Mem. Differential Equations Math. Phys. 31(2004), 149–152

M. Ashordia

ON LYAPUNOV STABILITY OF A CLASS OF LINEAR SYSTEMS OF DIFFERENCE EQUATIONS

(Reported on December 22, 2003)

In the present note we consider the linear system of difference equations

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1) + g(k) \quad (k = 1, 2, \dots), \tag{1}$$

where $G_j(k) \in \mathbb{R}^{n\times n}$ and $g(k) \in \mathbb{R}^n$ $(j = 1, 2, 3;\ k = 0, 1, \dots)$.

We give effective necessary and sufficient conditions guaranteeing the stability of the system (1) in Lyapunov sense with respect to small perturbations. They are the analogues of the well-know conditions for the stability of linear ordinary differential systems with constant coefficients (see, e.g., [1], [2]).

The following notation and definitions will be used in the paper.

$\mathbb{N} = \{1, 2, \dots\}$ is the set of all natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$; $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$. [t] is the integral part of $t \in \mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i=1}^{n}\sum_{j=1}^{m} |x_{ij}|;$$

$O_{n\times m}$ (or O) is the zero $n \times m$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant and the spectral radius of X; I_n is the identity $n \times n$-matrix; δ_{ij} is the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, 2, \dots)$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $J \subset \mathbb{N}_0$ and $Q \subset \mathbb{R}^{n\times m}$, then $E(J; Q)$ is the set of all matrix-functions $Y : I \to Q$.

Δ is the first order difference operator, i.e.,

$$\Delta y(k-1) = y(k) - y(k-1) \quad (k = 1, 2, \dots) \quad \text{for} \quad y \in E(\mathbb{N}_0; \mathbb{R}^n).$$

Let $y_0 \in E(\mathbb{N}_0; \mathbb{R}^n)$ be a solution of the difference system (1) and let $G \in E(\mathbb{N}_0; \mathbb{R}^{n\times n})$ be an arbitrary matrix-function.

Definition 1. A solution $y_0 \in E(\mathbb{N}_0; \mathbb{R}^n)$ of the system (1) is called G-stable if for every $\varepsilon > 0$ and $k_0 \in \mathbb{N}_0$ there exists $\delta(\varepsilon, k_0) > 0$ such that for every solution y of the system (1) for which

$$\|(I_n + G(k_0))(y(k_0) - y_0(k_0))\| + \|y(k_0 + 1) - y_0(k_0 + 1)\| < \delta$$

the estimate

$$\|(I_n + G(k))(y(k) - y_0(k))\| + \|y(k + 1) - y_0(k + 1)\| < \varepsilon \quad \text{for} \quad k \geq k_0$$

holds.

2000 *Mathematics Subject Classification.* 34K20.

Key words and phrases. Linear systems of difference equations, stability and asymptotic stability in Lyapunov sense.

Definition 2. A solution $y_0 \in E(\mathbb{N}_0;\mathbb{R}^n)$ of the system (1) is called G-asymptotically stable if it is G-stable and for every $k_0 \in \mathbb{N}_0$ there exists $\Delta = \Delta(k_0) > 0$ such that for every solution y of the system (1) for which

$$\|(I_n + G(k_0))(y(k_0) - y_0(k_0))\| + \|y(k_0+1) - y_0(k_0+1)\| < \Delta$$

the condition

$$\lim_{k\to\infty} \big(\|(I_n + G(k))(y(k) - y_0(k))\| + \|y(k+1) - y_0(k+1)\|\big) = 0$$

holds.

We say that y_0 is stable (asymptotically stable) if it is $O_{n\times n}$-stable ($O_{n\times n}$-asymptotically stable).

Definition 3. The system (1) is called G-stable (G-asymptotically stable) if every its solution is G-stable (G-asymtotically stable).

It is evident that the system (1) is G-stable (G-asymptotically stable) if and only if its corresponding homogeneous system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1) \quad (k = 1,2,\dots) \tag{1_0}$$

is G-stable (G-asymptotically stable). On the other hand, the system (1_0) is G-stable (G-asymptotically stable) if and only if its zero solution is G-stable (G-asymptotically stable). Thus the G-stability (G-asymptotic stability) of the system (1) is the common property of all solutions of this system and the vector-function g_0 does not affect this property. Therefore, it is the property of the triple (G_1, G_2, G_3). Hence the following definition is natural.

Definition 4. The triple (G_1, G_2, G_3) is G-stable (G-asymptotically stable) if the system (1_0) is G-stable (G-asymptotically stable).

Remark 1. It is evident that the triple (G_1, G_2, G_3) is G-stable if and only if every solution of the system (1_0) is G-bounded, i.e., there exists $M > 0$ such that

$$\|(I_n + G(k))y(k)\| + \|y(k+1)\| \le M \quad (k = 0,1,\dots).$$

Analogously, the triple (G_1, G_2, G_3) is G-asymptotically stable if and only if every solution y of the system (1_0) is G-convergent to zero, i.e.,

$$\lim_{k\to\infty} \big(\|(I_n + G(k))y(k)\| + \|y(k+1)\|\big) = 0.$$

Remark 2. If the matrix-function G is such that

$$\det(I_n + G(k)) \ne 0 \quad (k = 0,1,\dots)$$

and

$$\|G(k)\| + \|(I_n + G(k))^{-1}\| < M \quad (k = 0,1,\dots)$$

for some $M > 0$, then the triple (G_1, G_2, G_3) is G-stable (G-asymptotically stable) if and only if it is stable (asymptotically stable).

Theorem 1. *Let the matrix-functions $G_1, G_2, G_3 \in E(\mathbb{N}_0;\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + G_1(k)) \ne 0 \quad (k = 1,2,\dots)$$

and

$$G(k) = I_{2n} - \exp\Big(-\sum_{l=1}^{m} \Delta\beta_l(k-1)B_l\Big) \quad (k = 1,2,\dots),$$

where $G(k) = (G_{ij}(k))_{i,j=1}^2$,

$$G_{11}(k) \equiv (G_1(k) + G_2(k))(I_n + G_1(k))^{-1}, \quad G_{12}(k) \equiv G_3(k),$$
$$G_{21}(k) \equiv -(I_n + G_1(k))^{-1}, \quad G_{22}(k) \equiv O_{n\times n},$$

$B_l \in \mathbb{R}^{2n\times 2n}$ $(l = 1,\dots,m)$ are pairwise permutable constant matrices, and $\beta_l \in E(\widetilde{\mathbb{N}};\mathbb{R}_+)$ $(l = 1,\dots,m)$ are such that

$$\lim_{k\to+\infty} \beta_l(k) = +\infty \quad (l = 1,\dots,m).$$

Then:

a) *the triple (G_1, G_2, G_3) is G_1-stable of and only if every eigenvalue of the matrices B_l $(l = 1,\dots,m)$ has the nonpositive real part and, in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;*

b) *the triple (G_1, G_2, G_3) is G_1-asymptotically stable if and only if every eigenvalue of the matrices B_l $(l = 1,\dots,m)$ has the negative real part.*

Corollary 1. *Let $G_j(k) \equiv G_{0j}$ $(j = 1, 2, 3)$ be constant matrix-functions and*

$$\det(I_n + G_{01}) \neq 0, \quad \det G_{03} \neq 0,$$

where $G_{0j} \in \mathbb{R}^{n\times n}$ $(j = 1, 2, 3)$ are constant matrices. Let, moreover, λ_i $(i = 1,\dots,m)$ be pairwise different eigenvalues of the $2n \times 2n$-matrix $G_0 = (G_{0ij})_{i,j=1}^2$, where

$$G_{011} = (G_{01} + G_{02})(I_n + G_{01})^{-1}, \quad G_{012} = G_{03},$$
$$G_{021} = -(I_n + G_{01})^{-1}, \quad G_{022} = I_n$$

Then:

a) *the triple (G_{01}, G_{02}, G_{03}) is stable if and only if $|1 - \lambda_i| \geq 1$ $(i = 1,\dots,m)$ and, in addition, if $|1 - \lambda_i| = 1$ for some $i \in \{1,\dots,m\}$, then the elementary divisor corresponding to λ_i is simple;*

b) *the triple (G_{01}, G_{02}, G_{03}) is asymptotically stable if and only if $|1 - \lambda_i| > 1$ $(i = 1,\dots,m)$.*

Theorem 2. *Let $G_j(k) \equiv G_{0j}$ $(j = 1, 2, 3)$ be constant matrix-functions such that*

$$G_{01} = (A_1 - A_3)(I_n - A_1 + A_3)^{-1},$$
$$G_{02} = I_n + (A_1 + A_2 - 2I_n)(I_n + G_{01}), \quad G_{03} = (I_n - A_2),$$

where $A_j = (\alpha_{jil})_{i,l=1}^n$ $(j = 1, 2)$, are constant $n \times n$-matrices such that

$$\det(I_n - A_1 + A_3) \neq 0, \quad \det((I_n - A_2)) \neq 0.$$

Let, moreover,

$$\alpha_{jii} < 0 \quad (j = 1, 2; i = 1,\dots,n) \quad \text{and} \quad r(H) < 1, \tag{2}$$

where $H = (H_{mj})_{m,j=1}^2$,

$$H_{jj} = ((1 - \delta_{il})|\alpha_{jil}||\alpha_{jii}|^{-1})_{i,l=1}^n \quad (j = 1, 2),$$
$$H_{21} = (|\alpha_{3il}||\alpha_{2ii}|^{-1})_{i,l=1}^n, \quad H_{12} = (|\alpha_{2il}\mu_{2i} - \delta_{il}||\alpha_{1ii}|^{-1}\mu_{1i}^{-1})_{i,l=1}^n.$$

Then the triple (G_{01}, G_{02}, G_{03}) is asymptotically stable. Conversely, if this triple is asymptotically stable,

$$\alpha_{jil} \geq 0, \quad \alpha_{2ii} \geq 1 \quad (j = 1, 2, 3; \quad i \neq l; \quad i, l = 1,\dots,n)$$

and

$$\alpha_{j+1ii} - \delta_{2j} + \sum_{l=1,l\neq i}^{n} (\alpha_{jil} + \alpha_{j+1il}) <$$
$$< \min\{1 - \alpha_{jii}, |1 + \alpha_{jii}|\} \quad (j = 1, 2; \quad i = 1,\dots,n),$$

then the condition (2) holds as well.

To prove of these results we use the following concept.

Consider the system of the so-called generalized linear ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \quad \text{for} \quad t \in \mathbb{R}_+, \tag{3}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, the matrix and vector-functions with the components having bounded variation on every closed interval from $\mathbb{R}_+$ (see, i.e. [3]).

Under a solution of the system (2) we understand a vector-function $x : \mathbb{R}_+ \to \mathbb{R}^n$ with the components having bounded variations on every closed interval from $\mathbb{R}_+$ and such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for} \quad 0 \le t \le s,$$

where the integral is understood in Lebesgue–Stiltjes sense.

The difference system (1) is a particular case of the system (2). Namely, $y \in E(\mathbb{N}_0; \mathbb{R}^n)$ is a solution of the system (1) if and only if the vector-function $x(t) = (z_i([t]))_{i=1}^2$ for $t \in \mathbb{R}_+$, where $z_1([t]) \equiv (I_n + G_1([t]))y([t])$, $z_2([t]) = y([t]+1)$,is a solution of the $2n \times 2n$-system (2), where

$$A(t) = O_{2n\times 2n} \quad \text{for} \quad 0 \le t \le 1, \quad A(t) = \sum_{k=1}^{[t]} G(k) \quad \text{for} \quad t \ge 1,$$

$$f(t) = O_{2n} \quad \text{for} \quad 0 \le t \le 1, \quad f(t) = \sum_{k=1}^{[t]} G(k) \quad \text{for} \quad t \ge 1.$$

Thus Theorem 1 and its corollaries immediately follow from the corresponding results contained in [4] for the system (1).

As to the proof of Theorem 2, we use a system of the form (2) different from the one constructed above, in order to apply the analogous result from [4].

Acknowledgment

This work was supported by the grant CRDF–Georgia No. 3318.

References

1. B. P. Demidovich, Lectures on mathematical theory of stability. (Russian) *Nauka, Moscow*, 1967.

2. I. T. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations. Vol. I. Linear Theory. (Russian) *Metsniereba, Tbilisi*, 1997.

3. Š. Schwabik, M. Tvrdy and O. Vejvoda, Differential and integral equations: boundary value problems and adjoints. *Academia, Praha*, 1979.

4. M. Ashordia, On Lyapunov srability of a class of linear systems of generalized ordinary differential equations and linear impulsive systems. *Mem. Differential Equations Math. Phys.* **31**(2004), 139–144.

Author's address:

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0143
Georgia

Sukhumi Branch of I. Javakhishvili Tbilisi State University
12, Jikia St., Tbilisi 0186, Georgia
E-mail: ashordia@viam.hepi.edu.ge