient that*

$$\int\limits_{\Gamma} f(t)\psi(t)dt = 0, \tag{2.14}$$

where ψ *is an arbitrary solution of the class* $L_q(\Gamma,\rho^{1-q})$ *of the adjoint homogeneous equation*

$$K^{0\prime}\psi = A'(t_0)\psi(t_0) - \frac{1}{\pi i}\int\limits_{\Gamma}\frac{B'(t)\psi(t)}{t-t_0}dt = 0. \tag{2.15}$$

If the conditions (2.14) *are fulfilled, then all solutions of the equation* (2.8) *of the class* $L_p(\Gamma,\rho)$ *are given by the formula*

$$\begin{aligned}\varphi(t_0) = A^*(t_0)f(t_0) - \frac{B^*(t_0)Z(t_0)}{\pi i}\int\limits_{L}\frac{[Z(t)]^{-1}f(t)}{t-t_0}dt+\\ + B^*(t_0)Z(t_0)P(t_0),\end{aligned} \tag{2.16}$$

$$A^*(t_0) = \frac{1}{2}[S^{-1}(t)+D^{-1}(t)], \quad B^*(t) = -\frac{1}{2}[S^{-1}(t)-D^{-1}(t)],$$

$$Z(t_0) = S(t)\chi^+(t) = D(t)\chi^-(t),$$

where $\chi(z)$ *is a canonical matrix of the class* $E_p^{\pm}(\Gamma,\rho)$ *for the matrix* $a(t) = S^{-1}D$, $P(t)$ *is the vector*

$$P(t) = (P_{\Upsilon_1-1},\ldots,P_{\Upsilon_n-1}),$$

where $P_\alpha(t)$ *denotes arbitrary polynomials of order not exceeding* α, $P_\alpha(t) = 0$ *when* $\alpha < 0$.

The difference between the number l *of linearly independent solutions of the homogeneous equation* $K^0\varphi = 0$ (*in* $L_p(\Gamma,\rho)$) *and the number* l' *of linearly independent solutions of the adjoint homogeneous equation* $K^{0\prime}\psi = 0$ (*in* $L_q(\Gamma,\rho^{1-q})$) *is equal to the index of the matrix* $a = S^{-1}D$ *in the class* $E_p^{\pm}(\Gamma,\rho)$*:*

$$l - l' = \Upsilon.$$

This result is a cornerstone of solvability theory for singular integral equations. It is also necessary in our approach because in the next section we will reduce Riemann–Hilbert problems on a cut plane to systems of singular integral equations.

Remark 2.2. Using the properties of the solutions of the boundary value problem of linear conjugation, the following proposition can be proven:

if the coefficients and free terms of the equations mentioned in this section are Hölder-continuous, then the solutions of any class are also Hölder-continuous, and if the coefficients and free terms belong to the class $H_0(\Gamma)$, then the solutions of any class belong to the class $H^*(\Gamma)$.

2.3. **Linear conjugation problems with shift.** Let $\Gamma_k (k = 1, 2)$ be simple smooth curves bounding the finite and infinite domains D_k^+ and D_k^- in the plane of complex variable $z = x + iy$. The problem considered in this section is to find a vector $\varphi(z) = (\varphi_1, \dots, \varphi_n) \in E_p^\pm(\Gamma_1, \Gamma_2)$ satisfying the boundary condition

$$\varphi^+[a(t)] = a(t)\varphi^-(t) + b(t) \tag{2.17}$$

almost everywhere on Γ. Here $a(t)$ is a given continuous non-singular square matrix of order n, $b(t) = (b_1, \dots, b_n)$ is a given vector-function on Γ of the class $L_p(\Gamma_1), p > 1, \alpha(t)$ is a function which maps Γ_1 onto Γ_2 in one-to-one manner keeping the orientation, $\alpha(t)$ has the non-zero continuous derivative $\alpha'(t)$. We call a square matrix of order n a canonical matrix of the boundary problem (2.17) if the following properties hold:

1) $\chi(z), \chi^{-1}(z) \in E_\infty^\pm(\Gamma_1, \Gamma_2)$;
2) χ satisfies the homogeneous boundary condition

$$\chi^+[\alpha(t)] = \chi^-(t), \quad t \in \Gamma_1;$$

3) χ has a normal form at infinity with respect to columns.

The orders at infinity of the columns of the canonical matrix taken with opposite sign are the partial indices and the sum of the partial indices is the index of the problem (2.17). The following result is well-known [60], [70].

Theorem 2.4. *For the boundary value problem* (2.17) *with an arbitrary continuous non-singular matrix $a(t)$, there exists a canonical matrix as above.*

In case of piecewise-continuous coefficients an analog of Theorem 2.1 holds.

Theorem 2.5. *Let $a(t)$ be a piecewise continuous matrix with the discontinuity points $t_k (k = 1, \dots, r)$, $\inf|\det a(t)| > 0$ and let λ_{kj} $(k = 1, \dots, r,$ $j = 1, \dots, n)$ be the roots of the equation*

$$\det[a^{-1}(t_k - 0)a(t_k + 0) - I] = 0,$$
$$\mu_{kj} = \arg\lambda_{kj}/2\pi, \quad 0 \le \arg\lambda_{kj} < 2\pi.$$

If the inequalities

$$\frac{1 + \nu_k}{p} \neq \mu_{kj}$$

are valid, then there exists a canonical matrix of the problem (2.17) *of the class $E_p^\pm(\Gamma_1, \Gamma_2, \rho)$ and the index in this class is calculated by the formula*

$$\Upsilon = \frac{1}{2\pi}\left\{\arg[\prod_{k=1}^{r}(t - z_0)^{-\sigma_k}\det\, a(t)]\right\}_{\Gamma_1},$$

where $\sigma_k = \sum\limits_{j=1}^{n} \rho_{kj}$, $\rho_{kj} = -\dfrac{1}{2\pi i} \ln \lambda_{kj}$;

$$-1 < \operatorname{Re} \rho_{kj} \le 0 \text{ when } \mu_{kj} < (1+\nu_k)/p,$$
$$0 \le \operatorname{Re} \rho_{kj} < 1 \text{ when } \mu_{kj} > (1+\nu_k)/p.$$

All solutions of the class $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho)$ *of the problem* (2.17) *are given by the formula*

$$\Phi(z) = \chi(z)[\varphi_0(z) + P(\omega(z))], \tag{2.18}$$

where P *is an arbitrary polynomial vector,* $\varphi_0(z)$ *is a solution of the class* $E^{\pm}_{1+\varepsilon,0}(\Gamma_1, \Gamma_2)$ (ε *is a sufficiently small positive number*) *of the problem*

$$\varphi_0^+[\alpha(t)] = \varphi_0^-(t) + b_0(t), \ \ t \in \Gamma_1, \quad b_0(t) = \{\chi^+[\alpha(t)]\}^{-1} b(t). \tag{2.19}$$

The solutions vanishing at infinity are given by the same formula (2.18) *in which* $P = (P_{\Upsilon_1 - 1}, \dots, P_{\Upsilon_n - 1})$, $\Upsilon_1 \ge \cdots, \ge \Upsilon_s$ *are the positive indices of the problem* (2.17) *of the class* $E_p^{\pm}(\Gamma_1, \Gamma_2, \rho), P_j(z)$ *is an arbitrary polynomial of order* j ($P_j = 0$ *if* $j < 0$). *If all partial indices are non-negative, then vanishing solutions exist for any* $b(t) \in L_p(\Gamma_1, \rho)$; *if* $0 > \Upsilon_{s+1} \ge \cdots \ge \Upsilon_n$, *then the vector* $b(t)$ *has to satisfy the following conditions:*

$$\int\limits_{\Gamma_k} t^k \rho_j^0(t) dt = 0, \quad k = 0, \dots, |\Upsilon_j| - 1, \quad j = s+1, \dots, n, \tag{2.20}$$

where the vector $(\rho_1^0, \dots, \rho_n^0) = \rho_0$ *is a solution of the equation* $K(I)\rho_0 = \tilde{b_0}$ *of the class* $L_q(\Gamma, \rho^{1-q})$ *or* $\rho_0 = L_1 b_0$.

Remark 2.3. An analogous theorem holds in the case where the boundary condition contains complex conjugate values of the desired functions.

2.4. **Connection with generalized analytic functions.** In the rest of this section we set a relation between the problem of linear conjugation with displacement and the theory of generalized analytic functions. This will enable us to consider the problem of linear conjugation in somewhat different formulation than above.

Let Γ_1 and Γ_2 be the Lyapunov curves, $\alpha(t)$ be a function mapping Γ_1 onto Γ_2 in one-to-one manner preserving the orientation, $\alpha(t(s))$ be an absolutely continuous function, $M \ge |\alpha'(t)| \ge m > 0$ (M, m are constants), $a(t), b(t)$ be given matrices of the class $H^{\mu}(\Gamma_1)(\mu > \frac{1}{2})$, $a(t)$ be a nonsingular quadratic matrix of order n, $b(t)$ be an $(n \times l)$-matrix; we have to find a piecewise-holomorphic matrix $\varphi(z)$ having the finite order at infinity, satisfying $\varphi^+(t), \varphi^-(t) \in H(\Gamma)$ and the boundary condition

$$\varphi^+[\alpha(t)] = a(t)\varphi^-(t) + b(t), \quad t \in \Gamma_1. \tag{2.21}$$

We call the piecewise-holomorphic matrix $\chi(z)$ with a finite order at infinity the canonical matrix of the problem (2.1) if $\det \chi(z) \ne 0$ everywhere except perhaps at the point $z=\infty$, $\chi(z)$ has a normal form at infinity with respect to columns and

$$\chi^{+}[\alpha(t)] = a(t)\chi^{-}(t), \quad t \in \Gamma_1.$$

Mapping conformally D_2^+ and D_1^- into interior and exterior parts of the unit circle Γ respectively we get the same problem as (2.1), where $\alpha(t)$ has to map Γ onto Γ; the matrices $a(t)$, $b(t)$ and the function have the same properties. We will consider the problem in the case $\Gamma_1 = \Gamma_2 = \Gamma$.

First prove the following lemmas.

Lemma 2.1. *Let $\alpha(t)$ be a function satisfying the same conditions as mentioned above and $\omega(z)$ be a piecewise holomorphic function (bounded at infinity) $\omega^{+}[\alpha(t)] = \omega^{-}(t)$ on Γ, $\omega^{-}(t) \in H^{*}(\Gamma)$. Then $\omega(z)$ is a constant function.*

Proof. Consider the following function which is continuous on the whole plane

$$\Omega(z) = \begin{cases} \omega(\alpha(z)), & z \in \overline{D}^{+}, \\ \omega(z), & z \in D^{-}, \end{cases}$$

where

$$\alpha(z) = |z|\alpha\left(\frac{z}{|z|}\right). \tag{2.22}$$

On the basis of the Hardy–Littlewood theorem (see [40], [27]) we have

$$|\omega'(z)| \le A(1-|z|)^{\mu-1}, \quad z \in D^{+},$$
$$|\omega'(z)| \le A(|z|-1)^{\mu-1}, \quad z \in D^{-},$$

A is a constant.

Therefore,

$$\partial_z \Omega, \ \ \partial_{\bar z}\Omega \in L_p(\mathbb{C}), \quad 1 < p < (1-\mu)^{-1}.$$

Denoting by $w_0(z)$ the fundamental homeomorphism of the Beltrami equation

$$\begin{gathered} \partial_{\bar z} w - q(z)\partial_z w = 0, \\ q(z) = \partial_{\bar z}\alpha/\partial_z\alpha, \ \ z \in D^{+}, \quad q = 0, \quad z \in D^{-}, \end{gathered} \tag{2.23}$$

we obtain

$$\Omega(z) = \Phi(w_0(z)),$$

where $\Phi(z)$ is a holomorphic function on the whole finite plane. $\Omega(z)$ is a bounded function, that's why $\Phi(z) = \text{const}$, $\Omega(z) = \text{const}$ and the lemma is proved. In fact, if we replace the boundedness condition at infinity by the condition $\omega(z) = z + O(z^{-1})$, then we get a piecewise holomorphic function univalent in the domains D^{+} and D^{-}. □

Lemma 2.2. *Let Γ be a simple closed smooth curve, $a(t)$ be a nonsingular quadratic matrix of order n, $a(t) \in H^\mu(\Gamma)$, $\mu < 1$. If $a(t)$ is sufficiently close to the unit matrix I, i.e. if*

$$\|a_k\|_{H^\mu} \leq \varepsilon < \frac{1}{n(1+s_\mu)}, \quad k = 1, 2, \quad a_1 = \frac{1}{2}(a - I), \quad a_2 = \frac{1}{2}({a'}^{-1} - I),$$

where s_μ is a norm of the operator $\frac{1}{\pi i}\int\limits_\gamma \varphi(t)(t-t_0)^{-1}dt$ in the space $H^\mu(\Gamma)$, then for $a(t)$ there exists a canonical matrix $\chi(z)$ close to the unit matrix:

$$\chi^+(t) = a(t)\chi^-(t), \quad \chi(z) = I + \zeta_1(z), \quad \chi^{-1}(z) = I + \zeta_2(z),$$
$$\zeta_1(\infty) = \zeta_2(\infty) = 0, \quad |\zeta_k^+(t)|_\mu \leq C\varepsilon,$$

where the constant C depends only on n, μ and the curve Γ.
Proof. Consider the singular integral equations in $H^\mu(\Gamma)$:

$$(I + a_k)\varphi_k - a_k S\varphi_k = I + 2a_k, \quad k = 1, 2.$$

It is easy to see that these equations are solvable and also

$$\varphi_k = I + \varphi_k, \quad \|\varphi_k\|_{H^\mu} \leq \varepsilon + \frac{n\varepsilon(1+\varepsilon)(1+S_\mu)}{2 - n\varepsilon(1+S_\mu)} = \eta.$$

Introduce the piecewise holomorphic matrices

$$\chi_k(z) = \begin{cases} \rho_k(z), & z \in D^+, \\ \rho_k(z) + I, & z \in D^-, \end{cases} \quad \rho_k = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi_k(t)dt}{t - z}, \quad k = 1, 2,$$

where D^+, D^- are finite and infinite domains, bounded by Γ.

We have

$$\chi_1^+(t) = a(t)\chi_1^-(t), \quad \chi_2^+(t) = [a'(t)]^{-1}\chi_2^-(t),$$
$$\det \chi_1^+\chi_2^+ = \det \chi_1^-\chi_2^-, \quad \det[\chi_1\chi_2] \equiv 1, \quad \chi_1^{-1} = \chi_2^1.$$

Hence $\chi_1(z)$ is a canonical matrix for $a(t)$. Assuming

$$\chi_1 - I = \zeta_1, \quad \chi_1^{-1} - I = \zeta_2,$$

we obtain

$$|\zeta_k^\pm(t)| \leq \frac{1}{2}(1 + S_\mu).$$ □

Corollary. *In the particular case where γ is a unit circle, in virtue of the Hardy–Littlewood theorem, under the conditions of Lemma 2.2 for the canonical matrix $\chi(z) = \chi_{ik}(z)$ constructed above we have*

$$\left|\frac{d\chi_{ik}(z)}{dz}\right| \leq \frac{M_1\varepsilon}{(1-|z|)^{1-\mu}}, \quad z \in D^+, \quad \iint\limits_{D^+} \left|\frac{d\chi_{ik}(z)}{dz}\right| dxdy \leq M_2\varepsilon^p,$$
$$1 < p < (1-\mu)^{-1}, \quad i, k = 1, \ldots, n,$$

where the constants M_k depend only on n and μ.

Lemma 2.3. *If the matrix $a(t)$ is sufficiently close to the unit matrix, then there exists a canonical matrix for the problem* (2.1).

Proof. First show that we may construct one of the canonical matrices $\chi_\alpha(z)$ by the formulas

$$\chi_\alpha[\alpha(z)] = \chi(z)[If + I], \ \ z \in D^+, \ \ \chi_\alpha(z) = \chi(z)[If + I], \ \ z \in D^-,$$

where $\chi(z)$ is a canonical matrix when $\alpha(t) = t$, $\chi(\infty) = I$ and f is a solution (unique) of the two-dimensional singular integral equation

$$f(z) - q(z)\Pi f - ATf = A, \ \ f \in L_p(\overline{D^+}); \tag{2.24}$$

$A = q\chi^{-1}\dfrac{\partial\chi}{\partial z}$, $\alpha(z)$ and $q(z)$ are defined by the formulas (2.2) and (2.3).

If $\|a - I\|_{H^\mu} = \varepsilon$ is small, then there exists a matrix $\chi(z)$ with the properties from Lemma 3.2. Since $\mu > \frac{1}{2}$, we may take p from the interval $(2, (1-\mu)^{-1})$.

The operator ATf is a linear bounded operator transforming $L_p(D)$ into itself and also its norm is not greater than $M\varepsilon$, the constant M depends only on n and μ.

If we take ε sufficiently small, then the equation (2.4) has a unique solution $f \in L_p(D)$.

The matrix $w(z) = Tf$ is Hölder-continuous on the whole plane, vanishes at infinity and satisfies the following equation

$$\partial_{\bar z} w - q(z)\partial_z w - A(z)w = A(z).$$

Assuming $w_1(z) = \chi(z)[w(z) + I]$, $z \in D^+$, we obtain that $w_1(z)$ satisfies the equation

$$\partial_{\bar z} w_1 - q(z)\partial_z w_1 = 0$$

in D^+ and therefore

$$w_1(z) = \varphi_1[\alpha(z)], \quad z \in D^+,$$

where $\varphi_1(z)$ is a holomorphic matrix in D^+.

If we define the holomorphic matrix in D^- by the formula

$$\varphi_1(z) = \chi(z)[w(z) + I],$$

then we have

$$\varphi_1^+[\alpha(t)] = a(t)\varphi_1^-(t), \quad t \in \Gamma, \quad \varphi_1(\infty) = I.$$

We are able to construct the solution of the boundary value problem

$$\varphi_2^+[\alpha(t)] = a^{-1}(t)\varphi_2^-(t), \quad t \in \Gamma, \quad \varphi_2(\infty) = I$$

analogously since we have

$$\det[\varphi_1^+(\alpha(t))\varphi_2^+(\alpha(t))] = \det[\varphi_1^-(t)\varphi_2^-(t)], \quad t \in \Gamma,$$
$$\det[\varphi_1(z)\varphi_2(z)] \equiv 1,$$

and $\varphi_1(z)$ is a canonical matrix for the problem (2.1). □

Lemma 2.4. *There exists a canonical matrix of the problem* (2.1) *for an arbitrary matrix* $a(t)$ (*satisfying the above indicated conditions*)*; it is possible to construct one of them by the formulas*

$$\begin{aligned} \chi_\alpha[\alpha(z)] &= \chi^0_\alpha[\alpha(z)]R(w_0(z)), \quad z \in D^+, \\ \chi_\alpha(z) &= r(z)\chi^0_\alpha(z)R(w_0(z)), \quad z \in D^-, \end{aligned} \tag{2.25}$$

where $r(z)$ *and* $R(z)$ *are appropriately chosen matrices,* $\chi^0(z)$ *is a canonical matrix of the boundary condition*

$$\varphi^+[\alpha(t)] = a_0(t)\varphi^-(t), \quad a_0 = ar,$$

$w_0(z)$ *is the fundamental homeomorphism of the Beltrami equation*

$$\partial_{\bar z} w - q(z)\partial_z w = 0.$$

Proof. Let us choose the rational matrix $r(z)$ such that the matrix $a_0(t) = a(t)r(t)$ is close to the unit matrix; denote by $\chi_0(z)$ a canonical matrix of the problem

$$\varphi^+[\alpha(t)] = a_0(t)\varphi^-(t), \ \ t \in \Gamma.$$

Consider the piecewise-meromorphic matrix defined in the form

$$\begin{aligned} \chi_\alpha[\alpha(z)] &= \chi^0_\alpha[\alpha(z)]R(w_0(z)), \quad z \in D^+, \\ \chi_\alpha(z) &= r(z)\chi^0_\alpha(z)R(w_0(z)), \quad z \in D^-, \end{aligned} \tag{2.26}$$

where $R(z)$ is a rational matrix.

The boundary values of this matrix satisfy the homogeneous boundary condition; it is possible to choose the matrix R such that the matrix defined by (2.26) has to be a canonical matrix of the problem (2.21). □

The following theorem holds from these propositions.

Theorem 2.6. *All solutions of the problem* (2.21) *are given by the formulas*

$$\begin{aligned} \varphi[\alpha(z)] &= \chi_\alpha[\alpha(z)][Tf + h(z) + P(w_0(z))], \quad z \in D^+, \\ \varphi(z) &= \chi_\alpha(z)[Tf + h(z) + P(w_0(z))], \quad z \in D^-, \end{aligned} \tag{2.27}$$

where $P(z)$ *is an arbitrary polynomial vector and the vector* $f \in L_p(\bar{D^+})$, ($p > 2$) *is a solution (unique) of the equation*

$$Kf =: f(z) - q(z)\Pi f = g(z);$$

$$h(z) = \frac{1}{2\pi i}\int\limits_\gamma \frac{[\chi^+_\alpha(\alpha(t))]^{-1}b(t)}{t-z}dt,$$

$$g(z) = (g_1, \dots, g_n) = q(z)h'(z) \in L_p(\bar{D^+}).$$

The solutions vanishing at infinity are given by the formulas (2.27) *in which*

$$P(z) = (P_{\Upsilon_1-1}, \dots, P_{\Upsilon_n-1}), \quad \Upsilon_1 \ge \dots \ge \Upsilon_n,$$

are the partial indices of the problem (2.21), $P_j(z)$ is an arbitrary polynomial of order j ($P_j(z)=0$ if $j<0$); if $0\geq \Upsilon_{s+1}\geq\cdots\geq\Upsilon_n$, then the vector $b(t)$ has to satisfy the following conditions:

$$2i\int\int\limits_{D} g_j(\zeta)L(\zeta^k)d\zeta d\eta=\int\limits_{\gamma} t^k\{[\chi_\alpha^+(\alpha(t))]^{-1}b(t)\}_j dt,$$
$$j=s+1,\dots,n;\quad k=0,\dots,|\Upsilon_j|-1,$$

where L is the operator adjoint to K^{-1}:

$$Lf=f(z)-\Pi(qf).$$

Along with the problem (2.21), consider the set of problems

$$\begin{gathered}\varphi^+[\alpha_\lambda(t)]=a(t)\varphi^-(t)+b(t),\quad t\in\Gamma,\\ \alpha_\lambda(t)=\exp[iV_\lambda(\theta)],\quad V_\lambda(\theta)=(1-\lambda)\theta+\lambda V(\theta),\quad 0\leq\lambda<1,\end{gathered}\tag{2.28}$$

where $a(t),b(t)$ satisfy the conditions of the problem (2.21), $V(\theta)$ is a continuous strongly increasing function on $[0,2\pi]$ satisfying the conditions mentioned above.

Denote the partial indices of the problem (2.28) by

$$\Upsilon_1(\lambda)\geq\cdots\geq\Upsilon_\alpha(\lambda),$$

the sum of non-negative (non-positive) partial indices by $n^+(\lambda)(-n^-(\lambda))$ and also by

$$\delta_1\geq\cdots\geq\delta_s\geq 0>\delta_{s+1}\geq\cdots\geq\delta_n$$

the partial indices of the problem (2.28) in the case where $\lambda=0$.

We obtain

$$n^+(\lambda)-n^-(\lambda)=\frac{1}{2\pi}[\arg\delta(t)]_\Gamma.$$

Introduce the following vector

$$W(z)=\begin{cases}\chi^{-1}(z)\varphi[\alpha_\lambda(z)]-h(z), & z\in D^+,\\ \chi^{-1}(z)\varphi(z)-h(z), & z\in D^-,\end{cases}$$

where $\chi(z)$ denotes a canonical matrix of the problem (2.8) with $\lambda=0$,

$$h(z)=(h_1,\dots,h_n)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{[\chi^+(t)]^{-1}b(t)}{t-z}\,dt,$$
$$\alpha_\lambda(z)=\alpha_\lambda(e^{i\theta}).$$

The vector $W(z)$ is continuous on the whole plane, is holomorphic in D^- and may have a pole at infinity; $W(z)$ satisfies the equation

$$\begin{aligned}
&\partial_{(\overline{z})} w - q(z,\lambda)\partial_z w + A(z,\lambda)w = B(z,\lambda),\\
&q(z,\lambda) = \lambda e^{2\pi\theta}\frac{1-V'(\theta)}{2-\lambda+\lambda V(\theta)},\quad z\in D^+,\quad q(z,\lambda)=0,\quad z\in D^-,\\
&A(z,\lambda) = -q(z,\lambda)\chi^{-1}(z)\frac{d\chi}{dz},\\
&B(z,\lambda) = q(z,\lambda)[h'(z)+\chi^{-1}(z)\frac{d\chi}{dz}h(z)].
\end{aligned}\tag{2.29}$$

We have to find a vanishing at infinity solution of the problem (2.9). According to this, suppose the solution of the problem (2.9) has the form:

$$W(z) = P(z) + Tf,$$

where $f = (f_1,\ldots,f_n) \in L_p((\overline{D}))$, $p > 2$, $P(z) = (P_1,\ldots,P_n)$, $P_j(z)$ is an arbitrary polynomial of order n ($P_j(z) = 0$, if $j < 0$).

With respect to f we obtain the equation

$$\begin{aligned}
K_\lambda f \equiv f(z) - q(z,\lambda)\Pi f + A(z,\lambda)Tf &=\\
= B(z,\lambda) + q(z,\lambda)P(z) - A(z,\lambda)P(z)&,
\end{aligned}\tag{2.30}$$

and the following conditions

$$\iint\limits_{D^+} \xi^k f_j(\xi)d\xi d\eta + \pi a_{jk} = 0,\quad j = s+1,\ldots,;\quad k = 0,\ldots,|\delta_j|-1,\tag{2.31}$$

where a_{jk} are the coefficients of the expansion of $h_j(z)$ in the neighborhood of the point $z = \infty$:

$$h_j(z) = \sum_{k=0}^{\infty} a_{jk} z^{-k-1}.$$

In the case where the partial indices δ_j are non-negative, the conditions (2.11) are eliminated.

If for given λ the operator K_λ has the inverse operator K_λ^{-1}, then one may rewrite the conditions (2.11) in the following form:

$$\iint\limits_{D^+} \xi^k g_j(\xi,\lambda)d\xi d\eta + \pi a_{jk} = 0,\tag{2.32}$$

where $g_j(\zeta,\lambda)$ denotes the j-th component of the vector $K_\lambda^{-1}[B+qP'-AP]$.

The equality (2.32) is a linear algebraic system with respect to the coefficients of the polynomials $P_j(z)$.

It is easy to see that there exists a domain D_λ of the plane λ containing the segment $[0,1]$ of the real axis in which $q(z,\lambda)$ is a holomorphic function with respect to λ and in which the inequality

$$|q(z,\lambda)| \le q_0 < 1$$

is fulfilled. The operator K_λ analytically depends on λ in D_λ. As T is a completely continuous operator, therefore K_λ has an inverse operator for an arbitrary $\lambda \in D_\lambda$ except may be the points of some isolated set D^1_λ [36], [75]. For $\lambda \in D_\lambda \backslash D^1_\lambda$ the coefficients of the linear system (2.12) are analytic functions of λ; consequently the corresponding homogeneous system has the same number of linear independent solutions for all $\lambda \in D_\lambda \backslash D^1_\lambda$, except possibly the points of some isolated set. Hence, the following result takes place.

Theorem 2.7. *$n^+(\lambda)$ and $n^-(\lambda)$ have the same values for all $\lambda \in [0,1]$ except possibly the points of some finite set.*

Corollary. *The partial indices $\Upsilon_i(\lambda)$ are admitting constant values for all $\lambda \in [0,1]$ and*

$$\delta_1 \geq \Upsilon_1(\lambda) \geq \cdots \geq \Upsilon_n(\lambda) \geq \delta_n.$$

If $\delta_1 - \delta_n \leq 1$, then for all $\lambda \in [0,1]$ except possibly the points of some finite set there holds $\Upsilon_i(\lambda) = \delta_i$, $i = 1, \dots, n$.

Remark 2.4. As the following example shows, there exists an exceptional set. Suppose $a(t)$ has the form

$$a(t) = \begin{pmatrix} 1 + 2t^2 + 4t & 4t^2 \\ -2t & 1 - 2t \end{pmatrix},$$

$\alpha_1(t) = e^{i\nu(\theta)}$ is defined by the equality

$$\omega[\alpha_1(t)] = t + 1/4t,$$

where $\omega(z)$ conformally maps the circle $|z| < 1$ onto the interior of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{16}\,.$$

It is easy to verify that

$$\delta_1 = \delta_2 = 0, \quad \Upsilon_1(1) = 1, \quad \Upsilon_2(1) = -1.$$

3. Riemann–Hilbert Problems on a Cut Plane

Let D denote the plane of complex variable $z = x+iy$ cut along some non-intersecting simple open Lyapunov-smooth arcs $a_k b_k$, $k = 1, \dots, m$. Denote $\Gamma_k = a_k b_k$ and $\Gamma = \bigcup\limits_{k=1}^{m} \Gamma_k$.

Consider a function of the form

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(t)dt}{t - z} + P(z), \tag{3.1}$$

where $f(t) \in L_p(\Gamma, \rho)$, $p > 1$, the weight function

$$\rho(t) = \prod_{k=1}^{m} |t - c_k|^{\alpha_k}, \quad -1 < \alpha_k < p - 1, \quad c_{2k-1} = a_k, \qquad c_{2k} = b_k \quad (k = 1, \dots, m), \tag{3.2}$$

$P(z)$ is an arbitrary polynomial. Denote by $E^{\pm}_{p,0}(\Gamma, \rho)$ $(E^{\pm}_{p,c}(\Gamma, \rho))$ the subclass of this class containing the functions of the form (3.1) in case $P(z) \equiv 0$ $(P(z) \equiv \mathrm{const})$.

Consider the following version of the Hilbert problem (cf. Section 1). Find a vector $\Phi(z) \in E^{\pm}_{P,0}(\Gamma, \rho)$ satisfying the boundary condition

$$\mathrm{Re}[A_{\pm}(t)\Phi^{\pm}(t)] = f^{\pm}(t) \tag{3.3}$$

almost everywhere on Γ; here $A_+(t)$, $A_-(t)$ are given continuous non-singular quadratic matrices of order n on Γ, $f^+(t) = (f_1^+, \dots, f_n^+)$, $f^-(t) = (f_1^-, \dots, f_n^-)$ are given real vectors on Γ belonging to the class $L_p(\Gamma, \rho)$, $\Phi^+(t)$, $\Phi^-(t)$ denotes the boundary values of the vector $\Phi(z)$ on Γ from the left and from the right.

Along with this problem, let us consider the following homogeneous problem

$$\mathrm{Re}[A^*_{\pm}(t)\Psi^{\pm}(t)] = 0, \quad t \in \Gamma\,, \tag{3.4}$$

where $A^*_{\pm}(t) = t'(s)[A'_{\pm}(t)]^{-1}$. The solution of this problem $\Psi(z)$ will be sought in the class $E^{\pm}_q(\Gamma, \rho^{1-q})$, $q = \frac{p}{p-1}$. We call the problem (3.4) the conjugate problem to the problem (3.3). From the boundary condition we have

$$\Phi^{\pm}(t) = [A_{\pm}(t)]^{-1}[f^{\pm}(t) + i\mu^{\pm}(t)], \tag{3.5}$$

where $\mu^+(t)$, $\mu^-(t)$ are real vectors of the class $L_p(\Gamma, \rho)$ and

$$\Phi(z) = \frac{1}{2\pi} \int_{\Gamma} \frac{A_+^{-1}(t)\mu^+(t) - A_-^{-1}(t)\mu^-(t)}{t - z}\, dt + F(z). \tag{3.6}$$

Here

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{A_+^{-1}(t)f^+(t) - A_-^{-1}(t)f^-(t)}{t - z}\, dt. \tag{3.7}$$

Substituting the formulas (3.5), (3.6) in the boundary condition (3.3) and introducing the vector with $2n$ components $\mu(t) = (\mu^+(t), \mu^-(t))$, we get the following system of singular integral equations

$$a(t_0)\mu(t_0) + \frac{1}{\pi i} \int_{\Gamma} \frac{K(t_0, t)\mu(t)}{t - t_0}\, dt = g(t_0), \tag{3.8}$$

where

$$a = \mathrm{Im} \begin{pmatrix} A_- A_+^{-1} & 0 \\ 0 & -A_+ A_-^{-1} \end{pmatrix}, \quad K(t_0, t) = \frac{i}{2} \begin{pmatrix} N_+^1 & N_-^1 \\ N_+^2 & N_-^2 \end{pmatrix}, \tag{3.9}$$

and the matrices $N^1_\pm$, $N^2_\pm$ are defined by the formulas:

$$\begin{aligned} N^1_\pm(t_0,t) &= A_-(t_0)A^{-1}_\pm(t) + \overline{A_-(t_0)}\ \overline{A^{-1}_\pm(t)}h(t_0,t),\\ N^2_\pm(t_0,t) &= A_+(t_0)A^{-1}_\pm(t) + \overline{A_+(t_0)}\ \overline{A^{-1}_\pm(t)}h(t_0,t), \end{aligned} \tag{3.10}$$
$$g=(g^+,g^-),\ g^\pm = 2\operatorname{Re} f^\pm - 2\operatorname{Re}[A_\pm F^\pm],\ h(t_0,t)=\frac{t-t_0}{\overline{t}-\overline{t}_0}\,\overline{t}^{\,2}.$$

We have

$$a(t)+b(t) = i\begin{pmatrix} \overline{A_-}\ \overline{A^{-1}_+} & I \\ I & A_+A^{-1}_- \end{pmatrix},$$
$$a(t)-b(t) = -i\begin{pmatrix} A_-A^{-1}_+ & I \\ I & \overline{A_+}\ \overline{A^{-1}_-} \end{pmatrix}.$$

Denote

$$Q(t)\equiv \det(\overline{A_-}\ \overline{A^{-1}_+}A_+A^{-1}_- - I).$$

Then the following result takes place.

Theorem 3.1. *If $Q(t)\neq 0$, then the index of the problem* (3.3) *in the class $E^\pm_{p,o}(\Gamma,\rho)$ is equal to the index of the equation* (3.8) *of the class $L_p(\Gamma,\rho)$ (under the condition that $1+\alpha_k\neq p\,\mu^{(k)}_j$, where $\mu^{(k)}_j=\frac{\arg\lambda^{(k)}_j}{2\pi}$, $0\leq \arg\lambda^{(k)}_j<2\pi$, $\lambda^{(k)}_j$ are the roots of the equations $\det[H(a_k)-\lambda I]=0$ or $\det[H^{-1}(b_k)-\lambda I]=0$, for odd and even k correspondingly; $H(t)=[a(t)+b(t)]^{-1}[a(t)-b(t)]$). The necessary and sufficient solvability conditions for the problem* (3.3) *in the class $E_{p,o}(\Gamma,\rho)$ have the form*

$$\operatorname{Im}\int_\Gamma [A^{-1}_+ f^+\omega^+_k(t) - A^{-1}_-(t)f^-\omega^-_k(t)]dt = 0,\quad k=1,\dots,l',$$

where $\omega_k(z)$ is a complete system of linearly independent solutions of the problem (3) *in the class $E^\pm_{q,0}(\Gamma,\rho^{1-q})$.*

In the similar way one can treat some differential boundary value problem of the Riemann–Hilbert–Poincaré type [54], [55]. Let us add that the Riemann–Hilbert problems in a cut plane appear in some problems of elasticity theory [28], so our results can be applied, e.g., for explicating the asymptotics of solutions to those problems.

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Author's address:

Department of General Mathematics No. 63
Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: ninomanjavidze@yahoo.com