R. Gabasov and F. M. Kirillova

OPTIMAL ON-LINE CONTROL WITH DELAYS

Dedicated to the 70th anniversary of
Guram Levanovich Kharatishvili —
pioneer of the theory of
optimal processes with delays

Abstract. A linear optimal control problem is considered. Optimal feedback controls are realized by digital computers (microprocessors). Because of deficiency of microprocessor speed available, a number of microprocessors is used for forming control functions. This fact results in delays between the moments when information on current states of the system become available to Optimal Controller and the moments when controls are fed to the control system. An algorithm for Optimal Controller under such conditions is presented. Results of operating of Optimal Controller implemented on a number of slow microprocessors and on one fast microprocessor with delays in closed channel are compared.

რეზიუმე. განხილულია წრფივი ოპტიმალური მართვის ერთი ამოცანა. უკუკავშირზე დამყარებული ოპტიმალური მართვა ციფრული კომპიუტერების (მიკროპროცესორების) მიერ ხორციელდება. არსებული მიკროპროცესორების არასაკმარისი სიჩქარის გამო რამდენიმე მიკროპროცესორი გამოიყენება მართვის ფუნქციის განსახორციელებლად. ამის გამო წარმოიშვება დაგვიანება იმ მომენტებს შორის, როდესაც, ერთის მხრივ, ინფორმაცია სისტემის მიმდინარე მდგომარეობის შესახებ ცნობილი ხდება ოპტიმალური სამართავი მოწყობილობისათვის და, მეორეს მხრივ, როდესაც მართვა მიეწოდება სამართავ სისტემას. მოყვანილია ალგორითმი ოპტიმალური სამართავი მოწყობილობისათვის ამ პირობებში. ერთმანეთს შედარებულია შედეგები, რომლებიც მიიღწევა, ერთის მხრივ, რამდენიმე ნელ და, მეორეს მხრივ, ერთ ჩქარ, მაგრამ დაგვიანებულ მიკროპროცესორზე აწყობილი ოპტიმალური სამართავი მოწყობილებებით.

1. Introduction

There are different sources of delays in control systems. They can influence substantially the behaviour of control objects. In the mathematical theory of optimal control [1], G. L. Kharatishvili [2] was the first who solved optimal control problems with delays. These results were developed by his disciples and other authors [3].

The aim of this paper is to describe methods of optimal control in real-time for nonstationary linear systems when delays result from the slow processing of current information or from lags in the feedback loop. A similar problem without delays was studied in [4].

The paper is organized in the following way. Section 2 contains the statement of the problem. The behaviour of a nonstationary linear system with moving terminal state is optimized by bounded piecewise-continuous controls over linear terminal performance index . Notions of open-loop and closed-loop solutions are introduced. Difficulties are stressed that arise when classical approaches are used to solve the optimal synthesis problem. The presentation of a new approach to the optimal synthesis problem starts in Section 3. Here optimality and suboptimality criteria for the problem under consideration in the class of discrete controls are formulated. The notion of support is the main tool of the method used and support optimality criteria are also presented in Section 3. The method (Section 4) is a dynamic realization of the authors' dual adaptive method [5] of linear programming (LP). The efficiency of the dual method while calculating optimal open-loop controls is illustrated by the example of a model of the car. Section 5 deals with an algorithm for operating of Optimal Controller able to calculate the current value of the realization of the optimal feedback in real time. The base of the algorithm is again the dual method. Results of optimal on-line control are presented by means of the previous example. Section 6 concludes the paper describing the method of optimal control in real-time taking into account the delays. Influence of delays is studied on a numerical example.

2. Problem Statement

Let $T = [t_*, t^*]$, $-\infty < t_* < t^* < \infty$, be a control interval.

In the class of piecewise-continuous functions $u(\cdot) = (u(t), t \in T)$ consider the linear optimal control problem

$$\dot{x} = A(t)x + b(t)u, \ x(t_*) = x_0, \tag{1}$$

$$c'x(t^*) \to \max, \tag{2}$$

$$Hx(t^*) = g, \ |u(t)| \le L, t \in T. \tag{3}$$

Here, $x = x(t) \in R^n$ is the state vector of (2) at the moment t, $u = u(t) \in R$ is the control variable; $A(t), b(t), \ t \in T$, are piecewise-continuous $n \times n$-matrix and n-vector functions, $H \in R^{m \times n}$, rank $H = m < n$.

As is known, there are two types of solutions in the optimal control theory: 1) optimal open-loop controls and 2) feedback controls.

A piecewise-continuous function $u(\cdot)$ is called an admissible open-loop control if it satisfies the geometrical constraint $|u(t)| \leq L$, $t \in T$, and the corresponding trajectory $x(\cdot)$ of (2) at the moment t^* reaches the terminal set $X^* = \{x \in R^n : Hx = g\}$.

An admissible open-loop control $u^0(\cdot)$ is said to be an optimal open-loop control of the problem (1) – (3) if the optimal trajectory $x^0(\cdot)$ satisfies the equality $c'x^0(t^*) = \max c'x(t^*)$.

For any $\varepsilon > 0$, an admissible open-loop control $u^\varepsilon(\cdot)$ is called a sub-optimal open-loop control if it generates a trajectory $x^\varepsilon(\cdot)$ satisfying the inequality $c'x^0(t^*) - c'x^\varepsilon(t^*) \leq \varepsilon$.

To define optimal feedback control, imbed the problem (1)-(3) into the family of problems

$$c'x(t^*) \to \max, \dot{x} = A(t)x + b(t)u, \ x(\tau) = \xi,$$
$$x(t^*) \in X^*, \ |u(t)| \leq L, t \in T(\tau) = [\tau, t^*], \tag{4}$$

depending on the scalar $\tau \in T$ and n-vector ξ. Let $u^0(t|\tau, \xi)$, $t \in T(\tau)$, be an optimal open-loop control of the problem (2) for the values of the parameters (τ, ξ), X_τ be the set of vectors $\xi \in R^n$ for which the problem (2) has a solution at fixed $\tau \in T$.

The function

$$u^0(\tau, \xi) = u^0(\tau|\tau, \xi), \quad \xi \in X_\tau, \quad \tau \in T, \tag{5}$$

is said to be an optimal feedback control of the problem (1)–(3).

The problem under consideration is the simplest one in the mathematical theory of optimal processes. Without any of its elements it becomes trivial. It is substantially simpler than the time-optimal problem

$$t^* \to \min, ; \ \dot{x} = A(t)x + b(t)u, \ x(t_*) = x_0, \ x(t^*) = 0, \ |u(t)| \leq L, \ t \in T,$$

for which the Maximum Principle for the first time was proved by R.V.Gam-krelidze [6]. The result of R.V.Gamkrelidze together with some preliminary investigations allowed L.S.Pontryagin to state as a hypothesis his famous Maximum Principle for nonlinear optimal control problems. While qualitative theory for the problem (1)–(3) is developed in great detail, effective numerical methods are still in need. Numerous methods for calculating optimal open-loop controls are suggested but the optimal synthesis problem (construction of optimal feedback controls) has been open as yet.

The optimal synthesis problem appeared on the frontier of classical and modern theories of control. It is formulated in the terms of the classical theory as the problem of consrtucting optimal feedbacks realizing the classical principle of closed-loop control. Synthesis of optimal feedbacks, feedbacks with marginal properties appeared to be on the boundary of possibilities of the classical theory. In the frame of the theory it is realizable only for isolated examples. In this connection the classical theory hands over the optimal synthesis problem to the modern theory which possesses the modern theory of extremal problems and methods for their solution.

Principles of on-line control proved to be very important in the optimal synthesis problem. The new principle is not a surplus but the only tool of solving the classical problem.

The aim of both the classical principle of closed-loop controls and the modern one is construction of feedback controls. The difference between the two principles is that the classical approach implies the construction of feedbacks before the control process starts (off-line computations), while in the on-line control the values of the feedback are calculated in the course of the process (on-line control).

Obviously, realizability of the modern approach depends crucially on modern computers. However, it cannot solve the problems just by itself thanks to high velocity of modern digital devices.

During first years of development of the optimal control theory, the main hopes of synthesis constructions were pinned on dynamical programming [7]. It was thought that mathematical baselessness of the Bellman equation was the main obstacle. Nowadays, when the Bellman equation is justified in the frame of nonsmooth analysis, there still remains a basic difficulty — the curse of dimensionality — for the optimal synthesis problem to be solved effectively.

Our concept of positional solution began to be formed in the late 80s [8, 9]. Before that (starting from 70s) the authors were concentrated on effective methods for open-loop solution. Being aware that complicated extremal problems cannot be solved without skills to solve more simple ones and having analyzed the simplest nonclassical extremal problems, which were without doubt linear programming problems, the authors proposed new methods of linear programming and developed them for optimal control problems [10]. Besides, on the whole, only open-loop solutions were considered at that time.

As a rule, open-loop solutions are seldom used for real control. They allow to estimate the potentials of control systems but are not effective for real-time control since: 1) the behaviour of real systems differ from that of mathematical models used in construction of optimal open-loop controls; 2) real systems operate under disturbances which cannot be taken into account while modelling.

To our opinion, while studying optimal control problems one should keep in mind the words of Dantzig's fundamental work [11]: "The final test of a theory is its capacity to solve the problems which originated it". The optimal synthesis problem was the first one which stimulated the creation of the optimal control theory [12]. Therefore a value of the optimal control theory can be evaluated by effective solution to this problem.

The significance of methods for synthesis of optimal systems is determined not only by own needs of the optimal control theory. With the help of these methods classical problems of the control theory (stabilization, regulation, tracking problems) which have no extremal form can be effectively solved [13]. Methods of optimal control can be the core of the model

predictive control theory which is being intensively developed and used in applications [14].

Analysis of the known by the late 80s approaches to the synthesis of optimal systems and results obtained persuaded the authors that the problem in the classical statement is not solvable without computer tools. The classical statement requires the feedback to be constructed before the real control process starts. This is the essence of the classical closed-loop principle. For simple problems, where marginal abilities of feedbacks are not used, effective feedbacks can be constructed. But for optimal control problems the feedbacks as a rule are very complicated and success is possible only for particular examples*.

The solution to the problem was found by the authors after changing the classical feedback principle by the modern one (the control in the real-time mode). According to it, the optimal feedback is not constructed beforehand but the current values needed for the control are calculated in the course of the process. Implementation of that principle is supported by two facts: 1) the fast algorithms for calculating optimal open-loop controls created on the basis of the adaptive method of linear programming [5], 2) the use of modern computers.

Below we present principal elements of the new approach to the optimal synthesis problem.

3. Maximum and ε-Maximum Principles

A natural way to solve the problem (1)–(3) is based on the use of digital computers. Keeping in mind real applications, we narrow the class of admissible controls and consider the problem (1)–(3) in the class of discrete controls.

A function $u(\cdot)$ is said to be a discrete control (with the quantization period $h = (t^* - t_*/N)$), if

$$u(t) = u(t_k), \quad t \in [t_k, t_k + h[, \quad t_k = t_* + kh, \quad k = \overline{0, N-1},$$

where N is fixed.

The use of discrete controls dismiss some analytical problems but does not simplify the problem (1) – (3) for the constructive approach.

In the class of discrete controls the problem (1)–(3) is equivalent to LP problem

$$\sum_{t \in T_h} c_h(t)u(t) \to \max, \quad \sum_{t \in T_h} d_h(t)u(t) = \tilde{g},$$

$$|u(t)| \leq L, \quad t \in T_h = \{t_*, t_* + h, \ldots t^* - h\}. \tag{6}$$

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*The Kalman–Letov problem is an exception confirming the general rule. But it doesn't contain important for applications geometric constraints on controls, being more a problem of calculus of variations than that of optimal control.

Here $c_h(t) = \int\limits_t^{t+h} c(s)ds$, $c(t) = c'F(t^*)F^{-1}(t)$, $d_h(t) = \int\limits_t^{t+h} d(s)ds$, $d(t) = HF(t^*)F^{-1}(t)$; $\dot{F} = A(t)F$, $F(t_*) = E$, E is the identity matrix; $\tilde{g} = g - HF(t^*)x_0$.

For small quantization periods h the problem (6) has not many (m) basic constraints but it has a large number (N) of variables, and neighboring columns $d_h(t)$, $d_h(t + h)$ are almost collinear. The problem (6) can be solved by standard methods of LP, but in case of this approach dynamical specificity of the problem (6) can be missed. The situation resembles the one with transportation problems. Any transportation problem can be reduced to a general linear programming problem and solved by the simplex-method. But a special simplex-method being "transportation's realization" of it, which takes into account all features of the specific problem, proved to be more effective. Following this idea, the authors (together with N.V.Balashevich) justified a dynamical realization [4] of the adaptive method [5] for optimal open-loop solutions.

The main tool of the adaptive method is the notion of support. Its analog for dynamic problem (6) is the set $T_{\text{sup}} = \{t_i \in T_h, i = \overline{1, m}\}$ consisting of m moments such that the matrix

$$D_{\text{sup}} = (d_h(t), \quad t \in T_{\text{sup}}) \tag{7}$$

is nonsingular.

To construct the support matrix (7), it is sufficient to find m solutions $\psi_i(t)$, $t \in T$, $i = \overline{1, m}$, to the adjoint system

$$\dot{\psi} = -A'(t)\psi \tag{8}$$

with initial conditions $\psi(t^*) = h_i$, $i = \overline{1, m}$ (h_i is the ith row of the matrix H), and calculate the functions $d(t) = (\psi'_i(t)b(t), i = \overline{1, m})$, $d_h(t) = \int\limits_t^{t+h} d(s)ds$.

The support characterizes controllability of the output $y = Hx(t^*)$ by means of the values of the input $u(t)$ at the support moments $t \in T_{\text{sup}}$.

Every support is accompanied by:

1) an m-vector of Lagrange multipliers

$$\nu' = c'_{\text{sup}}D_{\text{sup}}^{-1}, \quad c_{\text{sup}} = \left(\int\limits_t^{t+h} c(s)ds, t \in T_{\text{sup}} \right), \quad c(t) = \psi'_c(t)b(t);$$

2) a co-control

$$\delta_h(t) = \int\limits_t^{t+h} \delta(s)ds, t \in T_h; \quad \delta(t) = \psi'(t)b(t), \quad t \in T,$$

where the co-trajectory $\psi(t)$, $t \in T$, is a solution of the equation (8) with the initial condition $\psi(t^*) = c - H'\nu$.

3) a pseudocontrol $\omega(t)$, $t \in T$. Nonsupport values $\omega(t)$, $t \in T_n = T_h \setminus T_{\sup}$ are defined as

$$\omega(t) = L \operatorname{sign} \delta_h(t), \text{ at } \delta_h(t) \neq 0; \quad \omega(t) \in [-L, L], \text{ at } \delta_h(t) = 0; t \in T_n.$$

Support values $\omega_{\sup} = (\omega(t), t \in T_{\sup})$, of the pseudocontrol are given by the formula

$$\omega_{\sup} = D_{\sup}^{-1}(g - H \ae_0(t^*)),$$

where $\ae_0(t^*)$ is the state at the moment t^* of the system (2) with the control $\omega_0(t) = \omega(t)$, $t \in T_n$, $\omega_0(t) = 0$, $t \in T_{\sup}$.

The support is the main tool for identifying the optimal controls.

Theorem 1 (Maximum Principle). *For an admissible open-loop control $u(t)$, $t \in T$, to be optimal it is necessary and sufficient that a support T_{sup} exists with the accompanying co-control $\delta_h(t)$, $t \in T$, satisfying the maximum condition*

$$\delta_h(t)u(t) = \max_{|u| \leq L} \delta_h(t)u, \quad t \in T_n.$$

A support T_{sup} which identifies an optimal control is said to be the optimal support.

Theorem 2 (Support Optimality Criteria). *A support $T_{\sup}$ is optimal if the accompanying pseudocontrol $\omega(t)$, $t \in T_h$, satisfies the inequality*

$$|\omega(t)| \leq L, \quad t \in T_{\sup}.$$

In this case the pseudocontrol $\omega(t)$, $t \in T$, is an optimal control of the problem (1)–(3): $u^0(t) = \omega(t)$, $t \in T$.

Suboptimality criteria can also be proved in terms of the support:

Theorem 3 (ϵ-maximum Principle). *For any $\epsilon \geq 0$, for ϵ-optimality of an admissible control $u(t)$, $t \in T$, it is necessary and sufficient the existence of such a support $T_{\sup}$ for which the following conditions hold:*

1) ϵ*-maximum condition*

$$\delta_h(t)u(t) = \max_{|u| \leq L} \delta_h(t)u - \epsilon(t), \quad t \in T_n;$$

2) ϵ*-accuracy condition*

$$\sum_{t \in T_n} \epsilon(t) \leq \epsilon.$$

4. A Dual Method for Program (Open-Loop) Solutions

On the basis of the results from Section 3, primal and dual methods for constructing optimal open-loop controls are elaborated [5]. Below the main elements of the dual method are presented. The method in question allows to construct program solution to the problem (1)–(3) very quickly. The significance of the method is revealed while solving the optimal synthesis problem (see Sections 5 and 6).

The dual method for the program solution is a dynamic realization of the dual adaptive method of LP [10]. It consists in consecutive change of supports

$$T^1_{\text{sup}} \to T^2_{\text{sup}} \to \ldots T^0_{\text{sup}},$$

which results in construction of support T^0_{sup}. The initial support T^1_{sup} can be arbitrary.

The method is finite if the supports used in iterations are all regular (with $\delta_h(t) \neq 0$, $t \in T_n$). There exists [13] a modification of the dual method which is finite for any problem (1)–(3).

The iteration of the dual method represents a movement by specified rules of one support and all nonsupport zeroes of the co-control until complete relaxation of the performance index of the dual to (6) problem. The details can be found in [4].

The effectiveness of the dual method is estimated as in [16]. According to the mentioned methodology of the estimate, the main time consuming operations are integrations of primal (2) and dual (8) systems. The method has a complexity equal to the unit if while constructing an optimal open-loop control the integrations were made on the intervals of summarized length equal to $t^* - t_*$. It is impossible to find explicit formulae to estimate the complexity but computer experiments can give a certain idea about it.
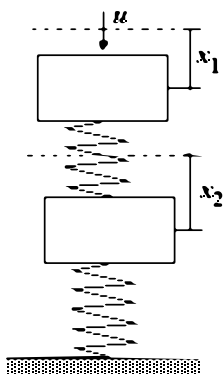

Fig. 1

The effectiveness of the dual method in consideration is illustrated on the following example which is a one-quarter model of a car (see Figure 1).

The mathematical model of the problem is as follows:

$$J(u) = \int_0^{25} u(t)dt \to \min, \quad \dot{x}_1 = x_3, \ \dot{x}_2 = x_4, \ x_1(0) = x_2(0) = 0,$$

$$\dot{x}_3 = -x_1 + x_2 + u, \ \dot{x}_4 = 0.1x_1 - 1.02x_2, \ x_3(0) = 2, \ x_4(0) = 1, \qquad (9)$$

$$x_1(25) = x_2(25) = x_3(25) = x_4(25) = 0, \quad 0 \leq u(t) \leq 1, \ t \in [0, 25[,$$

where $x_1 = x_1(t)$ is the deviation of the first mass from its equilibrium, $x_2 = x_2(t)$ is the deviation of the second mass, $x_3 = dx_1/dt$, $x_4 = dx_2/dt$.

If we interpret $u(t)$ as fuel consumption per second at the moment t, then the problem (4) is to damp oscillations of both masses with the minimal fuel consumption. The problem (4) is equivalent to (1) – (3) with $x_5(t) = \int_0^t u(s)ds$.

As an initial support, the set $T_{\sup} = \{5, 10, 15, 20\}$ of moments uniformly distributed on T was taken. The problem (4) was solved for different values of h. It was discovered that the complexity of the dual method almost doesn't depend on h (at $h = 0.0025$ the complexity was equal to 0.2018 with $J(u^0) = 6.330941$; at $h = 0.001$ the complexity was equal to 0.23564 with $J(u^0) = 6.330938$). In all cases the complexity of iterations did not exceed 0.25. That means that time spent on calculation of the optimal open-loop controls is not greater that 25 percent of time needed to perform one integration of the adjoint system (8) on the interval $[0, 25]$.
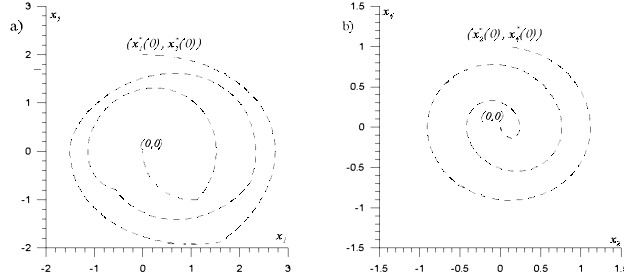

Fig. 2

Figure 2 presents the projections on the planes Ox_1x_2 and Ox_3x_4 of optimal open-loop trajectories of the system (2).

5. Optimal on-Line Control

In the previous section discrete controls were introduced. Let us modify the definition of the optimal feedback control according to the new class of admissible open-loop controls. Assume that during the control process the current states of the control system are measured at the moments $t \in T_h$ only.

Imbed the problem (1)–(3) into a new family of problems

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = \xi,$$
$$x(t^*) \in X^*, \quad |u(t)| \le L, \quad t \in T(\tau) = [\tau, t^*],$$

depending on a discrete moment $\tau \in T_h$ and an n-vector ξ.

A function

$$u^0(\tau, \xi) = u^0(\tau|\tau, \xi), \quad \xi \in X_\tau, \quad \tau \in T_h, \tag{10}$$

is said to be a discrete optimal feedback control of the problem (1)–(3).

While using discrete optimal feedbacks, the dynamic programming does not come across any problems of justification. However the "curse of dimensionality" is present as usual if the order of the control system is more than 2.

An analysis of optimal feedback controls led the authors to a new statement of the synthesis problem. First of all, the goal of feedbacks and the way they are used in optimal control processes were clarified: the system (2) represents a mathematical model of a real dynamic system. The behaviour of this real system differs from that of the model (2). Let a physical prototype of that system (2) behave according to the equation

$$\dot{x} = A(t)x + b(t)u + w, \quad x(t_*) = x_0, \tag{11}$$

where w is a totality of elements corresponding to mathematical modelling inaccuracies and unknown disturbances.

Optimal feedback (10) is determined from (2) but is intended for control of the real system (11). Close the system (11) by feedback (10)

$$\dot{x} = A(t)x + b(t)u^0(t, x) + w, \quad x(t_*) = x_0. \tag{12}$$

The equation (12) is a nonlinear differential equation with discontinuous righthand side. Basing on discrete feedback (10), define the solution of the equation (12) as a solution of the equation $\dot{x} = A(t)x + b(t)u(t) + w$, $x(t_*) = x_0$, where $u(t) = u^0(t|t_* + kh, x(t_* + kh))$, $t \in [t_* + kh, t_* + (k+1)h[$, $k = \overline{0, N-1}$.

Suppose that the optimal feedback (10) has been constructed. Consider the behaviour of the closed system (12) in a concrete control process where an unknown disturbance $w^* = w^*(t)$, $t \in T$, is realized. This disturbance generates a trajectory $x^*(t)$, $t \in T$, of (12) satisfying the identity

$$\dot{x}^*(t) \equiv A(t)x^*(t) + b(t)u^0(t, x^*(t)) + w^*(t), \quad t \in T.$$

From the identity one can observe that in the process in question the optimal feedback is not used as a whole (for all $x \in X_\tau$, $\tau \in T_h$). Only the signals $u^*(t) = u^0(t, x^*(t))$, $t \in T_h$, along the continuous trajectory $x^*(t)$, $t \in T$, are used in the control process. Moreover, it is not necessary to know beforehand the realization $u^*(t)$, $t \in T$, of optimal the feedback (10). It is sufficient to know $x^*(\tau)$ at each moment $\tau \in T_h$ to calculate the current value $u^*(\tau)$ in time $s(\tau)$ which does not exceed h.[†]

A device which is able to fulfill this work is called Optimal Controller.

Thus, the optimal synthesis problem is reduced to constructing algorithm for Optimal Controller.

[†] The delay $s(\tau)$ influences the optimal trajectory at control switching point only yielding minor variations.

Optimal Controller operates as follows. Before the process starts, it calculates the optimal open-loop control $u^0(t|t_*, x_0)$, $t \in T$, for the initial position (t_*, x_0). Any algorithm for program solution can be used as there are no restrictions on duration of calculations. Nevertheless, it is reasonable to use the above described dual method to construct the optimal support which plays a significant part in what follows. When the control process starts, Optimal Controller feeds to the input of the control system a signal $u^*(t) = u^0(t|t_*, x_0)$, $t \in [t_*, t_* + h + s(t_* + h)[$.

Suppose that Optimal Controller has been acting during the time $[t_*, \tau[$. At the moment $\tau - h + s(\tau + h)$ it finished constructing the optimal support $T_{\sup}^0(\tau - h)$ and the current value $u^*(\tau - h)$ of the realization of the optimal feedback. The signals $u^{**}(t) = u^*(\tau - 2h)$, $t \in [\tau - 2h, \tau - h + s(\tau - h)[$; $u^{**}(t) = u^*(\tau - h)$, $t \in [\tau - h + s(\tau - h), \tau - h[$; and realized disturbance $w^*(t)$, $t \in [\tau - h, \tau]$, transfer the system (2) at the moment τ into the state $x^*(\tau)$. Optimal Controller obtains the information about this state at instant τ. The task of Optimal Controller on the interval $[\tau, \tau + h[$ is to calculate the optimal open-loop control $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau)$.

Let $x^0(\tau)$ be a state of the system (2) achieved from the state $x^*(\tau - h)$ with the signal $u^{**}(t)$, $t \in [\tau - h, \tau[$. The vector $x^0(\tau)$ differs from the "true" state $x^*(\tau)$ by the quantity $\int_{\tau-h}^{\tau} F(\tau) F^{-1}(s) w^*(s) ds$. Under bounded disturbances $w^*(t)$, $t \in [\tau - h, \tau]$, the smaller is the quantization period h the smaller is the distance $\|x^*(\tau - x^0(\tau))\|$. In this situation the dual method proves to be very efficient. This method takes the optimal support $T_{\sup}^0(\tau - h)$ constructed during the previous interval $[\tau - h, \tau[$ as an initial support $T_{\sup}(\tau)$ to construct the optimal support $T_{\sup}^0(\tau)$ and the corresponding optimal open-loop control $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau)$. Starting from the moment $\tau + s(\tau)$, Optimal Controller feeds to the input of the dynamical system the signal $u^*(t) = u^0(t|t_*, x_0)$, $t \geq \tau + s(\tau)$.

Optimal Controller repeats the described operations at the moments $\tau + h \in T_h$. The control signal $u^{**}(t)$, $t \in T$, generated by Optimal Controller may only differ from the ideal realization $u^*(t)$, $t \in T$, of the optimal feedback in the neighborhoods of the switching points. Therefore the trajectories of the dynamical system (2) generated by these controls will be almost nondistinct.

Let us use the previous example to show how Optimal Controller operates.

Let the realized disturbance (unknown to Optimal Controller) be

$$w^*(t) \equiv 0.3 \sin 4t, \quad t \in [0, 9.75[, \quad w^*(t) \equiv 0, \quad t \geq 9.75.$$

It turned out that in the course of the control process the complexity of calculating the current values $u^*(\tau)$, $\tau \in T_h$, did no exceed 0.02. This means that for every $\tau \in T_h$, to calculate $u^*(\tau)$ it is used only 2 percent of time needed to integrate the adjoint system on the whole interval T. The realization of the optimal feedback $u^*(t)$, $t \in T$, is given in Figure 3. Figure

3 presents the projections on the planes $0x_1x_3$, $0x_2x_4$ of the trajectories of the closed system.
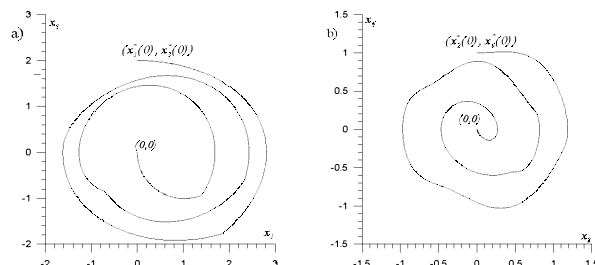

Fig. 3

If a given microprocessor integrates the adjoint system in time α and $0.02\alpha < h$, then the microprocessor can be used for optimal on-line control. It is clear that this inequality is fulfilled for high-order control systems.

Notes:

1. It was assumed above that the initial condition x_0 was known. The method can be developed in situation when x_0 becomes available at the instant t_* only but before that moment it is known that it belongs to a bounded set $X_0 \subset R^n$ [17].

2. The method described is developed for more complicated problems such as optimal control problem with intermediate state constraints [18], optimal control problem for piecewise-linear systems [19] and nonlinear systems [20], optimal control problem with parallelepiped restrictions on controls [21] and optimal control problem under uncertainty [22, 23].

6. Optimal Control in Real Time with Delays

Consider the situation[‡] which corresponds to the goal of the paper. First, define slow and fast microprocessors. Let the system (2), set of its admissible states $X(\tau)$, $\tau \in T_h$, optimal supports $T_{\sup}(\tau, x)$ for all possible positions (τ, x), $x \in X(\tau)$, $\tau \in T_h$, be given.

A microprocessor is said to belong to the class l if for a given level ρ of disturbance the dual method knowing the support $T_{\sup}(\tau, x)$ constructs in time not exceeding lh the optimal support $T_{\sup}(\tau, \bar{x})$ for all $\bar{x} \in X(\tau)$ such that $\|\bar{x} - x\| \leq \rho$.

Microprocessors with $l \leq 1$ are said to be fast microprocessors, with $l > 1$ are called slow ones.

[‡]The results were obtained together with N. N. Kavalionak.

The microprocessors of class l can be used for optimal on-line control of the dynamic system if it operates under disturbances satisfying the inequality $\left\| \int\limits_{\tau}^{\tau+lh} F(\tau + lh)F^{-1}(s)w(s)ds \right\| \leq \rho$ for all $\tau \in T_h$.

The procedure of optimal control in real-time is described in Section 5. Now we suppose that for control of the dynamic system in question, microprocessors of class l $(l > 1)$ only are available.

Under such condition the optimal control process is divided into stages: preliminary stage, first, second etc.

On the preliminary stage (before the control process), using the a priori information Optimal Controller constructs the optimal support $T_{\sup}(t_*, x_0)$ and the optimal open-loop control $u^0(t|t_*, x_0)$, $t \in T_h$, for the initial position (t_*, x_0). The optimal support $T_{\sup}(t_*, x_0)$ is corrected for the moments $\tau = t_* + ih$, $i = \overline{1, l}$, if there exists a support moment $t \in T_{\sup}(t_*, x_0)$ such that $t < \tau$. The correction is made according to the following rules. With the initial support $T_{\sup}(t_*, x_0)$ for every $\tau > t \in T_{\sup}(t_*, x_0)$, the following problem is solved by the dual method

$$c'x(t^*) \to \max; \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0; \quad x(t^*) \in X^*;$$
$$l_*(t) \leq u(t) \leq l^*(t), \quad t \in [t_*, \tau[; \quad |u(t)| \leq L, t \in T(\tau), \tag{13}$$

where $l_*(t) = l^*(t) = u^0(t|t_*, x_0)$, $t \in [t_*, \tau[$. Denote by $T_{\sup}(t_*, x_0|\tau)$ the modified support for the moment $\tau = t_* + ih$, $i = \overline{1, l}$, (the optimal support of problem (13)). On the preliminary stage, the time needed to perform these operations is not essential.

The control process starts at the moment t_*. The signal $u^*(t) = u^0(t|t_*, x_0)$, $t \in [t_*, t_* + (l+1)h[$ is fed to the input of the control system.

The first stage of operating Optimal Controller is the control of the system in question on the interval $[t_*, t_* + (l+1)h[$. At the moment $t_* + h$ Optimal Controller obtains the first measurement, the state $x^*(t_* + h)$ of the system generated by $u^*(t)$, $w^*(t)$, $t \in [t_*, t_* + h[$. The measurement $x^*(t_* + h)$ is transferred to the first microprocessor (M1) which using support $T_{\sup}(t_*, x_0|t_* + h)$ as the initial one solves by the dual method the following problem:

$$c'x(t^*) \to \max; \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = x^*(\tau); \quad x(t^*) \in X^*l;$$
$$l_*(t) \leq u(t) \leq l^*(t), \quad t \in [\tau, \tau + lh[; \quad |u(t)| \leq L, \quad t \in T(\tau + lh), \tag{14}$$

where $l_*(t) = l^*(t) = u^0(t|t_*, x_0)$, $t \in [\tau, \tau + lh[$, $\tau = t_* + h$. As a result, the microprocessor M1 constructs the optimal support $T_{\sup}(t_* + h, x^*(t_* + h)|t_* + (l+1)h)$ and the optimal open-loop control $u^0(t|t_* + h, x^*(t_* + h))$, $t \in T(t_* + h)$. The value of the obtained optimal open-loop control is fed into the input of the control system on the interval $[t_*(l+1)h, t_* + (l+2)h[$. A part $u^0(t|t_* + h, x^*(t_* + h))$, $t \in [t_*(l+1)h, t_* + (2l+1)h[$, together with the support $T_{\sup}(t_* + h, x^*(t_* + h)|t_* + (l+1)h)$ will be used by M1 for processing the measurement $x^*(t_* + (l+1)h))$ on the second stage.

Following measurements $x^*(t_* + ih)$, $i = \overline{2,l}$, are processed by microprocessors $M2, \ldots, Ml$ according to the scheme described for M1. By completing this task, the first stage of functioning Optimal Controller is finished.

The second stage starts at the $t_* + (l+1)h$ when Optimal Controller obtains the measurement $x^*(t_* + (l+1)h)$ transferred to the microprocessor M1 becoming free by this moment. Using the initial support $T_{\sup}(t_* + h, x^*(t_*+h)|t_* + (l+1)h)$, M1 solves problem (14) by the dual method with $\tau = t_* + (l+1)h$, $l_*(t) = l^*(t) = u^0(t|t_* + h, x^*(t_*+h))$, $t \in [\tau, \tau + lh[$. As a result, the optimal support $T_{\sup}(t_* + (l+1)h, x^*(t_* + (l+1)h)|t_* + (2l+1)h)$ and the optimal open-loop control $u^0(t|t_* + (l+1)h, x^*(t_* + (l+1)h))$, $t \in T(t_* + (l+1)h)$, are obtained. A part of the obtained optimal open-loop control on the interval $[t_*(2l+1)h, t_* + (2l+2)h[$ is fed into the input of the control system. The optimal support $T_{\sup}(t_* + (l+1)h, x^*(t_* + (l+1)h)|t_* + (2l+1)h)$ and the values $u^0(t|t_* + (l+1)h, x^*(t_* + (l+1)h))$, $t \in [t_*(2l+1)h, t_* + (3l+1)h[$ will be used by M1 for processing the measurement $x^*(t_* + (2l+1)h))$ on the third stage.

The described rules are applied to the rest of the microprocessors. All consequent stages are similar to the second stage.

Now consider the situation where the delay in the course of the optimal real-time control appears due to the other reason. Let every current measured state $x^*(\tau)$ be processed by a fast microprocessor but measurements become available to it in lh units of time. Describe an algorithm for optimal Controller in this case.

Before the control process starts, Optimal Controller constructs the optimal support $T_{\sup}(t_*, x_0)$ and the optimal open-loop control $u^0(t|t_*, x_0)$, $t \in T$, for the initial position (t_*, x_0).

On the interval $[t_*, t_* + (l+1)h[$ the object moves under the control $u^*(t) = u^0(t|t_*, x_0)$. At the moment $t_* + h$ Optimal Controller gets the first measurement $x^*(t_* + h)$. Using this measurement and the support $T_{\sup}(t_*, x_0)$, Optimal Controller constructs the optimal support $T_{\sup}(t_* + h, x^*(t_*+h))$ and the optimal open-loop control $u^0(t|t_* + h, x^*(t_*+h))$, $t \in T(t_* + h)$. The value $u^0(t_* + (l+1)h|t_* + h, x^*(t_*+h))$ sent to the input of the control system reaches it at the instant $t_* + (l+1)h$ and is used on the interval $[t_* + (l+1)h, t_* + (l+2)h[$.

Processing of the rest measurements $x^*(t_* + 2h)$, $x^*(t_* + 3h), \ldots$ are performed similarly.

The described method for control via Optimal Controllers can be called the optimal control in real-time with delays. In the method the rate of processing information on the behaviour of the control system and the rate at which this information becomes available are the same. But there exists a delay between moments when measurements are made and moments when the signals constructed on these measurements are fed to the control system.

The influence of delays during real-time control was performed by the use of the following example:

$$J(u) = \int\limits_0^{12} u(t)dt \to \min_u, \quad \ddot{y} = -y + u + w, \quad y(0) = 3, \quad \dot{y}(0) = 0,$$

$$y(12) = \dot{y}(12) = 0, \quad 0 \le u(t) \le 1, \quad t \in T = [0, 12], \tag{15}$$

where $w(t) = 0.3 \sin t$, $t \in [0, 6[$, $w(t) = 0$, $t \in [6, 12]$, $h = 0.12$.

If $u(t)$ is treated as consumption of fuel per second, then the problem (15) is to minimize fuel expenditures for damping the oscillator (15) during 12 units of time.
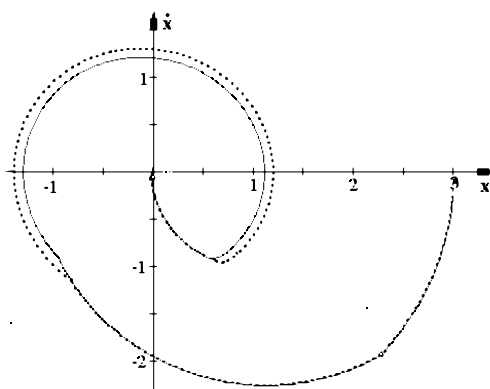

Fig. 4

Positional solutions for the problem (15) were constructed for slow ($l = 3$) and fast microprocessors.

Almost similar are the phase trajectories of the closed system for the fast microprocessor without delay (Section 5), $J(u) = 2.838$, the fast microprocessor with delay ($l = 3$), $J(u) = 2.840$, and slow microprocessors ($l = 3$), $J(u) = 2.842$, with similar input data. Figure 4 contains a phase trajectory of the closed system obtained with the help of slow microprocessors ($l = 3$) (dash line), and with the help of only one microprocessor (solid line) which obtains information on system state at instants lh, $J(u) = 2.885$.

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Authors' address:

18, Leninskii Pr., apt. 103
220050 Minsk
Belarus