Memoirs on Differential Equations and Mathematical Physics
Volume 33, 2004, 25–45

B. Bojarski and G. Khimshiashvili

THE GEOMETRY OF FREDHOLM PAIRS AND LINEAR CONJUGATION PROBLEMS

Abstract. We discuss Fredholm pairs and restricted Grassmannians in Banach spaces with a view towards developing geometric models of linear conjugation problems with discontinuous coefficients. It is shown that a considerable part of the classical theory of Fredholm pairs in Hilbert space can be extended to a wide class of Banach spaces with contractible general linear group. Some global geometric and topological properties of arising restricted Grassmannians are established and relations to the theory of Fredholm structures are described.

2000 Mathematics Subject Classification. 46L80, 58B15.

Key words and phrases. Fredholm operator, index, restricted Grassmanian, Fredholm structure.

რეზიუმე. ნაშრომში განხილულია ფრედჰოლმის წყვილები და შეზღუდული გრასმანიანები ბანახის სივრცეებში წყვეტილკოეფიციენტებიანი წრფივი შეუღლების ამოცანების გეომეტრიული მოდელების დამუშავების თვალსაზრისით. ნაჩვენებია, რომ ჰილბერტის სივრცეში ფრედჰოლმის წყვილების კლასიკური თეორიის მნიშვნელოვანი ნაწილი შეიძლება განზოგადდეს მოჭიმვადი ზოგადი წრფივი ჯგუფის მქონე ბანახის სივრცეების ფართო კლასისათვის. დადგენილია შეზღუდული გრასმანიანების ზოგიერთი ზოგადი გეომეტრიული და ტოპოლოგიური თვისება და აღწერილია კავშირები ფრედჰოლმური სტრუქტურების თეორიასთან.

Introduction

We present several basic results about Fredholm pairs of subspaces introduced by T.Kato [18] and discuss global geometric and topological properties of associated *restricted Grassmannians* which gained considerable interest in last two decades (see, e.g., [4]–[7], [32], [16], [21], [8]). As was shown in [4]–[7], many geometric aspects of classical linear conjugation problems with sufficiently regular (differentiable, Hölder) coefficients can be formulated and successfully studied in the framework of Fredholm pairs of subspaces and restricted Grassmannians in real or complex Hilbert space. The approach used in [4], [7], [8], [23] was based on consideration of Fredholm pairs in Hilbert spaces and in that setting it appeared possible to obtain a number of general results on the solvability and homotopy classification of linear conjugation problems with sufficiently regular coefficients (see, e.g., [23]).

However, when studying linear conjugation problems with discontinuous coefficients and families of linear conjugation problems , the context of Hilbert spaces appears insufficient and one is inevitably led to considering equations and operators in more general Banach spaces. For example, investigation of linear conjugation problems with discontinuous coefficients requires use of L_p-spaces and weighted L_p-spaces (see, e.g., [26], [28], [27], [25]). Linear conjugation problems for monogenic functions of one quaternion variable require considering Hilbert spaces over the quaternion algebra $\mathbb{H}$ [34]. Moreover, as was recently shown, families of linear conjugation problems can be successfully studied in the framework of operators and Grassmannians in Hilbert modules over C^*-algebras [22].

Thus it becomes desirable to develop a geometric theory of Fredholm pairs and associated Grassmannians in more general classes of Banach spaces than just real and complex Hilbert spaces. With this in mind, in the present paper we show that some fundamental properties of Fredholm pairs and related Grassmannians in Hilbert space can be established in more general contexts. This in principle enables one to develop a geometric approach to linear conjugation problems with discontinuous coefficients and families of linear conjugation problems by applying the paradigms and reasoning developed in [7], [8], [24]. The concrete applications to the theory of linear conjugation problems available on this way would have taken us too far away, so in this paper we confine ourselves to discussing the geometry of Fredholm pairs in Banach spaces.

One of our main aims is to indicate several new settings to which one can extend the basic results on Fredholm pairs of subspaces. In particular, we show that the homotopy type of the restricted Grassmannian can be described for certain splitting subspaces of a Banach space with contractible general linear group (Theorem 2.3). We also show that similar results can be obtained for Fredholm pairs in Hilbert modules (Theorems 3.1 and 3.2). Another general idea we wish to advocate in this paper, is that the restricted Grassmannians can be studied using the so-called *Fredholm structures*. To this end we construct a natural Fredholm structure on the restricted Grassmannian and outline some consequences and possible applications of its existence (Theorem 4.1).

Let us now say a few words about the structure of the paper. We begin by presenting the most essential geometric and topological properties of *Fredholm pair of subspaces* in a Hilbert space. The discussion in this section closely follows [4] and [8] (cf. also [1]). In the next section we give some relevant results on the geometry of restricted Grassmannians and show that they can be extended beyond the Hilbert space setting. The third section contains results on Fredholm pairs and restricted Grassmannians over C^*-algebras. Here we also suggest some conceptual and technical novelties. In the conclusion, we present an explicit construction of Fredholm structures on restricted Grassmannians.

It should be added that a number of important contributions to the theory of linear conjugation problems and Riemann-Hilbert problems with discontinuous coefficients were made by G. Manjavidze [28], [29]. The both authors were lucky to enjoy scientific and friendly communication with this wonderful person and cordially dedicate this paper to the memory of Giorgi Manjavidze.

1. Fredholm Pairs of Subspaces and Projections

In the sequel we freely use standard concepts and results of functional analysis and operator theory concerned with Fredholm operators, indices, resolvents, spectra, and Hilbert modules over C^*-algebras. All Banach spaces are supposed to be real or complex and all subspaces are supposed to be closed. We begin by recalling the general concept of Fredholm pair of subspaces.

Consider two (closed) subspaces L_1, L_2 of a Banach space E over the field K which can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1 ([18]). The pair (L_1, L_2) is called a Fredholm pair of subspaces (FPS) if their intersection $L_1 \cap L_2$ is finite dimensional and their sum $L_1 + L_2$ has finite codimension in E. Then the index of the pair (L_1, L_2) is defined as

$$\operatorname{ind}(L_1, L_2) = \dim_K(L_1 \cap L_2) - \operatorname{codim}_K(L_1 + L_2). \tag{1.1}$$

This concept was introduced by T.Kato who, in particular, proved that the index is invariant under homotopies [18]. To distinguish this concept from similar ones, we sometimes speak of a *Kato Fredholm pair*. Some modifications of the above definition are presented below and when no confusion is possible they all are referred to as *Fredholm pairs*.

As is well known, Fredholm pairs appear in a number of important problems of analysis and operator theory [18], [5], [6], [10], [32]. Global geometric and topological properties of the set of Fredholm pairs play essential role in some recent papers on differential equations and infinite dimensional Morse theory in the spirit of Floer approach (see, e.g., [1]).

Taking this into account it seems remarkable that one can relate the theory of Fredholm pairs with some topics of global analysis in the spirit of Fredholm structures theory [14], [15]. In line with that idea we present in the fourth section a natural construction of Fredholm structure on the set of Fredholm pairs with a fixed first component and indicate some corollaries and perspectives opened by this construction.

The theory of Fredholm pairs is especially rich and fruitful in the case of Hilbert space. The results available in that case serve as a pattern for our discussion, in particular, they suggest useful generalizations applicable to wider classes of Banach spaces. Therefore we begin by presenting the basic results in the case where E is a separable Hilbert space H. However, as was mentioned in the introduction, studying the geometry of linear conjugation problems with discontinuous coefficients requires considering Fredholm pairs in more general Banach spaces, in particular, in Hilbert modules over $C*$-algebras. For this reason, whenever possible we indicate possibilities of further generalizations.

There are two important peculiarities in the case of a Hilbert space. First of all, in a Hilbert space one can pass to the orthogonal complements of subspaces considered. Thus, for each pair of subspaces (L_1, L_2), one has a dual pair $P^\perp = (L_1^\perp, L_2^\perp)$ and it is easy to see that if one of them is a Fredholm pair then the second one also has this property. Since

$$\begin{aligned} \operatorname{ind}(L_1, L_2) &= \dim_K(L_1 \cap L_2) - \operatorname{cod}_K(L_1 + L_2) = \\ &= \dim_K(L_1 \cap L_2) - \dim_K(L_1^\perp \cap L_2^\perp), \end{aligned} \tag{1.2}$$

we also have

$$\operatorname{ind}(L_1, L_2) = -\operatorname{ind}(L_1^\perp, L_2^\perp). \tag{1.3}$$

Thus there exists a sort of duality for Fredholm pairs in Hilbert spaces.

Proposition 1.1. *A pair $P = (L_1, L_2)$ is a FPS if and only if $P^\perp = (L_1^\perp, L_2^\perp)$ is a FPS, and* $\operatorname{ind} P = -\operatorname{ind} P^\perp$.

This result means that the operation of passing to orthogonal complements acts on the set of Fredholm pairs $Fp(H)$. As we will see below, the set of Fredholm pairs can be endowed with a natural structure of infinite dimensional manifold and this map becomes a smooth self-mapping of $Fp(H)$. We wish to point out that one can easily generalize these observations to the case of Banach space E.

Indeed, if one considers the dual space E' consisting of continuous linear functionals on E then, for a subspace $L \subset E$, a natural analog of orthogonal complement is the annihilator $L^o = \{\phi \in E' : \phi|L = 0\}$. Replacing in the above reasoning $L^\perp$ by L^o, one can easily obtain an analog of Proposition 1.1 in this context. The analogy becomes especially far reaching in the case of a reflexive Banach space E. However we will not further develop this idea, the main aim of this remark being to show that results for the Hilbert space may really serve as a pattern and hint for the theory of Fredholm pairs in more general Banach spaces.

In the setting of Hilbert spaces one can equivalently deal with orthogonal projections on subspaces considered. This is convenient in many constructions and more appropriate for generalizations in the context of Banach algebras along the lines of [3] (cf. [8] and [13]). To make this idea more precise, we now present a counterpart of Definition 1.1 in the language of projections.

Definition 1.2 (cf. [2])**.** Let P_1 and P_2 be orthogonal projections in a separable Hilbert space. A pair (P_1, P_2) is called a Fredholm pair of projections (FPP) if the operator $C = P_2 P_1 | \operatorname{im} P_1$ considered as an operator from $\operatorname{im} P_1$ to $\operatorname{im} P_2$ is Fredholm, and if this is the case then the index $\operatorname{ind}(P_1, P_2)$ is defined as the Fredholm index of C_{21}.

It is remarkable that the two definitions are equivalent as is shown by the following simple but important proposition which is one of the key steps in developing generalizations we have in mind.

Proposition 1.2. *A pair of orthogonal projections* (P_1, P_2) *is a FPP if and only if* $(\operatorname{im} P_1, \operatorname{im} P_2^{\perp})$ *is a FPS. In such case*

$$\operatorname{ind}(P_1, P_2) = \operatorname{ind}(\operatorname{im} P_1, \operatorname{im} P_2^{\perp}). \tag{1.4}$$

Proof. First of all it is easy to verify that

$$\ker C = L_1 \cap L_2^{\perp}, (\operatorname{im} C)^{\perp} = L_1^{\perp} \cap L_2.$$

Thus fredholmness of C is equivalent to the fact that $L_1 \cap L_2^{\perp}$ and $L_1^{\perp} \cap L_2 = (L_1 + L_2^{\perp})^{\perp}$ are finite-dimensional, which in turn means that $(L_1, L_2^{\perp})$ is a FPS. Moreover,

$$\begin{aligned} \operatorname{ind}(P_1, P_2) &= \dim_K (L_1 \cap L_2^{\perp}) - \dim_K (L_1^{\perp} \cap L_2) = \\ &= \dim_K (L_1 \cap L_2^{\perp}) - \dim_K (L_1 + L_2^{\perp})^{\perp} = \\ &= \dim_K (L_1 \cap L_2^{\perp}) - \operatorname{codim}_K (L_1 + L_2^{\perp}) = \operatorname{ind}(L_1, L_2^{\perp}), \end{aligned}$$

which finishes the proof. □

Thus we already have two different interpretations of the concept of Fredholm pair each of which has its specific applications and suggests further generalizations. One of the most interesting possibilities is related to the abstract Fredholm theory in Banach algebras developed in [3]. In the setting of Banach algebras, projections should be substituted by idempotents and one should define which pairs of idempotents are considered as analogs of Fredholm pairs of projections. It is not quite clear how to do this in the most general case but we'll suggest one version which leads to a reasonable concept.

Recall that there also exist more restrictive notions of Fredholm pair which appear useful in operator theory and functional analysis. We describe here one of them which plays an important role in the study of linear conjugation problems and Grassmannian embeddings of loop groups. It is based on the concept of commensurability of projections which can be easily generalized to the Banach algebra context.

Definition 1.3. Let P_1, P_2 be orthogonal projections in a separable Hilbert space. They are called commensurable if their difference $P_1 - P_2$ is a compact operator and then it is also said that (P_1, P_2) is a commensurable pair of projections (CPP). A pair of subspaces (L_1, L_2) is called a commensurable pair of subspaces (CPS) if orthogonal projectors on these subspaces are commensurable.

Proposition 1.3. *Two subspaces L_1, L_2 are commensurable if and only if the operators $P_{L_1^\perp} P_{L_2}$ and $P_{L_2^\perp} P_{L_1}$ are compact.*

Proof. The result follows by an easy calculation:

$$P_1 - P_2 = (P_2 + P_2^\perp)P_1 - P_2(P_1 + P_1^\perp) =$$
$$= P_2P_1 + P_2^\perp P_1 - P_2P_1 - P_2P_1^\perp = P_2^\perp P_1 - P_2P_1^\perp,$$

where we used the evident fact that, for each orthogonal projection P, one has $I = P + P^\perp$.

Comparing this proposition with the definition of Fredholm pair , one sees that if L and M are commensurable then $(L, M^\perp)$ and $(L^\perp, M)$ are FPSs. Thus commensurable pairs give rise to a special class of Fredholm pairs which are sometimes called *strict* Fredholm pairs .

Notice that the definition of commensurability is applicable to arbitrary (not necessarily orthogonal) projections and so it makes sense for an arbitrary Banach space. This suggests a way of generalization to the case of idempotents in a Banach algebra. Recall that a general notion of compact element in a Banach algebra A was defined in [3] using the concept of *socle* $\operatorname{soc} A$. Thus one can call a pair of idempotents $(e_1, e_2) \in A^2$ a Fredholm pair of idempotents if their difference belongs to the socle $\operatorname{soc} A$.

It can be shown that if (e_1, e_2) is a Fredholm pair of idempotents in the algebra of bounded operators $B(E)$ in a Banach space E then such definition is consistent with Definition 1.3. One can use the results about compact elements in A in order to introduce a reasonable generalization of the index in the spirit of [3] and prove basic properties of FPSs and their indices in this context.

However we do not pursue this possibility here and return to FPSs in Hilbert space in order to investigate their geometry more closely. For doing so it appears extremely useful to consider certain Grassmannians which will be our main concern in the next section. To conclude this section, we want to point out that one can give a useful parameterization of Fredholm pairs in terms of an associated operator group which is a sort of analog of the classical singular integral operators and appears useful in the geometric study of linear conjugation problems [4] (cf. [33], [22]).

In particular, one can describe the set of all subspaces which form a commensurable Fredholm pair with a given subspace L using the following construction suggested in [4]. This construction is only interesting when $\dim L = \dim L^{\perp} = \infty$, so we suppose that this is the case. For a given operator $A \in B(H)$ denote by A'_e the essential commutant of A, i.e., the set of all operators T such that the commutator $[T, A]$ is compact. □

Definition 1.4. Let P be an orthogonal projection in a separable Hilbert space such that $\dim \operatorname{im} P = \dim \ker P = \infty$. The algebra P'_e is called the algebra of abstract singular operators associated with P.

Invertible operators from P'_e play an essential role in describing commensurable Fredholm pairs. Denote by $G_e(P)$ the set of all invertible operators from P'_e. Recall that the classical singular integral operators with Cauchy kernel arise if one takes in the role of P the Szegö projector on the Hardy space in $L_2(S^1, \mathbb{C}^n)$ [4]. Fix now a subspace L, denote by P the orthogonal projection on L and put $Q = I - P$. The following result which was established in [4] can be proved by a direct check.

Proposition 1.4 ([4]). *For each $A \in G_e(P)$, the pair $(L, AL^{\perp})$ is a Fredholm pair of subspaces. The operator $\Phi = P + AQ$ is a Fredholm operator and* $\operatorname{ind} \Phi = \operatorname{ind} (L, AL^{\perp})$.

Actually, in some sense converse is also true, i.e., all subspaces commensurable with a given one can be obtained using the action of $G_e(P)$ [4]. We do not give detailed formulation of the latter claim because more precise results will be presented in the next section in terms of the so-called restricted Grassmannians .

2. Fredholm Pairs and Restricted Grassmannians

In many geometric problems it becomes necessary to consider the set of all Fredholm pairs in some Banach space E with a fixed first subspace. In other words, one chooses a closed infinite dimensional and infinite codimensional subspace L and considers the so-called *Kato Grassmannian* $Gr_F(L, E)$ consisting of all subspaces $M \subset E$ such that (L, M) is a Kato Fredholm pair [4], [32]. This is actually just a "leaf" in the set $Fp(E)$ of all Fredholm pairs and in many cases one may represent Fp as a fibration over the set of all closed subspaces $Gr(E)$ with the fiber homeomorphic to $Gr_F(L, E)$. Thus the geometry and topology of $Fp(E)$ can be often understood by investigating Fredholm Grassmannian .

This definition permits several useful modifications which we present following [32] and [1]. We consider first the case where E is a Hilbert space H and it is convenient to introduce the Grassmannian $Gr(H)$ defined as the collection of all closed subspaces in H. The assignment $L \mapsto P_L$ defines an inclusion of $Gr(H)$ into $B(H)$ with the image equal to the set of all orthogonal projections in H. One can now define the metric on $Gr(H)$ by putting the distance between two subspaces to be the norm of the difference of orthogonal projections on these subspaces. This makes $Gr(H)$ into a complete metric space and it can be proved that it has a natural structure of analytic Banach submanifold induced from $B(H)$ [1].

The set of Fredholm pairs is an open subset of $Gr(H) \times Gr(H)$ so we endow it with the induced topology and then index is a continuous function on Fp. Denote by Fp^* the subset consisting of Fredholm pairs consisting of infinite dimensional subspaces. The connected components of Fp^* are exactly the subsets Fp^*_k consisting of Fredholm pairs with index k and it can be proved that all components are homeomorphic (see, e.g., [32]). For a real Hilbert space, each of them has the homotopy type of the classifying space of infinite orthogonal group $BO(\infty)$ and so its homotopy groups are well known. Thus Fp^* is homotopy equivalent to $\mathbb{Z} \times BO(\infty)$, its homotopy groups are 8-periodic by Bott periodicity, and the first 8 homotopy groups are: $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0$. In the complex case one has the same picture with 2-periodic homotopy groups beginning with $\mathbb{Z}, 0$.

We are now prepared to discuss restricted Grassmannians . Consider a complex Hilbert space decomposed into an orthogonal direct sum $H = H_+ \oplus H_-$ and choose a positive number s. For further use we need a family of subideals in the ideal of compact operators $K(H)$ which is defined as follows (cf. [16]).

Recall that for any bounded operator $A \in L(H)$ the product A^*A is a non-negative self-adjoint operator, so it has a well-defined square root $|A| = (A^*A)^{1/2}$ (see, e.g., [14]). If A is compact, then A^*A is also compact and $|A|$ has a discrete sequence of eigenvalues

$$\mu_1(A) \geq \mu_2(A) \geq \cdots$$

tending to zero. The $\mu_n(A)$ are called singular values of A. For a finite $s \geq 1$ one can consider the expression (sth norm of A)

$$||A||_s = \left[\sum_{j=1}^{\infty} (\mu_j(A))^s \right]^{1/s} \tag{2.1}$$

and define the sth Schatten ideal K_s as the collection of all compact operators A with a finite sth norm (s-summable operators) [14].

Using elementary inequalities, it is easy to check that K_s is really a two-sided ideal in $L(H)$. These ideals are not closed in $L(H)$ with its usual norm topology but if one endows K_s with the sth norm as above, then K_s becomes a Banach space [14]. Two special cases are well-known: K_1 is the ideal of trace class operators and K_2 is the ideal of Hilbert-Schmidt operators. For $s = 2$, the above norm is called the Hilbert-Schmidt norm of A and it is well known that $K_2(H)$ endowed with this norm becomes a Hilbert space (see, e.g., [14]). Obviously $K_1 \subset K_s \subset K_r$ for $1 < s < r$ so one obtains a chain of ideals starting with K_1. For convenience we set $K_\infty = K$ and obtain an increasing chain of ideals K_s with $s \in [1, \infty]$.

Of course one can introduce similar definitions for a linear operator A acting between two different Hilbert spaces, e.g., for an operator from one subspace M to another subspace N of a fixed Hilbert space H. In particular we can consider the classes $K_s(H_\pm, H_\mp)$. Let us also denote by $F(M,N)$ the space of all Fredholm operators from M to N.

Definition 2.1 ([32])**.** The sth restricted Grassmannian of a polarized Hilbert space H is defined as

$$Gr^s_F(H) = \{W \subset H: \ \pi_+|W \ \text{ is an operator from } \ F(W, H_+), \\ \pi_-|W \ \text{ is an operator from } \ K_s(W, H_-)\}.$$

In general, two subspaces are called s-commensurable if the difference of orthogonal projections on these subspaces belongs to K_s. Thus the sth Grassmannian is the collection of all subspaces s-commensurable with H_+. These Grassmannians are of the major interest for us. Actually, many of their topological properties (e.g., the homotopy type) do not depend on the number s appearing in the definition. On the other hand, more subtle geometric properties like manifold structures and characteristic classes of Gr^s_F do depend on s in a quite essential way. As follows from the discussion in [16], this is a delicate issue and we circumvent it by properly choosing the context.

As follows from the results of [32], it is especially convenient to work with the Grassmannian $Gr^2_F(H)$ defined by the condition that the second projection π_- restricted to W is a Hilbert-Schmidt operator. Following [32] we denote it by $Gr_r(H)$ and call the *restricted Grassmannian* of H.

The Kato Grassmannians appear to have interesting analytic and topological properties. It turns out that Grassmannian Gr^s_F can be turned into a Banach manifold modelled on Schatten ideal K_s. In particular $Gr_r(H)$ has a natural structure of a Hilbert manifold modelled on the Hilbert space $K_2(H)$ [32]. All these Grassmannians have the same homotopy type (see Theorem 2.2 below). Moreover, as we will see in the last section Grassmannians Gr^s_F can be endowed with so-called Fredholm structures [14], which suggests in particular that one can define various global topological invariants of $Gr^s_F(H)$.

Definition 2.1 also yields a family of subgroups $GL^s = GL(\pi_+, K_s)$ of $GL(\pi_+, K)$ $(s \geq 1)$. For our purposes the subgroup $GL(\pi_+, K_2)$ is especially important.

Definition 2.2 ([32])**.** The restricted linear group $GL_r(H)$ is defined as the subgroup of $GL(\pi_+, K)$ consisting of all operators A such that the commutator $[A, \pi_+]$ belongs to the Hilbert-Schmidt class $K_2(H)$.

From the very definition it follows that GL^s acts on Gr^s and it was shown in [8] that these actions are transitive (cf. also [32], Ch.7). In order to give a convenient description of the isotropy subgroups of these actions, we follow the presentation of [32] and introduce a subgroup $U^s(H) = U(H) \cap GL^s(H)$ consisting of all unitary operators from GL^s. For $s = 2$ this subgroup is denoted by U_r. Now the description of isotropy groups is available by the same way of reasoning which was applied in [32] for $s = 2$.

Theorem 2.1. *The subgroup $U^s(H)$ acts transitively on $Gr^s(H)$ and the isotropy subgroup of the subspace H_+ is isomorphic to $U(H_+) \times U(H_-)$.*

From the existence of a polar decomposition for a bounded operator on H it follows that the subgroup $U^s(H)$ is a retract of GL^s and it is straightforward to obtain similar conclusions for the actions of GL^s.

Corollary 2.1. *The group GL^s acts transitively on the Grassmannian $Gr^s(H)$ and the isotropy groups of this action are contractible.*

Thus such an action obviously defines a fibration with contractible fibers and it is well known that for such fibrations the total space (GL^s) and the base (Gr^s) are homotopy equivalent [14].

Corollary 2.2. *For any $s \geq 1$, the Grassmannian Gr^s and the group GL^s have the same homotopy type. In particular, GL_r is homotopy equivalent to Gr_r.*

Remark. All the groups $GL(\pi_+, J)$ have the same homotopy type for any ideal J between K_0 and K. In particular, this is true for every Schatten ideal K_s. Thus all the above groups and Grassmannians have the same homotopy type.

We are now ready to give more comprehensive results about the topology of Gr^s and GL^s. The homotopy type of GL_r and Gr_r over the field of complex numbers is described in the following statement which was obtained in [21], [10], [32]. This gives an answer to a question posed in [4].

Theorem 2.2. *For any $s \in [1, \infty]$, the homotopy groups of the group GL^s and Fredholm Grassmannian Gr^s are given by the formulae:*

$$\pi_0 \cong \mathbb{Z}; \;\; \pi_{2k+1} \cong \mathbb{Z}, \; \pi_{2k+2} = 0, \; k \geq 0. \tag{2.2}$$

This is an important result which has many applications to the homotopy classification of linear conjugation problems [22], [8]. It follows that, in the case of a Hilbert space, all restricted Grassmannians and Kato Grassmannian have the same topological structure. In particular, their topologicla properties do not depend on the choice of subspace H_+. As we will see below this is not the case for a general Banach space.

So let us now consider restricted Grassmannians in a Banach space E. Recall that Grassmannian $G(E)$ is defined as the set of all closed subspaces in H. As above we use the term Kato biGrassmannian to denote the collection of all Fredholm pairs of subspaces in E. Suppose that $\dim L = \operatorname{codim} L = \infty$. The following natural definition is implicitly contained in [6].

Definition 2.3. The collection of all closed subspaces $M \subset E$ such that (L, M) is a Fredholm pair of subspaces is called the Kato Grassmannian $Gr_F(L, E)$ associated with L.

For an arbitrary Banach space E there is no a priori reason why the topology and geometry of $Gr_F(L)$ should be the same for all infinite dimensional and infinite codimensional subspaces L. It may be seen in examples that the structure of restricted Grassmannian may depend on the isomorphy type and embedding of L and its complement. Therefore a general discussion of Kato Grassmannians seems difficult and not very useful. For this reason, in the sequel we impose some conditions on E and L which enable us to extend the preceding discussion to a more general context. The assumptions we make are fulfilled, for example, for the Hardy subspace in $L_p(S^1)$, which is important in the investigation of linear conjugation problems with discontinuous coefficients.

Thus we assume that L is a *bisplitting subspace*, i.e., it admits a closed complement L' such that $L' \cong L$ and $E \cong L \oplus L'$. In such situation it is easy to show that the topology of $Gr_F(L)$ depends only on the isomorphism class of L. Let us say that a Banach space E is *tame* if its general linear group $GL(E)$ is contractible (some authors say in such case that E has Kuiper property).

In the sequel we fix the above decomposition of E into the direct sum of two bisplitting subspaces and consider the Kato Grassmannian defined by such a decomposition. In this situation on has a natural analog of the restricted linear group which is defined in the same way as above and denoted $GL_F(L, E)$. The following result can be proved analogously to Theorem 2.1.

Theorem 2.3. *If L is a bisplitting tame subspace of a tame Banach space then the group of abstract singular operators $GL_F(L, E)$ acts transitively on the Kato Grassmannian $Gr_F(L, E)$ with contractible isotropy groups.*

It is possible to show that it is sufficient to require tameness of L since it can be shown that then E is automatically tame. We do not work out this point because the properties we need are fulfilled in many interesting cases.

We are now in a position to obtain some topological information on the Kato Grassmannian $Gr_F(L, E)$ of the above type. As was already mentioned, there is little hope that its topological type is always the same. However it is remarkable that one can still obtain rather precise information on its homotopy type. Namely, if one assumes that L is bisplitting and tame, then it turns out that the proof of Theorem 2.1 presented above remains valid in this case and one arrives at a similar conclusion. For brevity we formulate the final result in the case of a complex Banach space. The case of a real Banach space is completely analogous. The proof can be obtained applying the same argument as was used in [32] for proving Theorem 2.2. The necessary modifications are self-evident and therefore omitted.

Theorem 2.4. *Let L be a bisplitting tame subspace of a separable tame complex Banach space E. Then the homotopy groups of the restricted Grassmannian $Gr_F(L, E)$ are as follows:*

$$\pi_0 \cong \mathbb{Z}; \quad \pi_{2k-1} \cong \mathbb{Z}, \quad \pi_{2k} = 0, \quad k \geq 1. \tag{2.3}$$

This result can be applied to investigating the homotopy classes of families of linear conjugation problems in L_p-spaces by the same scheme which was used in [22], [8] in the setting of Hilbert spaces. Discussion of such applications is delayed for the future but in the next section we present some related results which are available for linear conjugation problems and restricted Grassmannians over C^*-algebras. It should be noted that the topological results presented in the next section do not follow from the above discussion since in the setting of C^*-algebras one uses an essentially different definition of restricted Grassmannian. We believe it might be instructive to mention the both settings within the present paper.

3. Linear Conjugation Problems over C^*-Algebras

In this section we introduce certain geometric objects over C^*-algebras which are relevant to the homotopy classification of abstract elliptic problems of linear conjugation. The abstract problem of linear conjugation was introduced by B. Bojarski [4] as a natural generalization of the classical linear conjugation problem for holomorphic vector-functions [31]. As was realized much later (see, e.g., [22], [23]), the whole issue fits nicely into the Fredholm structures theory [14], more precisely, into the homotopy theory of operator groups.

Recall that in 1979 B. Bojarski formulated a topological problem which appeared stimulating in the theory of operators and boundary value problems [4]. This problem was independently solved in [19] and [36] (cf. also [32]). Moreover, the results obtained on this way were used in studying several related topics of global analysis and operator theory [10], [36], [32].

An important advantage of the geometric formulation of elliptic transmission problems in terms of Fredholm pairs of subspaces of a Hilbert space given in [4] was that it permitted various modifications and generalizations. Thus it became meaningful to consider similar problems in more general situations [22], in particular, in the context of Hilbert C^*-modules [30], which led to some progress in the theory of generalized transmission problems [22], [23].

Such an approach enables one, in particular, to investigate elliptic transmission problems over an arbitrary C^*-algebra. Clearly, this gives a wide generalization of the original setting used in [4], [36], [19], since the latter corresponds to the case in which the algebra is taken to be the field of complex numbers $\mathbb{C}$. This also generalizes the geometric models for classical linear conjugation problems problems in terms of Grassmannian embeddings of loop groups [32].

Notice also that the setting of linear conjugation problems over C^*-algebras includes the investigation of families of elliptic transmission problems parameterized by a (locally) compact topological space X. In fact, this corresponds to considering linear conjugation problems over the algebra $C(X)$ of continuous functions on the parameter space, and classification of families of elliptic problems of such kind becomes a special case of our general results.

To make the presentation concise, we freely use the terms and constructions from the theory of Hilbert modules over C^*-algebras. A detailed exposition of necessary concepts and results is contained in [30].

We pass now to the precise definitions needed to formulate a generalization of a geometric approach to linear conjugation problems suggested in [4]. We use essentially the same concepts as in [4], but sometimes in a slightly different form adjusted to the case of Hilbert C^*-modules.

Let A be a unital C^*-algebra. Denote by H_A the standard Hilbert module over A, i.e.,

$$H_A = \left\{ \{a_i\}, a_i \in A, i = 1, 2, \dots \ : \ \sum_{i=1}^{\infty} a_i a_i^* \in A \right\}. \tag{3.1}$$

Since there exists a natural A-valued scalar product on H_A possessing usual properties [30], one can introduce direct sum decompositions and consider various types of bounded linear operators on H_A. Denote by $B(H_A)$ the collection of all A-bounded linear operators having A-bounded adjoints. This algebra is one of the most fundamental objects in Hilbert C^*-modules theory [30].

As is well known, $B(H_A)$ is a Banach algebra and it is useful to consider also its group of units $GB = GB(H_A)$ and the subgroup of unitaries $U = U(H_A)$. For our purpose it is important to have adjoints, which, as is explained, e.g., in [30], is not the case for an arbitrary bounded operator on the Hilbert A-module H_A. In particular, for this algebra we have an analog of the polar decomposition [30], which implies that $GB(H_A)$ is retractable to $U(H_A)$. Thus these two operator groups are homotopy equivalent, which is important for our consideration.

Compact linear operators on H_A are defined to be A-norm limits of finite rank linear operators [30]. Their collection is denoted by $K(H_A)$.

Recall that one of the central objects in B. Bojarski's approach [4] was a special group of operators associated with a fixed direct sum decomposition of a given complex Hilbert space which already appeared in the first section of this paper. We now give its generalization in the context of Hilbert modules. To this end, we fix a direct sum decomposition in the category of Hilbert A-modules of the form $H_A = H_+ + H_-$, where H_+ and H_- are both isomorphic to H_A as A-modules. As is well known, any operator on H_A can be written as a (2×2)-matrix of operators with respect to this decomposition. Denote by π_+ and π_- the natural orthogonal projections defined by this decomposition.

Introduce now the subgroup $GB_r = GB_r(H_A)$ of $GB(H_A)$ consisting of operators whose off-diagonal terms belong to $K(H_A)$. Let $U_r = U_r(H_A)$ denote the subgroup of its unitary elements. To relate this to transmission problems, we must have an analog of the Kato Grassmannian introduced above. In fact, this is practically equivalent to introducing the concept of

Fredholm pairs of submodules. However we do not wish to generalize the whole discussion in Section 1 and we introduce only the notions which are necessary for formulating the main result.

Recall that there is a well-defined notion of a finite rank A-submodule of a Hilbert A-module [30]. Then the notion of Fredholm operator in a Hilbert A-module is introduced by requiring that its kernel and image be finite-rank A-submodules [30]. It turns out that many important properties of usual Fredholm operators remain valid in this context, too. Thus, if the collection of all Fredholm operators on H_A is denoted by $F(H_A)$, then there exists a canonical homomorphism $\mathrm{ind} = \mathrm{ind}_A : F(H_A) \to K_0(A)$, where $K_0(A)$ is the usual topological K-group of the basic algebra A [30].

This means that Fredholm operators over C^*-algebras have indices obeying the usual additivity law. In the sequel, we freely refer to a detailed exposition of these results in [30].

Granted the above technicalities, we can now introduce a special Grassmannian $Gr_+ = Gr_+(H_A)$ associated with the given decomposition. It consists of all A-submodules V of H_A such that the projection π_+ restricted on V is Fredholm while the projection π_- restricted on V is compact. Using the analogs of the local coordinate systems for $Gr_+(H_{\mathbb{C}})$ constructed in [32], one can verify that $Gr_+(H_A)$ is a Banach manifold modelled on the Banach space $K(H_A)$. For our purpose it suffices to consider Gr_+ as a metrizable topological space with the topology induced by the standard one on the infinite Grassmannian $Gr^\infty(H_A)$.

Now the problem that we are interested in is to investigate the topology of $Gr_+(H_A)$ and $GB_r(H_A)$. Notice that for $A = \mathbb{C}$ this is exactly the problem formulated by B. Bojarski in [4]. The main topological results about these objects can be formulated as follows. By $K_*(A)$ we denote the topological K-groups of A.

Theorem 3.1. *The group $GB_r(H_A)$ acts transitively on $Gr_+(H_A)$ with contractible isotropy subgroups.*

Theorem 3.2. *All even-dimensional homotopy groups of $Gr_+(H_A)$ are isomorphic to the index group $K_0(A)$ while its odd-dimensional homotopy groups are isomorphic to the Milnor group $K_1(A)$.*

Of course, the same statements hold for the homotopy groups of $GB_r(H_A)$, since by Theorem 3.1 these two spaces are homotopy equivalent. We formulate the result for $Gr_+(H_A)$ because it is the space of interest in the theory of linear conjugation problems.

The homotopy groups of $GB_r(H_A)$ were first computed in [19] without considering Grassmannians. Later, similar results were obtained by S. Zhang [37] in the framework of K-theory. The contractibility of isotropy subgroups involved in Theorem 3.1 in the case $A = \mathbb{C}$ was established in [32].

Actually, one can obtain more precise information on the structure of isotropy subgroups. It should also be noted that the contractibility of isotropy subgroups follows from a fundamental result on C^*-modules called the generalization of Kuiper's theorem for Hilbert C^*-modules [30]. Particular cases of Theorem 3.2 for various commutative C^*-algebras A may be useful to construct classifying spaces for K-theory. The solution of Bojarski's problem formulated in [4] is now immediate (cf. [19], [36], [32]).

Corollary 3.1. *Even-dimensional homotopy groups of the collection of classical Riemann–Hilbert problems are trivial while odd-dimensional ones are isomorphic to additive group of integers* $\mathbb{Z}$.

Note that the above fundamental group can be interpreted in terms of the so-called spectral flow of order zero pseudo-differential operators, which has recently led to some interesting developments by B. Booss and K. Wojciechowsky concerned with the Atiyah–Singer index formula in the odd-dimensional case [10]. Similar results hold for abstract singular operators over A which are defined by an analogy with Definition 1.4 (cf. [19]).

Corollary 3.2. *Homotopy groups of invertible singular operators over a unital* C^**-algebra* A *are expressed by the relations* (n *is an arbitrary natural number*)

$$\begin{gathered}\pi_0 \cong K_0(A), \quad \pi_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus K_1(A); \\ \pi_{2n} \cong K_0(A), \quad \pi_{2n+1} \cong K_1(A).\end{gathered} \tag{3.2}$$

Specifying this result for the algebras of continuous functions one can, in particular, compute the homotopy classes of invertible classical singular integral operators on arbitrary regular closed curves in the complex plane $\mathbb{C}$ (see [19], [23] for the precise definitions).

Corollary 3.3. *If* $\Gamma \subset \mathbb{C}$ *is a smooth closed curve with* k *components, then homotopy groups of invertible classical singular integral operators on* Γ *are expressed by the relations* (n *is an arbitrary natural number*)*:*

$$\pi_0 \cong \mathbb{Z}, \quad \pi_1 \cong \mathbb{Z}^{2k+1}; \quad \pi_{2n} = 0, \quad \pi_{2n+1} \cong \mathbb{Z}. \tag{3.3}$$

There exist some other applications of the above results to linear conjugation problems and singular integral operators but they are not so much in the spirit of this paper. Therefore we switch to another general paradigm which emerged in the theory of restricted Grassmannians and Grassmannian embeddings of loop groups. As was shown in [16], [21], the topological study of loop groups can be performed in the framework of the theory of Fredholm structures [14]. Since the Grassmannian embeddings of loop groups establish a close relation between geometric properties of the loop groups and those of restricted Grassmannians , it became highly plausible that one should be able to construct geometrically meaningful Fredholm structures on restricted Grassmannians .

Indeed, it was proved in [22] that restricted Grassmannians can be endowed with natural Fredholm structures arising from the generalized linear conjugation problems . The main construction in [22] was given in the language of linear conjugation problems with coefficients in loop groups and required a lot of preliminary considerations. Recently, the second author found another construction of Fredholm structure on restricted Grassmannian which is more explicit and direct. As was explained in [23] and [8], Fredholm structures on restricted Grassmannians induce the ones on loop groups and vice versa. Therefore the results presented in the last section provide simultaneously an alternative way of introducing Fredholm structures on loop groups which seems simpler than the one used in [16], [21].

4. Fredholm Structures on Restricted Grassmannians

The aim of this section is to show that restricted Grassmannians can be studied using so-called *Fredholm structures* [14]. The main result is that they can be endowed with natural Fredholm structures.

Fredholm structures on loop groups have already been described in the literature (see, e.g., [16], [21]) and it has been observed in [23], [8] that this enables one to obtain Fredholm structures on restricted Grassmannians as well. However the construction presented below seems more simple and instructive.

Before passing to precise definitions, we recall necessary concepts from functional analysis. For a Banach space E, let $L(E)$ denote the algebra of bounded linear operators in E endowed with the norm topology. Let $F(E)(F_k(E))$ denote the subset of Fredholm operators (of index k). Let also $GL(E)$ stand for the group of units of $L(E)$ and denote by $GC(E)$ the so-called *Fredholm group* of E defined as the set of all invertible operators from $L(E)$ having the form "identity plus compact".

Recall that a Fredholm structure on a smooth manifold M modelled on a (infinite dimensional) Banach space E is defined as a reduction of the structural group $GL(E)$ of the tangent bundle TM to the subgroup $GC(E)$ [14]. In the sequel we only deal with the case where $E = H$ is a separable Hilbert space but much of the following discussion is valid in a more general context.

As $GL(H)$ is contractible, $F_0(H)$ is the classifying space for $GC(H)$ bundles [14]. For a Hilbert manifold M, defining a Fredholm structure on M is equivalent to constructing an index zero Fredholm map $M \to H$ [15]. It was also shown in [15] that a Fredholm structure on M can be constructed from a smooth map $\Phi : M \to F_0(H)$, i.e., from a smooth family of index zero Fredholm operators parameterized by points of M. This is actually the most effective way of constructing Fredholm structures which has already been used in [16], [21].

We are now going to present an explicit construction of such families on restricted Grassmannians . For simplicity we describe it for the restricted Grassmannian Gr_r. It is convenient to use the subgroup $U_r = GL_r(H) \cap U(H)$ of $GL_r(H)$ and recall that in virtue of Theorem 2.1 U_r transitively acts on Gr_r and the isotropy subgroup of H_+ in U_r is $U(H_+) \times U(H_-)$.

We will construct a family of zero index Fredholm operators in H_+ parameterized by points of Gr_r. For $V \in Gr_r$, we first construct an element $T \in U_r$ such that $T(H_+) = V$. Let $v : H_+ \to H$ be an isometry with image V and $w : H_- \to H$ be an isometry with image $W = V^\perp$. Then the mapping

$$A = u \oplus w : H_+ \oplus H_- \to H_+ \oplus H_-$$

is a unitary transformation in H such that $A(H_+) = V$. Let us write it in the block form (A_{ij}) corresponding to the fixed decomposition $H = H_+ \oplus H_-$. Then the left upper element $T = A_{11}$ is a linear operator in H_+. As $V \in Gr_r$, from the definition of Gr_r it follows that T is a Fredholm operator in H_+, in other words, $T \in F(H_+)$.

Consider now a component Gr_0 of Gr_r consisting of subspaces $L \in Gr_r$ such that $\operatorname{ind} P_+|L = 0$. From the description of the connected components of the set of Fredholm pairs Fp^* given in Section 1 it follows that Gr_0 is a connected component of Gr_r and all other connected components Gr_n are homeomorphic to Gr_0. Moreover, the group U_r permutes those components so that any geometric structure on one of the components can be transplanted to all of them. Thus it is sufficient to construct a Fredholm structure on Gr_0.

Notice that the operator T constructed above is not unique so we cannot a priori assign it to V in a well-defined way. However from the description of the isotropy subgroups of Gr_r in U_r it follows that the totality of all such operators T has the structure of smooth fibration over Gr_r with contractible fiber. Thus by general results of infinite dimensional topology it has a global section S which can be chosen to be smooth [14]. This means that the assignment $V \mapsto S(V)$ defines a smooth family of index zero Fredholm operators in $L(H_+)$.

Referring now to the aforementioned result from [15], we see that the family $S(V), V \in Gr_0$, defines a Fredholm structure on Gr_0. In virtue of the said above, this structure can be transplanted on all other connected components. Thus we have established the desired result.

Theorem 4.1. *The restricted Grassmannian of a polarized Hilbert space has a natural structure of smooth Fredholm manifold modelled on the ideal of Hilbert-Schmidt operators $K_2(H)$.*

By using a proper modification of the above construction one can show that a similar statement holds for each restricted Grassmannian Gr^s with $s > 1$. A similar result holds for Kato Grassmannians in many Banach spaces, in particular, in the tame setting described in Section 2. Moreover, using the construction of orientation bundle on the space of Fredholm pairs given in [1], one can show that, in the real case, the above Fredholm structure is orientable in the sense of [15]. This enables one to study the geometry and topology of restricted Grassmannians and Kato Grassmannians using methods of the theory of Fredholm structures which are nowadays sufficiently developed [14], [15]. For the reason of space we confine ourselves to a few short remarks in this spirit.

As was proved in [15], each Fredholm structure on manifold M induces a zero index Fredholm map of M into its model. It is now natural to conjecture that such a map of Gr_r into $K_2(H)$ can be obtained from our construction. It would be instructive to find an explicit description of that map. It would be also interesting to define the same Fredholm structure by an explicitly given atlas on Gr_r.

Using the general methods of Fredholm structures theory, one can derive a number of immediate consequences of the theorem. For example, one can define Chern classes, fundamental classes of submanifolds, and so on in the spirit of [14], [16], [22]. Over the field of reals one can consider the maps between restricted Grassmannians arising from Fredholm operators in the ambient space and try to obtain their topological invariants using the degree theory for Fredholm manifolds developed in [15].

It is impossible to deal with any of the mentioned topics in a short paper like this one and so in conclusion we just express an intent to continue research in this direction.

References

1. P. Abbondandolo and P. Majer, Morse homology on Hilbert spaces. *Comm. Pure Appl. Math.* **54**(2001), No. 6, 689–760.
2. J. Avron, R. Seiler, and B. Simon, The index of a pair of projections. *J. Funct. Anal.* **120**(1994), No. 1, 220–237.
3. B. A. Barnes, G. J. Murphym, M. R. F. Smyth, and T. T. West, Riesz and Fredholm theory in Banach algebras. *Research Notes in Mathematics,* 67. *Pitman* (*Advanced Publishing Program*), *Boston, Mass.-London,* 1982. 1982.
4. B. Bojarski, Abstract linear conjugation problems and Fredholm pairs of subspaces, (Russian) *Differential and Integral equations. Boundary value problems. Collection of papers dedicated to the memory of Academician I. Vekua,* 45–60, *Tbilisi University Press, Tbilisi,* 1979.
5. B. Bojarski, Connections between complex and global analysis: some analytical and geometrical aspects of the Riemann-Hilbert transmission problem. *Complex analysis,* 97–110, *Math. Lehrbücher Monogr.* II. *Abt. Math. Monogr.,* 61, *Akademie-Verlag, Berlin,* 1983.
6. B. Bojarski, The geometry of the Riemann-Hilbert problem. *Geometric aspects of partial differential equations* (*Roskilde,* 1998), 25–33, *Contemp. Math.,* 242, *Amer. Math. Soc., Providence, RI,* 1999.
7. B. Bojarski, The geometry of Riemann-Hilbert problem. II. *Boundary value problems, integral equations and related problems* (*Beijing/Chengde,* 1999), 41–48, *World Sci. Publishing, River Edge, NJ,* 2000.
8. *B. Bojarski and G. Khimshiashvili*, Global geometric aspects of Riemann-Hilbert problems. *Georgian Math. J.* **8**(2001), No. 4, 713–726.
9. B. Bojarski and A. Weber, Generalized Riemann-Hilbert transmission and boundary value problems, Fredholm pairs and bordisms. *Bull. Polish Acad. Sci. Math.* **50**(2002), No. 4, 479–496.
10. B. Booss-Bavnbek and K. Wojciechowski, Elliptic boundary problems for Dirac operators. *Mathematics: Theory* & *Applications. Birkhäuser Boston, Inc., Boston, MA,* 1993.
11. A. Connes, Noncommutative geometry. *Academic Press, Inc., San Diego, CA,* 1994.

12. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, Modern geometry. (Russian) *Nauka, Moscow,* 1984.
13. M. Dupré and J. Glazebrook, The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* **41**(2001), No. 3, 264–287.
14. J. Eells, Jr., Fredholm structures. *Nonlinear functional analysis* (*Proc. Sympos. Pure Math., Vol.* XVIII, *Part* 1, *Chicago, Ill.,* 1968), 62–85. *Amer. Math. Soc., Providence, R.I.,* 1970.
15. K. D. Elworthy and A. J. Tromba, Differential structures and Fredholm maps on Banach manifolds. *Global Analysis* (*Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif.,* 1968), 45–94. *Amer. Math. Soc., Providence, R.I.,* 1970.
16. D. Freed, The geometry of loop groups. *J. Differential Geom.* **28**(1988), No. 2, 223–276.
17. D. Freed, An index theorem for families of Fredholm operators parametrized by a group. *Topology* **27**(1988), No. 3, 279–300.
18. T. Kato, Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. *Springer-Verlag, Berlin,* 1995.
19. G. Khimshiashvili, On the topology of invertible linear singular integral operators [*Analysis on manifolds and differential equations* (Russian), 112–133, *Voronezh. Gos. Univ., Voronezh,* 1986]. *Global analysis—studies and applications,* II, 211–230, *Lecture Notes in Math.,* 1214, *Springer, Berlin,* 1986.
20. G. Khimshiashvili, The Riemann-Hilbert problem for a compact Lie group. (Russian) *Dokl. Akad. Nauk SSSR* **310**(1990), No. 5, 1055–1058; English transl.: *Soviet Math. Dokl.* **41**(1990), No. 1, 165–169.
21. G. Khimshiashvili, On Fredholmian aspects of linear transmission problems. *Global analysis — studies and applications,* V, 193–216, *Lecture Notes in Math.,* 1520, *Springer, Berlin,* 1992.
22. G. Khimshiashvili, Homotopy classes of elliptic transmission problems over C^*-algebras. *Georgian Math. J.* **5**(1998), No. 5, 453–468.
23. G. Khimshiashvili, Geometric aspects of Riemann-Hilbert problems. *Mem. Differential Equations Math. Phys.* **27**(2002), 1–114.
24. G. Khimshiashvili, Global geometric aspects of linear conjugation problems. Complex analysis. *J. Math. Sci. (N. Y.)* **118** (2003), No. 5, 5400–5466.
25. G. Khuskivadze, V. Kokilashvili, and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**(1998), 1–195.
26. D. Kveselava, Some boundary problems of the theory of functions. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **16**(1948), 39–80.
27. G. Litvinchuk and I. Spitkovski, Factorization of measurable matrix functions. (Translated from the Russian) *Operator Theory: Advances and Applications,* 25. *Birkhäuser Verlag, Basel,* 1987.
28. G. Manjavidze, On a system of singular integral equations with discontinuous coefficients. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **11** (1950), 351–356.
29. G. Manjavidze, The behavior of solutions of a boundary value problem of linear conjugacy. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **35**(1969), 173–182.
30. V. Manuilov and E. Troitsky, Hilbert C^*- and W^*-modules and their morphisms. *Functional analysis* 6. *J. Math. Sci.* (*New York*) **98**(2000), No. 2, 137–201.
31. N. I. Muskhelishvili, Singular integral equations. Boundary problems of function theory and their application to mathematical physics. (Translated from the Russian) *P. Noordhoff N. V., Groningen,* 1953.
32. A. Pressley and G. Segal, Loop groups. *Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York,* 1986.

33. S. Prössdorf, Einige Klassen singulärer Gleichungen. *Mathematische Reihe, Band* 46. *Birkhäuser Verlag, Basel-Stuttgart,* 1974.
34. M. V. Shapiro and N. I. Vasilevski, Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I, II. *Complex Variables Theory Appl.* **27**(1995), No. 1, 17–46, 67–96.
35. N. P. Vekua, Systems of singular integral equations. (Translated from the Russian) *P. Noordhoff, Ltd., Groningen,* 1967.
36. K. Wojciechowski, Spectral flow and the general linear conjugation problem. *Simon Stevin* **59**(1985), No. 1, 59–91.
37. S. Zhang, Factorizations of invertible operators and K-theory of C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)* **28**(1993), No. 1, 75–83.

(Received 20.09.2004)

Authors' addresses:

B. Bojarski
Institute of Mathematics
Polish Academy of Sciences
8, Sniadeckich St., Warsaw
Poland
E-mail: bojarski@impan.gov.pl

G. Khimshiashvili
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 0193
Georgia
E-mail: khimsh@rmi.acnet.ge