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NECESSARY CONDITIONS OF OPTIMALITY FOR SOME CLASS QUASI-LINEAR NEUTRAL OPTIMAL PROBLEMS WITH CONTINUOUS INITIAL CONDITION

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Let $O \subset R^n$, $G_i \subset R^n$, $i = 1, 2$, be open sets; the function $f : J = [a, b] \times O^s \times G_1 \to R^n$ be continuously differentiable; for any $(x_1, \dots, x_s, u) \in O^s \times G_1$ the functions $f(t, x_1, \dots, x_s, u)$, $f_{x_j}(\cdot)$, $i = 1, \dots, s$, be measurable on J; for arbitrary compacts $K \subset O$, $K_1 \subset G_1$ there exist a function $m_{K,K_1}(\cdot) \in L_1(J, [0, \infty))$, such that for any $(x_1, \dots, x_s, u) \in K^s \times K_1$ and for almost all $t \in J$, the following inequality is fulfilled

$$|f(t, x_1, \dots, x_s, u)| + \sum_{i=1}^{s} |f_{x_i}(\cdot)| \leq m_{K,K_1}(t).$$

Let the scalar functions $\tau_i(t)$, $i = 1, \dots, s$, $t \in R$ and $\eta_j(t)$, $j = 1, \dots, p$, $t \in R$, be absolutely continuous and continuously differentiable , respectively, and satisfy the conditions: $\tau_i(t) \leq t$, $\dot{\tau}_i(t) > 0$, $i = 1, \dots, s$; $\eta_j(t) < t$, $\dot{\eta}_j(t) > 0$, $j = 1, \dots, p$; $\gamma_i(t) = \tau_i^{-1}(t)$, $i = 1, \dots, s$; $\sigma_j(t) = \eta_j^{-1}(t)$, $j = 1, \dots, p$; Γ be the set of continuously differentiable functions $\varphi : J_1 = [\rho, b] \to N$, $\rho = \min\{\tau_1(a), \dots, \tau_s(a), \eta_1(a), \dots, \eta_p(a)\}$, $\|\varphi\| = \sup\{|\varphi(t)| + |\dot{\varphi}(t)| : t \in J_1\}$,where $N \subset O$is a convex set; Ω_1 be the set of measurable functions $u : J \to U$, such that $cl\{u(t) : t \in J\} \subset G_1$ is compact, where $U \subset G_1$ is an arbitrary set; Ω_2 be the set of piecewise continuous functions $v : J \to V$,where $V \subset G_2$ is a convex set; $A_j(t, \mu)$, $j = 1, \dots, p$ be $n \times n$-dimensional matrix functions, continuous on $J \times V$ and continuously differentiable with respect to $v \in V$; $q^i : J^2 \times O^2 \to R$, $i = 0, \dots, l$, be continuously differentiable functions.

To every element $\mu = (t_0, t_1, \varphi, u, v) \in B = J^2 \times \Gamma \times \Omega_1 \times \Omega_2$, $t_0 < t_1$, let us correspond the differential equation

$$\dot{x}(t) = \sum_{i=1}^{p} A_j(t, v(t))\dot{x}(\eta_j(t)) + f(t, x(\tau_1(t)), \dots, \tau_s(t), u(t)), \quad t \in [t_0, t_1], \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \quad t \in [\rho, t_0]. \tag{2}$$

Definition 1. The function $x(t) = x(t, \mu) \in O$, $t \in [\rho, t_1]$, is said to be a solution corresponding to the element $\mu \in B$, if on $[\rho, t_0]$ it satisfies the condition (2), while on the interval $[t_0, t_1]$ the function $x(t)$ is absolutely continuous and satisfies the equation (1) almost everywhere.

Definition 2. The element $\mu \in B$ is said to be admissible, if the corresponding solution $x(t) = x(t, \mu)$ is defined on $[t_0, t_1]$ and satisfies the conditions

$$q'(t_0, t_1, x(t_0), x(t_1)) = 0, \quad i = 1, \dots, l.$$

Denote by B_0 the set of the admissible elements.

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Definition 3. The element $\widetilde{\mu}=(\widetilde{t}_0,\widetilde{t}_1,\widetilde{\varphi},\widetilde{u},\widetilde{v})\in B_0$ is said to be locally optimal, if there exist a number $\delta>0$ and a compact set $K\subset O$ such that for an arbitrary element $\mu\in B$ satisfying

$$|t_0-\widetilde{t}_0|+|t_1-\widetilde{t}_1|+\|\varphi-\widetilde{\varphi}\|+\|f-\widetilde{f}\|_K+\sup_{t\in J}|v(t)-\widetilde{v}(t)|\le\delta,$$

the inequality

$$q^0(\widetilde{t}_0,\widetilde{t}_1,\widetilde{x}(\widetilde{t}_0),\widetilde{x}(\widetilde{t}_1))\le q^0(t_0,t_1,x(t_0),x(t_1))$$

is fulfilled.

Here

$$\|f-\widetilde{f}\|_K=\int_J H(t;f,K)dt,$$

$$H(t;f,K)=\sup\Big\{|f(t,x_1,\dots,x_s)-\widetilde{f}(t,x_1,\dots,x_s)|+$$
$$+\sum_{i=1}^{s}|f_{x_i}(\cdot)-\widetilde{f}_{x_i}(\cdot)|:(x_1,\dots,x_s)\in K^s\Big\};$$
$$f(t,x_1,\dots,x_s)=f(t,x_1,\dots,x_s,u(t)),\widetilde{f}(t,x_1,\dots,x_s)=$$
$$=f(t,x_1,\dots,x_s,\widetilde{u}(t)),\widetilde{x}(t)=x(t,\widetilde{\mu}).$$

The problem of optimal control consists in finding a locally optimal element.

Introduce the notation

$$\omega_0=(\widetilde{t}_0,\widetilde{\varphi}(\tau_1(\widetilde{t}_0)),\dots,\widetilde{\varphi}(\tau_s(\widetilde{t}_0))),\quad \omega_1=(\widetilde{t}_1,\widetilde{x}(\tau_1(\widetilde{t}_1)),\dots,\widetilde{x}(\tau_s(\widetilde{t}_1))),$$
$$\omega=(t,x_1,\dots,x_s);$$
$$\widetilde{f}_{x_i}[t]=\widetilde{f}_{x_i}(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t))),\quad \widetilde{f}[t]=\widetilde{f}(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t))),$$
$$\widetilde{A}_j(t)=A_j(t,\widetilde{v}(t)),\quad j=1,\dots,p.$$

Theorem 1. *Let $\widetilde{\mu}\in B_0$, $\widetilde{t}_i\in(a,b)$, $i=0,1$, be a locally optimal element and there exist the finite limits: $\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1-))$, $j=1,\dots,p$; $\lim\limits_{\omega\to\omega_0}\widetilde{f}(\omega)=f_0^-$, $\omega\in[a,\widetilde{t}_0]\times O^s$. $\lim\limits_{\omega\to\omega_1}\widetilde{f}(\omega)=f_1^-$, $\omega\in[a,\widetilde{t}_1]\times O^s$. Then there exist a non-zero vector $\pi=(\pi_0,\dots,\pi_l)$, $\pi_0\le 0$ and solutions $\chi(t)$, $\psi(t)$ of the system*

$$\begin{cases}\dot{\chi}(t)=-\sum\limits_{i=1}^{s}\psi(\gamma_i(t))\widetilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_i(t),\\ \psi(t)=\chi(t)+\sum\limits_{j=1}^{p}\psi(\sigma_j(t))\widetilde{A}_j(\sigma_j(t))\dot{\sigma}_j(t),\quad t\in[\widetilde{t}_0,\widetilde{t}_1],\quad \psi(t)=0,\quad t>\widetilde{t}_1,\end{cases}\tag{3}$$

such that the following conditions are fulfilled:

$$\chi(\widetilde{t}_0)\widetilde{\varphi}(\widetilde{t}_0)+\sum_{i=1}^{s}\int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\gamma_i(t))\widetilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_i(t)\widetilde{\varphi}(t)dt+$$
$$+\sum_{j=1}^{p}\int_{\eta_j(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\sigma_j(t))\widetilde{A}_j(\sigma_j(t))\dot{\sigma}_j(t)\dot{\widetilde{\varphi}}(t)dt\ge$$
$$\ge\chi(\widetilde{t}_0)\varphi(\widetilde{t}_0)+\sum_{i=1}^{s}\int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\gamma_i(t))\widetilde{f}_{x_i}[\gamma_i(t)]\dot{\gamma}_i(t)\widetilde{\varphi}(t)dt+$$

$$+\sum_{j=1}^{p}\int\limits_{\eta_j(\widetilde{t}_0)}^{\widetilde{t}_0}\psi(\sigma_j(t))\widetilde{A}_j(\sigma_j(t))\dot{\sigma}_j(t)\dot{\widetilde{\varphi}}(t)dt,\quad \forall\varphi\in\Gamma; \tag{4}$$

$$\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)\widetilde{f}[t]dt\geq\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)f(t,\widetilde{x}(t),\widetilde{\tau}_1(t)),\ldots,\widetilde{x}(\tau_s(t)),u(t)dt,\quad \forall u\in\Omega; \tag{5}$$

$$\sum_{j=1}^{p}\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)\Big[\frac{\partial}{\partial v}\widetilde{A}_j(t)\times\widetilde{v}(t)\Big]\dot{\widetilde{x}}(\eta_j(t))dt\geq$$

$$\geq\sum_{j=1}^{p}\int\limits_{\widetilde{t}_0}^{\widetilde{t}_1}\psi(t)\Big[\frac{\partial}{\partial v}A_j(v(t))\times v(t)\Big]\dot{\widetilde{x}}(\eta_j(t))dt,\quad \forall v\in\Omega_2; \tag{6}$$

$$\pi\widetilde{Q}_{x_1}=\psi(\widetilde{t}_1); \tag{7}$$

$$\pi\widetilde{Q}_{t_0}\geq\psi(\widetilde{t}_0-)\Big[\dot{\widetilde{\varphi}}(\widetilde{t}_0)-\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_0-)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))-f_0^-\Big],$$

$$\pi\widetilde{Q}_{t_1}\geq-\psi(\widetilde{t}_1)\Big[\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_1-)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1))+f_1^-\Big].$$

Here $Q=(q^0,\ldots,q^l)^*$, the tilde over Q means that the corresponding gradient is calculated at the point $(\widetilde{t}_0,\widetilde{t}_1,\widetilde{x}_0,\widetilde{x}(\widetilde{t}_1))$; $\dfrac{\partial}{\partial v}\widetilde{A}_j(t)\times\widetilde{v}(t)=\Big(\dfrac{\partial}{\partial v}\widetilde{a}_j^{im}(t)\cdot\widetilde{v}(t)\Big)_{i,m=1}^{n}$.

Theorem 2. *Let $\widetilde{\mu}\in B_0$, $\widetilde{t}_i\in(a,b)$, $i=0,1$, be a locally optimal element and there exist the finite limits: $\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1+))$, $j=1,\ldots,p$; $\lim\limits_{\omega\to\omega_0}\widetilde{f}(\omega)=f_0^+$, $\omega\in[\widetilde{t}_0,b]\times O^S$, $\lim\limits_{\omega\to\omega_1}\widetilde{f}(\omega)=f_1^+$, $\omega\in[\widetilde{t}_1,b]\times O^S$. Then there exist a non-zero vector $\pi=(\pi_0,\ldots,\pi_l)$, $\pi_0\leq 0$ and solutions $\chi(t)$, $\psi(t)$ of the system* (3) *such that the conditions* (4)–(7) *are fulfilled. Moreover,*

$$\pi\widetilde{Q}_{t_0}\leq\psi(\widetilde{t}_0)+\Big[\dot{\widetilde{\varphi}}(\widetilde{t}_0)-\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_0+)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))-f_0^+\Big],$$

$$\pi\widetilde{Q}_{t_0}\leq-\psi(\widetilde{t}_1)\Big[\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_1+)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1+))+f_1^+\Big].$$

Theorem 3. *Let $\widetilde{\mu}\in B_0$, $\widetilde{t}_i\in(a,b)$, $i=0,1$, be a locally optimal element and the assumptions of Theorems* 1, 2 *be fulfilled. Moreover,*

$$\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_0-)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))+f_0^-=\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_0+)\dot{\widetilde{\varphi}}(\eta_j(\widetilde{t}_0))+f_0^+=f_0;$$

$$\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_1-)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1-))+f_1^-=\sum_{j=1}^{p}\widetilde{A}_j(\widetilde{t}_1+)\dot{\widetilde{x}}(\eta_j(\widetilde{t}_1+))+f_1^+=f_1;$$

$\widetilde{A}_{k_j}(\sigma_{k_j}(\cdots(\sigma_{k_1}(\sigma_i(\widetilde{t}_0-)))\cdots))=\widetilde{A}_{k_j}(\sigma_{k_j}(\cdots(\sigma_{k_1}(\sigma_i(\widetilde{t}_0+)))\cdots))$, $k_j=1,\ldots,p$, $j=1,\ldots,n_i$, $i=1,\ldots,p$, *where $n_i\geq 0$, $i=1,\ldots,p$, are integer numbers such that* $\sigma_i(\widetilde{t}_0)\in(\eta^{n_i+1}(\widetilde{t}_1),\eta^{n_i}(\widetilde{t}_1))$, $i=1,\ldots,p$, $\eta(t)=\max\limits_{1\leq j\leq p}\{\eta_j(t)\}$, $t\in J$, $\eta^i(t)=\eta\ (\eta^{i-1}(t))$, $\eta^0(t)=t$.

Then there exist a non-zero vector $\pi=(\pi_0,\dots,\pi_l)$*,* $\pi_0\le 0$ *and solutions* $\chi(t)$*,* $\psi(t)$ *of the system* (3) *such that the conditions* (4)–(7) *are fulfilled. Moreover,*

$$\pi\widetilde{Q}_{t_0}=\psi(\widetilde{t}_0)[\dot{\widetilde{\varphi}}(\widetilde{t}_0)-f_0];\quad \pi\widetilde{Q}_{t_1}=-\psi(\widetilde{t}_1)f_1.$$

Finally we note that the optimal problems with non-fixed initial moment for various classes of delay and neutral differential equations are considered in [1]–[6].

References

1. G. L. Kharatishvili, A maximum principle in extremal problems with delays. *Mathematical Theory of Control* (*Proc. Conf., Los Angeles, Calif.,* 1967), 26–34. *Academic Press, New York,* 1967.
2. G. L. Kharatishvili and T. A. Tadumadze, A nonlinear optimal control problem with variable delay, nonfixed initial moment, and piecewise-continuous prehistory. (Russian) *Tr. Mat. Inst. Steklova* **220** (1998), *Optim. Upr., Differ. Uravn. i Gladk. Optim.,* 236–255; translation in *Proc. Steklov Inst. Math.* **1998**, No. 1 (220), 233–252.
3. N. Gorgodze, Necessary conditions of optimality in neutral type optimal problems with non-fixed initial moment. *Mem. Differential equations Math. Phys.* **19**(2000), 150–153.
4. N. Gorgodze, Necessary conditions of optimality for one class neutral type problems of optimal control. *Proc. I. Javakhishvili Tbilisi State Univ. Appl. Math. Inform.* **342(20)** (2000), 19–22.
5. T. Tadumadze and L. Alkhasishvili, Necessary conditions of optimality for optimal problems with delays and with a discontinuous initial condition. *Mem. Differential equations Math. Phys.* **22** (2001), 154–158.
6. L. Alkhazishvili, The linearized maximum principle for optimal problems with variable delays and continuous initial condition. *Mem. Differential equations Math. Phys.* **29** (2003), 153–155.

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