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A REPRESENTATION OF SOLUTIONS FOR A SYSTEM OF COMPLEX DIFFERENTIAL EQUATIONS IN THE PLANE AND PERIODIC SOLUTIONS

Abstract. In this article, first we will obtain a representation of the solutions for the system of complex differential equations

$$\begin{aligned} w_z &= A(z,\overline{z})w \\ w_{\overline{z}} &= B(z,\overline{z})w, \quad A, B \in C^1(G), \end{aligned}$$

which are defined in a simply-connected domain $G \subset \mathbb{C}$ containing $z_0 = 0$ and satisfying the functional relations

$$w(z_1 + z_2) = w(z_1) + w(z_2), \; w(0) = 1; \quad z_1, z_2, z_1 + z_2 \in G.$$

Then we will discuss the conditions under which the solutions of the system are periodic.

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რეზიუმე. ნაშრომში მიღებულია კომპლექსურ დიფერენციალურ განტოლებათა

$$\begin{aligned} w_z &= A(z,\overline{z})w \\ w_{\overline{z}} &= B(z,\overline{z})w, \quad A, B \in C^1(G) \end{aligned}$$

სისტემის ისეთ ამონახსნთა ერთი წარმოდგენა, რომლებიც განსაზღვრულია $z_0 = 0$ წერტილის შემცველ ცალადბმულ G არეზე და აკმაყოფილებს

$$w(z_1 + z_2) = w(z_1) + w(z_2), \; w(0) = 1; \quad z_1, z_2, z_1 + z_2 \in G,$$

ფუნქციონალურ დამოკიდებულებას. შემდეგ განხილულია პირობები, რომელთა შესრულებისას სისტემის ამონახსნები პერიოდულია.

1. Introduction

It is trivial that the function $w = e^z$ defined in a domain in $\mathbb{C}$ is a particular solution of the system of differential equations

$$\begin{aligned} w_z &= w, \\ w_{\overline{z}} &= 0 \end{aligned}$$

and the functional relations

$$w\left(z_1 + z_2\right) = w\left(z_1\right) + w\left(z_2\right), \quad w\left(0\right) = 1 \tag{1}$$

are satisfied for all $z_1, z_2 \in \mathbb{C}$. Tutschke [3] has considered a more general system

$$\begin{aligned} w_z &= A(z, \overline{z})w, \\ w_{\overline{z}} &= B(z, \overline{z})w, \end{aligned} \tag{2}$$

and obtained the necessary conditions for the solutions to satisfy (1), as

Theorem 1 ([3])**.** *If the coefficients A and B of the system* (2) *satisfy*

$$A\left(z_0 - z\right) = A\left(z\right), \quad B\left(z_0 - z\right) = B\left(z\right), \tag{3}$$

then every solution of (2) *satisfies the functional relation* (1) *for all $z_1, z_2 \in G$ with $z_1 + z_2 = z_0$.*

The condition (3) means that the coefficients A and B are symmetric with respect to the point $\frac{1}{2}z_0$. In that article, Tutschke has investigated the solutions of (2) satisfying (1) along the straight lines passing through the origin; thus the argument of the points is considered as constant.

Definition 1. The solutions of the system (2) satisfying the relations (1) are called *pseudoholomorphic exponential functions*.

2. A Representation of Solution

Let $G \subset \mathbb{C}$ be a simply connected domain with smooth boundary and let $z_0 = 0$ be a point in G. Now, let us look for a solution of the system (2) in G of the form

$$w(z) = \exp\left[H(z)\right], \quad H \in C^1\left(G\right). \tag{4}$$

Substituting (4) in (2), we find that $H(z, \overline{z})$ should satisfy

$$\begin{aligned} &H_z = A \\ &H_{\overline{z}} = B; \quad A, B \in C^1\left(G\right). \end{aligned} \tag{5}$$

If w is a solution of the system (2), then $A_{\overline{z}} = B_z$ should hold. Thus we can write

$$dw = w_z dz + w_{\overline{z}} d\overline{z} = \left(A\, dz + B\, d\overline{z}\right) w. \tag{6}$$

On the other hand, the exact differential of (4) is

$$dw = \exp\left[H(z)\right] dH = w\, dH. \tag{7}$$

Comparing (6) and (7), we get

$$dH = A\, dz + B\, d\overline{z}. \tag{8}$$

From (8) we find

$$H(z) = \int\limits_{\gamma(z_0,z)} (A\,dz + B\,d\overline{z}), \quad z \neq z_0, \tag{9}$$

where $\gamma(z_0, z)$ is a smooth curve in G connecting $z_0 = 0$ to z. Besides, $w(0) = 1$ corresponds to $H(0) = 0$. So we can identify H uniquely. Thus, we have obtained

Theorem 2. *Let $G \subset \mathbb{C}$ be a simply-connected domain containing the point $z_0 = 0$. If the function H defined by (9) satisfies the condition $H(0) = 0$, then*

$$w(z) = \exp\left[\int\limits_{\gamma} A\,dz + B\,d\overline{z}\right] \tag{10}$$

is a solution of the system (2) satisfying $w(0) = 1$. Furthermore, if the coefficients A, B satisfy (3), then $w(z)$ satisfies the functional relationship

$$w(z_1 + z_2) = w(z_1) + w(z_2).$$

Note. The similar complex system

$$w_{\overline{z}} = A \exp(-w),$$
$$w_z = B \exp(-w)$$

has been investigated previously [2], and a solution of the form

$$w(z, \overline{z}) = \log\left[\int\limits_{\gamma(z_0,z)} A\,dz + B\,d\overline{z}\right]$$

has been obtained, imposing the condition $w(1) = 0$.

3. Periodic Solutions

It is well known from complex analysis that the function $w(z) = e^z$ satisfies

$$w(z + 2n\pi i) = \exp[z + 2n\pi i] = w(z), \quad n \in \mathbb{Z}.$$

In this section, we will investigate the conditions under which the solution (10) satisfies

$$w(z + p) = w(z) \tag{11}$$

for some constant $p \in \mathbb{C}$. We have to assume that if $z \in G$, then $z + p \in G$ for $p \in \mathbb{C}$. From (10) we can write

$$w(z+p) = \exp\left\{\int\limits_{z_0}^{z+p} [A(z)dz + B(z)d\overline{z}]\right\} =$$

$$= \exp\left\{\int\limits_{z_0}^{z} [A(z)dz + B(z)d\overline{z}]\right\} \exp\left\{\int\limits_{z}^{z+p} [A(z)dz + B(z)d\overline{z}]\right\} =$$

$$= w(z)\exp\left[H\left(z+p\right)-H(z)\right].$$

Thus we can state the following theorem on periodic solutions of the system (2):

Theorem 3. *If the function $H(z)$ defined by* (9) *satisfies*

$$H\left(z+p\right)-H(z)=2\pi i, \tag{12}$$

then the solution $w(z)$ defined by (10) *satisfies the property* (11) *of periodicity. This solution is unique.*

Now let us consider the set

$$P=\left\{p_n\in\mathbb{C}: n\in\mathbb{Z},\ w\left(z+p_n\right)=w(z),\ z+p_n\in G\right\}.$$

Corollary 1. *If H satisfies the functional relation*

$$H\left(z+p_m\right)-H(z)=2m\pi i,\quad m\in\mathbb{Z},$$

then $p_m\in P$. In particular, if H is a linear function, then

$$H\left(p_m\right)=2m\pi i.$$

Corollary 2. *Let H satisfy* (12) *and let the coefficients A, B of the system* (2) *satisfy* (3). *Then $w(z)=\exp H(z)$ is a single valued solution satisfying the functional relation* (1) *and $w\left(z+p\right)=w(z)$. Conversely, if $w\left(z\right)=\exp\left[H(z)\right]$ is a solution of* (2) *satisfying $w\left(z+p\right)=w(z)$, then $H(z)$ satisfies the functional relation*

$$H\left(z+p_m\right)-H(z)=2m\pi i,\quad m\in\mathbb{Z}. \tag{13}$$

However, $H(z)$ cannot be determined uniquely from (13).

Theorem 4. *If $w(z)=\exp H(z)$ is a periodic solution of the system* (2), *then the coefficients A and B are periodic functions with period p.*

Proof. Let $w(z)=\exp H(z)$ be a periodic solution of (2). Then (13) holds. Let us differentiate both sides of (13) with respect to z and $\overline{z}$.

$$\begin{aligned}\frac{\partial H\left(z+p_m\right)}{\partial z}-\frac{\partial H(z)}{\partial z}=0,\\ \frac{\partial H\left(z+p_m\right)}{\partial \overline{z}}-\frac{\partial H(z)}{\partial \overline{z}}=0,\end{aligned} \tag{14}$$

which leads to

$$\begin{aligned}A\left(z+p_m\right)-A(z)=0,\\ B\left(z+p_m\right)-B(z)=0\end{aligned}$$

by (5). But these are the conditions for the coefficients A and B to be periodic with period p_m.

Example 1. Let the coefficients of A and B of (2) be complex constants subject to $|A| \neq |B|$. Then from (9) we have

$$H(z) = A\,z + B\,\overline{z}$$

and

$$w(z) = \exp\left(A\,z + B\,\overline{z}\right)$$

is obtained. This solution fulfills the requirements

$$\begin{aligned} w\left(z_1 + z_2\right) &= w\left(z_1\right) w\left(z_2\right), \\ w\left(0\right) &= 1. \end{aligned}$$

Also, we can find by (13) that

$$A\,p_m + B\,\overline{p_m} = 2m\pi i, \quad m \in \mathbb{Z},$$

that is,

$$p_m = \frac{2m\pi i\left(\overline{A} + B\right)}{|A|^2 - |B|^2}, \quad m \in \mathbb{Z}. \tag{15}$$

On the other hand, since we can write

$$p_m = \frac{m}{n}\,p_n$$

for every $m, n \in \mathbb{Z}$, $n \neq 0$, the period p_m is simple (see [1]).

Note. If $|A| = |B|$, we still can determine the period p_m by a simple computation as

$$p_m = \frac{im\pi\left(1 + e^{-i\theta}\right)A}{|A|^2\left(1 + \cos\theta\right)},$$

where $A \neq 0$, $m \in \mathbb{Z}$.

Example 2. Let h be a complex valued function of $y = \operatorname{Im} z$ subject to $h\left(-y\right) = h\left(y\right)$. Let us assume that the coefficients A and B of (2) are given as

$$\begin{aligned} A(z) &= c_1 + h\left(y\right), \\ B(z) &= c_2 - h\left(y\right), \end{aligned} \tag{16}$$

where $c_1, c_2 \in \mathbb{C}$ are constants. In this case the solubility condition $A_{\overline{z}} = B_z$ holds. Thus (9) yields

$$\begin{aligned} H(z) = \int_{z_0}^{z} \left\{\left[c_1 + h\left(y\right)\right]dz + \left[c_2 - h\left(y\right)\right]d\overline{z}\right\} = \\ = c_1 z + c_2\overline{z} + F\left(z - \overline{z}\right), \end{aligned} \tag{17}$$

where F is the primitive of h. Hence

$$w(z) = \exp\left[c_1 z + c_2\overline{z} + F\left(z - \overline{z}\right)\right] \tag{18}$$

is a solution of the system (2). The values p_n satisfying

$$c_1 p_n + c_2\overline{p_n} + F\left(z - \overline{z} + p_n - \overline{p}_n\right) = 2n\pi i + F\left(z - \overline{z}\right), \quad n \in \mathbb{Z}, \tag{19}$$

are the periods of the solutions of (18). Restricting ourselves to the real periods, we get

$$p_n = \frac{2n\pi i}{c_1 + c_2}, \quad n \in \mathbb{Z},$$

if $\operatorname{Re}(c_1 + c_2) = 0$.

Choosing $h(y) = i\, y^{2m}$, $m \in \mathbb{N}$, the function H satisfying $H(0) = 0$ will be obtained as

$$H(z) = c_1 z + c_2 \overline{z} + \frac{(-1)^m i}{2^m (2m+1)} (z - \overline{z})^{2m+1}.$$

So the solution of the system (2) with the coefficients defined by (16) is

$$w(z) = \exp\left[c_1 z + c_2 \overline{z} + \frac{(-1)^m i}{2^m (2m+1)} (z - \overline{z})^{2m+1}\right]$$

with the period

$$p_n = \frac{2n\pi i}{c_1 + c_2},$$

where $\operatorname{Re}(c_1 + c_2) = 0$.

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