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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. A functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1}$$

is considered with continuous $F : C(R_+; R) \to L_{loc}(R_+; R)$.

Oscillatory properties of proper solutions of (1) are studied. In particular sufficient conditions are given for equation (1) to have the property **A** or **B** ($\widetilde{\mathbf{A}}$ or $\widetilde{\mathbf{B}}$) which are optimal in a certain sense. Sufficient conditions for every solution of (1) to be oscillatory are obtained as well as existence conditions for an oscillatory solution.

Chapter 6 is dedicated to boundary value problem (16.1)-(16.2). Sufficient conditions are established for the existence of a unique solution, a unique oscillatory solution and a unique bounded oscillatory solution of this problem.

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Key words and Phrases. Functional differential equation, proper solution, equations with properties **A**, **B**, $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$, Kneser-type solution, oscillatory solution, bounded solution, boundary value problem.

რეზიუმე. განხილულია ფუნქციონალურ-დიფერენციალური განტოლება

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1}$$

სადაც $F : C(R_+; R) \to L_{loc}(R_+; R)$ უწყვეტი ასახვაა.

შრომაში შესწავლილია (1) განტოლების წესიერი ამონახსნების ოსცილაციური თვისებები. კერძოდ, მოყვანილია გარკვეული აზრით ოპტიმალური საკმარისი პირობები იმისა, რომ (1) განტოლებას გააჩნდეს **A** ან **B** ($\widetilde{\mathbf{A}}$ ან $\widetilde{\mathbf{B}}$) თვისება. გარდა ამისა დადგენილია (1) განტოლების ყოველი წესიერი ამონახსნის რხევადობისა და რხევადი ამონახსნის არსებობის საკმარისი პირობები.

ნაშრომის VI თავი ეძღვნება (16.1)-(16.2) სასაზღვრო ამოცანის შესწავლას. დადგენილია ამ ამოცანის ერთადერთი ამონახსნის, ერთადერთი რხევადი ამონახსნის, ერთადერთი რხევადი შემოსაზღვრული ამონახსნის არსებობის საკმარისი პირობები.

PREFACE

Let $\tau, \sigma \in C(\mathbb{R}_+; \mathbb{R}_+)$,[1] $\lim_{t\to+\infty} \tau(t) = +\infty$ and $\tau(t) \le \sigma(t)$ for $t \in \mathbb{R}_+$. Denote by $V(\tau)$ $(V(\tau,\sigma))$ the set of continuous mappings $F : C(\mathbb{R}_+; \mathbb{R}) \to L_{loc}(\mathbb{R}_+; \mathbb{R})$ satisfying the condition: $F(x)(t) = F(y)(t)$ holds for any $t \in \mathbb{R}_+$ and $x, y \in C(\mathbb{R}_+; \mathbb{R})$ provided that $x(s) = y(s)$ for $s \ge \tau(t)$ $(\tau(t) \le s \le \sigma(t))$. Obviously $V(\tau,\sigma) \subset V(\tau)$.

This work is dedicated to the study of oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + F(u)(t) = 0, \tag{0.1}$$

where $n \ge 1$ and $F \in V(\tau)$ $(F \in V(\tau,\sigma))$.

For any $t_0 \in \mathbb{R}_+$ we denote by $H_{t_0,\tau}$ the set of all functions $u \in C(\mathbb{R}_+; \mathbb{R})$ satisfying $u(t) \ne 0$ for $t \ge t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf(\tau(s) : s \ge t\}$.

Throughout the work whenever the notation $V(\tau)$, $V(\tau,\sigma)$ and $H_{t_0,\tau}$ occurs, it will be understood unless specified otherwise that the functions τ and σ satisfy the conditions stated above.

It will always be assumed that either the condition

$$F(u)(t)\, u(t) \ge 0 \ \text{ for } \ t \ge t_0, \ \ u \in H_{t_0,\tau} \tag{0.2}$$

or the condition

$$F(u)(t)\, u(t) \le 0 \ \text{ for } \ t \ge t_0, \ \ u \in H_{t_0,\tau} \tag{0.3}$$

is fulfilled.

Let $t_0 \in \mathbb{R}_+$. A function $u : [t_0, +\infty[\to \mathbb{R}$ is said to be a proper solution of equation (0.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive,

$$\sup\{|u(s)| : s \in [t, +\infty[\} > 0 \ \text{ for } \ t \ge t_0$$

and there exists a function $\overline{u} \in C(\mathbb{R}_+; \mathbb{R})$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty[$ and the equality $\overline{u}^{(n)}(t) + F(\overline{u}(t)) = 0$ holds for $t \in [t_0, +\infty[$. A proper solution $u : [t_0, +\infty[\to \mathbb{R}$ of equation (0.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

We say that equation (0.1) has property if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$|u^{(i)}(t)| \downarrow 0 \ \text{ as } \ t \uparrow +\infty \ \ (i = 0, \dots, n-1) \tag{0.4}$$

when n is odd.

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[1]For the notation see Subsection 0.2.

We say that equation (0.1) has property if any of its proper solutions either is oscillatory or satisfies either (0.4) or

$$|u^i(t)| \uparrow +\infty \text{ for } t \uparrow +\infty \ (i = 0, \dots, n-1) \tag{0.5}$$

when n is even, and either is oscillatory or satisfies (0.5) when n is odd.

We say that equation (0.1) has property $\tilde{}$ if any of its proper solutions is oscillatory when n is odd, and either is oscillatory or satisfies (0.5) when n is even.

We say that equation (0.1) has property $\tilde{}$ if any of its proper solutions is oscillatory when n is odd, and either is oscillatory or satisfies (0.4) when n is even.

At the close of the last centure A. Kneser [38] posed the problem of finding conditions for an equation

$$u^{(n)}(t) + p(t)u(t) = 0 \tag{0.6}$$

to have properties similar to either $u^{(n)} + u = 0$ or $u^{(n)} - u = 0$. Using the well-known comparison theorem of Strum, he proved that if $n = 2$ and $\varliminf_{t\to+\infty} t^2 p(t) > 1/4$, then (0.6) has property , i.e. any of its solutions is oscillatory. This result for second order linear differential equations was further improved and generalized by quite a number of authors (see the survey of M. Ráb [94]).

Sufficient conditions for an ordinary differential equation to have property or can be found in A. Kneser [38], W.B. Fite [22], I. Mikusinski [84], F.W. Atkinson [4], V.A. Kondratyev [39, 40], G.V. Ananyeva and V.I. Balaganskii [2], Š. Belohorec [6], I.T. Kiguradze [31 – 37], L. Ličko and M. Švec [81], D.V. Izyumova [28], T.A. Chanturia [8 – 15], M. Bartušek [5] and in other papers. The monographs by O. Borůvka [7] and F. Neuman [90] should also be mentioned as they deal with the problem of global (in particular, oscillatory) behaviour of solutions of second [7] and n-th order [90] linear differential equations.

Analogous results for differential equations with deviating arguments and functional-differential equations are obtained in [20, 23-26, 29, 41-50, 53, 61-64, 69, 71, 73, 79, 83, 85, 91, 92, 96-101, 103].

If $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ ($n \geq 3$ and $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_-)$) and (0.6) has property ( ), then it has solutions of all the types mentioned in Definitions 0.1 and 0.2 [32, 40]. However for a differential equation with a delayed argument

$$u^{(n)}(t) + p(t)u(t - \Delta) = 0$$

($\Delta = const > 0$, $(-1)^{n+1} p(t) \geq 0$ for $t \in \mathbb{R}_+$) properties and do not guarantee the existence of proper solutions satisfying (0.4), while for a differential equation with an advanced argument

$$u^{(n)}(t) + p(t)u(t + \Delta) = 0$$

($\Delta = const > 0,\ p(t) \le 0$ for $t \in \mathbb{R}_+$) property does not guarantee the existence of proper solutions satisfying (0.5). As for a functional differential equation of the form

$$u^{(n)}(t) + p(t)\int_{t-\Delta}^{t+\Delta} u(s)ds = 0$$

($\Delta = const > 0,\ p(t) \le 0$ for $t \in \mathbb{R}_+$) property does nor guarantee the existence of proper solutions of forms (0.4) and (0.5).

The above examples show that the set of nonoscillatory solutions of functional differential equations has a structure differing from that of the set of solutions of ordinary differential equations.

Oscillation criteria specific of differential equations with delay were for the first time suggested by A. Myshkis [86]. Subsequently analogous problems were studied in [3, 19, 27, 42, 43, 54, 55, 70, 78, 87, 106] for first and second order linear and nonlinear differential equations.

Sufficient conditions for higher order differential equations with deviating arguments to have property $\tilde{}$ or $\tilde{}$ can be found in [16, 17, 48, 50, 65-68, 79, 80, 89, 97, 104, 105, 107]. Sufficient (necessary and sufficient) conditions for every proper solution of a higher order functional differential equation to be oscillatory are given in [72].

Chapter 1 of this work is concerned with equations having property or . Some basic definitions and auxiliary statements are formulated in §1. Comparison theorems are proved in §2, thereby making it possible to derive property or of the considered equations. Based on these theorems, sufficient (necessary and sufficient) conditions are established in §3 (in §4) for an essentially nonlinear functional differential equation to have property or .

Chapter 2 deals with analogous problems for equation (0.1) with the operator F admitting a linear minorant. In §5 we prove some auxiliary lemmas for linear differential inequalities with deviating arguments, which in §§6 and 7 are used to derive sufficient conditions for a functional differential equation with a linear minorant to have property or .[2]

Not only the results obtained in §§6 and 7 are the new ones for equation (0.1). They also improve some of the previous well-known results for equation (0.6). Chapter 2 concludes with some sufficient conditions for equation (0.1) not to have property ( ), thereby illustrating to what extent the results of §§6 and 7 are precise.

Chapters 3 and 4 are concerned with solutions satisfying conditions (0.4) and (0.5). In §8 auxiliary lemmas are formulated which enable one to establish the asymptotic behaviour near $+\infty$ of solutions satisfying (0.4) of

[2] Alongside with the problem of equation (0.1) having property or , there arises a need to study whether all solutions of (0.1) are or are not oscillatory. Interesing results in this direction were obtained by N.V. Azbelev [1].

differential equations and inequalities with a delayed argument. These lemmas are used in §§9 and 11 to find the sufficient or necessary and sufficient conditions for equation (0.1) not to have a solution satisfying (0.4) or (0.5).

In §§12 and 13 of Chapter 5 the previously obtained results are used to find the sufficient or necessary and sufficient conditions for equation (0.1) to have property $\widetilde{\ }$ or $\widetilde{\ }$, while in §14 the sufficient or necessary and sufficient conditions are established for any solution of (0.1) to be oscillatory. The results presented in this chapter are specific of functional differential equations and have no analogues for ordinary differential equations.

Chapter 6 is dedicated to second order differential equations with a delayed argument. The question of the existence and uniqueness of a solution of the linear problem (16.1), (16.2) is studied in §16, while in §17 we discuss the boundedness of all oscillatory solutions of both linear and nonlinear equations. In §18 sufficient conditions are given for the unique solution of problem (16.1), (16.2) to be oscillatory or bounded oscillatory.

The results of this work make it possible to extend a number of the earlier results concerning the oscillatory behaviour of differential equations with deviating arguments to the case of general functional differential equations. Some of them improve well-known results not only for differential equations with deviating arguments but for ordinary differential equations as well. Besides, the work presents essentially new results specific of functional differential equations.

The following notation will be used throughout the work:

N is the set of natural numbers;

$$\mathbb{R} =]-\infty, +\infty[; \quad \mathbb{R}_+ = [0, +\infty[; \quad \mathbb{R}_- =]-\infty, 0]; \quad \mathbb{R}^k = \mathbb{R} \underbrace{\times \cdots \times}_{k \text{ times}} \mathbb{R}.$$

If $I \subset \mathbb{R}$ is an interval and $E \subset \mathbb{R}$ is a set, then:

$C(I; E)$ denotes the space of all continuous functions with the topology of uniform convergence on any finite subsegment of I. $L_{loc}(I; E)$ denotes the space of all locally integrable functions $u : I \to E$ with the topology of convergence in mean on any finite subsegment of I.

$\widetilde{C}^k_{loc}(I; E)$ denotes the space of all functions $u : I \to E$ which are absolutely continuous on any finite subsegment of I together with their derivatives up to order k inclusive.

$K_{loc}(I \times \mathbb{R}^k; E)$ denotes the set of functions $f : I \times \mathbb{R}^k \to E$ satisfying the local Carathéodory conditions, i.e. on any finite subsegment of I the function $f(\cdot, x)$ is measurable for every $x \in \mathbb{R}^k$, $f(t, \cdot)$ is continuous for almost every $t \in I$ and for any $r > 0$

$$\sup\{|f(t, x)| : \|x\| \le r\} \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+).$$

CHAPTER I

§ 1. Some Auxiliary Statements

Let $u \in \widetilde{C}^{n-1}_{loc}([t_0,+\infty[;\mathbb{R})$ *and*

$$u(t) > 0, \quad u^{(n)}(t) \le 0 \ \ (u^{(n)}(t) \ge 0) \ \ for \ \ t \ge t_0. \tag{1.1}$$

Then there exist $t_1 \in [t_0,+\infty[$ *and* $l \in \{0,\dots,n\}$ *such that* $l+n$ *is odd* (*even*) *and*

$$\begin{gathered} u^{(i)}(t) > 0 \ \ for \ \ t \ge t_0 \ \ (i=0,\dots,l-1), \\ (-1)^{i+l}u^{(i)}(t) \ge 0 \ \ for \ \ t \ge t_0 \ \ (i=l,\dots,n). \end{gathered} \tag{1.2}$$

The proof of this lemma can be found in [34] (Lemmas 14.1 and 14.2).

Let $u \in \widetilde{C}^{n-1}_{loc}([t_0,+\infty[;\mathbb{R})$ *and*

$$(-1)^i u^{(i)}(t) > 0 \ \ (i=0,\dots,n-1), \ \ (-1)^n u^{(n)}(t) \ge 0 \ \ for \ \ t \ge t_0.$$

Then

$$\int_{t_0}^{+\infty} t^{n-1}|u^{(n)}(t)|dt < +\infty, \tag{1.3}$$

$$|u^{(i)}(t)| \ge \frac{1}{(n-i-1)!}\int_t^{+\infty}(s-t)^{n-i-1}|u^{(n)}(s)|ds \quad for \ \ t \ge t_0 \ \ (i=0,\dots,n-1) \tag{1.4}$$

and

$$u(t) \ge \sum_{i=0}^{n-1}\frac{|u^{(i)}(s)|(s-t)^i}{i!} \ \ for \ \ s \ge t \ge t_0. \tag{1.5}$$

Proof. Using the signs of the derivatives of u, we can readily obtain (1.3) and (1.5) from the identity

$$u^{(i)}(t) = \sum_{j=i}^{k-1}\frac{u^{(j)}(s)}{(j-i)!}(t-s)^{j-i} + \\ + \frac{1}{(k-i-1)!}\int_s^t (t-\xi)^{k-i-1}u^{(k)}(\xi)d\xi \tag{1.6$_{ik}$}$$

with $i=0$, $k=n$, $s \ge t$. As to (1.4), it is an immediate consequence of (1.6_{in}). ■

Let $u \in \widetilde{C}^{n-1}_{loc}([t_0,+\infty[)$ and (1.2) be fulfilled for some $l \in \{1,\dots,n-1\}$. Then

$$\int_{t_0}^{+\infty} t^{n-l-1}|u^{(n)}(t)|dt < +\infty, \tag{1.7}$$

$$u^{(i)}(t) \geq u^{(i)}(t_0) + \frac{1}{(l-i-1)!(n-l-1)!}\int_{t_0}^{t}(t-\xi)^{l-i-1} \times$$
$$\times \int_{\xi}^{+\infty}(s-\xi)^{n-l-1}|u^{(n)}(s)|ds\, d\xi \text{ for } t \geq t_0 \ \ (i=0,\dots,l-1), \tag{1.8}$$

$$|u^{(i)}(t)| \geq \frac{1}{(n-i-1)!}\int_{t}^{+\infty}(s-t)^{n-i-1}|u^{(n)}(s)|ds \tag{1.9}$$
$$\text{for } t \geq t_0 \ \ (i=l,\dots,n-1).$$

If, in addition,

$$\int_{t_0}^{+\infty} t^{n-l}|u^{(n)}(t)|dt = +\infty, \tag{1.10}$$

then

$$\frac{u(t)}{t^l}\downarrow, \ \ \frac{u(t)}{t^{l-1}}\uparrow +\infty \quad \text{as } t\uparrow +\infty, \tag{1.11}$$

$$u(t) \geq \frac{t^l}{l!(n-l)!}\int_{t}^{+\infty} s^{n-l-1}|u^{(n)}(s)|ds \text{ for large } t \tag{1.12}$$

and for any nondecreasing $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfying

$$\beta(t) \leq t \text{ for } t \in \mathbb{R}_+, \ \ \lim_{t\to+\infty}\beta(t) = +\infty \tag{1.13}$$

there exists $t_* \in [t_0,+\infty[$ such that

$$u(\beta(t)) \geq \frac{\beta^{l-1}(t)}{l!(n-l)!}\int_{t_*}^{t}\beta^{n-l}(s)|u^{(n)}(s)|ds \text{ for } t \geq t_*. \tag{1.14}$$

Proof. By virtue of (1.2) condition (1.7) readily follows from the identity

$$\sum_{j=i}^{k-1}\frac{(-1)^j t^{j-i}u^{(j)}(t)}{(j-i)!} = \sum_{j=i}^{k-1}\frac{(-1)^j t_0^{j-i}u^{(j)}(t_0)}{(j-i)!} +$$
$$+\frac{(-1)^{k-1}}{(k-i-1)!}\int_{t_0}^{t}s^{k-i-1}u^{(k)}(s)ds \tag{1.15$_{ik}$}$$

with $i=l$, $k=n$. The same identity also implies the inequality

$$\sum_{j=l}^{n-1}\frac{t^{j-l}|u^{(j)}(t)|}{(j-l)!} \geq \frac{1}{(n-l-1)!}\int_{t}^{+\infty}s^{n-l-1}|u^{(n)}(s)|ds \text{ for } t \geq t_0. \tag{1.16}$$

On account of (1.2) and (1.7), from (1.6_{in}) with $s \to +\infty$ we obtain (1.9). Analogously (1.6_{il}) with $s = t_0$ gives

$$u^{(i)}(t) \geq u^{(i)}(t_0) + \frac{1}{(l-i-1)!}\int_{t_0}^{t}(t-\xi)^{l-i-1}u^{(l)}(\xi)d\xi$$
$$\text{for } t \geq t_0 \ \ (i = 0, \dots, l-1).$$

Hence by (1.9) we obtain (1.8).

Assume now that (1.10) is fulfilled. Using (1.2), from $(1.15_{l-1\ n})$ we have

$$\lim_{t\to+\infty}(u^{(l-1)}(t) - tu^{(l)}(t)) = +\infty \tag{1.17}$$

and

$$u^{(l-1)}(t) \geq \sum_{j=l}^{n-1}\frac{t^{j-l+1}|u^{(j)}(t)|}{(j-l+1)!} \ \text{ for large } t. \tag{1.18}$$

For any $t \geq t_0$ and $i \in \{1, \dots, l\}$ put

$$\rho_i(t) = iu^{(l-i)}(t) - tu^{(l-i+1)}(t) = -t^{i+1}(t^{-i}u^{(l-i)}(t))', \tag{1.19}$$

$$r_i(t) = tu^{(l-i+1)}(t) - (i-1)u^{(l-i)}(t) = t^i(t^{1-i}u^{(l-i)}(t))'. \tag{1.20}$$

Applying (1.17) and L'Hospital's rule, we have

$$\lim_{t\to+\infty} t^{1-i}u^{(l-i)}(t) = +\infty \ \ (i = 1, \dots, l), \tag{1.21}$$

so that in view of (1.20) there exist $t_l \geq \dots \geq t_1 \geq t_0$ such that $r_i(t_i) > 0$ $(i = 1, \dots, l)$. Since by (1.17) $\rho_1(t) \to \infty$ as $t \to +\infty$ $\rho'_{i+1}(t) = \rho_i(t)$, $r'_{i+1}(t) = r_i(t)$ and $r_1(t) = tu^{(l)}(t) > 0$ for $t \geq t_0$ $(i = 1, \dots, l-1)$, we find that $\rho_i(t) \to +\infty$ as $t \to +\infty$ and $r_i(t) > 0$ for $t \geq t_i$ $(i = 1, \dots, l)$. These facts along with (1.19)–(1.21) prove (1.11). On the other hand, by (1.19) we have

$$iu^{(l-i)}(t) \geq tu^{(l-i+1)}(t) \ \text{ for large } t \ \ (i = 1, \dots, l)$$

which implies

$$u(t) \geq \frac{t^{l-1}}{l!}u^{(l-1)}(t) \ \text{ for large } t. \tag{1.22}$$

Inequalities (1.16), (1.18) and (1.22) imply (1.12).

It remains to prove (1.14). Let $t_* > t_1$ be such that $\beta(t) \geq t_1$ for $t \geq t_*$. From $(1.15_{l-1\ n-1})$ we have

$$u^{(l-1)}(\beta(t)) \geq u^{(l-1)}(\beta(t_*)) - \sum_{j=l}^{n-2}\frac{\beta^{j-l+1}(t_*)|u^{(j)}(\beta(t_*))|}{(j-l+1)!} +$$
$$+\frac{1}{(n-l-1)!}\int_{\beta(t_*)}^{\beta(t)} s^{n-l-1}|u^{(n-1)}(s)|ds. \tag{1.23}$$

On the other hand, changing the order of integration and taking into account (1.13) and the fact that β is monotone, we obtain

$$\frac{1}{(n-l-1)!}\int_{\beta(t_*)}^{\beta(t)} s^{n-l-1}|u^{(n-1)}(s)|ds \geq \frac{1}{(n-l-1)!}\int_{\beta(t_*)}^{\beta(t)} s^{n-l-1} \times$$
$$\times \int_s^t |u^{(n)}(\xi)|d\xi\, ds \geq \frac{1}{(n-l-1)!}\int_{t_*}^{t} |u^{(n)}(\xi)| \times$$
$$\times \int_{\beta(t_*)}^{\beta(\xi)} s^{n-l-1} ds\, d\xi \geq \frac{1}{(n-l)!}\int_{t_*}^{t} \beta^{n-l}(\xi)|u^{(n)}(\xi)|d\xi -$$
$$-\frac{1}{(n-l)!}|u^{(n-1)}(\beta(t_*))|\beta^{n-l}(t_*) \ \text{ for } \ t \geq t_*,$$

so that, applying (1.20), we arrive at

$$u^{(l-1)}(\beta(t)) \geq c_0 + \frac{1}{(n-l)!}\int_{t_0}^{t} \beta^{n-l}(s)|u^{(n)}(s)|ds \ \text{ for } \ t \geq t_0, \qquad (1.24)$$

where

$$c_0 = u^{(l-1)}(\beta(t_*)) - \sum_{j=l}^{n-1} \frac{\beta^{j-l+1}(t_*)}{(j-l+1)!}|u^{(j)}(\beta(t_*))|.$$

By (1.18) t_* can be assumed to be large enough for the inequality $c_0 \geq 0$ to hold. Therefore (1.22) and (1.24) immediately imply (1.14). ■

$C(\mathbb{R}_+;\mathbb{R})$ $L_{loc}(\mathbb{R}_+;\mathbb{R})$ Let $\tau,\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the condions given at the begining of Subsection 0.1.

Denote by $M(\tau)$ ($M(\tau,\sigma)$) the set of all $\varphi \in V(\tau)$ ($\varphi \in V(\tau,\sigma)$) satisfying the condition: for any $t_0 \in \mathbb{R}_+$, $t \in [t_0,+\infty[$ and $x,y \in H_{t_0,\tau}$ such that $x(s)y(s) > 0$, $|x(s)| \geq |y(s)|$ for $s \geq \tau(t)$ ($\tau(t) \leq s \leq \sigma(t)$) one has $\varphi(x)(t)\operatorname{sign} x(t) \geq \varphi(y)(t)\operatorname{sign} y(t) \geq 0$.

Denote by $M_1(\tau)$ ($M_1(\tau,\sigma)$) the set of all $\varphi \in M(\tau)$ ($\varphi \in M(\tau,\sigma)$) satisfying the condition: for any $t_0 \in \mathbb{R}_+$ and $m \in \{1,2\}$ the integral inequality

$$y(t) \geq \int_t^{+\infty} |\varphi((-1)^m y)(s)|ds \qquad (1.25)$$

has no nonincreasing positive solution $y : [t_0,+\infty[\to]0,+\infty[$.

For any nondecreasing function $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ satisfying (1.13) denote by $M_2^{\beta}(\tau)$ ($M_2^{\beta}(\tau,\sigma)$) the set of all $\varphi \in M(\tau)$ ($\varphi \in M(\tau,\sigma)$)

satisfying the condition: for any $t_0 \in \mathbb{R}_+$ and $m \in \{1,2\}$ the integral inequality

$$y(\beta(t)) \geq \int_{t_0}^{t} |\varphi((-1)^m y)(s)|ds \tag{1.26}$$

has no nondecreasing positive solution $y : [t_0, +\infty[\to]0, +\infty[$.

Remark 1.1. As in the case of equation (0.1), a function $y \in C([t_0, +\infty[; \mathbb{R})$ is said to be a solution of (1.25) ((1.26)) if there exists $\overline{y} \in C(\mathbb{R}_+; \mathbb{R})$ which satisfies (1.25) ((1.26)) for any $t \in \mathbb{R}_+$ and whose restriction to $[t_0, +\infty[$ coincides with y.

Remark 1.2. As in the case of $V(\tau)$ ($V(\tau,\sigma)$), whenever in the sequel the notations $M(\tau)$, $M_1(\tau)$ and $M_2^{\beta}(\tau)$ ($M(\tau,\sigma)$, $M_1(\tau,\sigma)$ and $M_2^{\beta}(\tau,\sigma)$) are used, τ and σ will be assumed, unless stated otherwise, to satisfy the conditions given at the beginning of Subsection 0.1 and β will be assumed to be nondecreasing and to satisfy (1.13).

Obviously, $M(\tau,\sigma) \subset M(\tau)$, $M_1(\tau,\sigma) \subset M_1(\tau)$ and $M_2^{\beta}(\tau,\sigma) \subset M_2^{\beta}(\tau)$.

On account of the definitions of the sets $M_1(\tau)$ and $M_2^{\beta}(\tau)$, one can easily ascertain that the following lemma is valid.

If $\varphi \in M_1(\tau)$ ($\varphi \in M_2^{\beta}(\tau)$), then

$$\int^{+\infty} |\varphi(c)|(t)dt = +\infty.^3$$

for any $c \neq 0$.

If $\alpha \in C(\mathbb{R}_+;]0, +\infty[)$ is nondecreasing and $\varphi \in M_1(\tau)$, then $\psi \in M_1(\tau)$ where

$$\psi(y)(t) = \alpha(t)\varphi(y/\alpha)(t) \text{ for } y \in C(\mathbb{R}_+;\mathbb{R}), \quad t \in \mathbb{R}_+.$$

.

Proof. Assume the contrary: $\psi \notin M_1(\tau)$. In that case there exist $t_0 \in \mathbb{R}_+$ and $m \in \{1.2\}$ such that the inequality

$$y(t) \geq \int_{t}^{+\infty} \alpha(s)|\varphi((-1)^m y/\alpha)(s)|ds$$

has a nonincreasing solution $y : [t_0, +\infty[\to]0, +\infty[$. Since α is nondecreasing, we find that the nonincreasing function $z = y/\alpha$ satisfies

$$z(t) \geq \int_{t}^{+\infty} |\varphi((-1)^m z)(s)|ds \text{ for } t \geq t_0.$$

Therefore $\varphi \notin M_1(\tau)$ which is a contradiction. ■

[3] We usually do not distinguish between the notations of a constant and a function identically equal to this constant.

Let $\varphi \in M(\tau)$ and there exist $t_0 \in \mathbb{R}_+$ such that for any $m \in \{1,2\}$ we have

$$|\varphi((-1)^m y)(t)| \geq p(t)\omega(y(t)) \quad \text{for } y \in C(\mathbb{R}_+;]0,+\infty[), \quad t \geq t_0, \quad (1.27)$$

where $p \in L_{loc}([t_0,+\infty[;\mathbb{R}_+)$, $\omega \in C(\mathbb{R}_+;\mathbb{R}_+)$ is nondecreasing, $\omega(s) > 0$ for $s > 0$ and

$$\int_0^1 \frac{ds}{\omega(s)} < +\infty, \quad \int^{+\infty} p(t)dt = +\infty. \quad (1.28)$$

Then $\varphi \in M_1(\tau)$.

Proof. Assume the contrary: $\varphi \notin M_1(\tau)$. Then there exist $t_1 \in [t_0,+\infty[$ and $m \in \{1,2\}$ such that (1.25) has a nonincreasing solution $y : [t_1,+\infty[\to]0,+\infty[$. According to (1.27) we have

$$y(t) \geq \int_t^{+\infty} p(s)\omega(y(s))ds \quad \text{for } t \geq t_1.$$

Therefore, since ω is nondecreasing,

$$p(t)\omega(y(t)) \geq p(t)\omega\Big(\int_t^{+\infty} p(s)\omega(y(s))ds\Big) \quad \text{for } t \geq t_1.$$

This inequality implies

$$\int_{x(t)}^{x(t_1)} \frac{ds}{\omega(s)} \geq \int_{t_1}^t p(s)ds \quad \text{for } t \geq t_1$$

with $x(t) = \int_t^{+\infty} p(s)\omega(y(s))ds$, which contradicts (1.28). ■

Let $\varphi \in M(\tau)$ there exists and $t_0 \in \mathbb{R}_+$ such that for any $m \in \{1,2\}$ we have

$$|\varphi((-1)^m y)(t)| \geq p(t)y^\lambda(t) \quad \text{for } y \in C(\mathbb{R}_+;]0,+\infty[), \quad t \geq t_0,$$

where $\lambda \in]0,1[$ and $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ satisfies the second condition of (1.28). Then $\varphi \in M_1(\tau)$.

Let $\varphi \in M_2^\beta(\tau)$ and $\alpha \in C(\mathbb{R}_+;]0,+\infty[)$ be nonincreasing. Then $\psi \in M_2^\beta(\tau)$, where

$$\psi(y)(t) = \alpha(\beta(t))\varphi(y/\alpha)(t) \quad \text{for } y \in C(\mathbb{R}_+;\mathbb{R}), \quad t \in \mathbb{R}_+.$$

Proof. Assume the contrary: $\psi \notin M_2^\beta(\tau)$. Then there exist $t_0 \in \mathbb{R}_+$ and $m \in \{1,2\}$ such that the inequality

$$y(\beta(t)) \geq \int_{t_0}^t \alpha(\beta(s))|\varphi((-1)^m y/\alpha)(s)|ds$$

has a nondecreasing solution $y : [t_0, +\infty[\to]0, +\infty[$. Since α is nonincreasing, we find that the nondecreasing function $z = y/\alpha$ satisfies

$$z(\beta(t)) \geq \int_{t_0}^{t} |\varphi((-1)^m z)(s)| ds \quad \text{for} \quad t \geq t_0.$$

Therefore $\varphi \notin M_2^{\beta}(\tau)$ which is a contradiction. ■

Let $\varphi \in M(\tau)$, a nondecreasing function $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ satisfy (1.13) *and there exist $t_0 \in \mathbb{R}_+$ such that*

$$|\varphi((-1)^m y)(t)| \geq p(t)\omega(y(\beta(t))) \quad \textit{for} \quad y \in C(\mathbb{R}_+;]0, +\infty[), \quad t \geq t_0, \tag{1.29}$$

where $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ satisfies the second condition of (1.28), *$\omega \in C(\mathbb{R}_+; \mathbb{R}_+)$ is nondecreasing, $\omega(s) > 0$ for $s > 0$ and*

$$\int_{1}^{+\infty} \frac{ds}{\omega(s)} < +\infty. \tag{1.30}$$

Then $\varphi \in M_2^{\beta}(\tau)$.

Proof. Assume the contrary: $\varphi \notin M_2^{\beta}(\tau)$. Then there exist $t_1 \in [t_0, +\infty[$ and $m \in \{1, 2\}$ such that (1.26) has a nondecreasing solution $y : [t_1, +\infty[\to]0, +\infty[$. By (1.29)

$$z(t) \geq \int_{t_1}^{t} p(s)\omega(z(s)) ds \quad \text{for} \quad t \geq t_1,$$

where $z(t) = y(\beta(t))$. Therefore, since ω is nondecreasing, we have

$$p(t)\omega(z(t)) \geq p(t)\omega\Big(\int_{t_1}^{t} p(s)\omega(z(s)) ds\Big) \quad \text{for} \quad t \geq t_1. \tag{1.31}$$

Choose $t_2 \in]t_1, +\infty[$ such that $\int_{t_1}^{t_2} p(s)\omega(z(s)) ds > 0$. Then (1.31) implies

$$\int_{x(t_2)}^{x(t)} \frac{ds}{\omega(s)} \geq \int_{t_2}^{t} p(s) ds \quad \text{for} \quad t \geq t_0,$$

where $x(t) = \int_{t_1}^{t} p(s)\omega(z(s)) ds$. But this contradicts the second condition of (1.28) and (1.30). ■

Let $\varphi \in M(\tau)$, a nondecreasing function $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ satisfy (1.13) *and there exist $t_0 \in \mathbb{R}_+$ such that for any $m \in \{1, 2\}$ we have*

$$|\varphi((-1)^m y)(t)| \geq p(t) y^{\lambda}(\beta(t)) \quad \textit{for} \quad y \in C(\mathbb{R}_+;]0, +\infty[), \quad t \geq t_0,$$

where $\lambda \in]1, +\infty[$ and $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ satisfies the second condition of (1.28). *Then $\varphi \in M_2^{\beta}(\tau)$.*

§ 2. Comparison Theorems

In this section we shall prove the comparison theorems for functional differential equations and inequalities. The essence of the comparison principle consists in the following: $\varphi, \Phi : C(\mathbb{R}_+;\mathbb{R}) \to L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ being somehow related continuous operators, the fact that the equation

$$v^{(n)}(t) + \varphi(v)(t) = 0 \tag{2.1}$$

$$(v^{(n)}(t) - \varphi(v)(t) = 0) \tag{2.2}$$

has property (property) implies that the inequality

$$\big[u^{(n)}(t) + \Phi(u)(t)\big] \operatorname{sign} u(t) \le 0 \tag{2.3}$$

$$\big(\big[u^{(n)}(t) - \Phi(u)(t)\big] \operatorname{sign} u(t) \ge 0\big) \tag{2.4}$$

also possesses the same property.

We shall consider here relations of two types existing between φ and Φ: 1) φ is a minorant of Φ (2.1. Minorant Case); 2) φ is a superposition of the form

$$\varphi(u)(t) = \Phi(\psi_t(u))(t) \quad \text{for} \quad u \in C(\mathbb{R}_+;\mathbb{R}), \quad t \in \mathbb{R}_+, \tag{2.5}$$

where $\{\psi_t : C(\mathbb{R}_+;\mathbb{R}) \to C(\mathbb{R}_+;\mathbb{R})\}_{t\in\mathbb{R}_+}$ is a family of operators of either type (2.2. Superposition Case).

The results obtained will enable us in §§3–4 to derive the sufficient and necessary conditions for equation (0.1) to possess property (property).

We begin by considering a lemma which is a special case of the Schauder-Tikhonoff theorem (see, for example, [18, p. 227]).

Let $t_0 \in \mathbb{R}$, U be a closed bounded convex subset of $C([t_0,+\infty[;\mathbb{R})$, and let $T : U \to U$ be a continuous mapping such that the set $T(U)$ is equicontinuous on every finite subsegment of $[t_0,+\infty[$. Then T has a fixed point.

Let $\Phi \in V(\tau)$, $t_0 \in \mathbb{R}_+$, and assume that for any $u,v \in H_{t_0,\tau}$ satisfying $|u(t)| \ge |v(t)|$, $u(t)v(t) > 0$ for $t \ge t_0$ the inequality

$$\Phi(u)(t) \operatorname{sign} u(t) \ge \varphi(v)(t) \operatorname{sign} v(t) \ge 0 \quad \text{for} \quad t \ge t_0 \tag{2.6}$$

holds, where $\varphi \in V(\tau)$. Let, moreover, equation (2.1) have property . Then inequality (2.3) also has the property .

Proof. Let $u : [t_0,+\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (2.3). It can be assumed without loss of generality that

$$u(t) > 0, \quad u^{(i)}(t) \ne 0 \quad \text{for} \quad t \ge t_0 \quad (i = 1,\dots,n-1). \tag{2.7}$$

By (2.6) and Lemma 1.1 there exist $t_1 \in [t_0, +\infty[$ and $l \in \{0, \dots, n-1\}$ such that $l+n$ is odd and

$$u^{(i)}(t) > 0 \quad (i = 0, \dots, l-1), \quad (-1)^{i+l} u^{(i)}(t) > 0 \qquad (2.8_l)$$
$$(i = l, \dots, n-1) \ \text{ for } \ t \geq t_1.$$

First we assume that $l \in \{1, \dots, n-1\}$. In that case (2.8_l) and Lemma 1.3 imply

$$u(t) \geq u(t_2) + \frac{1}{(l-1)!(n-l-1)!} \int_{t_2}^{t} (t-\xi)^{l-1} \times$$
$$\times \int_{\xi}^{+\infty} (s-\xi)^{n-l-1} \Phi(u)(s) ds \, d\xi \ \text{ for } \ t \geq t_2, \qquad (2.9)$$

where $t_2 \in [t_1, +\infty[$ is a sufficiently large number to be chosen such that $t_* = \min(t_2, \inf\{\tau(t) : t \geq t_2\}) \geq t_1$.

Let U be the set of all $v \in C([t_*, +\infty[; \mathbb{R})$ satisfying $u(t_1) \leq v(t) \leq u(t)$ for $t \geq t_2$, $v(t) = u(t)$ for $t_* \leq t \leq t_2$. Define $T : u \to C([t_*, +\infty[; \mathbb{R})$ by

$$T(v)(t) = \begin{cases} u(t_2) + \frac{1}{(l-1)!(n-l-1)!} \int_{t_2}^{t} (t-\xi)^{l-1} \int_{\xi}^{+\infty} (s-\xi)^{n-l-1} \times \\ \quad \times \varphi(v)(\xi) ds \, d\xi \quad \text{for} \quad t \geq t_2 \\ u(t) \quad \text{for} \quad t_* \leq t \leq t_2. \end{cases} \qquad (2.10)$$

By virtue of (2.6) and (2.8_l)–(2.10) we have $T(U) \subset U$.

Let $v_k \in U$ $(k = 1, 2, \dots)$ and $\lim_{k \to \infty} v_k = v_0$ uniformly on every finite subsegment of $[t_*, +\infty[$. Take arbitrarily $\varepsilon > 0$ and $t^* > t_2$, and choose $t_3 \in]t^*, +\infty[$ and $k_0 \in \mathbb{N}$ such that

$$2(t^* - t_2)^l \int_{t_3}^{+\infty} s^{n-l-1} \Phi(u)(s) ds < \varepsilon,$$
$$(t_3 - t_2)^{n-1} \int_{t_2}^{t_3} |\varphi(v_k)(s) - \varphi(v_0)(s)| ds < \varepsilon \ \text{ for } \ k = k_0, k_0 + 1, \dots .$$

Then since

$$|T(v_k)(t) - T(v_0)(t)| \leq \int_{t_2}^{t^*} (t^* - \xi)^{l-1} \int_{\xi}^{t_3} (s-\xi)^{n-l-1} \times$$
$$\times |\varphi(v_k)(s) - \varphi(v_0)(s)| ds \, d\xi + 2 \int_{t_2}^{t^*} (t^* - \xi)^{l-1} \times$$
$$\times \int_{t_3}^{+\infty} (s-\xi)^{n-l-1} \Phi(u)(s) ds \, d\xi < 2\varepsilon \ \text{ for } \ t_* \leq t \leq t^*,$$

we find that $T(v_k)(t) \to T(v_0)(t)$ uniformly on $[t_*, t^*]$. As t^* is arbitrary, this implies the continuity of T.

Let $v \in U$, $t', t'' \in [t_2, t^*]$ and $t'' > t'$. Then

$$|T(v)(t'') - T(v)(t')| \leq \int_{t_2}^{t'} [(t''-\xi)^{l-1} - (t'-\xi)^{l-1}] \int_{\xi}^{+\infty} s^{n-l-1} \times$$
$$\times \Phi(u)(s)ds\, d\xi + \int_{t'}^{t''} (t''-\xi)^{l-1} \int_{\xi}^{+\infty} s^{n-l-1} \Phi(u)(s)ds\, d\xi.$$

Thus the set $T(U)$ is equicontinuous on every finite subsegment of $[t_*, +\infty[$. Since U is closed, bounded and convex, by Lemma 2.1 there exists $v \in U$ such that $v = T(v)$. The function v is obviously a solution of (2.1) on $[t_0, +\infty[$, satisfying $u(t_1) \leq v(t) \leq u(t)$ for $t \geq t_2$. This however contradicts the fact that equation (2.1) possesses property . The obtained contradiction proves that $l \notin \{1, \dots, n-1\}$. If n is even, then $l = 0$ cannot take place either, which proves the theorem in this case.

Let now n be odd and $l = 0$. Then (2.8_l) implies

$$\lim_{t \to +\infty} u^{(i)}(t) = 0 \;\; (i = 1, \dots, n-1), \;\; \lim_{t \to +\infty} u(t) = c_0 \geq 0.$$

Assume that $c_0 > 0$. Let U be the set of all $v \in C([t_*, +\infty[; \mathbb{R})$ satisfying $c_0 \leq v(t) \leq u(t)$ for $t \geq t_*$. Using the above reasoning, we can show that the operator $T : U \to U$ defined by

$$T(v)(t) = \begin{cases} c_0 + \frac{1}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \varphi(v)(s)ds & \text{for } t \geq t_2 \\ T(v)(t_2) \quad \text{for} \quad t_* \leq t < t_2 \end{cases}$$

has a fixed point v which is a solution of (2.1) on $[t_2, +\infty[$, satisfying $v(t) \geq c_0$ for $t \geq t_2$. But this contradicts property of (2.1). Therefore (0.4) is fulfilled when n is odd and $l = 0$. ■

Let $\Phi \in V(\tau)$, $t_0 \in \mathbb{R}$ and assume that for any $u, v \in H_{t_0,\tau}$ satisfying $|u(t)| \geq |v(t)|$, $u(t)v(t) > 0$ for $t \geq t_0$ inequality (2.6) hold, where $\varphi \in V(\tau)$. Let, moreover, equation (2.2) have property . Then inequality (2.4) also has property .

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (2.4). Assume that (2.7) is fulfilled. Then (2.6) and Lemma 1.1 imply that there exist $t_1 \in [t_0, +\infty[$ and $l \in \{0, \dots, n\}$ such that $l + n$ is even and (2.8_l) holds. Like in proving Theorem 2.1, it can be shown that $l \notin \{1, \dots, n-2\}$ and (0.4) is fulfilled when n is even and $l = 0$. To complete the proof it suffices to show that (0.5) is valid when $l = n$.

Thus assuming that $l = n$, from (2.8_n) and (2.4) we obtain

$$u^{(i-1)}(t) \geq \frac{(t-t_1)^{n-i} u^{(n-1)}(t_1)}{(n-i)!} + \frac{1}{(n-i)!} \int_{t_1}^{t} (t-s)^{n-i} \times$$
$$\times \Phi(u)(s)ds \;\; \text{for} \;\; t \geq t_1 \;\; (i = 1, \dots, n), \tag{2.11}$$

where $t_1 \in [t_0, +\infty[$ is such that $t_* = \min(t_1, \inf\{\tau(t) : t \geq t_1\}) \geq t_0$.

Let U be the set of all $v \in C([t_*, +\infty[; \mathbb{R})$ satisfying

$$\frac{u^{(n-1)}(t_1)(t-t_1)^{n-1}}{(n-1)!} \le v(t) \le u(t). \tag{2.12}$$

As in proving Theorem 1.2, by (2.6), (2.11), (2.12) and Lemma 2.1 we find that the operator $T : U \to U$ defined by

$$T(v)(t) = \begin{cases} \frac{u^{(n-1)}(t_1)(t-t_1)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}\times \\ \quad \times\varphi(v)(s)ds \quad \text{for} \quad t \ge t_1 \\ 0 \quad \text{for} \quad t_* \le t \le t_1. \end{cases}$$

has a fixed point v which is a solution of (2.2) satisfying

$$\frac{u^{(n-1)}(t_1)(t-t_1)^{n-i}}{(n-i)!} \le v^{(i-1)}(t) \le u^{(i-1)}(t) \ \text{ for } \ t \ge t_1 \ \ (i = 1, \dots, n).$$

Since (2.2) has property , $v^{(i)}(t) \uparrow +\infty$ as $t \uparrow +\infty$ $(i = 0, \dots, n-1)$ and therefore (0.5) is fulfilled. ■

Remark 2.1. By Theorem 2.1 (Theorem 2.2) it is obvious that if $\varphi \in M(\tau)$ and equation (2.1) (equation (2.2)) has property (property), then the inequality

$$\begin{gathered} [u^{(n)}(t) + \varphi(u)(t)] \operatorname{sign} u(t) \le 0 \\ ([u^{(n)}(t) - \varphi(u)(t)] \operatorname{sign} u(t) \ge 0) \end{gathered} \tag{2.13}$$

also has property (property).

Remark 2.2. If $\varphi \in M(\tau)$ and inequality (2.13) has a solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying

$$u^{(i)}(t)u(t) > 0 \ \ (i = 0, \dots, l-1), \ \ (-1)^{i+l}u^{(i)}(t)u(t) \ge 0 \quad (2.14_l)$$
$$(i = l, \dots, n), \ \text{ for } \ t \ge t_0,$$

where $l = \{1, \dots, n-1\}$ and $l+n$ is odd (even), then equation (2.1) (equation (2.2)) also has a solution of the same type.

If condition (0.2) ((0.3)) is fulfilled and $u : [t_0, +\infty[\to \mathbb{R}$ is a proper solution of equation (0.1) not satisfying (0.4) (satisfying neither (0.4) nor (0.5)), then there exists $c \in]0, +\infty[$ such that $1/c \le u(t) \le ct^{n-1}$ for $t \ge t_1$ with $t_1 \in [t_0, +\infty[$ sufficiently large. Taking this fact into account and repeating the arguments we used in proving Theorems 2.1, 2.2, we can easily ascertain that the following result is valid.

Let $F \in V(\tau)$, condition (0.2) ((0.3)) *be fulfilled, and for any sufficiently large $c \in]0, +\infty[$ let there exist $\varphi_c \in M(\tau)$ and $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying*

$$\frac{1}{c} \le |u(t)| \le ct^{n-1}, \ \ \textit{for} \ \ t \ge t_c \tag{2.15}$$

we have the inequality

$$F(u)(t)\operatorname{sign}u(t)\geq\varphi_c(u)(t)\operatorname{sign}u(t)\ \ \textit{for}\ \ t\geq t_c \tag{2.16}$$

$$(F(u)(t)\operatorname{sign}u(t)\leq-\varphi_c(u)(t)\operatorname{sign}u(t)\ \ \textit{for}\ \ t\geq t_c) \tag{2.17}$$

Then if the equation

$$v^{(n)}+\varphi_c(v)(t)\operatorname{sign}v(t)=0\ \ (v^{(n)}-\varphi_c(v)(t)\operatorname{sign}v(t)=0) \tag{2.18}$$

has property (property), equation (0.1) *will also have property (property).*

Assume that $\Phi\in M(\tau)$, $l\in\{1,\dots,n-1\}$, $l+n$ *is odd*

$$\mu\in]0,1],\ \ \tau_0\in C(\mathbb{R}_+;\mathbb{R}_+),\ \ \tau_0(t)\leq\tau(t)\ \ \textit{for}\ \ t\in\mathbb{R}_+,\quad \lim_{t\to+\infty}\tau_0(t)=+\infty \tag{2.19}$$

and φ *is defined by* (2.5)*, where for any* $t\in\mathbb{R}_+$ *a function* $\psi_t:C(\mathbb{R}_+;\mathbb{R})\to C(\mathbb{R}_+;\mathbb{R})$ *is given by*

$$\psi_t(v)(s)=[\tau_0(t)]^{-\mu(l-1)}|v(\tau_0(t))|^{\mu}\operatorname{sign}v(\tau_0(t))s^{l-1}\ \ \textit{for}\ \ s\in\mathbb{R}_+. \tag{2.20$_l$}$$

Then if inequality (2.3) *has a proper nonoscillatory solution* $u:[t_0,+\infty[\to\mathbb{R}$ *satisfying* (2.14$_l$)*, then equation* (2.1) *also has a solution of the same type.*

Proof. It can be assumed without loss of generality that $u(t)>0$ for $t\geq t_0$. Then either

$$\int_{t_0}^{+\infty}s^{n-l}|u^{(n)}(s)|ds<+\infty\ \ \text{and}\ \ u(t)\geq c_0t^{l-1}\ \ \text{for}\ \ t\geq t_1 \tag{2.21}$$

with $t_1\in[t_0,+\infty[$ and $c_0\in]0,1]$, or by Lemma 1.3

$$\frac{u(t)}{t^{l-1}}\uparrow+\infty\quad\text{as}\quad t\uparrow+\infty. \tag{2.22}$$

Let (2.21) hold. Then by (2.3) there exists $t_2\in[t_1,+\infty[$ such that $t_*=\min(\{t_2,\inf\{\tau(t):t\geq t_2\}\})\geq t_1$ and

$$\int_{t_2}^{+\infty}t^{n-l}\Phi(\theta)(t)dt<\frac{c_0^{1/\mu}}{2}, \tag{2.23}$$

where $\theta(s)=c_0s^{l-1}$.

Let further U be the set of all $v\in C([t_*,+\infty[;\mathbb{R})$ satisfying

$$\frac{c_0^{1/\mu}}{2}t^{l-1}\leq v(t)\leq c_0^{1/\mu}t^{l-1}\ \ \text{for}\ \ t\geq t_*. \tag{2.24}$$

Define $T : U \to C([t_*, +\infty[; \mathbb{R})$ by

$$T(v)(t) = \begin{cases} c_0^{1/\mu} t^{l-1} - \frac{1}{(l-2)!(n-l)!} \int_{t_2}^{t} (t-s)^{l-2} \times \\ \quad \times \int_s^{+\infty} (\xi - s)^{n-l} \varphi(v)(\xi) d\xi \, ds \ \text{ for } \ t \geq t_2 \\ c_0^{1/\mu} t^{l-1} \quad \text{ for } \ t_* \leq t \leq t_2 \end{cases}$$

if $l > 1$ and by

$$T(v)(t) = \begin{cases} c_0^{1/\mu} - \frac{1}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \varphi(v)(s) ds \ \text{ for } \ t \geq t_2 \\ T(v)(t_2) \quad \text{ for } \ t_* \leq t \leq t_2 \end{cases}$$

if $l = 1$. According to (2.24) $\frac{c_0}{2^\mu} \leq [\tau_0(t)]^{-\mu(l-1)} [v(\tau_0(t))]^\mu \leq c_0$ for $t \geq t_2$, so that $T(U) \subset U$ by (2.5), (2.20_l), (2.23). As in proving Theorem 1.2, we find that T has a fixed point which is a solution of (2.1) on $[t_2, +\infty[$ satisfying (2.14_l).

If (2.22) is fulfilled, then for any sufficiently large t we have $u^{(n)}(t) + \varphi(u)(t) \leq 0$. By Remark 2.2 we conclude that (2.1) has a solution satisfying (2.14_l). ■

Similar arguments can be used to prove

Assume that $\Phi \in M(\tau)$, $l \in \{1, \ldots, n-2\}$, $l + n$, *is even,* μ *and* τ_0 *satisfy* (2.19) *and* φ *is defined by* (2.5), *where* $\psi_t : C(\mathbb{R}_+; \mathbb{R}) \to C(\mathbb{R}_+; \mathbb{R})$ *is given by* (2.20_l) *for any* $t \in \mathbb{R}_+$. *Then if inequality* (2.4) *has a proper nonoscillatory solution* $u : [t_0, +\infty[\to \mathbb{R}$ *satisfying* (2.14_l), *then equation* (2.2) *also has a solution of the same type.*

Assume that $\Phi \in M(\tau, \sigma)$, $l \in \{1, \ldots, n-1\}$, $l + n$ *is odd,*

$$\mu \in [1, +\infty[, \ \ \tau_0 \in C(\mathbb{R}_+; \mathbb{R}_+), \ \ \tau_0(t) \leq \sigma(t) \\ \textit{for } \ t \in \mathbb{R}_+, \ \ \lim_{t \to +\infty} \tau_0(t) = +\infty \tag{2.25}$$

and φ *is defined by* (2.5) *where for any* $t \in \mathbb{R}_+$

$$\psi_t(v)(s) = [\sigma(t)]^{-\mu l} |v(\tau_0(t))|^\mu \operatorname{sign} v(\tau_0(t)) s^l \ \ \textit{for} \ \ s \in \mathbb{R}_+. \tag{2.26_l}$$

Then the conclusion of Lemma 2.2 *is true.*

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (2.3) satisfying (2.14_l). It can be assumed that $u(t) > 0$ for $t \geq t_0$. Then either (2.21) is fulfilled with $t_1 \in [t_0, +\infty[$ and $c_0 \in]0, 1]$ or by Lemma 1.3

$$\frac{u(t)}{t^l} \downarrow c_1 \geq 0 \quad \text{as} \quad t \uparrow +\infty. \tag{2.27}$$

Let (2.21) hold. Obviously by (2.3)

$$\int_{t_0}^{+\infty} t^{n-l} \Phi(\theta_t)(t) dt < +\infty, \tag{2.28}$$

where $\theta_t(s)=c_0 s^l/[\sigma(t)]^\mu$, $s\in\mathbb{R}_+$, for any $t\geq t_0$. Using (2.8) and repeating arguments from the proof of Lemma 2.2, we find that the equation

$$v^{(n)}(t)+\Phi(\overline{\xi}_t(v))(t)=0, \tag{2.29}$$

where

$$\overline{\xi}_t(v)(s)=[\sigma(t)]^{-\mu l}|v(\sigma(t))|^\mu \operatorname{sign} v(\sigma(t))s^l \quad \text{for} \quad s\in\mathbb{R}_+ \tag{2.30}$$

for any $t\in\mathbb{R}_+$, has a solution $v:[t_2,+\infty[\to]0,+\infty[$ satisfying (2.14_l), where $t_2\in[t_1,+\infty[$ is sufficiently large.

On the other hand, since $\tau_0(t)\leq\sigma(t)$ for $t\in\mathbb{R}_+$, by (2.5) and (2.26_l) v satisfies

$$v^{(n)}(t)+\varphi(v)(t)\leq 0 \tag{2.31}$$

on $[t_2,+\infty[$. Following Remark 2.2 equation (2.1) has a solution of type (2.14_l).

Now consider the case when (2.27) is fulfilled. Assume at first that $c_1>0$. Then there exists $t_1\in[t_0,+\infty[$ such that $u(t)\geq c_1 t^l/2$ for $t\geq t_1$. Therefore by (2.1) and Lemma 1.3 we have

$$\int_{t_1}^{+\infty} t^{n-l-1}\Phi(\theta)(t)dt<+\infty,$$

where $\theta(s)=c_1 s^l/2$ for $s\in\mathbb{R}_+$. Choose $t_2\in[t_1,+\infty[$ such that $t_*=\min(t_2,\inf\{\tau(t):t\geq t_2\})\geq t_1$ and

$$\int_{t_2}^{+\infty} t^{n-l-1}\Phi(\theta)(t)dt<\frac{c_1^{1/\mu}}{4}. \tag{2.32}$$

Let U be the set of all $v\in C([t_*,+\infty[;\mathbb{R})$ satisfying

$$\frac{c_1^{1/\mu}}{4}t^l\leq v(t)\leq\frac{c_1^{1/\mu}}{2}t^l \quad \text{for} \quad t\geq t_*. \tag{2.33}$$

Define $T:U\to C([t_*,+\infty[;\mathbb{R})$ by

$$T(v)(t)=\begin{cases}\frac{c_1^{1/\mu}}{4}t^l-\frac{1}{(l-1)!(n-l-1)!}\int_{t_2}^t(t-s)^{l-1}\int_s^{+\infty}(\xi-s)^{n-l-1}\times\\ \quad\times\varphi(\overline{\psi}_\xi(v))(\xi)d\xi\, ds \quad \text{for} \quad t\geq t_2\\ \frac{c_1^{1/\mu}}{4}t^l \qquad \text{for} \quad t_*\leq t\leq t_2,\end{cases} \tag{2.34}$$

where for any $t\in\mathbb{R}_+$, $\overline{\psi}_t$ is given by (2.30).

By (2.33) it is clear that for any $v\in U$ we have

$$\frac{c_1}{4^\mu}\leq[\sigma(t)]^{-\mu l}[v(\sigma(t))]^\mu\leq\frac{c_1}{2^\mu} \quad \text{for} \quad t\geq t_2,$$

and therefore (2.32) and (2.34) imply that $T(U)\subset U$. As we did previously in this section, by Lemma 2.1 we ascertain that T has a fixed point v which is a solution of (2.29) of type (2.14_l) on $[t_2,+\infty[$. Thus v satisfies (2.31) on

$[t_2, +\infty[$. Hence on account of Remark 2.2 equation (2.1) also has a solution of type (2.14_l).

Now we shall consider the last remaining case $c_1 = 0$. Since we have $u(t)/t^l \le 1$ for any sufficiently large t, u satisfies inequality (2.31) for large t. By Remark 2.2 equation (2.1) has a solution of type (2.14_l). ■

In a mannar similer to the above we can prove

Assume that $\Phi \in M(\tau, \sigma)$, $l \in \{1, \dots, n-2\}$, $l+n$ is even, μ and τ_0 satisfy (2.25) *and φ is defined by* (2.5), *where $\psi_t : C(\mathbb{R}_+; \mathbb{R}) \to C(\mathbb{R}_+; \mathbb{R})$ is given by* (2.26) *for any $t \in \mathbb{R}_+$. Then the conclusion of Lemma* 2.3 *is true.*

Assume that $\Phi \in M(\tau)$, μ and τ_0 satisfy (2.19), *φ is defined by* (2.5) *where ψ_t is given by* (2.20_1) *for any $t \in \mathbb{R}_+$, and equation* (2.1) *has property . Then inequality* (2.3) *also has property .*

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (2.3). Then by Lemma 1.1 there exists $l \in \{0, \dots, n-1\}$ such that $l+n$ is odd and (2.14_l) holds. Assume that $u(t) > 0$ for $t \ge t_0$.

Let $l \in \{1, \dots, n-1\}$. Then by Lemma 2.2 equation (2.1) with φ defined by (2.5) and (2.20_l) has a solution $v : [t_1, +\infty[\to]0, +\infty[$ of type (2.14_l), where $t_1 \in [t_0, +\infty[$ is sufficiently large. Since

$$\left(\frac{s}{[\tau_0(t)]^\mu}\right)^{l-1} \ge 1 \quad \text{for} \quad s \ge \tau(t),$$

v satisfies (2.31) on $[t_1, +\infty[$ with φ defined by (2.5) and (2.20_1). Following Remark 2.2, equation (2.1) with φ defined by (2.5) and (2.20_1) also has a solution of type (2.14_l). But this contradicts property of this equation stated in the conditions of the theorem. Therefore we have proved that $l \notin \{1, \dots, n-1\}$.

Now assume that $l = 0$, n is odd and $u(t) \downarrow c > 0$ as $t \uparrow +\infty$. By (2.3) and Lemma 1.2 we have

$$\int^{+\infty} t^{n-1} \Phi\left(\frac{c}{2}\right)(t) dt < +\infty.$$

Therefore by Lemma 4.1 to be proved later in §4 equation (2.1) with φ defined by (2.5) and (2.20_1) has a solution v of type (2.14_0) such that $\lim_{t \to +\infty} v(t) \ne 0$. But this again contradicts property of this equation. The obtained contradiction proves that (0.4) holds for $l = 0$. ■

′ *Assume that $\Phi \in M(\tau, \sigma)$, μ and τ_0 satisfy* (2.25) *φ is defined by* (2.5), *where ψ_t is given by* (2.26_{n-1}) *for any $t \in \mathbb{R}_+$ and equation* (2.1) *has property . Then inequality* (2.3) *also has property .*

Proof. Let $u : [t_0, +\infty[$ be a proper nonoscillatory solution of (2.3). Then by Lemma 1.1 there exists $l \in \{0, \dots, n-1\}$ such that $l + n$ is odd and (2.14_l) holds. It can be assumed that $u(t) > 0$ for $t \geq t_0$.

Assume that $l \in \{1, \dots, n-1\}$. By Lemma 2.4 equation (2.1) with φ defined by (2.5) and (2.26_l) has a solution $v : [t_1, +\infty[\to]0, +\infty[$ of type (2.14_l), where $t_1 \in [t_0, +\infty[$ is sufficiently large. Since

$$\left(\frac{s}{[\sigma(t)]^{\mu}}\right)^{n-1} \leq \left(\frac{s}{[\sigma(t)]^{\mu}}\right)^{l} \quad \text{for} \quad \tau(t) \leq s \leq \sigma(t),$$

v satisfies (2.31) on $[t_1, +\infty[$ with φ defined by (2.5) and (2.26_{n-1}). By Remark 2.2 equation (2.1) with φ defined by (2.5) and (2.26_{n-1}) also has a solution of type (2.14_l). But this contradicts property of this equation stated in the conditions of the theorem.

Assuming that $l = 0$ and n is odd and applying the arguments from the proof of Theorem 2.4, we can show that condition (0.4) is fulfilled. ■

Assume that $\Phi \in M(\tau)$, μ and τ_0 satisfy (2.19), *φ is defined by* (2.5) *where ψ_t is given by* (2.20_1) *for any $t \in \mathbb{R}_+$, and equation* (2.2) *has property . Then inequality* (2.4) *also has property .*

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (2.4). By Lemma 1.1 there exists $l \in \{0, \dots, n\}$ such that $l + n$ is even and (2.14_l) holds. As in the proof of Theorem 2.4, we can show that $l \notin \{1, \dots, n-2\}$, and if n is even and $l = 0$, then (0.4) is fulfilled.

Assume now that $l = n$ and $\lim_{t \to +\infty} |u^{(n-1)}(t)| < +\infty$. By (2.14_n) there exist $c_0 \in]0, +\infty[$ and $t_2 \in]t_1, +\infty[$ such that $|u(t)| \geq c_0 t^{n-1}$ for $t \geq t_2$, and therefore (2.4) implies

$$\int^{+\infty} |\varphi(\theta)(t)| dt < +\infty \tag{2.35}$$

with $\theta(s) = c_0 \operatorname{sign} u(t_2) s^{n-1}$ for $s \in \mathbb{R}_+$.

On the other hand, by Lemma 4.1 and (2.35) equation (2.2) with φ defined by (2.5) and (2.20_1) has a solution v of type (2.14_n) satisfying $\lim_{t \to +\infty} |v^{(n-1)}(t)| < +\infty$. But this contradicts property of this equation. ■

The proof of Theorem 2.4 has been a guide for us in proving Theorem 2.4′. In the same way we shall be guided by the proof of Theorem 2.5 to show that the next theorem is valid.

′ *Assume that $\varphi \in M(\tau, \sigma)$, μ and τ_0 satisfy* (2.25), *φ is defined by* (2.5), *where ψ_t is given by* (2.26_{n-1}) *for any $t \in \mathbb{R}_+$, and equation* (2.2) *has property . Then inequality* (2.4) *also has property .*

§ 3. Sufficient Conditions

In this subsection we shall derive the ineffective sufficient conditions for the equations

$$u^{(n)} + \varphi(u)(t) = 0 \tag{3.1}$$

$$u^{(n)} - \varphi(u)(t) = 0 \tag{3.2}$$

to have property and , respectively. These conditions will be stated in terms of the classes introduced in 1.2 and be used in 3.2 to obtain the effective sufficient conditions for (0.1).

Let $\varphi \in M(\tau)$, $l \in \{1, \dots, n-1\}$ and $l+n$ be odd. Then the condition

$$\widetilde{\varphi}_l \in M_1(\tau) \tag{3.3$_l$}$$

is sufficient for equation (3.1) *not to have a proper solution satisfying* (2.14$_l$), *where for any $u \in C(\mathbb{R}_+;\mathbb{R})$ and $t \in \mathbb{R}_+$*

$$\widetilde{\varphi}_l(u)(t) = \frac{t^{n-l-1}}{l!(n-l)!}\varphi(\alpha u)(t) \ \ \textit{with} \ \ \alpha(s) = s^l \ \ \textit{for} \ \ s \in \mathbb{R}_+. \tag{3.4$_l$}$$

Proof. Let, on the contrary, (3.1) have a solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying (2.14$_l$). It can be assumed without loss of generality that (2.8$_l$) is fulfilled, where t_1 is sufficiently large. By (3.3$_l$), (3.4$_l$) and Lemmas 1.4, 1.5 for any $c \neq 0$ we have

$$\int^{+\infty} t^{n-l}\varphi(\theta)(t)dt = +\infty, \ \ \text{with } l; \theta(s) = cs^{l-1} \ \ \text{for} \ \ s \in \mathbb{R}_+. \tag{3.5}$$

(2.8$_l$) implies that there exist $c_0 \in]0, +\infty[$ and $t_2 \in [t_1, +\infty[$ such that $u(t) \geq c_0 t^{l-1}$ for $t \geq t_2$. Therefore by (3.1), (3.5) we have

$$\int^{+\infty} t^{n-l}|u^{(n)}(t)|dt = +\infty. \tag{3.6}$$

Hence by Lemma 1.3 $u(t)/t^l \downarrow$ as $t \uparrow +\infty$ and

$$u(t) \geq \frac{t^l}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}|u^{(n)}(s)|ds \ \ \text{for} \ \ t \geq t_*,$$

where $t_* \in [t_2, +\infty[$ is sufficiently large, so that by (3.1), (3.4$_l$) we obtain

$$x(t) \geq \int_t^{+\infty} \widetilde{\varphi}_l(x)(s)ds \quad \text{for} \ \ t \geq t_*,$$

where $x(t) = u(t)/t^l$ is nonincreasing. This means that $\widetilde{\varphi}_l \notin M_1(\tau)$, which contradicts (3.3$_l$). ■

In a similar way we can prove

′ *Let* $\varphi \in M(\tau)$, $l \in \{1,\dots,n-2\}$ *and* $l+n$ *be even. Then condition* (3.3_l) *with* $\widetilde{\varphi}_l$ *defined by* (3.4_l) *is sufficient for equation* (3.2) *not to have a proper solution satisfying* (2.14_l).

Let $\varphi \in M(\tau)$, $l \in \{1,\dots,n-1\}$ *and* $l+n$ *be odd. Let, moreover,* $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *be a nondecreasing function satisfying* (1.13). *Then the condition*

$$\widetilde{\varphi}_l \in M_2^{\beta}(\tau) \tag{3.7_l}$$

is sufficient for equation (3.1) *not to have a proper solution satisfying* (2.14_l), *where for any* $u \in C(\mathbb{R}_+;\mathbb{R})$ *and* $t \in \mathbb{R}_+$ *we have*

$$\widetilde{\varphi}_l(u)(t) = \frac{\beta^{n-l}(t)}{l!(n-l)!}\varphi(\alpha u)(t) \quad \textit{with} \quad \alpha(s) = s^{l-1} \quad \textit{for} \quad s \in \mathbb{R}_+. \tag{3.8_l}$$

Proof. Let, on the contrary, (3.1) have a solution $u : [t_0,+\infty[\to \mathbb{R}$ satisfying (2.14_l). It can be assumed that (2.8_l) is fulfilled. Condition (3.6) is valid on account of (3.1), (3.7_l), (3.8_l) and Lemma 1.4, so that Lemma 1.3 implies

$$u(\beta(t)) \geq \frac{\beta^{l-1}(t)}{l!(n-l)!}\int_{t_1}^{t} \beta^{n-l}(s)|u^{(n)}(s)|ds \quad \textit{for} \quad t \geq t_1,$$

where $t_1 \in [t_0,+\infty[$ is sufficiently large. Hence by (3.1) and (3.8_l) we have

$$x(\beta(t)) \geq \int_{t_1}^{t} \widetilde{\varphi}_l(x)(s)ds \quad \text{for} \quad t \geq t_1,$$

where $x(t) = u(t)/t^{l-1}$ is a nondecreasing function by Lemma 1.3. But this means that $\widetilde{\varphi}_l \notin M_2^{\beta}(\tau)$, which contradicts (3.7_l). ∎

The next lemma can be proved similarly.

′ *Let* $\varphi \in M(\tau)$, $l \in \{1,\dots,n-2\}$ *and* $l+n$ *be even. Then condition* (3.7_l), *where* $\widetilde{\varphi}_l$ *is defined by* (3.8_l), *is sufficient for equation* (3.2) *not to have a proper solution satisfying* (2.14_l).

Let $\varphi \in M(\tau)$ *and* (3.3_{n-1}) *be fulfilled, where* $\widetilde{\varphi}_{n-1}$ *is defined by* (3.4_{n-1}). *Then equation* (3.1) *has property* .

Proof. According to Lemma 1.5 condition (3.3_l) is fulfilled for any $l \in \{0,\dots,n-1\}$ with $\widetilde{\varphi}_l$ defined by (3.4_l).

Now assume that $u : [t_0,+\infty[\to \mathbb{R}$ is a proper nonoscillatory solution of (3.1). By Lemma 1.1 there exists $l \in \{0,\dots,n-1\}$ such that $l+n$ is odd and (2.14_l) holds. On account of Lemma 3.1 we have $l \notin \{1,\dots,n-1\}$. Thus it can be assumed that $l = 0$ and n is odd.

Let $u(t) \downarrow c_0 \neq 0$ as $t \uparrow +\infty$. Since $\widetilde{\varphi}_0 \in M_1(\tau)$, by Lemma 1.4 we have

$$\int^{+\infty} |\widetilde{\varphi}_0(c)(t)|dt = +\infty \quad \text{for any} \quad c \neq 0. \tag{3.9}$$

Obviously there exists $t_1 \in [t_0, +\infty[$ such that $|u(t)| \geq c_0/2$ for $t \geq t_1$ and therefore by (3.1) and (3.9)

$$\int_{t_2}^{+\infty} t^{n-1} |u^{(n)}(t)| dt \geq \int_{t_2}^{+\infty} |\widetilde{\varphi}_0(\frac{c_0}{2})(t)| dt = +\infty.$$

This however contradicts (1.3). The obtained contradiction proves that (0.4) holds if n is odd and $l = 0$. ■

Let $\varphi \in M(\tau)$ and (3.3_{n-2}) be satisfied, where for any $u \in C(\mathbb{R}_+; \mathbb{R})$ and $t \in \mathbb{R}_+$

$$\widetilde{\varphi}_{n-2}(u)(t) = \frac{t}{(n-1)!} \varphi(\alpha u)(t) \;\; \textit{with} \;\; \alpha(s) = s^{n-2} \;\; \textit{for} \;\; s \in \mathbb{R}_+. \quad (3.10)$$

Moreover, let for any $c \neq 0$

$$\int^{+\infty} |\varphi(\theta)(t)| dt = +\infty \;\; \textit{with} \;\; \theta(s) = cs^{n-1} \;\; \textit{for} \;\; s \in \mathbb{R}_+. \quad (3.11)$$

Then equation (3.2) *has property* .

Proof. By Lemma 1.5, (3.3_{n-2}) and (3.10) condition (3.3_l) is fulfilled for any $l \in \{0, \dots, n-2\}$ with $\widetilde{\varphi}_l$ defined by (3.4_l).

Assume now that $u : [t_0, +\infty[\to \mathbb{R}$ is a proper nonoscillatory solution of (3.2). Then by Lemma 1.1 there exists $l \in \{0, \dots, n\}$ such that $l + n$ is even and (2.14_l) holds. By Lemma 3.1′ $l \notin \{1, \dots, n-2\}$. As while proving Theorem 3.1, it can be shown that (0.4) holds if n is even and $l = 0$.

To complete the proof it remains to show that (0.5) is fulfilled for $l = n$. Indeed, by (2.14_n) there exists $t_1 \in [t_0, +\infty[$ and $c_0 \in]0, +\infty[$ such that $|u(t)| \geq c_0 t^{n-1}$ for $t \geq t_1$. Therefore from (3.2) we have

$$|u^{(n-1)}(t)| \geq \int_{t_2}^{t} |\varphi(\theta)(s)| ds,$$

where $\theta(s) = c_0 \operatorname{sign} u(t_1) s^{n-1}$ and $t_2 \in [t_1, +\infty[$ is chosen such that $\inf\{\tau(t) : t \geq t_2\} \geq t_1$. By (2.14_n) and (3.11) the latter inequality obviously implies (0.5). ■

′ *Let $\varphi \in M(\tau)$, τ_0 satisfy* (2.19) *and (3.3_{n-1}) be fulfilled, where for any $u \in C(\mathbb{R}_+; \mathbb{R})$ and $t \in \mathbb{R}_+$*

$$\begin{gathered} \widetilde{\varphi}_{n-1}(u)(t) = \frac{1}{(n-1)!} \varphi(\psi_t(u))(t) \\ \textit{with} \;\; \psi_t(u)(s) = \tau_0^{n-1}(t) u(\tau_0(t)) \;\; \textit{for} \;\; s \in \mathbb{R}_+. \end{gathered} \quad (3.12)$$

Then equation (3.1) *has property* .

Proof. By Theorem 3.1, (3.3_{n-1}) and (3.12) the equation

$$v^{(n)} + \varphi(\psi_t(v))(t) = 0, \tag{3.13}$$

where $\psi_t(v)(s) = v(\tau_0(t))$, for any $v \in C(\mathbb{R}_+;\mathbb{R})$ and $t \in \mathbb{R}_+$, has property . Therefore by Theorem 2.4 ($\mu = 1$) equation (3.1) also has property . ■

′ *Let* $\varphi \in M(\tau)$, τ_0 *satisfy* (2.19), *let* (3.11) *hold for any* $c \neq 0$ *and* (3.3_{n-2}) *be fulfilled, where for any* $u \in C(\mathbb{R}_+;\mathbb{R})$ *and* $t \in \mathbb{R}_+$

$$\begin{gathered}\widetilde{\varphi}_{n-2}(u)(t) = \frac{t}{(n-1)!}\varphi(\psi_t(u))(t) \\ \text{with } \psi_t(u)(s) = \tau_0^{n-2}(t)u(\tau_0(t)) \text{ for } s \in \mathbb{R}_+.\end{gathered} \tag{3.14}$$

Then equation (3.2) *has property* .

Proof. By Theorem 3.2, (3.3_{n-2}) and (3.11) the equation

$$v^{(n)} - \varphi(\psi_t(v))(t) = 0, \tag{3.15}$$

where ψ_t is defined by (2.20_1) ($\mu = 1$) has property . Therefore according to Theorem 2.5 equation (3.2) also has property . ■

″ *Let* $\varphi \in M(\tau,\sigma)$, τ_0 *satisfy* (2.25) *and*

$$\widetilde{\varphi}_{n-1} \in M_1(\tau,\sigma), \tag{3.16}$$

where for any $u \in C(\mathbb{R}_+;\mathbb{R})$ *and* $t \in \mathbb{R}_+$

$$\begin{gathered}\widetilde{\varphi}_{n-1}(u)(t) = \frac{1}{(n-1)!}\varphi(\psi_t(u))(t) \\ \text{with } \psi_t(u)(s) = [\sigma(t)]^{1-n}[\tau_0(t)]^{n-1}u(\tau_0(t))s^{n-1}.\end{gathered} \tag{3.17}$$

Then equation (3.1) *has property* .

Proof. By Theorem 3.1, (3.16) and (3.17) equation (3.13) with ψ_t defined by (2.26_{n-1}) ($\mu = 1$) has property . Therefore by Theorem 2.4′ equation (3.1) also has property . ■

″ *Let* $\varphi \in M(\tau,\sigma)$, τ_0 *satisfy* (2.25) *and*

$$\widetilde{\varphi}_{n-2} \in M_1(\tau,\sigma), \tag{3.18}$$

where for any $u \in C(\mathbb{R}_+;\mathbb{R})$ *and* $t \in \mathbb{R}_+$

$$\begin{gathered}\widetilde{\varphi}_{n-2}(u)(t) = \frac{t}{(n-1)!}\varphi(\psi_t(u))(t) \\ \text{with } \psi_t(u)(s) = [\sigma(t)]^{2-n}[\tau_0(t)]^{n-2}u(\tau_0(t))s^{n-2} \text{ for } s \in \mathbb{R}_+.\end{gathered} \tag{3.19}$$

Moreover, let (3.11) *hold for any* $c \neq 0$. *Then equation* (3.2) *has property* .

Proof. By Theorem 3.2, (3.18) and (3.19) equation (3.15) with ψ_t defined by (2.26_{n-2}) $(\mu = 1)$ has property . Therefore by Theorem 2.5′ equation (3.2) also has property . ■

Let $\varphi \in M(\tau)$ *and* (3.7_1) *be fulfilled with* $\widetilde{\varphi}_1$ *defined by* (3.8_1). *Then equation* (3.1) *has property* .

Proof. By Lemma 1.7 and (3.7_1) condition (3.7_l) is fulfilled for any $l \in \{1,\dots,n-1\}$, where $\widetilde{\varphi}_l$ is defined by (3.8_l).

Assume that $u : [t_0, +\infty[\to \mathbb{R}$ is a proper nonoscillatory solution of (3.1). Then by Lemma 1.1 there exists $l \in \{0,\dots,n-1\}$ such that $l+n$ is odd and (2.14_l) is satisfied. According to Lemma 3.2 $l \notin \{1,\dots,n-1\}$. Condition (3.7_1) and Lemma 1.4 imply that

$$\int^{+\infty} |\widetilde{\varphi}_1(c)(t)|dt = +\infty \tag{3.20}$$

for any $c \neq 0$. Therefore, as while proving Theorem 3.1, it can be shown that (0.4) is satisfied if n is odd and $l = 0$. ■

Let $\varphi \in M(\tau)$ *and* (3.7_1) *be fulfilled with* $\widetilde{\varphi}_1$ *defined by* (3.8_1). *Then equation* (3.2) *has property* .

Proof. As above, by Lemma 1.7 and (3.7_1) condition (3.7_l) is fulfilled for any $l \in \{1,\dots,n\}$, where $\widetilde{\varphi}_l$ is defined by (3.8_l).

Assume that $u : [t_0, +\infty[\to \mathbb{R}$ is a proper nonoscillatory solution of (3.2). Then by Lemma 1.1 there exists $l \in \{0,\dots,n\}$ such that $l+n$ is even and (2.14_l) holds. By Lemma 3.1′ we have $l \notin \{1,\dots,n-2\}$.

On the other hand, since $\widetilde{\varphi}_l \in M_2^{\beta}(\tau)$ $(l = 1,\dots,n)$, by Lemma 1.4 condition (3.20) holds and

$$\int^{+\infty} |\widetilde{\varphi}_n(c)(t)|dt = +\infty$$

for any $c \neq 0$.

As while proving Theorem 3.2, it can be shown that (0.4) ((0.5)) is satisfied if n is even and $l = 0$ $(l = n)$. ■

′ *Let* $\varphi \in M(\tau)$, τ_0 *satisfy* (2.19) *and* (3.7_1) *be fulfilled, where*

$$\widetilde{\varphi}_1(u)(t) = \frac{[\beta(t)]^{n-1}}{(n-1)!}\varphi(\psi_t(u))(t) \quad \text{with} \quad \psi_t(u)(s) = u(\tau_0(t)) \ \text{for } s \in \mathbb{R}_+ \tag{3.21}$$

for any $u \in C(\mathbb{R}_+;\mathbb{R})$ *and* $t \in \mathbb{R}_+$. *Then equation* (3.1) *has property* .

Proof. By Theorem 3.3, (3.7_1) and (3.21) the equation $v^{(n)}(t)+\varphi(\psi_t(v))(t) = 0$ with ψ_t defined by (2.20_1) $(\mu = 1)$ has property . Therefore by Theorem 2.4 equation (3.1) also has property . ■

In a similar manner one can prove

′ *Let* $\varphi \in M(\tau)$, τ_0 *satsify* (2.19) *and condition* (3.7_1) *be fulfilled with* φ_1 *defined by* (3.21). *Then equation* (3.2) *has property* .

Let $\varphi \in M(\tau, \sigma)$, τ_0 *satisfy* (2.25) *and*

$$\widetilde{\varphi}_1 \in M_2^{\beta}(\tau, \sigma), \tag{3.22}$$

where for any $u \in C(\mathbb{R}_+; \mathbb{R})$ *and* $t \in \mathbb{R}_+$

$$\begin{gathered} \widetilde{\varphi}_1(u)(t) = \frac{[\beta(t)]^{n-1}}{(n-1)!} \varphi(\psi_t(u))(t) \\ \textit{with} \quad \psi_t(u)(s) = [\sigma(t)]^{1-n} u(\tau_0(t)) s^{n-1} \quad \textit{for} \quad s \in \mathbb{R}_+ . \end{gathered} \tag{3.23}$$

Then equation (3.1) *has property* .

Proof. The equation $v^{(n)}(t) + \varphi(\psi_t(u))(t) = 0$ with ψ_t defined by (2.26_{n-1}) $(\mu = 1)$ has property by virtue of Theorem 3.3, (3.22) and (3.23). Therefore by Theorem 2.4′ equation (3.1) also has property . ■

Quite similarly one can prove

Let $\varphi \in M(\tau, \sigma)$, τ_0 *satisfy* (2.25) *and let* (3.22) *be fulfilled, where*

$$\begin{gathered} \widetilde{\varphi}_1(u)(t) = \frac{[\beta(t)]^{n-1}}{(n-1)!} \varphi(\psi_t(u))(t) \\ \textit{with} \ \psi_t(u)(s) = [\sigma(t)]^{2-n} u(\tau_0(t)) s^{n-2} \quad \textit{for} \quad s \in \mathbb{R}_+ \end{gathered}$$

for any $u \in C(\mathbb{R}_+; \mathbb{R})$ *and* $t \in \mathbb{R}_+$. *Then equation* (3.2) *has property* .

We conclude this subsection by a general theorem concerning equation (0.1).

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled, and for any sufficiently large* $c > 0$ *let there exist* $t_c \in \mathbb{R}_+$ *and* $\varphi_c \in M(\tau)$ *such that inequality* (2.16) ((2.17)) *holds for any* $u \in H_{t_c,\tau}$ *satisfying* (2.15). *Then for equation* (0.1) *to have property* ( ) *it is sufficient that* φ_c *satisfy the conditions of anyone of Theorems* 3.1, 3.1′, 3.1″, 3.3, 3.3′, 3.5 (3.2, 3.2′, 3.2″, 3.4, 3.4′, 3.5).

This theorem immediately follows from Theorems 2.1 and 2.3 (2.2 and 2.3).

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *and* $a_c \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *such that for any* $u \in H_{t_0,\tau}$ *satisfying* $1/c \le |u(t)| \le ct^{n-1}$ *for* $t \ge t_c$

$$|F(u)(t)| \ge a_c(t) \quad \textit{for} \quad t \ge t_c. \tag{3.24}$$

Then the condition

$$\int^{+\infty} a_c(t)dt = +\infty \tag{3.25}$$

is sufficient for equation (0.1) *to have property* ( ).

Proof. Assume that $c > 0$ is sufficiently large. Then by (0.2), (3.24) ((0.3), (3.24)) inequality (2.16) ((2.17)) holds with

$$\varphi_c(u)(t) = \frac{a_c(t)|u(t)|^\lambda \operatorname{sign} u(t)}{c^\lambda t^{\lambda(n-1)}}, \quad \lambda \in]0,1[.$$

By Corollary 1.1 of Lemma 1.6 and (3.25) φ_l satisfies the conditions of Theorem 3.1′ (Theorem 3.2′). Therefore by Theorem 3.7 equation (0.1) has property ( ). ■

In the theorems throughout this and next subsections the following conditions will appear:

$$\begin{gathered} m \in \mathbb{N}, \quad \tau_i, \sigma_i \in C(\mathbb{R}_+;\mathbb{R}_+), \quad \tau_i(t) \le \sigma_i(t) \ \text{ for } \ t \in \mathbb{R}_+, \\ \lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, \dots, m); \end{gathered} \tag{3.26}$$

$$\begin{gathered} r_{ic} : \mathbb{R}_+^2 \to \mathbb{R} \ \text{ is measurable}, \ \ r_i(\cdot, t) \ \text{ is nondecreasing} \\ \text{for } \ t \in \mathbb{R}_+ \quad (i = 1, \dots, m); \end{gathered} \tag{3.27}$$

$$\begin{gathered} \omega_i \in C(\mathbb{R}_+;\mathbb{R}_+), \quad \omega_i(0) = 0, \quad \omega_i(s) > 0 \\ \text{for } \ s > 0, \ \ \omega_i \ \text{ is nondecreasing} (i = 1, \dots, m). \end{gathered} \tag{3.28}$$

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *such that for any* $u \in H_{t_c,\tau}$ *satisfying* $1/c \le |u(t)| \le ct^{n-1}$ *for* $t \ge t_c$ *we have*

$$|F(u)(t)| \ge \prod_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(\tau_*^{1-n}(t)|u(s)|) d_s r_{ic}(s,t) \ \text{ for } \ t \ge t_c, \tag{3.29}$$

where (3.26)–(3.28) *hold,* $\tau_*(t) = \min\{\tau_i(t), t \ (i = 1, \dots, m)\}$ *and*

$$\int_0^1 \frac{ds}{\prod_{i=1}^m \omega_{ic}(s)} < +\infty, \tag{3.30}$$

$$\int^{+\infty} \prod_{i=1}^m [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)] dt = +\infty. \tag{3.31}$$

Then equation (0.1) *has property* ( ).

Proof. By (0.2), (3.29) ((0.3), (3.29)) inequality (2.16) ((2.17)) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(\tau_*^{1-n}(t)|u(s)|) \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, by (3.30), (3.31) and Lemma 1.6 we have $\widetilde{\varphi}_{c\,n-1} \in M_1(\tau)$, where

$$\widetilde{\varphi}_{c\,n-1}(u)(t) = \frac{1}{(n-1)!} \prod_{i=1}^{m} \omega_{ic}(u(\tau_*(t))) \operatorname{sign} u(\tau_*(t)) \times \\ \times \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)]. \tag{3.32}$$

Since φ_c satisfies the conditions of Theorem 3.1′ (Theorem 3.2′), equation (0.1) has property ( ) on account of Theorem 3.7. ■

Let $F \in V(\tau)$, condition (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_{ic}} d_s r_{ic}(s,t) \tag{3.33}$$

where (3.26) *and* (3.27) *hold, $\lambda_{ic} \in]0,1[$ $(i = 1, \dots, m)$ and*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} < 1, \\ \int^{+\infty} \tau_*^{(n-1)\lambda_c}(t) \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)] dt = +\infty \tag{3.34}$$

with $\tau_*(t) = \min\{\tau_i(t), t \ (i = 1, \dots, m)\}$. Then equation (0.1) *has property* ( ).

Proof. It suffices to note that by (3.33) and (3.34) all the conditions of Theorem 3.8 are satisfied with $\omega_{ic}(s) = s^{\lambda_{ic}}$ for $s \in \mathbb{R}_+$ $(i = 1, \dots, m)$. ■

Let $F \in V(\sigma)$, condition (0.2) ((0.3)) *be fulfilled, and there exist $t_0 \in \mathbb{R}_+$ such that for any $u \in H_{t_0,\tau}$ we have*

$$|F(u)(t)| \ge p(t)|u(\tau(t))|^{\lambda} \ \text{ for } t \in [t_0, +\infty[, \tag{3.35}$$

where $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ and $\lambda \in]0,1[$. Then the condition

$$\int^{+\infty} \tau_*^{(n-1)\lambda}(t) p(t) dt = +\infty,$$

where $\tau_*(t) = \min\{t, \tau(t)\}$, is sufficient for equation (0.1) *to have property* ( ).

Let $F \in V(\tau)$, condition (0.2) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-1}[\sigma_*(t)]^{1-n}s^{1-n}|u(s)|)d_s r_{ic}(s,t) \quad (3.36)$$
$$\text{for} \quad t \ge t_c,$$

where (3.26)–(3.28) *hold,* $\sigma^*(t) = \max\{\sigma_i(t) : i = 1,\dots,m\}$, $\sigma_*(t) = \min\{\sigma^*(t), t\}$ *and* (3.30), (3.31) *are satisfied. Then equation* (0.1) *has property* .

Proof. By (0.2) and (3.36) inequality (2.16) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-1}[\sigma_*(t)]^{1-n}s^{1-n}|u(s)|)\operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, by (3.30), (3.31) and Lemma 1.6 we have $\widetilde{\varphi}_{c\,n-1} \in M_1(\tau_*, \sigma^*)$, where $\tau_* = \min\{\tau_i(t) : i = 1,\dots,m\}$ and $\widetilde{\varphi}_{c\,n-1}$ is defined by (3.32). Since φ_c satisfies the conditions of Theorem 3.1$''$, equation (0.1) has property on account of Theorem 3.7. ■

Let $F \in V(\tau)$, condition (0.2) *be satisfied, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ inequality* (3.33) *is fulfilled, where* (3.26) *and* (3.27) *hold,* $\lambda_i \in]0,1[$ $(i = 1,\dots,m)$ *and*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} < 1,$$

$$\int^{+\infty} [\sigma_*(t)]^{\lambda_c(n-1)}[\sigma^*(t)]^{\lambda_c(1-n)} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-1)} d_s r_{ic}(s,t)dt = +\infty, \quad (3.37)$$

with $\sigma^*(t) = \max\{\sigma_i(t) : i = 1,\dots,m\}$, $\sigma_*(t) = \min\{\sigma^*(t), t\}$. *Then equation* (0.1) *has property* .

Proof. By (0.2) and (3.33) inequality (2.16) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_{ic}} \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, by (3.37) and Corollary 1.1 of Lemma 1.6 we have $\widetilde{\varphi}_{c\,n-1} \in M_1(\tau_*, \sigma^*)$, where $\tau_* = \min\{\tau_i(t) : i = 1,\dots,m\}$ and

$$\widetilde{\varphi}_{c\,n-1}(v)(t) = [\sigma_*(t)]^{\lambda_c(n-1)}[\sigma^*(t)]^{\lambda_c(1-n)} \times$$
$$\times \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |v(s)|^{\lambda_{ic}} \operatorname{sign} v(s) d_s r_{ic}(s,t).$$

Thus we see that φ_c satisfies the conditions of Theorem 3.1″. Therefore equation (0.1) has property on account of Theorem 3.7. ■

Let $F \in V(\tau)$, *condition* (0.3) *be fulfilled, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *such that for any* $u \in H_{t_c,\tau}$ *satisfying* $1/c \le |u(t)| \le ct^{n-1}$ *for* $t \ge t_c$ *we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-2}[\sigma_*(t)]^{2-n}s^{2-n}|u(s)|)d_s r_{ic}(s,t) \quad (3.38)$$
$$\text{for} \quad t \in [t_c, +\infty[,$$

where (3.26)–(3.28) *hold and* $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$, $\sigma_*(t) = \min\{\sigma^*(t), t\}$. *Moreover, let conditions* (3.30) *and*

$$\int^{+\infty} t \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)]dt = +\infty, \quad (3.39)$$

$$\int^{+\infty} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-2}[\sigma_*(t)]^{2-n}s)d_s r_{ic}(s,t)dt = +\infty \quad (3.40)$$

be fulfilled. Then equation (0.1) *has property* .

Proof. By (0.3) and (3.38) inequality (2.17) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-2}[\sigma_*(t)]^{2-n}s^{2-n}|u(s)|) \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, (3.30) and (3.39) imply $\widetilde{\varphi}_{c\, n-2} \in M_1(\tau_*, \sigma^*)$, where $\tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$ and

$$\widetilde{\varphi}_{c\, n-2}(v)(t) = \frac{t}{(n-1)!} \prod_{i=1}^{m} \omega_{ic}(v(\sigma_*(t))) \operatorname{sign} v(\sigma_*(t)) \times$$
$$\times \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)].$$

Taking into account (3.40), we see that φ_c satisfies the conditions of Theorem 3.2″. Therefore by Theorem 3.7 equation (0.1) has property . ■

Repeating the arguments given in Theorem 3.10 and Corollary 3.4, we easily ascertain that the corollary below is true.

Let $F \in V(\tau)$, *condition* (0.3) *be satisfied, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *such that for any* $u \in H_{t_c,\tau}$ *satisfying* $1/c \le |u(t)| \le ct^{n-1}$ *for* $t \ge t_c$ *we have inequality* (3.33), *where*

(3.26) *and* (3.27) *hold and* $\lambda_{ic} \in]0,1[$ $(i = 1,\dots,m)$. *Moreover, let the conditions*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} < 1,$$

$$\int^{+\infty} t[\sigma_*(t)]^{\lambda_c(n-2)}[\sigma^*(t)]^{\lambda_c(2-n)} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-2)} d_s r_{ic}(s,t)dt = +\infty,$$

$$\int^{+\infty} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-1)} d_s r_{ic}(s,t)dt = +\infty$$

be fulfilled. Then equation (0.1) *has property* .

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *such that for any* $u \in H_{t_0,\tau}$ *satisfying* $1/c \le |u(t)| \le ct^{n-1}$ *for* $t \ge t_c$ *we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(|u(s)|) d_s r_{ic}(s,t), \tag{3.41}$$

where (3.26)–(3.28) *hold. Moreover, let*

$$\int_1^{+\infty} \frac{ds}{\prod_{i=1}^{m} \omega_{ic}(s)} < +\infty, \tag{3.42}$$

$$\int^{+\infty} \beta^{n-1}(t) \prod_{i=1}^{m} [r_{ic}(\sigma_i(t),t) - r_{ic}(\tau_i(t),t)]dt = +\infty, \tag{3.43}$$

where $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *is a nondecreasing function satisfying*

$$\beta(t) \le \min\{\tau_i(t), t \ (i = 1,\dots,m)\}, \quad \lim_{t\to+\infty} \beta(t) = +\infty. \tag{3.44}$$

Then equation (0.1) *has property* ( ).

Proof. By (0.2) and (3.41) ((0.3) and (3.41)) inequality (2.16) ((2.17)) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(|u(s)|) \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, by (3.42), (3.43) and Lemma 1.8 we have $\widetilde{\varphi}_{1c} \in M_2^{\beta}(\tau)$, where

$$\widetilde{\varphi}_{1c}(u)(t) = \frac{\beta^{n-1}(t)}{(n-1)!} \prod_{i=1}^{m} \omega_{ic}(|u(\beta(t))|) \operatorname{sign} u(\beta(t)) \times$$
$$\times \prod_{i=1}^{m} [r_{ic}(\sigma_i(t),t) - r_{ic}(\tau_i(t),t)].$$

Since φ_c satisfies all the conditions of Theorem 3.3′ (Theorem 3.4′), equation (0.1) has property ( ) by virtue of Theorem 3.7. ■

Let $F \in V(\tau)$, condition (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have inequality* (3.33) *where* (3.26) *and* (3.27) *hold, $\lambda_{ic} \in]0,+\infty[$, $(i = 1,\dots,m)$ and*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} > 1,$$
$$\int^{+\infty} \beta^{n-1}(t) \prod_{i=1}^{m} [r_{ic}(\sigma_i(t),t) - r_{ic}(\tau_i(t),t)]dt = +\infty, \tag{3.45}$$

where $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ is a nondecreasing function satisfying (3.44). *Then equation* (0.1) *has property* ( ).

Proof. By (0.2), ((0.3)), (3.33), (3.45) and Corollary 1.2 all the conditions of Theorem 3.11 are satisfied with $\omega_{ic}(s) = s^{\lambda_{ic}}$ for $s \in \mathbb{R}_+$ $(i = 1,\dots,m)$. ■

Let $F \in V(\tau)$ and conditions (0.2), (3.35) ((0.3), (3.35)) *be fulfilled, where $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ and $\lambda \in]1,+\infty[$. Then the condition*

$$\int^{+\infty} \beta^{n-1}(t)p(t)dt = +\infty$$

is sufficient for equation (0.1) *to have property* ( ), *where $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ is a nondecreasing function satisfying*

$$\beta(t) \le \min\{t, \tau(t)\}, \ \lim_{t\to+\infty} \beta(t) = +\infty.$$

Let $F \in V(\tau)$, condition (0.2) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-1}s^{1-n}|u(s)|)d_s r_{ic}(s,t), \tag{3.46}$$

where (3.26)–(3.28) *hold and $\sigma^*(t) = \max\{\sigma_i(t) : i = 1,\dots,m\}$. Moreover, let* (3.42) *and* (3.43) *hold, where $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ is a nondecreasing function satisfying*

$$\beta(t) \le \min\{\sigma^*(t), t\}, \ \lim_{t\to+\infty} \beta(t) = +\infty, \tag{3.47}$$

Then equation (0.1) *has property* .

Proof. By (0.2) and (3.46) inequality (2.16) holds with

$$\varphi_c(u)(t) = \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-1} s^{1-n} |u(s)|) \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, according to (3.42), (3.43) and Lemma 1.8 we have $\widetilde{\varphi}_{1c} \in M_2^{\beta}(\tau_*, \sigma^*)$, where $\tau_* = \min\{\tau_i(t) : i = 1, \dots, m\}$ and

$$\widetilde{\varphi}_{1c}(u)(t) = \frac{\beta^{n-1}(t)}{(n-1)!} \prod_{i=1}^{m} \omega_{ic}(|u(\beta(t))|) \operatorname{sign} u(\beta(t)) \times$$
$$\times \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)].$$

Since φ_c satisfies all the conditions of Theorem 3.5, equation (0.1) has property   by Theorem 3.7. ■

Let $F \in V(\tau)$, condition (0.2) *be fulfilled and let for any sufficiently large $c > 0$ let there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have inequality* (3.33)*, where* (3.26) *and* (3.27) *hold and $\lambda_{ic} \in]0, +\infty[$ $(i = 1, \dots, m)$. Moreover, let*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} > 1,$$
$$\int^{+\infty} \beta^{n-1}(t) [\sigma^*(t)]^{\lambda_c(1-n)} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-1)} d_s r_{ic}(s,t) dt = +\infty, \tag{3.48}$$

where $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$ and $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ is a nondecreasing function satisfying (3.47)*. Then equation* (0.1) *has property* .

Proof. By (0.2), (3.33), (3.48) and Corollary 1.2 of Lemma 1.8 all the conditions of Theorem 3.12 are satisfied with $\omega_{ic}(s) = s^{\lambda_{ic}}$ for $s \in \mathbb{R}_+$ $(i = 1, \dots, m)$. ■

Let $F \in V(\tau)$, condition (0.3) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have*

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-2} s^{2-n} |u(s)|) d_s r_{ic}(s,t), \tag{3.49}$$

where (3.26)–(3.28) *hold and $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$. Moreover, let* (3.42) *and* (3.43) *be fulfilled, where $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ is a nondecreasing function satisfying* (3.47)*. Then equation* (0.1) *has property* .

Proof. By (0.3) and (3.49) inequality (2.17) holds with

$$\varphi_c(u)(t) = -\prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}([\sigma^*(t)]^{n-2} s^{2-n} |u(s)|) \operatorname{sign} u(s) d_s r_{ic}(s,t).$$

On the other hand, according to (3.42), (3.43) and Lemma 1.8 we have $\widetilde{\varphi}_{1c} \in M_1^{\beta}(\tau_*, \sigma^*)$, where $\tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$ and

$$\widetilde{\varphi}_{1c}(u)(t) = \frac{\beta^{n-1}(t)}{(n-1)!} \prod_{i=1}^{m} \omega_{ic}(|u(\beta(t))|) \operatorname{sign} u(\beta(t)) \times$$
$$\times \prod_{i=1}^{m} [r_{ic}(\sigma_i(t), t) - r_{ic}(\tau_i(t), t)].$$

Since φ_c satisfies all the conditions of Theorem 3.6, equation (0.1) has property on account of Theorem 3.7. ■

Let $F \in V(\tau)$, condition (0.3) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le ct^{n-1}$ for $t \ge t_c$ we have inequality* (3.33), *where* (3.26) *and* (3.27) *hold and $\lambda_{ic} \in]0, +\infty[$ $(i = 1, \dots, m)$. Moreover, let*

$$\lambda_c = \sum_{i=1}^{m} \lambda_{ic} > 1,$$
$$\int^{+\infty} \beta^{n-1}(t) [\sigma^*(t)]^{\lambda_c(2-n)} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-2)} d_s r_{ic}(s,t) dt = +\infty, \tag{3.50}$$

where $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$ and $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ is a nondecreasing function satisfying (3.48). *Then equation* (0.1) *has property* .

Proof. It suffices to note that by (0.3), (3.33), (3.50) and Corollary 1.2 all the conditions of Theorem 3.13 are satisfied with $\omega_{ic}(s) = s^{\lambda_{ic}}$. ■

In Theorems 3.14–3.17 below the following condition will be imposed in F:

$$|F(u)(t)| \ge \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_i} d_s r_i(s,t), \quad \text{for } t \ge t_0, \ \ u \in H_{t_0,\tau}, \tag{3.51}$$

where (3.26) and (3.27) hold and $\lambda_i \in]0, +\infty[$ $(i = 1, \dots, m)$.

Let $F \in V(\tau)$, (0.2), (3.51), ((0.3), (3.51)) *hold, n be odd (even), and $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nondecreasing solution of* (0.1) *satisfying* (2.14_0). *Then the condition*

$$\int^{+\infty} t^{n-1} \prod_{i=1}^{m} [r_i(\sigma_i(t), t) - r_i(\tau_i(t), t)] dt = +\infty \tag{3.52}$$

is sufficient for (0.4) *to hold.*

Proof. Assuming that (0.4) is not fulfilled, according to Lemma 1.2 we obtain the contradiction. ■

In a similar simple way one can prove

Let $F \in V(\tau)$, (0.3) and (3.51) be fulfilled and

$$\int^{+\infty} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_i(n-1)} d_s r_i(s,t) dt = +\infty. \tag{3.53}$$

Moreover, let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (0.1) satisfying (2.14_n). Then (0.5) holds.

Let $F \in V(\tau)$, conditions (0.2), (3.51), (3.52) hold and $\sum_{i=1}^{m} \lambda_i = 1$. Moreover, let there exist nondecreasing functions $\psi_i \in C(\mathbb{R}_+; \mathbb{R}_+)$ such that $\psi_i(s) > 0$ for $s > 0$, functions $x \to x^{\lambda_i} \psi_i\big(\frac{1}{x}\big)$ are nondecreasing on $]0, +\infty[$ $(i = 1, \dots, m)$,

$$\int_0^1 \frac{dx}{x \prod_{i=1}^{m} \psi_i\big(\frac{1}{x}\big)} < +\infty, \tag{3.54}$$

and for any $l \in \{1, \dots, n-1\}$ such that $l + n$ is odd we have

$$\int^{+\infty} t^{n-l-1} \Big(\frac{\tau_0(t)}{\sigma^*(t)}\Big)^l \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \frac{s^{\lambda_i l} d_s r_i(s,t)}{\psi_i(s[\tau_0(t)]^l [\sigma^*(t)]^{-l})} dt = +\infty, \tag{3.55}$$

where $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$ and $\tau_0 \in C(\mathbb{R}_+; \mathbb{R}_+)$ is a function satisfying $\tau_0(t) \le \min\{t, \sigma^*(t)\}$ for $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \tau_0(t) = +\infty$. Then equation (0.1) has property .

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (0.1). By Lemma 1.1 there is $l \in \{0, \dots, n-1\}$ such that $l + n$ is odd and (2.14_l) holds. It can be assumed without loss of generality that

$$u^{(i)}(t) > 0 \ \ (i = 0, \dots, l), \ \ (-1)^{i+l} u^{(i)}(t) > 0 \qquad (i = l, \dots, n-1), \ \ t \ge t_0. \tag{3.56}$$

Let $l \in \{1, \dots, n-1\}$. Then by (3.56) there are $c_0 \in]0, +\infty[$ and $t_1 \in [t_0, +\infty[$ such that

$$u(t) \ge c_0 t^{l-1} \quad \text{for} \ \ t \ge t_1. \tag{3.57}$$

Using (3.51) and (3.56), from (0.1) we obtain

$$u^{(n)}(t) + \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} u^{\lambda_i}(s) d_s r_i(s,t) \le 0,$$

which on account of the nondecreasing character of functions ψ_i $(i = 1, \dots, m)$ implies

$$u^{(n)}(t) + \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \frac{u^{\lambda_i}(s)\psi_i\Big(\frac{c_0 s^l}{[\sigma^*(t)]^l u(s)}\Big)}{\psi_i([\sigma^*(t)]^{-l}\tau_0^l(t)s)} d_s r_i(s,t) \le 0 \ \text{ for } \ t \ge t_*,$$

where $t_* \in [t_1, +\infty[$ is sufficiently large. Therefore u satisfies the differential inequality

$$u^{(n)}(t) + c_0\Big(\frac{\tau_0(t)}{\sigma^*(t)}\Big)^l \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \frac{s^{l\lambda_i}\omega_i\Big(\frac{[\sigma^*(t)]^l u(s)}{c_0[\tau_0(t)]^l s^l}\Big)}{\psi_i\Big(\big(\frac{\tau_0(t)}{\sigma^*(t)}\big)^l s\Big)} d_s r_i(s,t) \le 0 \quad (3.58)$$

on $[t_*, +\infty[$ with $\omega_i(x) = x^{\lambda_i}\psi_i\big(\frac{1}{x}\big)$ $(i = 1, \dots, m)$. Following Remark 2.2 and Lemma 2.3 $(\mu = 1)$ the equation

$$v^{(n)}(t) + \Big(\frac{\tau_0(t)}{\sigma^*(t)}\Big)^l \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \frac{s^{l\lambda_i} d_s r_i(s,t)}{\psi_i([\tau_0(t)]^l[\sigma^*(t)]^{-l}s)} \times$$

$$\times \prod_{i=1}^{m} \omega_i(\tau_0^{-l}(t) v(\tau_0(t))) = 0 \quad (3.59)$$

has a solution of type (3.56).

On the other hand, by (3.54), (3.55) and Lemma 1.6 we have

$$\frac{t^{n-l-1}}{l!(n-l)!}\Big(\frac{\tau_0(t)}{\sigma^*(t)}\Big)^l \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \frac{s^{l\lambda_i} d_s r_i(s,t)}{\psi_i([\tau_0(t)]^l[\sigma^*(t)]^{-l}s)} \prod_{i=1}^{m} \omega_i(v(\tau_0(t))) \in M_1(\tau).$$

Therefore according to Lemma 3.1 equation (3.59) has no solution of type (3.56). The obtained contradiction proves that $l \notin \{1, \dots, n-1\}$.

If n is odd and $l = 0$, then condition (0.4) is satisfied by (3.52) and Lemma 3.3. ■

Let $F \in V(\tau)$, $\sigma_i(t) \le t$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$, (0.2) *and* (3.51) *hold, $\sum_{i=1}^m \lambda_i = 1$, $\varepsilon \in]0,1[$ and*

$$\int^{+\infty} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_i(n-1)-\varepsilon} d_s r_i(s,t) dt = +\infty.$$

Then equation (0.1) *has property* .

Similarly to Theorem 3.14 one can prove

Let $F \in V(\tau)$, conditions (0.3), (3.51)–(3.53) *be fulfilled and $\sum_{i=1}^m \lambda_i = 1$. Moreover, let there exist nondereasing functions $\psi_i \in C(\mathbb{R}_+; \mathbb{R}_+)$ such that $\psi_i(s) > 0$ for $s > 0$, functions $x \to x^{\lambda_i}\psi_i\big(\frac{1}{x}\big)$ are nondecreasing on $]0, +\infty[$ $(i = 1, \dots, m)$,* (3.55) *holds and for any $l \in \{1, \dots, n-2\}$ such that $l + n$ is even we have* (3.55), *where $\sigma^*(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$ and $\tau_0 \in C(\mathbb{R}_+; \mathbb{R}_+)$ is a function satisfying*

$\tau_0(t) \le \min\{t, \sigma^*(t)\}$ *for* $t \in \mathbb{R}_+$ *and* $\lim_{t\to+\infty} \tau_0(t) = +\infty$. *Then equation* (0.1) *has property* .

Let $F \in V(\tau)$, $\sigma_i(t) \le t$ *for* $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$ *and conditions* (0.3), (3.51), (3.53) *hold*, $\sum_{i=1}^{m} \lambda_i = 1$ *and*

$$\int^{+\infty} t \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_i(n-2)-\varepsilon} d_s r_i(s,t) dt = +\infty.$$

Then equation (0.1) *has property* .

Remark. One cannot take $\varepsilon = 0$ in Corollaries 3.10 and 3.11 because in that case equation (0.1) does not have, in general, property ( ). In this sense the corresponding theorems are the exact ones.

Let $F \in V(\tau)$, *conditions* (0.2) *and* (3.51) *hold*, $\sum_{i=1}^{m} \lambda_i > 1$ *and let for any* $l \in \{1, \dots, n-1\}$ *such that* $l+n$ *is odd*

$$\int^{+\infty} [t^{n-l-1}[\tau_*(t)]^{1-\varepsilon} + [\beta(t)]^{n-l}] \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_i(l-1)} d_s r_i(s,t) dt = +\infty, \tag{3.60}$$

where $\varepsilon \in]0,1[$, $\tau_*(t) = \min\{t, \tau_i(t) : i = 1, \dots, m\}$ *and* $\beta \in C(\mathbb{R}_+; \mathbb{R}_+)$ *is a nondecreasing function satisfying* $\beta(t) \le \tau_*(t)$ *for* $t \in \mathbb{R}_+$ *and* $\lim_{t\to+\infty} \beta(t) = +\infty$. *Then equation* (0.1) *has property* .

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a proper nonoscillatory solution of (0.1). By Lemma 1.1 there exists $l \in \{0, \dots, n-1\}$ such that $l+n$ is odd and (2.14_l) is fulfilled. It is obvious that condition (3.52) holds by (3.60). Therefore if n is odd and $l = 0$, then (0.4) is satisfied by Lemma 3.3.

Let us now assume that $l \in \{1, \dots, n-1\}$ and

$$\int^{+\infty} t^{n-l-1} \tau_*^{1-\varepsilon}(t) \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_i(l-1)} d_s r_i(s,t) dt = +\infty. \tag{3.61}$$

By virtue of Lemma 2.2 with $\mu = (1-\varepsilon)/\lambda$ the equation

$$v^{(n)}(t) + \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{(l-1)\lambda_i} d_s r_i(s,t) [\tau_*(t)]^{-(1-\varepsilon)(l-1)} \times$$
$$\times |v(\tau_*(t))|^{1-\varepsilon} \operatorname{sign} v(\tau_*(t)) = 0 \tag{3.62}$$

has a solution of type (2.14_l).

On the other hand, by (3.61) and Corollary 1.1 of Lemma 1.6 we have

$$\frac{t^{n-l-1}}{l!(n-l)!} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{(l-1)\lambda_i} d_s r_i(s,t) [\tau_*(t)]^{1-\varepsilon} \times$$
$$\times |v(\tau_*(t))|^{1-\varepsilon} \operatorname{sign} v(\tau_*(t)) \in M_1(\tau).$$

Therefore by Lemma 3.1 equation (3.62) has no solution of type (2.14_l). This contradiction proves that $l \notin \{1,\dots,n-1\}$.

If the condition

$$\int^{+\infty} \beta^{n-l}(t) \prod_{i=1}^{m} \int_{\tau(t)}^{\sigma_i(t)} s^{\lambda_i(l-1)} d_s r_i(s,t) dt = +\infty$$

is fulfilled, then we can prove $l \notin \{1,\dots,n-1\}$ using Lemmas 2.2 and 3.2. ■

Our next theorem is proved similarly.

Let $F \in V(\tau)$, conditions (0.3) and (3.51) hold, $\sum_{i=1}^{m} \lambda_i > 1$, and for any $l \in \{1,\dots,n-2\}$ such that $n+l$ is even let condition (3.60) be fulfilled, where $\varepsilon \in]0,1[$, $\tau_*(t) = \min\{t, \tau_i(t) : i = 1,\dots,m\}$ and $\beta \in C(\mathbb{R}_+;\mathbb{R}_+)$ is a nondecreasing function satisfying $\beta(t) \le \tau_*(t)$ for $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \beta(t) = +\infty$. Then equation (0.1) has property .

§ 4. Necessary and Sufficient Conditions

In this section we shall establish the classes of equations for which the sufficient conditions obtained in 3.2 turn out to be the necessary ones as well.

Let $F, \varphi \in V(\tau)$, $l \in \{0,\dots,n-1\}$, $c_1, c \in]0,+\infty[$, $c_1 < c$ and assume that for any $u \in C(\mathbb{R}_+;\mathbb{R})$, satisfying $c_1 t^l \le |u(t)| \le ct^l$ for $t \ge t_0$ we have

$$|F(u)(t)| \le \varphi(|u|)(t) \quad \text{for} \quad t \ge t_0. \tag{4.1}$$

Moreover, let

$$\int^{+\infty} t^{n-l-1} \varphi(\theta)(t) dt < +\infty, \tag{4.2}$$

where $\theta(s) = cs^l$ for $s \in \mathbb{R}_+$ and

$$\varphi(x)(t) \ge \varphi(y)(t) \ge 0 \ \text{ for } x(s) \ge y(s) \ge 0, \quad s \in [\tau(t), +\infty[, \tag{4.3}$$

Then for any $c_0 \in \mathbb{R}$ satisfying $l!c_1 < |c_0| < l!c$ equation (0.1) has a proper solution $u : [t_*, +\infty[\to \mathbb{R}$ such that

$$\lim_{t\to+\infty} u^{(l)}(t) = c_0. \tag{4.4}$$

Proof. Using (4.2) we can choose $t_* \in [t_0, +\infty[$ such that $\inf\{\tau(t) : t \ge t_*\} \ge t_0$ and

$$\int_{t_*}^{+\infty} t^{n-l-1} \varphi(\theta)(t) dt \le \min\Big\{c - \frac{|c_0|}{l!}, \frac{|c_0|}{l!} - c_1\Big\}. \tag{4.5}$$

Let U be the set of all $u \in C([t_0,+\infty[;\mathbb{R})$ satisfying

$$c_1 t^l \le u(t)\operatorname{sign} c_0 \le c t^l \quad \text{for} \ \ t \ge t_1. \tag{4.6}$$

Define $T : U \to C([t_0,+\infty[;\mathbb{R})$ by

$$T(u)(t) = \begin{cases} \frac{c_0}{l!}t^l + \frac{(-1)^{n-l-1}}{(l-1)!(n-l-1)!}\int_{t_*}^{t}(t-s)^{l-1}\int_s^{+\infty}(\xi-s)^{n-l-1}\times \\ \quad \times F(u)(s)d\xi\; ds \ \ \text{for} \ \ t \ge t_* \\ \frac{c_0}{l!}t^l \quad \ \text{for} \quad t_0 \le t < t_* \end{cases} \tag{4.7}$$

if $l \in \{1,\dots,n-1\}$, and by

$$T(u)(t) = \begin{cases} c_0 + \frac{(-1)^{n+1}}{(l-1)!}\int_t^{+\infty}(s-t)^{n-1}F(u)(s)ds \ \ \text{for} \ \ t \ge t_* \\ T(u)(t_*) \quad \text{for} \quad t_0 \le t < t_* \end{cases} \tag{4.8}$$

if $l = 0$. By virtue of (4.1), (4.3), (4.5)–(4.8) we have $T(U) \subset U$. It is easy to verify that the operator T satisfies all the conditions of Lemma 2.1. Therefore T has a fixed point u which is obviously a solution of (0.1) on $[t_*,+\infty[$ satisfying (4.4). ■

Let $F \in V(\tau)$, condition (0.2) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \le |u(t)| \le c t^{n-1}$ for $t \ge t_c$ we have*

$$\prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(s^{1-n}|u(s)|)d_s r_{ic}(s,t) \le |F(u)(t)| \le$$
$$\le \delta \prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(s^{1-n}|u(s)|)d_s r_{ic}(s,t), \tag{4.9}$$

where $\delta \in [1,+\infty[$, (3.26)–(3.28) *hold and $\sigma_i(t) \le t$ for $t \in \mathbb{R}_+$ $(i = 1,\dots,m)$. Moreover, let* (3.30) *be valid. Then condition* (3.31) *is necessary and sufficient for equation* (0.1) *to have property .*

Proof. Sufficiency. By virtue of (0.2), (3.30), (3.31) and (4.9) the conditions of Theorem 3.9 are obviously satisfied with $\sigma_*(t) = \sigma^*(t) = \max\{\sigma_i(t) : i = 1,\dots,m\} \le t$ for $t \in \mathbb{R}_+$. Therefore according to the same theorem equation (0.1) has property .

Necessity. Assume that equation (0.1) has property and for some c

$$\int^{+\infty}\prod_{i=1}^{m}[r_{ic}(\sigma_i(t),t) - r_{ic}(\tau_i(t),t)]dt < +\infty. \tag{4.10}$$

Conditions (4.1) and (4.2), where $l = n-1$ and

$$\varphi(|u|)(t) = \delta\prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(s^{1-n}|u(s)|)d_s r_{ic}(s,t), \ \ \theta(s) = c s^{n-1}$$

are obviously fulfilled on account of (4.9) and (4.10). Therefore, following Lemma 4.1, there exists $c_0 \neq 0$ such that equation (0.1) has a proper solution $u : [t_*, +\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u^{(n-1)}(t) = c_0$. But this contradicts the fact that equation (0.1) has property . ■

Let $F \in V(\tau)$, condition (0.2) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \leq |u(t)| \leq ct^{n-1}$ for $t \geq t_c$ we have*

$$\prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_{ic}} d_s r_{ic}(s,t) \leq |F(u)(t)| \leq$$
$$\leq \delta \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_{ic}} d_s r_{ic}(s,t), \tag{4.11}$$

where $\delta \in [1, +\infty[$, (3.26) *and* (3.27) *hold, $\lambda_{ic} \in]0,1[$ $(i = 1, \dots, m)$, $\sum_{i=1}^{m} \lambda_{ic} = \lambda < 1$ and $\sigma_i(t) \leq t$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$. Then the condition*

$$\int^{+\infty} \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda_{ic}(n-1)} d_s r_i(s,t) = +\infty \tag{4.12}$$

is necessary and sufficient for equation (0.1) *to have property* .

Proof. Sufficiency. Since in the case under consideration $\sigma_*(t) = \sigma^*(t)$ for $t \in \mathbb{R}_+$, (4.12) coincides with (3.37) and thus the sufficiency follows from Corollary 3.4.

Necessity. Assume that equation (0.1) has property and (4.12) is not fulfilled for some c. Then by (4.1) and Lemma 4.1 there exists $c_0 \neq 0$ such that equation (0.1) has a proper solution $u : [t_*, +\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u^{(n-1)}(t) = c_0$. But this contradicts the fact that equation (0.1) has property . ■

Let $F \in V(\tau)$, condition (0.3) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \leq |u(t)| \leq ct^{n-1}$ for $t \geq t_c$ we have*

$$\prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(s^{2-n}|u(s)|) d_s r_{ic}(s,t) \leq |F(u)(t)| \leq$$
$$\leq \delta \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \omega_{ic}(s^{2-n}|u(s)|) d_s r_{ic}(s,t), \tag{4.13}$$

where $\delta \in [1,+\infty[$, (3.26)–(3.28) *hold and* $\sigma_i(t) \leq t$ *for* $t \in \mathbb{R}_+$ $(i = 1,\dots,m)$. *Moreover, let* (3.30) *be valid. Then the conditions*

$$\int^{+\infty} t\prod_{i=1}^{m}[r_{ic}(\sigma_i(t)) - r_{ic}(\tau_i(t),t)]dt = +\infty, \tag{4.14}$$

$$\int^{+\infty}\prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}\omega_{ic}(s)d_s r_{ic}(s,t)dt = \infty \tag{4.15}$$

are necessary and sufficient for equation (0.1) *to have property* .

Proof. Sufficiency. Since in the case under consideration $\sigma_*(t) = \sigma^*(t)$ for $t \in \mathbb{R}_+$, (4.14), (4.15) coincide with (3.39), (3.40). Therefore the sufficiency follows from Theorem 3.10.

Necessity. Assume that equation (0.1) has property and condition (4.14) ((4.15)) is not fulfilled for some $c > 0$. Then by (4.13) and Lemma 4.1 equation (0.1) has a proper solution $u : [t_*,+\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u^{(n-2)}(t) = c_0$ $(\lim_{t\to+\infty} u^{(n-1)}(t) = c_0)$ where $c_0 \neq 0$. But this contradicts the fact that equation (0.1) has property . ■

Let all the conditions of Corollary 4.1 *be fulfilled except* (0.2) *which is to be replaced by* (0.3). *Then condition* (4.12) *is necessary and sufficient for equation* (0.1) *to have property* .

Proof. Since in the case under consideration $\sigma_*(t) = \sigma^*(t)$ for $t \in \mathbb{R}_+$, the sufficiency follows from Corollary 3.5. Assuming that equation (0.1) has property and (4.12) is not fulfilled, we can show, as while proving Corollary 4.1, that equation (0.1) has a proper solution $u : [t_*,+\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u^{(n-1)}(t) = c_0$, where $c_0 \neq 0$, which is the contradiction. ■

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large* $c > 0$ *there exist* $t_c \in \mathbb{R}_+$ *such that for any* $u \in H_{t_c,\tau}$ *satisfying* $1/c \leq |u(t)| \leq ct^{n-1}$ *for* $t \geq t_c$ *we have*

$$\prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}\omega_{ic}(|u(s)|)d_s r_{ic}(s,t) \leq |F(u)(t)| \leq$$
$$\leq \delta\prod_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}\omega_{ic}(|u(s)|)d_s r_{ic}(s,t) \ \ \text{for} \ \ t \geq t_c, \tag{4.16}$$

where $\delta \in [1,+\infty[$, (3.26)–(3.28) *hold and*

$$\varliminf_{t\to+\infty}\frac{\tau_i(t)}{t} > 0 \quad (i = 1,\dots,m). \tag{4.17}$$

Moreover, let (3.42) *be valid. Then the condition*

$$\int^{+\infty} t^{n-1}\prod_{i=1}^{m}[r_{ic}(\sigma_i(t),t) - r_{ic}(\tau_i(t),t)]dt = +\infty \tag{4.18}$$

is necessary and sufficient for equation (0.1) *to have property* ( ).

Proof. Sufficiency. By (4.17) there exist $\alpha \in]0,+\infty[$ and $t_0 \in \mathbb{R}_+$ such that

$$\tau_i(t) \geq \alpha t \quad \text{for} \quad t \geq t_0 \quad (i = 1, \dots, m). \tag{4.19}$$

It is obvious that condition (3.43), where $\beta(t) = \alpha t$, is fulfilled by virtue of (4.18) and (4.19). Therefore all the conditions of Theorem 3.11 are satisfied by (0.2) ((0.3)) and (4.16)–(4.18), thereby implying the sufficiency of (4.18).

Necessity. Assume that equation (0.1) has property ( ) and (4.18) is not fulfilled for some $c > 0$. Then it can be shown by (4.16) and Lemma 4.1 that equation (0.1) has a proper solution $u : [t_*, +\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u(t) = c_0$, where $c_0 \neq 0$. But this contradicts the fact that (0.1) has property ( ). ■

Let $F \in V(\tau)$, condition (0.2) ((0.3)) *be fulfilled, and let for any sufficiently large $c > 0$ there exist $t_c \in \mathbb{R}_+$ such that for any $u \in H_{t_c,\tau}$ satisfying $1/c \leq |u(t)| \leq ct^{n-1}$ for $t \geq t_c$ we have inequality* (4.11)*, where $\delta \in [1, +\infty[$,* (3.26) *and* (3.27) *hold, $\lambda_{ic} \in]0, +\infty[$ $(i = 1, \dots, m)$ and $\sum_{i=1}^{m} \lambda_{ic} > 1$. Then condition* (4.18) *is necessary and sufficient for equation* (0.1) *to have property* ( ).

Proof. The sufficiency follows from Corollary 3.6. The necessity can be proved similarly to Theorem 4.3. ■

Let $F \in V(\tau)$, condition (0.2) ((0.3)) *be fulfilled and let there exist $t_0 \in \mathbb{R}_+$ such that for any $u \in H_{t_0,\tau}$ we have*

$$p(t) \int_{\tau_1(t)}^{\sigma_1(t)} |u(s)|^{\lambda} ds \leq |F(u)(t)| \leq$$
$$\leq \delta p(t) \int_{\tau_1(t)}^{\sigma_1(t)} |u(s)|^{\lambda} ds \;\; for \;\; t \geq t_0, \tag{4.20}$$

where $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, $\lambda > 1$, $\delta \in [1, +\infty[$, $\tau_1, \sigma_1 \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\tau_1(t) \leq \sigma_1(t)$ for $t \in \mathbb{R}_+$, $\lim_{t\to+\infty} \tau_1(t) = +\infty$ and

$$\overline{\lim_{t\to+\infty}} \frac{\tau_1(t)}{\sigma_1(t)} < 1, \quad \underline{\lim_{t\to+\infty}} \frac{\sigma_1(t)}{t} > 0. \tag{4.21}$$

Then the condition

$$\int^{+\infty} \sigma_1(t) t^{n-1} p(t) dt = +\infty \tag{4.22}$$

is necessary and sufficient for equation (0.1) *to have property* ( ).

Proof. Sufficiency. By (4.21) there exist $\alpha \in]0, +\infty[$ and $t_1 \in \mathbb{R}_+$ such that $\sigma_1(t) \geq \alpha t$ for $t \geq t_1$. Therefore by (4.20) and (4.21) condition (3.45) holds with $m = 1$, $r_{1c}(s,t) = p(t)s$ and $\beta(t) = \alpha t$. Due to Corollary 3.6 we easily ascertain that (4.22) is sufficient for equation (0.1) to have property ( ).

Necessity. Assume that equation (0.1) has property ( ) and (4.22) is not fulfilled. Then by Lemma 4.1 and (4.20), (4.21) we find that for any $c \neq 0$ equation (0.1) has a proper solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying $\lim_{t\to+\infty} u(t) = c$. But this contradicts the fact that (0.1) has property ( ). ■

CHAPTER II

In this chapter we derive sufficient conditions for functional differential equations with a linear minorant to have property or . The results obtained are new not only for equations of form (0.1) but can be regarded as improving to a certain extent the well-known earlier results for linear ordinary differential equations [11, 12].

§ 5. Linear Differential Inequalities with a Deviating Argument

Let us consider linear differential inequalities with a deviating argument

$$u^{(n)}(t)\,\mathrm{sign}\,u(\tau(t)) + p(t)|u(\tau(t))| \le 0 \tag{5.1}$$

and

$$u^{(n)}(t)\,\mathrm{sign}\,u(\tau(t)) - p(t)|u(\tau(t))| \ge 0, \tag{5.2}$$

where $n \ge 2$, $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\tau \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\lim_{t\to+\infty}\tau(t) = +\infty$. Throughout this section it will be assumed that the following condition is fulfilled:

$$\int_0^{+\infty} \tau_0^{n-1}(t)p(t)dt = +\infty, \tag{5.3}$$

where

$$\tau_0(t) = \min\{t, \tau(t)\}. \tag{5.4}$$

Let $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ and $\int_t^{+\infty} p(s)ds > 0$ for $t \in R_+$. Then for the equation

$$u^{(n)}(t) - p(t)u(t) = 0 \tag{5.5}$$

to have a solution satisfying (2.14_{n-2}) it is necessary and sufficient that the equation

$$u^{(n)}(t) - (-1)^n p(t)u(t) = 0 \tag{5.6}$$

have a solution satisfying (2.14_2).

The validity of Lemma 5.1 is proved similarly to that of Lemma 1.4 [11].

Let $n \ge 4$ and equation (5.5) have a solution satisfying (2.14_l) where $l \in \{2,\dots,n-2\}$ and $l+n$ is even. Then it has a solution satisfying (2.14_{n-2}).

Proof. Let equation (5.5) have a solution satisfying (2.14_l), where $l \in \{2, \dots, n-2\}$ and $l+n$ is even. Assume that $l \in \{3, \dots, n-4\}$[4]. Then by Lemma 1.3 and (2.14_l) we have

$$|u^{(l-2)}(t)| \geq |u^{(l-2)}(t_*)| + \frac{1}{(n-3)!} \int_{t_*}^{t} (t-s) \times$$
$$\times \int_{s}^{+\infty} (\xi - s)^{n-3} p(\xi) |u^{(l-2)}(\xi)| d\xi \, ds \quad \text{for} \quad t \in [t_*, +\infty[,$$

where t_* sufficiently large. Hence applying Lemmaa 2.1 it is easy to show that there exists a continuous function $v : [t_*, +\infty[\to R$ such that

$$v(t) = |u^{(l-2)}(t_*)| + \frac{1}{(n-3)!} \int_{t_*}^{t} (t-s) \int_{s}^{+\infty} (\xi - s)^{n-3} p(\xi) v(\xi) d\xi \, ds,$$
$$|u^{(l-2)}(t_*)| \leq v(t) \leq |u^{(l-2)}(t)| \quad \text{for} \quad t \in [t_*, +\infty[.$$

It is clear that v is a solution of equation (5.6) satisfying (2.14_2). Thus by Lemma 5.1 equation (5.5) has a solution satisfying (2.14_{n-2}). ■

Let $\tau(t) \leq t$ for $t \in R_+$. For the differential inequality (5.1) *to have property it is necessary and sufficient that it have no solution satisfying* (2.14_{n-1}).

Proof. Since the necessity is obvious, we shall prove the sufficiency. Let (5.1) have no property and $u_0(t)$ be its nonoscillatory proper solution. By Lemma 1.2 there exists $l \in \{0, \dots, n-1\}$ such that $l+n$ is odd and (2.14_l) is fulfilled. When n is odd and $l = 0$, (0.4) holds by (5.3), (5.4). Assume that $l \in \{1, \dots, n-3\}$. Then by Remark 2.2 the equation

$$u^{(n)}(t) + q(t)u(t) = 0 \tag{5.7}$$

has a solution satisfying (2.14_l) where

$$q(t) = p(t) \frac{u_0(\tau(t))}{u_0(t)}.$$

Therefore by [9, Lemmas 1.3 and 1.5] equation (5.7) has a solution $u_1(t)$ satisfying (2.14_{n-1}). Thus there exists $t_* \in R_+$ such that on the interval $[t_*, +\infty[$ $u_1(t)$ is a solution of the equation

$$u^{(n)}(t) + q_1(t)u(\tau(t)) = 0,$$

where

$$q_1(t) = p(t) \, \frac{u_0(\tau(t))u_1(t)}{u_0(t)u_1(\tau(t))}.$$

On the other hand, by Lemma 1.3 we have

$$\frac{u_0(t)}{t^l} \downarrow \quad \text{and} \quad \frac{u_1(t)}{t^l} \uparrow \quad \text{as} \quad t \uparrow +\infty.$$

[4] for $l = 2$ the validity of this corollary follows from Lemma 5.1.

Thus it is clear that there exists $t^* \in [t_*, +\infty[$ such that

$$\frac{u_0(\tau(t))u_1(t)}{u_0(t)u_1(\tau(t))} \geq 1 \quad \text{for} \quad t \in [t^*, +\infty[.$$

Therefore on the interval $[t^*, +\infty[$ differential inequality (5.1) has a solution satisfying (2.14_{n-1}). The obtained contradiction proves the sufficiency. ■

Let $\tau(t) \leq t$ for $t \in R_+$. For (5.2) *to have property it is necessary and sufficient when n is even* (*when n is odd*) *that it have no solution satisfying* (2.14_{n-2}) ((2.14_1) *and* (2.14_{n-2})).

Proof. The necessity is obvious. By virtue of Corollary 5.1 we shall can prove the sufficiency likewise to Lemma 5.2. ■

Similarly to Lemmas 5.2 and 5.3 one can prove

Let $\tau(t) \geq t$ for $t \in R_+$. Then for (5.1) *to have property it is necessary and sufficient when n is even* (*when n is odd*) *that it have no solution saatisfying* (2.14_1) ((2.14_2) *and* (2.14_{n-1})).

Let $\tau(t) \geq t$ for $t \in R_+$. Then for (5.2) *to have property it is necessary and sufficient when n is even* (n *is odd*) *that it have no solution satisfying* (2.14_2) ((2.14_1)).

Denote

$$\tau_*(t) = \inf\{\tau_0(s) : s \geq t\}, \quad \eta_1(t) = \max\{s : \tau_*(s) \leq t\},$$
$$\eta_i(t) = \eta_1(\eta_{i-1}(t)) \quad (i = 2, 3, \dots).$$

Let $l \in \{1, \dots, n-1\}$, $l+n$ be odd ($l+n$ *be even*) *and $u : [t_0, +\infty[\to R$ be a nonoscillatory proper solution of* (5.1) ((5.2)) *satisfying* (2.14_l). *Then there exists $t_1 \in [t_0, +\infty[$ such that for any $k \in \mathbb{N}$ we have*

$$|u^{(l)}(s)| \geq \exp\left\{\frac{1}{l!(n-l-1)!}\int_s^t p(\xi)(\xi - s)^{n-l-1}\tau_*^l(\xi) \times \right.$$
$$\left. \times \varphi_{lk}(\xi, t_1)d\xi\right\} u_l(t,s) \quad \textit{for} \quad t \geq s \geq \eta_k(t_1), \tag{5.8}$$

where

$$u_l(t,s) = \sum_{i=l}^{n-1} \frac{(t-s)^{i-l}}{(i-l)!}|u^{(i)}(t)|, \quad \varphi_{l1}(t, t_1) = 0,$$
$$\varphi_{li}(t,t_1) = \exp\left\{\frac{1}{l!(n-l-1)!}\int_{\tau_*(t)}^t p(\xi)(\xi - \tau_*(t))^{n-l-1} \times \right. \tag{5.9$_l$}$$
$$\left. \times \tau_*(\xi)\tau^{l-1}(\xi)\varphi_{l\ i-1}(\xi, t_1)d\xi\right\} \quad \textit{for} \quad t \geq \eta_i(t_1) \quad (i = 2, \dots, k).$$

Proof. It can be assumed without loss of generality that $u(t) > 0$ for $t \in [t_0, +\infty[$. Since

$$\frac{du_l(t,s)}{dt} = -\frac{(t-s)^{n-l-1}}{(n-l-1)!}|u^{(n)}(t)| \text{ and } u_l(s,s) = u^{(l)}(s),$$

we obtain

$$u^{(l)}(s) = \exp\Big\{\int_s^t \frac{(\xi-s)^{n-l-1}}{(n-l-1)!}\frac{|u^{(n)}(\xi)|}{u_l(\xi,s)}d\xi\Big\}u_l(t,s) \text{ for } t \geq t_1 \quad (5.10)$$

and

$$u^{(l)}(\tau_*(t)) = \exp\Big\{\int_{\tau_*(t)}^t \frac{(\xi-\tau_*(t))^{n-l-1}}{(n-l-1)!}\frac{|u^{(n)}(\xi)|}{u_l(\xi,\tau_*(t))}d\xi\Big\}u_l(t,\tau_*(t)) \text{ for } t \geq t_1,$$

where $t_1 = \eta_1(t_0)$. Keeping in mind that $u_l(\xi,\cdot)$ is a nonincreasing function, the latter equality gives us

$$u^{(l)}(\tau_*(t)) \geq \exp\Big\{\int_{\tau_*(t)}^t \frac{(\xi-\tau_*(t))^{n-l-1}}{(n-l-1)!}\frac{|u^{(n)}(\xi)|}{u_l(\xi,\tau_*(\xi))}d\xi\Big\}u_l(t,\tau_*(t)) \quad \text{for } t \in [t_1,+\infty[.$$

Hence by virtue of (5.10) and (5.1) ((5.2)) we immediately obtain (5.8) where the function φ_{lk} is defined by means of (5.9_l). ■

Let $l \in \{1,\dots,n-1\}$, $l+n$ be odd ($l+n$ be even), and $u : [t_0,+\infty[\to R$ be a nonoscillatory proper solution of (5.1) ((5.2)) *satisfying* (2.14_l). *Then there exists $t_1 \in [t_0,+\infty[$ such that for any $k \in \mathbb{N}$ we have*

$$|u^{(l-1)}(t)| \geq \Big(t + \frac{1}{(n-l-1)!}\int_{\eta_{k+1}(t_1)}^t s^{n-l}\psi_{lk}(s,t_1)p(s)ds\Big)|u^{(l)}(t)| \quad (5.11_l)$$
$$\text{for } t \geq \eta_{k+1}(t_1),$$

where for $l = n-1$ the function $\psi_{n-1\,k}$ is defined by

$$\psi_{n-1\,1}(t,t_1) = 0, \;\; \psi_{n-1\,i}(t,t_1) = \frac{1}{(n-2)!}\int_{\eta_i(t_1)}^{\tau_*(t)}(\tau_*(t)-s)^{n-2} \times$$
$$\times \exp\Big\{\int_s^t p(\xi)\psi_{n-1\,i-1}(\xi,t_1)d\xi\Big\}ds \quad (5.12)$$
$$\text{for } t \geq \eta_{i+1}(t_1) \quad (i = 2,\dots,k),$$

while for $l < n-1$ the function ψ_{lk} is defined by

$$\psi_{l1}(t,t_1)=0,\ \ \psi_{li}(t,t_1)=\frac{1}{(l-1)!}\int_{\eta_i(t_1)}^{\tau_*(t)}(\tau_*(t)-s)^{l-1}\times$$
$$\times\exp\Big\{\frac{1}{(n-l-2)!}\int_s^t\psi_{l\ i-1}(\xi,t_1)\tau_*^{1-l}(\xi)\int_\xi^{+\infty}(\xi_1-\xi)^{n-l-2}\times$$
$$\times\tau^{l-1}(\xi_1)p(\xi_1)d\xi_1\ d\xi\Big\}ds\ \ for\ \ t\geq\eta_{i+1}(t)\ \ (i=2,\dots,k). \quad (5.13_l)$$

Proof. It can be assumed without loss of generality that $u(t)>0$ for $t\in[t_0,+\infty[$. Then the equality

$$u^{(l)}(t)=u^{(l)}(t_1)\exp\Big\{-\int_{t_1}^t\frac{|u^{(l+1)}(s)|}{u^{(l)}(s)}ds\Big\}, \quad (5.14)$$

where $t_1\in[t_0,+\infty[$ is sufficiently large, implies

$$u(t)\geq\frac{u^{(l)}(t_1)}{(l-1)!}\int_{t_1}^t(t-s)^{l-1}\exp\Big\{-\int_{t_1}^s\frac{|u^{(l+1)}(\xi)|}{u^{(l)}(\xi)}d\xi\Big\}ds \quad (5.15)$$
$$\text{for}\quad t\in[t_1,+\infty[.$$

Consider the case $l=n-1$. By (5.14), (5.15) and (5.1) we obtain

$$\frac{u(\tau_*(t))}{u^{(n-1)}(t)}\geq\frac{1}{(n-2)!}\int_{t_1}^{\tau_*(t)}(\tau_*(t)-s)^{n-2}\times$$
$$\times\exp\Big\{\int_s^t p(\xi)\frac{u(\tau_*(\xi))}{u^{(n-1)}(\xi)}d\xi\Big\}ds\quad\text{for}\quad t\geq\eta(t_1),$$

from which it follows

$$u(\tau_*(t))\geq\psi_{n-1\ k}(t,t_1)u^{(n-1)}(t)\ \ \text{for}\ \ t\geq\eta_k(t_1). \quad (5.16)$$

On the other hand, from $(1.14_{n-2\ n})$ we obtain

$$u^{(n-2)}(t)\geq tu^{(n-1)}(t)+\int_{\eta_{k+1}(t_1)}^t sp(s)u(\tau_*(s))ds\ \ \text{for}\ \ t\in[\eta_{k+1}(t_1),+\infty[.$$

Therefore (5.16) implies that inequality (5.11_{n-1}) is valid.

Now consider the case $l\in\{1,\dots,n-2\}$. By (5.14), (5.15)

$$\frac{u(\tau_*(t))}{u^{(l)}(t)}\geq\frac{1}{(l-1)!}\int_{t_1}^{\tau_*(t)}(\tau_*(t)-s)^{l-1}\times$$
$$\times\exp\Big\{\int_s^t\frac{|u^{(l+1)}(\xi)|}{u^{(l)}(\xi)}d\xi\Big\}ds\ \ \text{for}\ \ t\geq\eta(t_1).$$

Taking into account (1.9) and the fact that $u(t)/t^{l-1}$ is a nondecreasing function, from the latter inequality we have

$$\frac{u(\tau_*(t))}{u^{(l)}(t)} \geq \frac{1}{(l-1)!}\int_{t_1}^{\tau_*(t)}(\tau_*(t)-s)^{l-1}\exp\Big\{\frac{1}{(n-l-2)!}\int_s^t \frac{u(\tau_*(\xi))}{u^{(l)}(\xi)} \times$$
$$\times\tau_*^{1-l}(\xi)\int_\xi^{+\infty}(\xi_1-\xi)^{n-l-2}\tau^{l-1}(\xi_1)p(\xi_1)d\xi_1\, d\xi\Big\}ds \ \text{ for } \ t \geq \eta(t_1).$$

So that

$$u(\tau_*(t)) \geq \psi_{lk}(t,t_1)u^{(l)}(t) \quad \text{for} \quad t \geq \eta_k(t_1), \tag{5.17}$$

where the function $\psi_{lk}(t,t_1)$ is defined by (5.13_l).

On the other hand, according to $(1.14_{l-1\ n})$ and (2.14_l) we have

$$u^{(l-1)}(t) \geq tu^{(l)}(t) + \frac{1}{(n-l-1)!}\int_{\eta_{k+1}(t_1)}^t s^{n-l}p(s)u(\tau_*(s))ds$$
$$\text{for} \quad t \geq \eta_{k+1}(t_1).$$

which by virtue of (5.17) implies that inequality (5.11_l) is valid. ■

For (5.1) ((5.2)) *not to have a solution satisfying* (2.14_l) *where* $l \in \{1,\dots,n-1\}$ *aand* $l+n$ *is odd* ($l+n$ *is even*), *it is sufficient that for some* $k_0 \in \mathbb{N}$

$$\varlimsup_{t\to+\infty}\Big[\int_{\tau_*(t)}^t (s-\tau_*(t))^{n-l-1}\tau_*^l(s)\widetilde{\varphi}_{lk_0}(s,t,0)p(s)ds +$$
$$+\widetilde{\psi}_{lk_0}(t,0)\int_t^{+\infty}(s-\tau_*(t))^{n-l-1}\tau^{l-1}(s)p(s)ds\Big] > l!(n-l-1)!, \tag{5.18}$$

where

$$\widetilde{\varphi}_{lk_0}(s,t,t_1) = \exp\Big\{\frac{1}{l!(n-l-1)!}\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)(\xi-\tau_*(s))^{n-l-1} \times$$
$$\times\tau_*^l(\xi)\varphi_{lk_0}(\xi,t_1)d\xi\Big\}ds, \tag{5.19_l}$$

$$\widetilde{\psi}_{lk_0}(t,t_1) = \tau_*(t) + \frac{1}{(n-l-1)!}\int_{\eta_{k_0+1}(t_1)}^{\tau_*(t)} s^{n-l}\psi_{lk_0}(s,t_1)p(s)ds, \tag{5.20_l}$$

$\varphi_{lk_0}(t,t_1)$ *is defined by* (5.9_l), *while* $\psi_{lk_0}(t,t_1)$ *is given by* (5.12) *and* (5.13_l).

Proof. Assume the contrary. Let (5.1) ((5.2)) have a solution $u : [t_0,+\infty[\to R$ satisfying (2.14_l) where $l \in \{1,\dots,n-1\}$ and $l+n$ is odd ($l+n$ is even). Since condition (1.10) is fulfilled due to (5.3) ((5.4)), by Lemmas 1.3, 5.6 and 5.7 there exists $t_1 \in [t_0,+\infty[$ such that

$$|u(\tau(t))| \geq \frac{\tau^{l-1}(t)}{l!}|u^{(l-1)}(\tau(t))| \ \text{ for } \ t \in [t_1,+\infty[, \tag{5.21}$$

$$|u^{(l-1)}(\tau(t))| \geq |u^{(l-1)}(\tau_*(t))| \geq \tau_*(t)|u^{(l)}(\tau_*(t))| \ \text{ for } \ t \geq t_1, \tag{5.22}$$

$$|u^{(l)}(\tau_*(s))| \geq \widetilde{\varphi}_{lk_0}(s,t,t_1)|u^{(l)}(\tau_*(t))| \ \text{ for } \ t \geq \eta_{k_0+1}(t_1), \tag{5.23}$$

$$|u^{(l-1)}(\tau_*(t))| \geq \widetilde{\psi}_{lk_0}(t,t_1)|u^{(l)}(\tau_*(t))| \ \text{ for } \ t \geq \eta_{k_0+1}(t_1). \tag{5.24}$$

By virtue of (5.18), (5.19_l) and (5.20_l) it is clear that

$$\varlimsup_{t\to+\infty}\Big[\int_{\tau_*(t)}^{t}(s-\tau_*(t))^{n-l-1}\tau_*^l(s)p(s)\widetilde{\varphi}_{lk_0}(s,t,t_1)ds+\widetilde{\psi}_{lk_0}(t,t_1)\times$$
$$\times\int_t^{+\infty}(s-\tau_*(t))^{n-l-1}\tau^{l-1}(s)p(s)ds\Big] > l!(n-l-1)!. \tag{5.25}$$

On the other hand, by (2.14_l), (5.21)–(5.24) we obtain from (5.1) ((5.2))

$$|u^{(l)}(\tau_*(t))| \geq \frac{1}{(n-l-1)!}\Big(\int_{\tau_*(t)}^{t}(s-\tau_*(t))^{n-l-1}p(s)|u(\tau(s))|ds+$$
$$+\int_t^{+\infty}(s-\tau_*(t))^{n-l-1}p(s)|u(\tau(s))|ds\Big) \geq \frac{|u^{(l)}(\tau_*(t))|}{l!(n-l-1)!}\times$$
$$\times\Big(\int_{\tau_*(t)}^{t}(s-\tau_*(t))^{n-l-1}\tau_*^l(s)p(s)\widetilde{\varphi}_{lk_0}(s,t,t_1)ds+$$
$$+\widetilde{\psi}_{lk_0}(t,t_1)\int_t^{+\infty}(s-\tau_*(t))^{n-l-1}\tau^{l-1}(s)p(s)ds\Big)$$

which contradicts (5.25). The obtained contradiction proves the validity of the lemma. ■

For (5.1) ((5.2)) not to have a proper solution satisfying (2.14_l) where $l \in \{1,\dots,n-1\}$ and $l+n$ is odd ($l+n$ is even), it is sufficient that

$$\varlimsup_{t\to+\infty}\int_{\tau_*(t)}^{t}\tau_*^l(s)(s-\tau_*(t))^{n-l-1}p(s)\exp\Big\{\frac{1}{l!(n-l-1)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)}p(s)(\xi-\tau_*(s))^{n-l-1}\tau_*^l(\xi)d\xi\Big\}ds > l!(n-l-1)!\,.$$

For (5.1) ((5.2)) not to have a proper solution satisfying (2.14_l) where $l \in \{1,\dots,n-1\}$ and $l+n$ is odd ($l+n$ is even), it is sufficient that

$$\varlimsup_{t\to+\infty}\Big(\tau_*(t)+\frac{1}{l!(n-l-1)!}\int_0^{\tau_*(t)}s^{n-l}\tau_*^l(s)p(s)ds\Big)\times$$
$$\times\int_t^{+\infty}(s-\tau_*(t))^{n-l-1}\tau^{l-1}(s)p(s)ds > l!(n-l-1)!\,.$$

Let $l \in \{1, \dots, n-1\}$, $l+n$ be odd ($l+n$ be even), $\tau(t) \le t$ for $t \in R_+$ and

$$\varlimsup_{t\to+\infty} \int_{\tau_*(t)}^{t} p(s)(s-\tau_*(t))^{n-l-1}\tau_*^l(s)ds > \frac{l!(n-l-1)!}{e}. \qquad (5.26)$$

Then (5.1) ((5.2)) *has no proper solution satisfying* (2.14_l).

Proof. To prove the lemma it suffices to show that the conditions of Lemma 5.8 are fulfilled. By (5.26) there exist $t_1 \in R_+$ and $c \in \left]\frac{l!(n-l-1)!}{e}, +\infty\right[$ such that

$$\int_{\tau_*(t)}^{t} p(s)(s-\tau_*(t))^{n-l-1}\tau_*^l(s)ds \ge c \ \text{ for } \ t \ge t_1. \qquad (5.27)$$

Choose $k_0 \in \mathbb{N}$ such that

$$\left(\frac{ec}{l!(n-l-1)!}\right)^{k_0} > [(n-l-1)!\, l!]^2 \frac{4}{c^2}. \qquad (5.28)$$

By virtue of (5.9_l), (5.27) we have

$$\varphi_{li}(t,t_1) \ge \left(\frac{ec}{l!(n-l-1)!}\right)^{i} \ \text{ for } \ t \ge \eta_i(t_1) \ \ (i=1,\dots,k_0).$$

Therefore

$$\int_{\tau_*(t)}^{t} \tau_*^l(s)s^{n-l-1}p(s)\exp\left\{\frac{1}{l!(n-l-1)!}\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)(\xi-\tau_*(s))^{n-l-1}\tau_*^l(s)\,\times\right.$$
$$\left.\times\varphi_{lk_0}(\xi,t_1)d\xi\right\}ds \ge \left[\frac{ec}{l!(n-l-1)!}\right]^{k_0}\frac{1}{l!(n-l-1)!}\int_{\tau_*(t)}^{t}\tau_*^l(s)s^{n-l-1}\,\times$$
$$\times p(s)\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)\tau_*^l(\xi)(\xi-\tau_*(s))^{n-l-1}d\xi\, ds \ \text{ for } \ t \ge \eta_{k_0}(t_1). \quad (5.29)$$

On the other hand, by (5.27) for any $t \ge \eta_{k_0}(t_1)$ there is $t_* \in [\tau_*(t), t]$ such that

$$\int_{\tau_*(t)}^{t_*} p(s)(s-\tau_*(t))^{n-l-1}\tau_*^l(s)ds = \frac{c}{2},$$
$$\int_{\tau_*(t_*)}^{\tau_*(t)} p(s)(s-\tau_*(t))^{n-l-1}\tau_*^l(s)ds \ge \frac{c}{2}.$$

Now using (5.28) and (5.29), we obtain

$$\int_{\tau_*(t)}^{t} \tau_*^l(s)(s-\tau_*(t))^{n-l-1}p(s)\exp\Big\{\frac{1}{l!(n-l-1)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)(\xi-\tau_*(s))^{n-l-1}\tau_*^l(\xi)\varphi_{lk_0}(\xi,t_1)d\xi\Big\}ds\geq$$
$$\geq\Big[\frac{ec}{l!(n-l-1)!}\Big]^{k_0}\frac{c^2}{4l!(n-l-1)!}>l!(n-l-1)! \text{ for } t\geq\eta_{k_0}(t_1).$$

Therefore (5.18) is fulfilled. ■

Remark 5.1. One cannot replace (5.26) by

$$\varliminf_{t\to+\infty}\int_{\tau_*(t)}^{t} p(s)\tau_*^l(s)(s-\tau_*(t))^{n-l-1}ds>\frac{l!(n-l-1)!}{e}-\varepsilon, \quad (5.30)$$

where ε is an arbitrarily small positive number.

Indeed, let $\varepsilon\in\Big]0,\frac{l!(n-l-1)!}{e}\Big[$. Choose $\beta\in[l-1,l]$ such that

$$|\beta(\beta-1)\cdots(\beta+1-l)(1-\gamma_l(\beta-l))(\beta-l-1)\cdots(\beta+1-n)|>l!(n-l-1)!-\varepsilon e,$$

where

$$\gamma_l=\begin{cases}0 & \text{for } \ l=n-1\\ \sum_{i=1}^{n-l-1}\frac{1}{n-l-i} & \text{for } \ l<n-1.\end{cases}$$

Clearly the equation

$$u^{(n)}(t)+p(t)u(\tau(t))=0 \quad (u^{(n)}(t)-p(t)u(\tau(t))=0),^5$$

where $\tau(t)=\alpha t$, $\alpha=e^{\frac{1}{\beta-l}}$, $p(t)=|\beta(\beta-1)\dots(\beta-(l-1))(\beta-l)\dots(\beta+1-n)|\alpha^{-\beta}t^{-n}$, has a solution $u(t)=t^\beta$ and, moreover, condition (5.30) is fulfilled.

Let $l\in\{1,\dots,n-1\}$, $l+n$ be odd ($l+n$ be even), $\sigma(t)\leq t$ for $t\in R_+$ and

$$\varliminf_{t\to+\infty}\int_{\tau_*(t)}^{t} p(s)(s-\tau_*(t))^{n-l-1}\tau_*^l(s)ds=c>0.^6 \quad (5.31)$$

[5]it is assumed that $l+n$ is odd ($l+n$ is even).

Then for (5.1) ((5.2)) *not to have a proper solution satisfying* (2.14_l), *it is sufficient that*

$$\varlimsup_{t\to+\infty}\int_{\tau_*(t)}^{t}\tau_*^l(s)(s-\tau_*(t))^{n-l-1}p(s)\exp\Big\{\frac{x_0}{l!(n-l-1)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)}p(\xi)(\xi-\tau_*(s))^{n-l-1}\tau_*^l(\xi)d\xi\Big\}ds>l!(n-l-1)!, \qquad (5.32)$$

where x_0 *is the smallest root of the equation* $\exp\Big\{\frac{c}{l!(n-l-1)!}x\Big\}=x$.

Proof. By (5.32) there exists $\varepsilon\in]0,x_0[$ such that

$$\varlimsup_{t\to+\infty}\int_{\tau_*(t)}^{t}\tau_*^l(s)(s-\tau_*(t))^{n-l-1}p(s)\exp\Big\{\frac{x_0-\varepsilon}{l!(n-l-1)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)}p(\xi)(\xi-\tau_*(s))^{n-l-1}\tau_*^l(\xi)d\xi\Big\}ds>l!(n-l-1)!.$$

Therefore by virtue of Lemma 5.8 it suffices for us to show that there exists $k_0\in\mathbb{N}$ such that

$$\varliminf_{t\to+\infty}\varphi_{lk_0}(t,0)>x_0-\varepsilon. \qquad (5.33)$$

By (5.29) there are numbers $c^*\in]0,c]$ and $t_0\in R_+$ such that

$$\int_{\tau_*(t)}^{t}\tau_*^l(s)(s-\tau_*(t))^{n-l-1}p(s)ds\geq c^* \ \text{ for } \ t\geq t_0,\ \ x_0^*>x_0-\varepsilon, \quad (5.34)$$

where x_0^* is the smallest root of the equation

$$\exp\Big\{\frac{c^*x}{l!(n-l-1)!}\Big\}=x. \qquad (5.35)$$

According to (5.9_l) and (5.34) we have $\varphi_{li}(t,0)\geq\alpha_i$ for $t\geq\eta_i(t_0)$ ($i=1,2,\dots$), where $\alpha_1=0$, $\alpha_i=\exp\{\frac{c^*\alpha_{i-1}}{l!(n-l-1)!}\}$ ($i=2,3,\dots$).

Denote $x^*=\lim_{i\to+\infty}\alpha_i$. Since x^* is a solution of equation (5.35), there exists $k_0\in\mathbb{N}$ such that (5.33) is fulfilled. ■

Assume that

$$\varliminf_{t\to+\infty}\tau_*(t)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds>\beta>0, \qquad (5.36)$$

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[6] it is assumed that $c\leq\frac{l!(n-l-1)!}{e}$. Otherwise condition (5.26) is fulfilled and condition (5.32) becomes unnecessary.

there exists $\varepsilon > 0$ *such that*

$$\varliminf_{t\to+\infty} \tau_*^{\lambda-(n-1)}(t)\int_{\eta_1(1)}^{\tau_*(t)} \frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda}(s)}ds > \frac{\lambda(n-2)!}{\beta}+\varepsilon \tag{5.37}$$
$$\text{for all } \lambda\in\Big[\frac{\beta}{(n-1)!},1\Big]$$

and for some natural m

$$\varlimsup_{t\to+\infty}\Big(\tau_*(t)+\frac{1}{(n-1)!}\int_0^{\tau_*(t)} p(s)s\tau_*^m(s)ds\Big)\times$$
$$\times\int_t^{+\infty}\tau^{n-2}(s)p(s)ds > (n-1)!. \tag{5.38}$$

Then (5.1) *has no proper solution satisfying* (2.14_{n-1}).

Proof. By Lemma 5.8 it suffices to show that there is a number $k_0\in\mathbb{N}$ such that

$$\psi_{n-1\ k_0}(t,0)\ge\tau_*^m(t) \quad \text{for} \quad t\ge t^*, \tag{5.39}$$

where t^* is sufficiently large.

Put

$$\varliminf_{t\to+\infty}\frac{\psi_{n-1\ i}(t,0)}{\tau_*^{n-2}(t)}\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds=\lambda_i \;\; (i=1,2,\dots),$$
$$\lim_{i\to+\infty}\lambda_i=\lambda^*. \tag{5.40}$$

By (5.12) and (5.36) we readily find that $\lambda^*\ge\frac{\beta}{(n-1)!}$. Show that $\lambda^*>1$. Assume the contrary, i.e. $\lambda^*\le 1$. By (5.36), (5.37) and (5.40) there exist $t_0\in[\eta(1),+\infty[$, $\varepsilon_0\in]0,\lambda^*[$ and $k\in\mathbb{N}$ such that

$$\varliminf_{t\to+\infty}\tau_*^{\lambda^*-(n-1)-\varepsilon_0}(t)\int_{\eta_1(1)}^{\tau_*(t)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds>\frac{\lambda^*(n-2)!}{\beta}, \tag{5.41}$$
$$\tau_*(t)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds>\beta \quad \text{for} \quad t\ge t_0,$$
$$\psi_{n-1\ i}(t,0)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds\ge(\lambda^*-\varepsilon_0)\tau_*^{n-2}(t)$$
$$(i=k,k+1,\dots) \quad \text{for} \quad t\ge t_0.$$

Therefore

$$\psi_{n-1\ i}(t,0) \geq \frac{1}{(n-2)!}\int_{\eta_i(t_0)}^{\tau_*(t)} (\tau_*(t)-s)^{n-2} \exp\Big\{\int_s^t (\lambda^*-\varepsilon_0)\tau_*^{n-2}(\xi)\times$$
$$\times p(\xi)\Big[\int_\xi^{+\infty}\tau_*^{n-2}(\xi_1)p(\xi_1)d\xi_1\Big]^{-1}d\xi\Big\}ds \geq \frac{\beta^{\lambda^*-\varepsilon_0}}{(n-2)!}\times$$
$$\times\Big(\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds\Big)^{\varepsilon_0-\lambda^*}\Big(\int_{\eta_1(1)}^{\tau_*(t)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds-$$
$$-\int_{\eta_1(1)}^{\eta_i(t_0)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds\Big)\ \text{ for }\ t\geq t_0\ \ (i=k,k+1,\dots).$$

Hence by (5.41) we obtain

$$\frac{\psi_{n-1\ i}(t,0)}{\tau_*^{n-2}(t)}\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds \geq \frac{\beta^{\lambda^*-\varepsilon_0}}{(n-2)!}\Big(\tau_*(t)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds\Big)^{1-\lambda^*+\varepsilon_0}\times$$
$$\times\tau_*^{\lambda^*-(n-1+\varepsilon_0)}(t)\Big(\int_{\eta_1(1)}^{\tau_*(t)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds-\int_{\eta_1(1)}^{\eta_i(t_0)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds\Big)\geq$$
$$\geq\frac{\beta\tau_*^{\lambda^*-(n-1+\varepsilon_0)}(t)}{(n-2)!}\Big(\int_{\eta_1(1)}^{\tau_*(t)}\frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda^*-\varepsilon_0}(s)}ds-$$
$$-\int_{\eta_1(1)}^{\eta_i(t_0)}\frac{(\tau_*(t)-s)^{n-2}}{\tau^{\lambda^*-\varepsilon_0}(s)}ds\Big)\ \text{ for }\ t\geq t_0\ \ (i=k,k+1,\dots).$$

Therefore by (5.41)

$$\varliminf_{t\to+\infty}\frac{\psi_{n-1\ i}(t,0)}{\tau_*^{n-2}(t)}\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds>\lambda^*$$

which contradicts the definition of λ^*. The obtained contradiction proves that $\lambda^*>1$. On the other hand, since we can assume that

$$\varlimsup_{t\to+\infty}\tau_*(t)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds\leq n-1$$

(otherwise we have (5.18) with $l=n-1$), we easily find that there exists $k_0\in\mathbb{N}$ such that (5.39) holds. ■

Let

$$\varliminf_{t\to+\infty}t\int_t^{+\infty}\frac{\tau_*^{n-1}(s)}{s}p(s)ds>M_n, \tag{5.42}$$

where $M_n=\max\{x(1-x)\dots(n-1-x):x\in[0,1]\}$. *Then* (5.1) *has no proper solution satisfying* (2.14_{n-1}).

Proof. Let (5.1) have a proper solution $u : [t_0, +\infty[\to R$ satisfying (2.14_{n-1}). By Lemma 1.3 there is $t_1 \in [t_0, +\infty[$ such that

$$\frac{|u(\tau(t))|}{\tau_*^{n-1}(t)} \geq \frac{|u(\tau_*(t))|}{\tau_*^{n-1}(t)} \geq \frac{|u(t)|}{t^{n-1}} \quad \text{for} \quad t \in [t_1, +\infty[.$$

Thus it is clear that on the interval $[t_1, +\infty[$ the function u is a solution of the differential inequality

$$u^{(n)}(t) \operatorname{sign} u(t) + q(t)|u(t)| \leq 0, \tag{5.43}$$

where

$$q(t) = \frac{\tau_*^{n-1}(t)}{t^{n-1}} p(t).$$

On the other hand, by (5.42)

$$\varliminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} q(s) ds > \beta > M_n. \tag{5.44}$$

Thus, clearly, there exists $\varepsilon > 0$ such that

$$\varliminf_{t \to +\infty} t^{\lambda-(n-1)} \int_1^t \frac{(t-s)^{n-2}}{s^\lambda} ds > \frac{\lambda(n-2)!}{\beta} + \varepsilon, \tag{5.45}$$
$$\text{for all} \quad \lambda \in \Big[\frac{\beta}{(n-1)!}, 1\Big].$$

Assume that

$$\varlimsup_{t \to +\infty} t \int_t^{+\infty} s^{n-2} q(s) ds \leq (n-1)!.$$

(otherwise (5.43) has no solution satisfying (2.14_{n-1})). Then we have

$$\int_0^t s^{n+1} q(s) ds \geq \int_{t^{\frac{1}{2}}}^t s^2 s^{n-1} q(s) ds \geq -t \int_{t^{\frac{1}{2}}}^t s \, d \int_s^{+\infty} \xi^{n-2} q(\xi) d\xi \geq$$
$$\geq t\Big(-t \int_t^{+\infty} s^{n-2} q(s) ds + \int_{t^{\frac{1}{2}}}^t \int_s^{+\infty} \xi^{n-2} q(\xi) d\xi \, ds\Big) \geq t\Big(-n! + \frac{M_n}{4} \ln t\Big).$$

Therefore by (5.44), (5.45) and Lemma 5.11 ($\tau_*(t) = t$) (5.43) has no solution satisfying (2.14_{n-1}). The obtained contradiction proves the validity of the corollary. ■

Let $\alpha \in]0, 1]$ and $\sigma(t) \geq \alpha t$ for sufficiently large t,

$$\varliminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds > M_n(\alpha),$$

where

$$M_n(\alpha) = \max\{\alpha^{x-n+1} x(1-x) \cdots (n-1-x) : x \in [0,1]\}. \tag{5.46}$$

Then (5.1) *has no solution satisfying* (2.14_{n-1}).

To prove the corollary it suffices to note that all the conditions of Lemma 5.11 with $\tau_*(t) = \alpha t$ are fulfilled.

Let

$$\lim_{t\to+\infty} \frac{t}{\tau_*(t)} = +\infty, \quad \varliminf_{t\to+\infty} \tau_*(t) \int_t^{+\infty} \tau_*^{n-2}(s)p(s)ds > 0 \tag{5.47}$$

and (5.38) *be fulfilled for some natural* m. *Then* (5.1) *has no proper solution satisfying* (2.14_{n-1}).

Proof. By (5.47) there is $\beta \in]0, +\infty[$ such that

$$\varliminf_{t\to+\infty} \tau_*(t) \int_t^{+\infty} \tau_*^{n-2}(s)p(s)ds > \beta > 0.$$

Thus due to Lemma 5.11 is suffices to show that (5.37) is fulfilled. Indeed, by (5.47) there is $t_0 \in [\eta(1), +\infty[$ such that

$$t \geq \Big(\frac{2}{\beta} \max\Big\{\lambda(1-\lambda)\cdots(n-1-\lambda) : \lambda \in \Big[\frac{\beta}{(n-1)!}, 1\Big]\Big\}\Big)^{\frac{1}{\lambda}} \tau_*(t) \quad \text{for } t \geq t_0. \tag{5.48}$$

Assuming that $\lambda \in \Big[\frac{\beta}{(n-1)!}, 1\Big[$, by (5.48) we obtain

$$\frac{\beta \tau_*^{\lambda-(n-1)}(t)}{(n-2)!} \int_{\eta_1(1)}^{\tau_*(t)} \frac{(\tau_*(t)-s)^{n-2}}{\tau_*^{\lambda}(s)} ds \geq \frac{2}{(n-2)!} \max\Big\{\lambda((1-\lambda)\cdots$$
$$\cdots(n-1-\lambda) : \lambda \in \Big[\frac{\beta}{(n-1)!}, 1\Big]\Big\} \tau_*^{\lambda-(n-1)}(t) \int_{t_0}^{\tau_*(t)} \frac{(\tau_*(t)-s)^{n-2}}{s^{\lambda}} ds >$$
$$> \frac{\max\Big\{\lambda(1-\lambda)\cdots(n-1-\lambda) : \lambda \in \Big[\frac{\beta}{(n-1)!}, 1\Big]\Big\}}{(1-\lambda)(2-\lambda)\cdots(n-1-\lambda)} + \varepsilon \geq \lambda + \varepsilon, \quad \text{for } t \geq t_1,$$

where $t_1 \in [t_0, +\infty[$ is sufficiently large, while ε is a sufficiently small positive number. For $\lambda = 1$ it is easy to find that (5.37) is satisfied. ■

Let

$$\varliminf_{t\to+\infty} t^{-\alpha}\tau(t) > 0, \quad \varliminf_{t\to+\infty} t^{\alpha} \int_t^{+\infty} s^{\alpha(n-2)}p(s)ds > 0,$$

where $\alpha \in]0, 1[$. *Then the* (5.1) *has no solution saatisfying* (2.14_{n-1}).

If the inequality $\tau(t) \geq \alpha t$ *is fulfilled for large* t,

$$\varliminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2}p(s)ds = \beta > 0^7 \tag{5.49}$$

and

$$\varlimsup_{t\to+\infty}\Big(\alpha t + x_0\int_0^{\alpha t} s^{n-1}\Big(\int_s^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}\times$$
$$\times p(s)ds\Big)\int_t^{+\infty}\tau^{n-2}(s)p(s)ds > (n-1)!. \tag{5.50}$$

where $\alpha\in]0,1]$ *and* x_0 *is the smallest root of the equation*

$$x(1-x)\cdots(n-1-x)\alpha^{x-(n-1)}=\beta, \tag{5.51}$$

then (5.1) *has no proper solution satisfying* (2.14_{n-1}).

Proof. By (5.49)–(5.51) there exist $\beta^*\in]0,\beta[$, $\varepsilon\in]0,\beta^*[$ and $t_0\in R_+$ such that

$$t\int_t^{+\infty}s^{n-2}p(s)ds\ge\beta^* \ \text{ for } \ t\in[t_0,+\infty[, \tag{5.52}$$

$$\varlimsup_{t\to+\infty}\Big(\alpha t + (x_0^*-\varepsilon)\int_0^{\alpha t} s^{n-1}\Big(\int_s^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}\times$$
$$\times p(s)ds\Big)\int_t^{+\infty}\tau^{n-2}(s)p(s)ds > (n-1)!, \tag{5.53}$$

where x_0^* is the smallest root of the equation

$$\alpha^{x-(n-1)}x(1-x)\cdots(n-1-x)=\beta^*-\varepsilon. \tag{5.54}$$

By Lemma 5.7 and (5.53) it suffices to show that there exists $k_0\in\mathbb{N}$ such that

$$\psi_{n-1k_0}(t,0)\ge t^{n-2}\Big(\int_t^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}(x_0^*-\varepsilon) \ \text{ for } \ t\ge t_*, \tag{5.55}$$

where $t_*\in[t_0,+\infty[$ is sufficiently large.

By (5.12) and (5.52) there exist $\beta_1\in]0,\beta^*]$ and $t_1\in[t_0,+\infty[$ such that

$$\psi_{n-1\,2}(t,0)t^{2-n}\int_t^{+\infty}\xi^{n-2}p(\xi)d\xi\ge\beta_1 \ \text{ for } \ t\in[t_0,+\infty[.$$

Taking into account (5.52), we now obtain

$$\psi_{n-1\,3}(t,0)\ge\frac{1}{(n-2)!}\int_{t_1}^{\tau_*(t)}(\tau_*(t)-s)^{n-2}\exp\Big\{\beta_1\int_s^t\xi^{n-2}p(\xi)\times$$
$$\times\Big(\int_\xi^{+\infty}\xi_1^{n-2}p(\xi_1)d\xi_1\Big)^{-1}d\xi\Big\}ds=\frac{\Big(\int_t^{+\infty}s^{n-2}p(s)ds\Big)^{-\beta_1}}{(n-2)!}\times$$

[7]it is assumed that $\beta\in]0,M_n(\alpha)]$ where $M_n(\alpha)$ is defined by equality (5.46).Otherwise the conditions of Corollary 5.5 are fulfilled and condition (5.50) becomes unnecessary.

$$\times \int_{t_1}^{\tau_*(t)} (\tau_*(t)-s)^{n-2}\Big(\int_s^{+\infty} \xi^{n-2}p(\xi)d\xi\Big)^{\beta_1} ds \geq$$

$$\geq \frac{\alpha^{n-1-\beta_1}(\beta^*-\varepsilon)^{\beta_1} t^{n-1-\beta_1}}{(1-\beta_1)\cdots(n-1-\beta_1)}\Big(\int_t^{+\infty} s^{n-2}p(s)ds\Big)^{-\beta_1} \quad \text{for } \ t\in[t_2,+\infty[,$$

where $t_2 \in [t_1,+\infty[$ is sufficiently large. Therefore

$$\psi_{n-1\,3}(t,0)t^{2-n}\int_t^{+\infty} s^{n-2}p(s)ds \geq \frac{(\beta^*-\varepsilon)\alpha^{n-1-\beta_1}}{(1-\beta_1)\cdots(n-1-\beta_1)} = \beta_2$$
$$\text{for} \quad t\in[t_2,+\infty[.$$

In a similar manner we show that

$$\psi_{n-1\,i}(t,0)t^{2-n}\int_t^{+\infty} s^{n-2}p(s)ds \geq \frac{(\beta^*-\varepsilon)\alpha^{n-1-\beta_{i-1}}}{(1-\beta_{i-1})\cdots(n-1-\beta_{i-1})} =$$
$$= \beta_i \ \text{ for } \ \ t\in[t_{i-1},+\infty[\quad (i=3,4,\dots), \tag{5.56}$$

where t_i $(i=2,3,\dots)$ is sufficiently large.

Putting

$$\lim_{i\to+\infty}\beta_i = x_0^*,$$

from the equalities

$$\beta_i = \frac{(\beta^*-\varepsilon)\alpha^{n-1-\beta_{i-1}}}{(1-\beta_{i-1})\cdots(n-1-\beta_{i-1})} \quad (i=2,3,\dots)$$

we have

$$\alpha^{x_0^*-(n-1)}x_0^*(1-x_0^*)\cdots(n-1-x_0^*) = \beta^*-\varepsilon.$$

Thus x_0^* is a solution of equation (5.54). Therefore by (5.56) there is $k_0\in\mathbb{N}$ such that inequality (5.55) is fulfilled for $t\in[t_*,+\infty[$, where t_* is sufficiently large. ■

′ *Let for sufficiently large t there hold*

$$\tau(t)\geq \alpha t, \quad p(t)\geq \frac{\beta}{t^n} \tag{5.57}$$

and

$$\overline{\lim_{t\to+\infty}}\, t\int_t^{+\infty}\tau^{n-2}(s)p(s)ds > \frac{(n-1)!}{\alpha(1+x_0)}, \tag{5.58}$$

where $\alpha \in]0,1]$, $\beta\in]0,M_n(\alpha)]$ ($M_n(\alpha)$ is defined by (5.46)), and x_0 is the smallest root of equation (5.51). Then (5.1) has no proper solution satisfying (2.14_{n-1}).

Proof. By (5.51) and (5.58) there exist $\beta^* \in]0,\beta[$ and $\varepsilon \in]0,\beta^*[$ such that

$$\varlimsup_{t\to+\infty} t\int_t^{+\infty} \tau^{n-2}(s)p(s)ds > \frac{(n-1)!}{\alpha(1+x_0^*)}, \tag{5.59}$$

where x_0^* is the smallest root of (5.54). Due to (5.59) and Lemma 5.8 it suffices to show that there exists $k_0 \in \mathbb{N}$ such that

$$\psi_{n-1\ k_0}(t,0) \geq \frac{t^{n-1}x_0^*}{\beta} \quad \text{for} \quad t \in [t_*,+\infty[, \tag{5.60}$$

where $t_* \in R_+$ is some number.

By (5.12), (5.57) there exist $\beta_1 \in]0,\beta]$ and $t_0 \in R_+$ such that

$$\psi_{n-1\ 2}(t,0) \geq \beta_1 t^{n-1} \quad \text{for} \quad t \in [t_0,+\infty[.$$

Since $\beta\beta_1 < 1$ (it is clear that (5.60) is fulfilled for $\beta\beta_1 \geq 1$), by (5.57) we obtain

$$\psi_{n-1\ 3}(t,0) \geq \frac{1}{(n-2)!}\int_{t_0}^{\tau_*(t)} (\tau_*(t)-s)^{n-2} \exp\Big\{\int_s^t \beta\beta_1\frac{d\xi}{\xi}\Big\}ds =$$
$$= \frac{t^{\beta\beta_1}}{(n-2)!}\int_t^{\tau_*(t)} \frac{(\tau_*(t)-s)^{n-2}}{s^{\beta\beta_1}}ds \geq \frac{t^{n-1}\alpha^{n-1-\beta^*\beta_1}}{(1-\beta^*\beta_1)\cdots(n-1-\beta^*\beta_1)}$$
$$\text{for} \quad t \in [t_1,+\infty[,$$

where $t_1 \in [t_0,+\infty[$ is sufficiently large. Therefore

$$t^{1-n}\psi_{n-1\ 3}(t,0) \geq \frac{\alpha^{n-1-\beta^*\beta_1}}{(1-\beta^*\beta_1)\cdots(n-1-\beta^*\beta_1)} = \beta_2 \ \text{ for } \ t \in [t_1,+\infty[.$$

Similarly we have

$$t^{1-n}\psi_{n-1\ i}(t,0) \geq \frac{\alpha^{n-1-\beta^*\beta_{i-2}}}{(1-\beta^*\beta_{i-2})\cdots(n-1-\beta^*\beta_{i-2})} =$$
$$= \beta_{i-1} \ \ (i=3,4,\dots) \ \text{ for } \ t \in [t_{i-2},+\infty[,$$

where t_i $(i=1,2,\dots)$ is sufficiently large.

Introducing the notation

$$\lim_{i\to\infty} \beta^*\beta_i = x_0,$$

from the equalities

$$\beta_i = \frac{\alpha^{n-1-\beta^*\beta_{i-1}}}{(1-\beta^*\beta_{i-1})\cdots(n-1-\beta^*\beta_{i-1})} \quad (i=2,3,\dots)$$

we find that x_0 is the root of the equation

$$x(1-x)\cdots(n-1-x)\alpha^{x-(n-1)} = \beta^*.$$

Hence we easily conclude that there exist numbers $t_* \in R_+$ and $k_0 \in \mathbb{N}$ for which (5.60) is fulfilled. ■

The validity of Lemmas 5.13 and 5.14 below can be proved similarly to that of Lemmas 5.11 and 5.12.

Let $l \in \{1, \dots, n-2\}$, $l+n$ *be odd* ($l+n$ *be even*),

$$\varliminf_{t \to +\infty} \tau_*(t) \int_t^{+\infty} (s-t)^{n-l-1} \tau^{l-1}(s) p(s)\,ds > \beta > 0,$$

there exist $\varepsilon > 0$ *such that*

$$\varliminf_{t \to +\infty} \tau_*^{\lambda-l}(t) \int_{\eta_1(1)}^{\tau_*(t)} \frac{(\tau_*(t)-s)^{l-1}}{\tau_*^{\lambda}(s)}\,ds >$$
$$> \frac{\lambda(l-1)!(n-l-1)!}{\beta} + \varepsilon \;\; \text{for all} \;\; \lambda \in \Big[\frac{\beta}{l!(n-l-1)!}, 1\Big]$$

and

$$\varlimsup_{t \to +\infty} \Big(\tau_*(t) + \frac{1}{(n-l-1)!} \int_0^{\tau_*(t)} s^{n-l} \tau_*^m(s) p(s)\,ds\Big) \times$$
$$\times \int_t^{+\infty} (s-\tau_*(t))^{n-l-1} \tau^{l-1}(s) p(s)\,ds > l!(n-l-1)! \tag{5.61}$$

for some natural m. *Then* (5.1) ((5.2)) *has no solution satisfying* (2.14_l).

Let $l \in \{1, \dots, n-2\}$, $l+n$ *be odd* ($l+n$ *be even*), *the inequality* $\sigma(t) \ge \alpha t$ *with* $\alpha \in]0,1]$ *hold for sufficiently large* t *and*

$$\varliminf_{t \to +\infty} t \int_t^{+\infty} (s-t)^{n-l-1} s^{l-1} p(s)\,ds > M_l(\alpha)(n-l-1)!,$$

where

$$M_l(\alpha) = \max\{x(1-x)\cdots(l-x)\alpha^{x-l} : x \in [0,1]\}. \tag{5.62}$$

Then (5.1) ((5.2)) *has no solution satisfying* (2.14_l).

Let $l \in \{1, \dots, n-2\}$, $l+n$ *be odd* ($l+n$ *be even*) *and*

$$\lim_{t \to +\infty} \frac{t}{\tau_*(t)} = +\infty, \quad \varliminf_{t \to +\infty} \tau_*(t) \int_t^{+\infty} (s-t)^{n-l-1} \tau^{l-1}(s) p(s)\,ds > 0.$$

Then for (5.1) ((5.2)) *not to have a solution satisfying* (2.14_l), *it is sufficient that* (5.61) *be fulfilled for some natural* m.

Let $l \in \{1, \dots, n-2\}$, $l+n$ *be odd* ($l+n$ *be even*) *and*

$$\varliminf_{t \to +\infty} t^{-\alpha} \tau(t) > 0, \quad \varliminf_{t \to +\infty} t^{\alpha} \int_t^{+\infty} (s-t)^{n-l-1} s^{\alpha(l-1)} p(s)\,ds > 0,$$

where $\alpha \in]0,1[$. *Then* (5.1) ((5.2)) *has no solution satisfying* (2.14_l).

Let $l \in \{1,\dots,n-2\}$, $l+n$ be odd ($l+n$ be even), for sufficiently large t inequality (5.57) *be fulfilled which $\alpha \in]0,1]$, $\beta \in]0, M_l(\alpha)(n-l)!]$ which $M_l(\alpha)$ is defined by* (5.62) *and*

$$\varlimsup_{t\to+\infty} t\int_t^{+\infty}(s-\alpha t)^{n-l-1}\tau^{l-1}(s)p(s)ds > \frac{(l-1)!(n-l-1)!}{\alpha(1+(n-l)x_0)},$$

where x_0 is the smallest root of the equation

$$x(1-x)\cdots(l-x)\alpha^{x-l} = \frac{\beta}{(n-l)!}.$$

Then (5.1) ((5.2)) *has no solution satisfying* (2.14_l).

§ 6. Linear Differential Inequalities with a Deviating Argument and Property ( )

The results obtained in §5 for the linear differential inequalities (5.1) and (5.2) will be used in this section to derive sufficient conditions for equation (0.1) to have property ( ).

Let $F \in V(\tau)$, condition (0.2) *be fulfilled for some $t_0 \in \mathbb{R}_+$, and*

$$|F(u)(t)| \ge \int_{\tau(t)}^{\sigma(t)} |u(s)|d_s r(s,t) \ \ \textit{for}\ \ t\in[t_0,+\infty[,\ \ u\in H_{t_0,\tau}, \tag{6.1}$$

where

$$\begin{gathered}\tau,\sigma\in C(\mathbb{R}_+;\mathbb{R}_+),\ \ \tau(t)\le\sigma(t)\ \ \textit{for}\ \ t\in\mathbb{R}_+,\ \ \lim\tau(t)=+\infty,\\ r(s,\cdot)\quad \textit{is a measurable function},\\ r(\cdot,t)\quad \textit{is a nondecreasing function.}\end{gathered}\tag{6.2}$$

Let, in addition to the above, for some $k_0 \in \mathbb{N}$ the condition

$$\begin{aligned}\varlimsup_{t\to+\infty}\Big[&\int_{\tau_*(t)}^{t}\tau_*^{n-1}(s)p(s)\widetilde{\varphi}_{n-1\,k_0}(s,t,0)ds+\\ &+\widetilde{\psi}_{n-1\,k_0}(t,0)\int_t^{+\infty}\tau_*^{n-2}(s)p(s)ds\Big] > (n-1)!\end{aligned}\tag{6.3}$$

hold, where

$$p(t) = r(\sigma(t),t) - r(\tau(t),t), \tag{6.4}$$

$$\tau_*(t) = \inf\{\tau_0(s) : s\ge t\},\quad \tau_0(t)=\min\{t,\tau(t)\} \tag{6.5}$$

and the functions $\widetilde{\varphi}_{n-1\,k_0}$ and $\varphi_{n-1\,k_0}$ ($\widetilde{\psi}_{n-1\,k_0}$ and $\psi_{n-1\,k_0}$) are defined by (5.19_{n-1}) *and* (5.9_{n-1}) ((5.20_{n-1}) *and* (5.12)). *Then equation* (0.1) *has property* .

Proof. First of all note that condition (5.3) with the functions p and τ_0 defined by (6.4) and (6.5), respectively, is fulfilled by virtue of (6.3).

Let us assume that equation (0.1) has no property . Then by (0.2) and (6.1) the differential inequality

$$u^{(n)}(t)\,\mathrm{sign}\,u(\sigma(t)) + \int_{\tau(t)}^{\sigma(t)} |u(s)|d_s r(s,t) \le 0 \tag{6.6}$$

has no property . Following Theorem 2.3, the equation

$$u^{(n)}(t) + p(t)u(\tau_*(t)) = 0 \tag{6.7}$$

with p and τ_* defined by (6.4) and (6.5), respectively, has no property . Therefore (6.7) has a nonoscillatory proper solution $u : [t_1, +\infty[\to R$ satisfying (2.14_l) where $l \in \{0, \dots, n-1\}$ ($l+n$ is odd). Assuming now that n is odd and $l = 0$, by (5.3) we crearly see that (0.4) is fulfilled. Therefore $l \in \{1, \dots, n-1\}$ and thus by Lemma 5.1 equation (6.7) has a proper solution satisfying (2.14_{n-1}).

But by (6.3) and Lemma 5.8 equation (6.7) has no proper solution satisfying (2.14_{n-1}). The obtained contradiction proves the validity of the theorem. ■

If $F \in V(\tau)$ and conditions (0.2), (6.1), (6.2) *are fulfilled, then for equation* (0.1) *to have property it is sufficient that*

$$\varlimsup_{t\to+\infty} \int_{\tau_*(t)}^{t} p(s)\tau_*^{n-1}(s)\exp\Big\{\frac{1}{(n-1)!}\int_{\tau_*(s)}^{s} p(\xi)\tau_*^{n-1}(\xi)d\xi\Big\}ds > (n-1)!,$$

where p and τ_* are defined by (6.4) *and* (6.5), *respectively.*

If $F \in V(\tau)$ and conditions (0.2), (6.1), (6.2) *are fulfilled, then for equation* (0.1) *to have property it is sufficient that*

$$\varlimsup_{t\to+\infty}\Big(\tau_*(t) + \frac{1}{(n-1)!}\int_0^{\tau_*(t)} s\tau_*^{n-1}(s)p(s)ds\Big)\int_t^{+\infty} \tau_*^{n-2}(s)p(s)ds > (n-1)!,$$

where p and τ_* are defined by (6.4) *and* (6.5), *respectively.*

′ *If $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ and*

$$\varlimsup_{t\to+\infty}\Big(t + \frac{1}{(n-1)!}\int_0^{t} s^n p(s)ds\Big)\int_t^{+\infty} s^{n-2}p(s)ds > (n-1)!,$$

then the equation

$$u^{(n)}(t) + p(t)u(t) = 0 \tag{6.8}$$

has property .

The particular case of Corollary 6.2′ is Theorem 2.3 in [11].

If $F \in V(\tau)$, conditions (0.2), (6.1), (6.2) *are fulfilled, and $\tau(t) < t$ for $t \in R_+$, then the condition*

$$\underline{\lim}_{t \to +\infty} \int_{\tau_*(t)}^{t} p(s)\tau^{n-1}(s)ds > \frac{(n-1)!}{e}, \tag{6.9}$$

where p and τ_* are defined by (6.4) *and* (6.5), *respectively, is sufficient for equation* (0.1) *to have property* .

Proof. By Lemma 5.9 and (6.9) there exists $k_0 \in \mathbb{N}$ such that (6.3) holds. Therefore the conditions of Theorem 6.1 are fulfilled, which proves that Theorem 6.2 is valid. ■

Remark 6.1. Remark 5.1 clearly implies that (6.9) cannot be replaced by the condition

$$\underline{\lim}_{t \to +\infty} \int_{\tau_*(t)}^{t} p(s)\tau^{n-1}(s)ds \geq \frac{(n-1)!}{e} - \varepsilon,$$

where ε is an arbitrarily small positive number.

Let $F \in V(\tau)$, conditions (0.2), (6.1), (6.2) *be fulfilled,*

$$\underline{\lim}_{t \to +\infty} \int_{\tau_*(t)}^{t} p(s)\tau_*^{n-1}(s)ds \geq c \tag{6.10}$$

and

$$\overline{\lim}_{t \to +\infty} \int_{\tau_*(t)}^{t} \tau_*^{n-1}(s)p(s)\exp\Big\{\frac{x_0}{(n-1)!} \times \times \int_{\tau_*(s)}^{\tau_*(t)} p(\xi)\tau_*^{n-1}(\xi)d\xi\Big\}ds > (n-1)!, \tag{6.11}$$

where $c \in \big]0, \frac{(n-1)!}{e}\big]$, p and τ_* are defined by (6.4) *and* (6.5)[8] *and x_0 is the smallest root of the equation*

$$\exp\{\frac{c}{(n-1)!}x\} = x. \tag{6.12}$$

Then equation (0.1) *has property* .

Proof. By (6.10)–(6.12) there exists $\varepsilon \in]0,1[$ such that

$$\overline{\lim}_{t \to +\infty} \int_{\tau_*(t)}^{t} \tau_*^{n-1}(s)p(s) \times \times \exp\Big\{\frac{x_0 - \varepsilon}{(n-1)!}\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)\tau_*^{n-1}(\xi)d\xi\Big\}ds > (n-1)!\,. \tag{6.13}$$

[8]it is obvious that for $c > \frac{(n-1)!}{e}$ the conditions of Theorem 6.2 are fulfilled and condition (6.10) becomes unnecessary.

On the other hand, as shown while proving Lemma 5.10, by (6.10)–(6.12) there exists $k_0 \in \mathbb{N}$ such that

$$\varliminf_{t\to+\infty} \varphi_{n-1\ k_0}(t,0) > x_0 - \varepsilon,$$

where $\varphi_{n-1\ k_0}$ is defined by (5.9_{n-1}). On account of (6.13) it is obvious that condition (6.3) is fulfilled where $\widetilde{\varphi}_{n-1\ k_0}$ is defined by (5.19_{n-1}). Therefore the conditions of theorem 6.1 are fulfilled, which proves that Theorem 6.3 is valid. ■

Let $F \in V(\tau)$ *and conditions* (0.2), (6.1), (6.2), (5.36) − −(5.38) *be fulfilled, where* p *and* τ_* *are defined by* (6.4) *and* (6.5), *respectively. Then equation* (0.1) *has property* .

Proof. By analogy with the reasoning used while proving Lemma 5.11, there exists $k_0 \in \mathbb{N}$ such that (5.39) holds. Therefore by (5.38) it is clear that (6.3) is fulfilled. Thus the conditions of Theorem 6.1 are satisfied, which proves the validity of Theorem 6.4. ■

Let $F \in V(\tau)$, *conditions* (0.2), (6.1) *and* (6.2) *be fulfilled, and*

$$\varliminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau_*^{n-1}(s)}{s} p(s)ds > M_n,$$

where

$$M_n = \max\{x(1-x)\cdots(n-1-x) : x \in [0,1]\}, \tag{6.14}$$

p *and* τ_* *are defined by equalities* (6.4) *and* (6.5), *respectively. Then equation* (0.1) *has property* .

Proof. The validity of the corollary follows from Corollary 5.4 and Lemma 5.1. ■

Let $F \in V(\tau)$, *conditions* (0.2), (6.1) *and* (6.2) *be fulfilled, and* $\tau(t) \geq \alpha t$, *for* $t \in R_+$, *where* $\alpha \in]0,1]$. *Then the condition*

$$\varliminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2} p(s)ds > M_n(\alpha),$$

where $M_n(\alpha)$ *and* p *are defined by* (5.46) *and* (6.4), *respectively, is sufficient for equation* (0.1) *to have property* .

Proof. The validity of the corollary follows from Corollary 5.5 and Lemma 5.1. ■

′ [12]. *Let* $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ *and*

$$\varliminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2} p(s)ds > M_n,$$

where M_n *is defined by* (6.14). *Then equation* (6.7) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.2), (6.1), (6.2) *and* (5.47) *hold. If besides there exists* $m \in \mathbb{N}$ *such that* (5.38) *is fulfilled with* p *and* τ_* *defined by* (6.4) *and* (6.5), *then equation* (0.1) *has property* .

Proof. The validity of the corollary follows from Corollary 5.6 and Lemma 5.1. ■

Let $F \in V(\tau)$, *conditions* (0.2), (6.1) *and* (6.2) *hold and*

$$\varliminf_{t\to+\infty} t^{-\alpha}\tau(t) > 0, \quad \varliminf_{t\to+\infty} t^{\alpha}\int_t^{+\infty} s^{\alpha(n-2)}p(s)ds > 0,$$

where $\alpha \in]0,1[$ *and the function* p *is defined by* (6.4). *Then equation* (0.1) *has property* .

Proof. The validity of the corollary follows from Corollary 5.7 and Lemma 5.1. ■

Let $F \in V(\tau)$, $\tau(t) \geq \alpha t$ *for* $t \in R_+$, *conditions* (0.2), (6.1), (6.2) *and* (5.49) *be fulfilled and*

$$\varlimsup_{t\to+\infty}\Big(\alpha t + x_0\int_0^{\alpha t} s^{n-1}\Big(\int_s^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}p(s)ds\Big)\times \\ \times\int_t^{+\infty} s^{n-2}p(s)ds > \frac{(n-1)!}{\alpha^{n-2}}, \tag{6.15}$$

where $\alpha \in]0,1]$, $\beta \in]0, M_n(\alpha)]$, $M_n(\alpha)$ *is defined by* (5.46), x_0 *is the smallest root of equation* (5.51) *and* p *is defined by* (6.4). *Then equation* (0.1) *has property* .

Proof. By (5.49), (6.15) and the same arguments as used in proving Lemma 5.12 there exist $\varepsilon \in]0,1[$ and $k_0 \in \mathbb{N}$ such that

$$\varlimsup_{t\to+\infty}\Big(\alpha t + (x_0-\varepsilon)\int_0^{\alpha t} s^{n-1}\Big(\int_s^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}p(s)ds\Big)\times \\ \times\int_t^{+\infty} s^{n-2}p(s)ds > \frac{(n-1)!}{\alpha^{n-2}}, \tag{6.16}$$

$$\psi_{n-1\,k_0}(t,0) \geq t^{n-2}\Big(\int_t^{+\infty}\xi^{n-2}p(\xi)d\xi\Big)^{-1}(x_0-\varepsilon) \quad \text{for } t \geq t_*, \tag{6.17}$$

where t_* is sufficiently large and $\psi_{n-1\,k_0}$ is defined by (5.12). Clearly condition (6.3), where $\tau_*(t) = \alpha t$, holds by virtue of (6.16), (6.17) and (5.20_{n-1}). Therefore the conditions of Theorem 6.1 are fulfilled, which proves the validity of Theorem 6.5. ■

Using Lemmas 5.1 and 5.12, one can easily prove

′. If $F \in V(\tau)$*, conditions* (0.2), (6.1) *and* (5.57) *are fulfilled and*

$$\varliminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}p(s)ds > \frac{(n-1)!}{\alpha^{n-1}(1+x_0)},$$

where $\alpha \in]0,1]$, $\beta \in]0, M_n(\alpha)]$, $M_n(\alpha)$ *is defined by* (5.46), x_0 *is the smallest root of equation* (5.51) *and* p *is given by* (6.4)*, then equation* (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.2), (6.1)–(6.3) *be fulfilled, where*

$$p(t) = \sigma^{1-n}(t)\int_{\tau(t)}^{\sigma(t)} s^{n-1}d_s r(s,t), \tag{6.18}$$

$$\tau_*(t) = \inf\{\tau_0(s) : s \geq t\}, \quad \tau_0(t) = \min\{t, \sigma(t)\} \tag{6.19}$$

and $\widetilde{\varphi}_{n-1\;k_0}$ *and* $\varphi_{n-1\;k_0}$ $(\widetilde{\psi}_{n-1\;k_0}$ *and* $\psi_{n-1\;k_0})$ *are defined by* (5.19_{n-1}) *and* (5.9_{n-1}) $((5.20_{n-1})$ *and* (5.12))*. Then equation* (0.1) *has property* .

Proof. Clearly condition (5.3), where p and τ_* are defined by (6.18) and (6.19), respectively, is fulfilled by virtue of (6.3).

Assume that (0.1) has no property . Then by (0.2) and (6.1) differential inequality (6.6) has no property . Following Theorem 2.4′, equation (6.7) with the functions p and τ_* defined by (6.18) and (6.19) has no property .

Let $u : [t_0, +\infty[\to R$ be a nonoscillatory proper solution of equation (6.7). By Lemma 1.2 there exists $l \in \{0, \dots, n-1\}$ such that $l+n$ is odd and condition (2.14_l) is fulfilled. Assuming that n is odd and $l = 0$, it is easy to show that (0.4) is fulfilled. Therefore $l \in \{1, \dots, n-1\}$. Thus by Lemma 5.1 equation (6.7) has a proper solution satisfying (2.14_{n-1}).

But by Lemmas 6.3 and 5.8 (as has been said several times, the functions p and τ_* are defined by (6.18) and (6.19), respectively) equation (6.7) has no proper solution satisfying (2.14_{n-1}). The obtained contradiction proves the validity of the theorem. ■

Let $F \in V(\tau)$*. Then for equation* (0.1) *to have property it is sufficient that the conditions of Corollary* 6.1 *or* 6.2 *be fulfilled, where the functions* p *and* τ_* *are defined by* (6.18) *and* (6.19)*, respectively.*

Using Theorems 6.2 – 6.5, 6.5′ and 6.6 one can easily prove

Let $F \in V(\tau)$ *and the conditions of anyone of Theorems* 6.2–6.5, 6.5′ *be fulfilled, where the functions* p *and* τ_* *are defined by* (6.18) *and* (6.19)*, respectively. Then equation* (0.1) *has property* .

Using Lemma 5.3, by a reasoning similar to that used in proving Theorem 6.1 one can prove

Let $F \in V(\tau)$ and conditions (0.3), (6.1), (6.2) *be fulfilled. Moreover, let there exist $k_0 \in \mathbb{N}$ such that*

$$\varlimsup_{t\to+\infty}\Big[\int_{\tau_*(t)}^{t}(s-\tau_*(t))\tau_*^{n-2}(s)\widetilde{\varphi}_{n-2\ k_0}(s,t,0)p(s)ds+$$
$$+\widetilde{\psi}_{n-2\ k_0}(t,0)\int_{t}^{+\infty}(s-\tau_*(t))\tau_*^{n-3}(s)p(s)ds\Big]>(n-2)!, \qquad (6.20)$$

for even n, while for odd n (6.20) *is fulfilled along with*

$$\varlimsup_{t\to+\infty}\Big[\int_{\tau_*(t)}^{t}(s-\tau_*(t))^{n-2}\tau_*(s)\widetilde{\varphi}_{1k_0}(s,t,0)p(s)ds+$$
$$+\widetilde{\psi}_{1k_0}(t,0)\int_{t}^{+\infty}(s-\tau_*(t))^{n-2}p(s)ds\Big]>(n-2)!, \qquad (6.21)$$

where p and τ_* are defined by (6.4) *and* (6.5), *respectively, while $\widetilde{\varphi}_{lk_0}$ and φ_{lk_0} ($\widetilde{\psi}_{lk_0}$ and ψ_{lk_0}) are defined by* (5.19_l) *and* (5.9_l) $((5.20_l)$ *and* $(5.13_l))$. *Then equation* (0.1) *has property* .

Let $F \in V(\tau)$ and conditions (0.3), (6.1), (6.2) *be fulfilled. Then for equation* (0.1) *to have property when n is even it is sufficient that*

$$\varlimsup_{t\to+\infty}\int_{\tau_*(t)}^{t}(s-\tau_*(t))\tau_*^{n-2}(s)p(s)\exp\Big\{\frac{1}{(n-2)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)}p(\xi)(\xi-\tau_*(s))\tau_*^{n-2}(\xi)d\xi\Big\}ds>(n-2)!, \qquad (6.22)$$

while for odd n it is sufficient that (6.22) *be fulfilled and*

$$\varlimsup_{t\to+\infty}\int_{\tau_*(t)}^{t}(s-\tau_*(t))^{n-2}\tau_*(s)p(s)\exp\Big\{\frac{1}{(n-2)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)}p(\xi)(\xi-\tau_*(s))^{n-2}\tau_*(\xi)d\xi\Big\}ds>(n-2)!,$$

where p and τ_* are defined by (6.4) *and* (6.5).

Let $F \in V(\tau)$ and conditions (0.3), (6.1), (6.2) *be fulfilled. Then for equation* (0.1) *to have property when n is even it is sufficient* *that*

$$\overline{\lim_{t\to+\infty}}\Big(\tau_*(t)+\frac{1}{(n-2)!}\int_0^{\tau_*(t)} s^2\tau_*^{n-2}(s)p(s)ds\Big)\times$$
$$\times\int_t^{+\infty}(s-\tau_*(t))\tau_*^{n-3}(s)p(s)ds>(n-2)!, \tag{6.23}$$

while for odd n it is sufficient that (6.23) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}}\Big(\tau_*(t)+\frac{1}{(n-2)!}\int_0^{\tau_*(t)} s^{n-1}\tau_*(s)p(s)ds\Big)\times$$
$$\times\int_t^{+\infty}(s-\tau_*(t))^{n-2}p(s)ds>(n-2)!,$$

where p and τ_* are defined by (6.4) *and* (6.5).

Let $F\in V(\tau)$ and conditions (0.3), (6.1), (6.2) *be fulfilled. Then for equation* (0.1) *to have property when n is even it is sufficient that*

$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))\tau_*^{n-2}(s)ds>\frac{(n-2)!}{e}, \tag{6.24}$$

while for odd n it is sufficient that (6.24) *be fulfilled and*

$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))^{n-2}\tau_*(s)ds>\frac{(n-2)!}{e}, \tag{6.25}$$

where the functions p and τ_* are defined by (6.4) *and* (6.5).

Proof. The validity of the theorem follows from Lemmas 5.3 and 5.9. ■

Remark 6.2. Condition (6.24) or (6.25) cannot be replaced by

$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))\tau_*^{n-2}(s)ds>\frac{(n-2)!}{e}-\varepsilon,$$
$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))^{n-2}\tau_*(s)ds>\frac{(n-2)!}{e}-\varepsilon,$$

where ε is an arbitrarily small positive number.

Let $F\in V(\tau)$, conditions (0.3), (6.1), (6.2) *be fulfilled, and*

$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))\tau_*^{n-2}(s)ds=c_1>0,$$
$$\underline{\lim}_{t\to+\infty}\int_{\tau_*(t)}^t p(s)(s-\tau_*(t))^{n-2}\tau_*(s)ds=c_2>0.$$

Then for equation (0.1) *to have property* *when* n *is even it is sufficient that*

$$\overline{\lim_{t\to+\infty}} \int_{\tau_*(t)}^{t} (s-\tau_*(t))\tau_*^{n-2}(s)p(s)\exp\Big\{\frac{x_1}{(n-2)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)(\xi-\tau_*(s))\tau_*^{n-2}(\xi)d\xi\Big\}ds > (n-2)!, \tag{6.26}$$

while for odd n *it is sufficient that* (6.26) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \int_{\tau_*(t)}^{t} (s-\tau_*(t))^{n-2}\tau_*(s)p(s)\exp\Big\{\frac{x_2}{(n-2)!}\times$$
$$\times\int_{\tau_*(s)}^{\tau_*(t)} p(\xi)(\xi-\tau_*(s))^{n-2}\tau_*(\xi)d\xi\Big\}ds > (n-2)!,$$

where x_i $(i=1,2)$ *is the smallest root of the equation*

$$\exp\Big\{\frac{c_i}{(n-2)!}x\Big\} = x \quad (i=1,2)$$

and p *and* τ_* *are defined by* (6.4) *and* (6.5), *respectively.*

Proof. The validity of the theorem follows from Lemmas 5.3 and 5.10. ■

Let $F\in V(\tau)$, *conditions* (0.3), (6.1), (6.2) *be fulfilled and*

$$\lim_{t\to+\infty}\frac{t}{\tau_*(t)} = +\infty.$$

Moreover, let for even n

$$\underline{\lim}_{t\to+\infty}\, \tau_*(t)\int_t^{+\infty}(s-t)\tau^{n-3}(s)p(s)ds > 0 \tag{6.27}$$

and for some $m\in\mathbb{N}$

$$\overline{\lim_{t\to+\infty}}\Big(\tau_*(t)+\int_0^{\tau_*(t)} s^2\tau_*^m(s)p(s)ds\Big)\times$$
$$\times\int_t^{+\infty}(s-\tau_*(t))\tau^{n-3}(s)p(s)ds > (n-2)!, \tag{6.28}$$

while for odd n (6.27) *and* (6.28) *be fulfilled,*

$$\underline{\lim}_{t\to+\infty}\, \tau_*(t)\int_t^{+\infty}(s-t)^{n-2}p(s)ds > 0$$

and for some $m \in \mathbb{N}$

$$\overline{\lim_{t\to+\infty}} \Big(\tau_*(t) + \int_0^{\tau_*(t)} s^{n-1}\tau_*^m(s)p(s)ds\Big) \times$$
$$\times \int_t^{+\infty} (s-\tau_*(t))^{n-2}p(s)ds > (n-2)!$$

with p *and* τ_* *defined by* (6.4) *and* (6.5), *respectively. Then equation* (0.1) *has property* .

Proof. The validity of the theorem follows from Lemma 5.3 and Corollary 5.9. ■

Let $F \in V(\tau)$, *conditions* (0.3), (6.1), (6.2) *be fulfilled and*

$$\underline{\lim_{t\to+\infty}} \frac{\tau(t)}{t^\alpha} > 0,$$

where $\alpha \in]0,1[$. *Besides, if for even* n

$$\underline{\lim_{t\to+\infty}} t^\alpha \int_t^{+\infty} (s-t)s^{\alpha(n-3)}p(s)ds > 0, \tag{6.29}$$

while for odd n (6.29) *is fulfilled and*

$$\underline{\lim_{t\to+\infty}} t^\alpha \int_t^{+\infty} (s-t)^{n-2}p(s)ds > 0,$$

where p *is defined by* (6.4), *then equation* (0.1) *has property* .

Proof. The validity of the corollary readily follows from Lemma 5.3 and Corollary 5.10. ■

Let $F \in V(\tau)$, *conditions* (0.3), (6.1), (6.2) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \tau_*(t) \int_t^{+\infty} \tau_*^{n-2}(s)p(s)ds > (n-1)!, \tag{6.30}$$

where p *and* τ_* *are defined by* (6.4) *and* (6.5), *respectively. Then equation* (0.1) *has property* .

Proof. First of all we note that (5.3) is fulfilled by virtue of (6.30). Assume that equation (0.1) has no property . Then by virtue of (0.3) and (6.1) the differential inequality

$$u^{(n)}(t)\operatorname{sign} u(\tau(t)) \geq \int_{\tau(t)}^{\sigma(t)} |u(s)|d_s r(s,t)$$

has no property . Therefore, following Theorem 2.4, the equation

$$u^{(n)}(t) - p(t)u(\tau_*(t)) = 0 \tag{6.31}$$

with p and τ_* defined by (6.4) and (6.5), respectively, has no property B.

Let $u : [t_0, +\infty[\to R$ be a nonoscillatory solution of equation (6.31). By Lemma 1.1 there is $l \in \{0, \dots, n\}$, $l + n$ is even, such that (2.14_l) is fulfilled. Assuming that $l = n$ (n is even and $l = 0$), we can easily to show that (0.5) ((0.4)) is fulfilled. Therefore $l \in \{1, \dots, n-2\}$.

By Lemma 1.3 we have

$$|u(t)| \geq \frac{t^l}{l!(n-l)!} \int_t^{+\infty} s^{n-l-1} p(s) |u(\tau_*(s))| ds \ \text{ for } \ t \geq t_*,$$

where t_* is sufficiently large. Since $t^{1-l}|u(t)|$ is a nondecreasing function, we now obtain

$$y(t) \geq \frac{\tau_*(t) y(t)}{l!(n-l)!} \int_t^{+\infty} s^{n-l-1} \tau_*^{l-1}(s) p(s) ds \ \text{ for } \ t \geq t_*,$$

where $y(t) = |u(\tau_*(t))| [\tau_*(t)]^{1-l}$.

But this result contradicts (6.30). The obtained contradiction proves the theorem. ■

Using Theorem 2.4′ and the above reasoning one can easily prove

Let $F \in V(\tau)$ *and conditions* (0.3), (6.1), (6.2) *be fulfilled. If in addition to this it is assumed that the conditions of anyone of Theorems* 6.8–6.12 *is fulfilled, where*

$$p(t) = \sigma^{2-n}(t) \int_{\tau(t)}^{\sigma(t)} s^{n-2} d_s r(s, t)$$

and τ_* *is defined by* (6.19), *then equation* (0.1) *has property* .

§ 7. Equations with a linear minorant having Properties and

Let $t_0 \in R_+$, $\varphi, \psi \in C([t_0, +\infty[;]0, +\infty[)$, ψ *be a nonincreasing function, and*

$$\lim_{t \to +\infty} \varphi(t) = +\infty, \tag{7.1}$$

$$\varliminf_{t \to +\infty} \psi(t) \cdot \widetilde{\varphi}(t) = 0, \tag{7.2}$$

where $\widetilde{\varphi}(t) = \inf\{\varphi(s) : s \geq t \geq t_0\}$. *Then there exists a sequence of* $\{t_k\}$ *such that* $t_k \uparrow +\infty$ *as* $k \uparrow +\infty$ *and*

$$\widetilde{\varphi}(t_k) = \varphi(t_k), \ \ \psi(t)\widetilde{\varphi}(t) \geq \psi(t_k)\widetilde{\varphi}(t_k) \quad \text{for } \ t_0 \leq t \leq t_k \ \ (k = 1, 2, \dots). \tag{7.3}$$

Proof. Let $t \in]t_0, +\infty[$. Introduce the sets E_i $(i = 1, 2)$ by:

$$t \in E_1 \Leftrightarrow \widetilde{\varphi}(t) = \varphi(t), \quad t \in E_2 \Leftrightarrow \widetilde{\varphi}(s)\psi(s) \geq \widetilde{\varphi}(t)\psi(t) \text{ for } s \in [t_0, t].$$

It is clear that by (7.1) and (7.2) $\sup E_i = +\infty$ $(i = 1, 2)$. Show that

$$\sup E_1 \cap E_2 = +\infty. \tag{7.4}$$

Indeed, if we assume that $t_* \in E_2$ and $t_* \notin E_1$, by (7.1) there exists $t^* > t_*$ such that $\widetilde{\varphi}(t) = \widetilde{\varphi}(t_*)$ for $t \in [t_*, t^*]$ and $\widetilde{\varphi}(t^*) = \varphi(t^*)$. On the other hand, since ψ is a nonincreasing function, we have $\psi(t)\widetilde{\varphi}(t) \geq \psi(t^*)\widetilde{\varphi}(t^*)$ for $t \in [t_0, t^*]$. Therefore $t^* \in E_1 \cap E_2$. By the above reasoning we easily ascertain that (7.4) is fulfilled. Thus there exists a sequence of points $\{t_k\}$ such that $t_k \uparrow +\infty$ for $k \uparrow +\infty$ and (7.3) holds. ■

Let $\sigma \in C(\mathbb{R}_+; \mathbb{R}_+)$ and $\lim_{t \to +\infty} \sigma(t) = +\infty$. Denote by $M^+(\sigma)$ the set of continuous mappings $\varphi : C(\mathbb{R}_+; \mathbb{R}_+) \to L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ which satisfy the conditions

$$\begin{gathered} \varphi(x)(t) \geq \varphi(y)(t) \text{ for } t \in \mathbb{R}_+ \text{ and if } x(s) \geq y(s) \geq 0 \text{ for } s \geq \sigma(t) \\ \varphi(xy)(t) \geq x(\sigma(t))\varphi(y)(t) \text{ for } t \in \mathbb{R}_+ \\ \text{if } x(t) \uparrow +\infty \text{ as } t \uparrow +\infty \text{ and } y(s) > 0 \text{ for } s \geq \sigma(t). \end{gathered} \tag{7.5}$$

Let $F \in V(\tau)$ and condition (0.2) ((0.3)) *be fulfilled, $l \in \{1, \dots, n-1\}$, $l + n$ be odd ($l + n$ be even) and for some $t_0 \in \mathbb{R}_+$*

$$|F(u)(t)| \geq \varphi(|u|)(t) \text{ for } u \in H_{t_0,\tau}, \quad t \in [t_0, +\infty[, \tag{7.6}$$

$$\varliminf_{t \to +\infty} \frac{\sigma(t)}{t} > 0, \tag{7.7}$$

where $\varphi \in M^+(\sigma)$. Besides, if it is assumed that for any $\lambda \in [l-1, l[$ and there exists $\varepsilon \in]0, 1[$ such that

$$\varliminf_{t \to +\infty} t^{l-\lambda} \int_t^{+\infty} s^{n-l-1} \varphi(\theta_\lambda)(s) ds \geq \prod_{i=0; i \neq l}^{n-1} |\lambda - i| + \varepsilon \tag{7.8$_l$}$$

and

$$\int^{+\infty} t^{n-1} \varphi(c)(t) dt = +\infty \text{ for all } c \in]0, +\infty[, \tag{7.9}$$

where $\theta_\lambda(t) = t^\lambda$, then equation (0.1) *has no proper solution satisfying* (2.14_l).

Proof. Assume the contrary, i.e. that (0.1) has a proper solution satisfying (2.14_l) where $l \in \{1, \dots, n-1\}$ and $l + n$ is odd ($l + n$ is even). By virtue of (7.5), (7.7) and (7.9) we have

$$\int^{+\infty} t^{n-l} \varphi(c\theta_{l-1})(t) dt = +\infty \text{ for all } c \in]0, +\infty[, \tag{7.10$_l$}$$

where $\theta_{l-1}(t)=t^{l-1}$. By (7.6) and (7.10) we clearly obtain

$$\int^{+\infty} t^{n-l}|u^{(n)}(t)|dt=+\infty.$$

Therefore by Lemma 1.4 and (7.6)

$$\lim_{t\to+\infty} t^{1-l}|u(t)|=+\infty,\quad \lim_{t\to+\infty} t^{-l}|u(t)|<+\infty, \tag{7.11}$$

$$|u(t)|\geq \frac{1}{(l-1)!(n-l-1)!}\int_{t_1}^{t}(t-s)^{l-1}\int_{s}^{+\infty}(\xi-s)^{n-l-1}\times$$
$$\times\varphi(|u|)(s)d\xi\,ds \quad \text{for} \quad t\geq t_1, \tag{7.12}$$

where $t_1\in R_+$ is sufficiently large.

Denote by Λ the set of $\lambda\in R_+$ satisfying

$$\lim_{t\to+\infty} t^{-\lambda}|u(t)|=+\infty.$$

Let $\lambda_0=\sup\Lambda$. By (7.11) it is clear that $\lambda_0\in[l-1,l]$ and if it is assumed that $\lambda_0=l-1$, then $\lambda_0\in\Lambda$. Therefore on account of (7.7) and (7.8_l) there exist $t_*\in[t_1+\infty[$, $\varepsilon_0\in]0,\varepsilon[$ and $\lambda^*\in[l-1,\lambda_0]\cap[l-1,l[$ such that

$$\lim_{t\to+\infty} t^{-\lambda^*}|u(t)|=+\infty,\quad \lim_{t\to+\infty} t^{-\lambda^*-\varepsilon_0}|u(t)|=0, \tag{7.13}$$

$$t^{l-\lambda^*}\int_t^{+\infty} s^{n-l-1}\varphi(\theta_{\lambda^*})(s)ds>\prod_{i=0}^{l-1}(\lambda^*+\varepsilon_0-i)\times$$
$$\times\prod_{i=l+1}^{n-1}(i-\lambda^*+\varepsilon_0)\Big(\frac{\alpha}{2}\Big)^{-\varepsilon_0} \quad \text{for } t\geq t_*,\text{[9]} \tag{7.14}$$

where

$$\alpha=\varliminf_{t\to+\infty} t^{-1}\widetilde{\tau}(t),\quad \widetilde{\tau}(t)=\inf\big\{\min\{s,\sigma(s)\}:s\geq t\big\},\quad \theta_{\lambda^*}(t)=t^{\lambda^*}.$$

Introducing the notation

$$\widetilde{u}(t)=\inf\{s^{-\lambda^*}|u(s)|:s\geq t\geq t_*\}. \tag{7.15}$$

by (7.13) we obtain

$$\widetilde{u}(t)\uparrow+\infty \quad \text{as} \quad t\uparrow+\infty, \tag{7.16}$$

$$\varliminf_{t\to+\infty} t^{-\varepsilon_0}\widetilde{u}(t)=0. \tag{7.17}$$

[9]For $l=n-1$ we have $\prod_{i=l+1}^{n-1}(i-\lambda^*+\varepsilon_0)=1$.

By virtue of (7.15)–(7.17) and Lemma 7.1 there exists an increasing sequence of points $\{t_k\}$ such that

$$\lim_{k\to+\infty} t_k=+\infty,\ \ (\widetilde{\tau}(t_k))^{-\varepsilon_0}\widetilde{u}(\widetilde{\tau}(t_k))\le \le(\widetilde{\tau}(t))^{-\varepsilon_0}\widetilde{u}(\widetilde{\tau}(t))\ \text{ for }\ t_*\le t\le t_k, \tag{7.18}$$

$$\widetilde{u}(\widetilde{\tau}(t_k))=(\widetilde{\tau}(t_k))^{-\lambda^*}|u(\widetilde{\tau}(t_k))|\ \ (k=k_0,k_0+1,\dots), \tag{7.19}$$

where k_0 is sufficiently large. Using (7.5), from (7.12) we obtain

$$|u(\widetilde{\tau}(t_k))|\ge\frac{1}{(l-1)!(n-l-1)!}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{l-1}\widetilde{u}(\widetilde{\tau}(s))\times \times\int_s^{+\infty}(\xi-s)^{n-l-1}\varphi(\theta_{\lambda^*})(\xi)d\xi\ ds\ \ (k=k_0,k_0+1,\dots). \tag{7.20}$$

It can be assumed without loss of generality that $\frac{\widetilde{\tau}(t)}{t}\ge\frac{2}{3}\alpha$ for $t\ge t_*$.

We shall first consider the case $l=n-1$. By (7.14), (7.18) and (7.19) it follows from (7.20) that

$$|u(\widetilde{\tau}(t_k))|\ge\frac{\prod_{i=0}^{n-2}(\lambda^*+\varepsilon_0-i)\left(\frac{\alpha}{2}\right)^{-\varepsilon_0}}{(n-2)!}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{n-2}\times$$
$$\times s^{\lambda^*+1-n}\widetilde{u}(\widetilde{\tau}(s))ds\ge\frac{\prod_{i=0}^{n-2}(\lambda^*+\varepsilon_0-i)\left(\frac{\alpha}{2}\right)^{-\varepsilon_0}}{(n-2)!}\frac{\widetilde{u}(\widetilde{\tau}(t_k))}{\widetilde{\tau}^{\varepsilon_0+\lambda^*}(t_k)}\times$$
$$\times\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{n-2}s^{\lambda^*+1+\varepsilon_0-n}\Big(\frac{\widetilde{\tau}(s)}{s}\Big)^{\varepsilon_0}ds\ge\frac{\left(\frac{4}{3}\right)^{\varepsilon_0}\prod_{i=0}^{n-2}(\lambda^*+\varepsilon_0-i)}{(n-2)!}\times$$
$$\times\frac{\widetilde{u}(\widetilde{\tau}(t_k))}{(\widetilde{\tau}(t_k))^{\varepsilon_0+\lambda^*}}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{n-2}s^{\lambda^*+1+\varepsilon_0-n}ds\ \ (k=k_0,k_0+1,\dots),$$

which implies

$$|u(\widetilde{\tau}(t_k))|>\frac{\prod_{i=0}^{n-2}(\lambda^*+\varepsilon_0-i)(\widetilde{\tau}(t_k))^{\varepsilon_0+\lambda^*}}{\prod_{i=0}^{n-2}(\lambda^*+\varepsilon_0-i)(\widetilde{\tau}(t_k))^{\varepsilon_0+\lambda^*}}|u(\widetilde{\tau}(t_k))|=$$
$$=|u(\widetilde{\tau}(t_k))|\ \ \ (k=k_1,k_1+1,\dots),$$

where $k_1>k_0$ is a sufficiently large number. The obtained contradiction proves the validity of the lemma for $l=n-1$.

Now consider the case $l \in \{1\dots,n-2\}$. From (7.20) we obtain

$$|u(\widetilde{\tau}(t_k))| \geq -\frac{1}{(l-1)!(n-l-1)!}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{l-1}\times$$
$$\times\widetilde{u}(\widetilde{\tau}(s))\int_s^{+\infty}(\xi-s)^{n-l-1}\xi^{-n+l+1}\times$$
$$\times d\int_\xi^{+\infty}\xi_1^{n-l-1}\varphi(\theta_{\lambda^*})(\xi_1)d\xi_1\ ds\ \ (k=k_0,k_0+1,\dots).$$

Therefore

$$|u(\widetilde{\tau}(t_k))| \geq \frac{1}{(l-1)!(n-l-1)!}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{l-1}\widetilde{u}(\widetilde{\tau}(s))\times$$
$$\times\int_s^{+\infty}\int_\xi^{+\infty}\xi_1^{n-l-1}\varphi(\theta_{\lambda^*})(\xi_1)d\xi_1\Big(\Big(\frac{\xi-s}{\xi}\Big)^{n-l-1}\Big)'d\xi\ ds.$$

Since $\Big(\Big(\frac{\xi-s}{\xi}\Big)^{n-l-1}\Big)' \geq 0$ for $\xi \geq s \geq t_*$, by (7.14) the latter inequality yields

$$|u(\widetilde{\tau}(t_k))| \geq \frac{\big(\frac{\alpha}{2}\big)^{-\varepsilon_0}\prod_{i=0}^{l-1}|\lambda^*+\varepsilon_0-i|\prod_{i=l+1}^{n-1}|\lambda^*+\varepsilon_0-i|}{(l-1)!(n-l-1)!}\times$$
$$\times\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{l-1}\widetilde{u}(\widetilde{\tau}(s))\int_s^{+\infty}\xi^{\lambda^*-l}\Big(\Big(\frac{\xi-s}{\xi}\Big)^{n-l-1}\Big)'d\xi\ ds=$$
$$=\frac{(l-\lambda^*)\big(\frac{\alpha}{2}\big)^{-\varepsilon_0}\prod_{i=0}^{l-1}|\lambda^*+\varepsilon_0-i|}{(l-1)!(n-l-1)!}\int_{t_*}^{\widetilde{\tau}(t_k)}(\widetilde{\tau}(t_k)-s)^{l-1}\times$$
$$\times\widetilde{u}(\widetilde{\tau}(s))\int_s^{+\infty}(\xi-s)^{n-l-1}\xi^{\lambda^*-n}d\xi\ ds\ \ (k=k_0,k_0+1,\dots).$$

Hence due to (7.18), (7.19) we have

$$|u(\widetilde{\tau}(t_k))| > \frac{\prod_{i=0}^{l-1}|\lambda^*+\varepsilon_0-i|\prod_{i=l+1}^{n-1}|\lambda^*+\varepsilon_0-i|(\widetilde{\tau}(t_k))^{\varepsilon_0+\lambda^*}}{\prod_{i=0}^{l-1}|\lambda^*+\varepsilon_0-i|\prod_{i=l+1}^{n-1}|i-\lambda^*|(\widetilde{\tau}(t_k))^{\varepsilon_0+\lambda^*}}\times$$
$$\times|u(\widetilde{\tau}(t_k))| \geq |u(\widetilde{\tau}(t_k))|\quad(k=k_1,k_1+1,\dots),$$

where $k_1 > k_0$ is sufficiently large. The obtained contradiction proves the validity of the lemma. ■

$'$ *Let* $F \in V(\tau)$ *and conditions* (0.2), (7.6), (7.7), (7.9) ((0.3), (7.6), (7.7), (7.9)) *be fulfilled, where* $\varphi \in M^+(\tau)$. *Besides, if it is assumed* *that* $l \in \{1,\dots,n-1\}$ $l+n$ *is odd* ($l+n$ *is even*), *and for any* $\lambda \in [l-1,l[$ *there exists* $\varepsilon \in]0,1[$ *such that*

$$\varliminf_{t\to+\infty} t\int_t^{\infty} s^{n-2-\lambda}\varphi(\theta_\lambda)(s)ds \ge \prod_{i=0}^{n-1}|\lambda-i|+\varepsilon, \tag{7.21$_l$}$$

where $\theta_\lambda(t) = t^\lambda$, *then equation* (0.1) *has no proper solution satisfying* (2.14$_l$).

Proof. To prove the lemma it is sufficient to show that condition (7.21$_l$) implies the validity of (7.8$_l$).

By (7.21) there exist $t_0 \in R_+$ and $\varepsilon_0 \in]0,\varepsilon[$ such that for any $\lambda \in [l-1,l[$ we have

$$t\int_t^{\infty} s^{n-2-\lambda}\varphi(\theta_\lambda)(s)ds \ge \prod_{i=0}^{n-1}|\lambda-i|+\varepsilon_0 \quad \text{for} \quad t\in[t_0,+\infty[,$$

so that

$$t^{l-\lambda}\int_t^{\infty} s^{n-1-\lambda}\varphi(\theta_\lambda)(s)ds = -t^{l-\lambda}\int_t^{+\infty} s^{\lambda-l+1}\times$$
$$\times d\int_s^{+\infty}\xi^{n-2-\lambda}\varphi(\theta_\lambda)(\xi)d\xi = t\int_t^{+\infty} s^{n-2-\lambda}\varphi(\theta_\lambda)(s)ds+$$
$$+(\lambda+1-l)t^{l-\lambda}\int_t^{+\infty} s^{\lambda-l}\int_s^{\infty}\xi^{n-2-\lambda}\varphi(\theta_\lambda)(\xi)d\xi\, ds \ge$$
$$\ge \Big(\prod_{i=0}^{n-1}|\lambda-i|+\varepsilon_0\Big)\Big(1+(\lambda+1-l)t^{l-\lambda}\int_t^{+\infty}\xi^{\lambda-l-1}d\xi\Big) =$$
$$= \Big(\prod_{i=0}^{n-1}|\lambda-i|+\varepsilon_0\Big)\Big(1+\frac{\lambda+1-l}{l-\lambda}\Big) \ge \prod_{i=0;i\ne l}^{n-1}|\lambda-i|+\varepsilon_0$$
$$\text{for} \quad t\in[t_0,+\infty[, \quad \lambda\in[l-1,l[.$$

Therefore (7.8$_l$) is fulfilled. ■

Let $F \in V(\tau)$ *and conditions* (0.2), (7.6), (7.7), (7.9) *be fulfilled, where* $\varphi \in M^+(\tau)$. *Besides, if condition* (7.8$_l$) *holds for any* $l \in \{1,\dots,n-1\}$ *and* $\lambda \in [l-1,l[$, *where* $l+n$ *is odd, then equation* (0.1) *has property* .

Proof. Let $u : [t_0,+\infty[\to R$ be a nonoscillatory proper solution of equation (0.1). Then by Lemma 1.1 there exists $l \in \{0,\dots,n-1\}$ such that $l+n$ is odd (2.14$_l$) is fulfilled. By Lemma 7.2 we have $l \notin \{1,\dots,n-1\}$. Assuming that n is odd and $l=0$ and using (7.9), we can easily show that (0.4) holds. ■

$'$ *Let $F \in V(\tau)$ and conditions* (0.2), (7.6), (7.7), (7.9) *be fulfilled, where $\varphi \in M^+(\tau)$. Besides, if condition* (7.21_l), *where $l+n$ is odd, is fulfilled for any $l \in \{1,\dots,n-1\}$ and $\lambda \in [l-1, l[$, then equation* (0.1) *has property .*

Proof. Since all the conditions of Theorem 7.1 are fulfilled on account of Lemma 7.2$'$, this proves the validity of Theorem 7.1$'$. ■

Let $F \in V(\tau)$, and conditions (0.3), (7.6), (7.7), (7.9), *where $\varphi \in M^+(\tau)$, be fulfilled. Besides, if condition* (7.8_l) *holds for any $l \in \{1,\dots,n-1\}$ and $\lambda \in [l-1, l[$, where $l+n$ is even, then equation* (0.1) *has property .*

Proof. Let $u : [t_0, +\infty[\to R$ be a nonoscillatory proper solution of equation (0.1). By Lemma 1.1 there exists $l \in \{0,\dots,n\}$ such that $l+n$ is even and (2.14_l) is fulfilled. By Lemma 7.2 we have $l \notin \{1,\dots,n-1\}$. If n is even and $l = 0$, then (0.4) is fulfilled.

Let $l = n$. By (7.5), (7.7) and (7.8_l) we obtain

$$\int^{+\infty} \varphi(c\theta_{n-1})(t)dt = +\infty, \ \text{ for all } \ c > 0, \tag{7.22}$$

where $\theta_{n-1}(t) = t^{n-1}$.

On the other hand, by (2.14_n) there exist $t_0 \in R_+$ and $c \in]0, +\infty[$ such that $|u(t)| \geq ct^{n-1}$ for $t \geq t_0$. Thus by (7.6) and (7.22) we find from (0.1) that

$$|u^{(n-1)}(t)| \geq \int_{t_0}^{t} \varphi(c\theta_{n-1})(s)ds \to +\infty \ \text{ for } \ t \to +\infty.$$

By (2.14_n) it is now clear that (0.5) is fulfilled. ■

$'$. *Let $F \in V(\tau)$ and conditions* (0.3), (7.6), (7.7), (7.9) *hold, where $\varphi \in M^+(\tau)$. Besides, if* (7.21_l) *is fulfilled for any $l \in \{1,\dots,n-1\}$ and $\lambda \in [l-1, l[$ where $l+n$ is even, then equation* (0.1) *has property .*

Proof. The Theorem is valid because all the conditions of Theorem 7.2 are fulfilled by Lemma 7.2$'$. ■

Let $F \in V(\tau)$, condition (0.2) *be fulfilled and let for any $t_0 \in R_+$*

$$|F(u)(t)| \geq \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)| d_s r_i(s,t) \ \text{ for } \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{7.23}$$

where

$$\begin{gathered} \tau_i; \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+), \ \ \tau_i(t) \leq \sigma_i(t) \ \text{ for } \ t \in R_+, \\ r_i(s,t), \ \ r_i \ \text{ is measurable}, \ \ r_i(\cdot, t) \uparrow, \ \text{ as } s \uparrow \ \ (i = 1,\dots,m), \end{gathered} \tag{7.24}$$

$$\varliminf_{t \to +\infty} \frac{\tau_i(t)}{t} > 0 \quad (i = 1,\dots,m). \tag{7.25}$$

Besides, if for any $l \in \{1,\dots,n-1\}$ *and* $\lambda \in [l-1,l[$, *where* $l+n$ *is odd, there exists* $\varepsilon \in]0,1[$ *such that*

$$\varliminf_{t\to+\infty} t^{l-\lambda}\int_t^{+\infty}\xi^{n-l-1}\sum_{i=1}^m\int_{\tau_i(\xi)}^{\sigma_i(\xi)} s^\lambda d_s r_i(s,\xi)d\xi > \prod_{i=0;i\neq l}^{n-1}|\lambda-i|+\varepsilon \quad (7.26_l)$$

and

$$\int^{+\infty} t^{n-1}\sum_{i=1}^m \big(r_i(\sigma_i(t),t)-r_i(\tau_i(t),t)\big)dt=+\infty, \tag{7.27}$$

then equation (0.1) *has property* .

Proof. To prove the theorem it suffices to note that the conditions of Theorem 7.1 are fulfilled by virtue of (7.23) and (7.24)

$$\begin{gathered}\varphi(x)(t)=\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)} x(s)d_s r_i(s,t),\\ \tau(t)=\min\{\tau_i(t): i=1,\dots,m\}.\end{gathered} \tag{7.28}$$ ■

Let $F \in V(\tau)$, *condition* (0.2) *be fulfilled and let for any* $t_0 \in R_+$

$$|F(u)(t)| \geq \sum_{i=1}^m p_i(t)|u(\tau_i(t))| \ \textit{ for } \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{7.29}$$

where

$$\begin{gathered} p_i \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+), \quad \tau_i \in C(\mathbb{R}_+;\mathbb{R}_+),\\ \varliminf_{t\to+\infty}\frac{\tau_i(t)}{t}>0 \quad (i=1,\dots,m).\end{gathered} \tag{7.30}$$

Besides, if for any $l \in \{1,\dots,n-1\}$ *and* $\lambda \in [l-1,l[$, *where* $l+n$ *is odd, there exists* $\varepsilon \in]0,1[$ *such that*

$$\varliminf_{t\to+\infty} t^{l-\lambda}\int_t^{+\infty} s^{n-l-1}\sum_{i=1}^m p_i(s)\tau_i^\lambda(s)ds \geq \prod_{i=0;i\neq l}^{n-1}|\lambda-i|+\varepsilon \quad (7.31_l)$$

and

$$\int^{+\infty} t^{n-1}\sum_{i=1}^m p_i(t)dt=+\infty, \tag{7.32}$$

then equation (0.1) *has property* .

If $F \in V(\tau)$, *condition* (0.2) *is fulfilled and for any* $t_0 \in \mathbb{R}_+$ *we have*

$$|F(u)(t)| \geq \frac{c}{t^{n+1}} \int_{\alpha t}^{\overline{\alpha} t} |u(s)| ds \ \text{ for } \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{7.33}$$

where $0 < \alpha < \overline{\alpha}$ *and*

$$\begin{aligned} c > \max\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1) \times \\ \times(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0, n-1]\}, \end{aligned} \tag{7.34}$$

then equation (0.1) *has property* .

Let $c > 0$, $0 < \alpha < \overline{\alpha}$. *Then for the equation*

$$u^{(n)}(t) + \frac{c}{t^{n+1}} \int_{\alpha t}^{\overline{\alpha} t} u(s) ds = 0 \tag{7.35}$$

to have property *it is necessary and sufficient that* (7.34) *be fulfilled.*

′ *Let* $F \in V(\tau)$ *and conditions* (0.2), (7.23)–(7.25), (7.27) *be fulfilled. Besides, if for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, *where* $l + n$ *is odd, there exists* $\varepsilon \in]0, 1[$ *such that the inequality*

$$\varliminf_{t \to +\infty} t \int_t^{+\infty} \xi^{n-2-\lambda} \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{\lambda} d_s r_i(s, \xi) d\xi \geq \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon, \tag{7.36}$$

holds, then equation (0.1) *has property* .

Proof. It suffices to note that the conditions of Theorem 7.1′ are fulfilled with φ and τ defined by (7.28). ■

Let $F \in V(\tau)$ *and conditions* (0.2), (7.29), (7.30), (7.32) *be fulfilled. Besides, if for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, *where* $l + n$ *is odd, there exists* $\varepsilon \in]0, 1[$ *such that*

$$\varliminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2-\lambda} \sum_{i=1}^{m} p_i(s) \tau_i^{\lambda}(s) ds \geq \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon, \tag{7.37}$$

then equation (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.3), (7.23)–(7.25), (7.27) *be fulfilled. Besides, if* (7.26_l) *holds for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$ *where* $l + n$ *is even, then equation* (0.1) *has property* .

Proof. It suffices to note that the conditions of Theorem 7.2 are fulfilled with φ and τ defined by (7.28). ■

Let $F \in V(\tau)$ *and conditions* (0.3), (7.29), (7.30), (7.32) *be fulfilled. Besides, if for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, *where* $l+n$ *is even, there exists* $\varepsilon \in]0,1[$ *such that* (7.31_l) *holds, then equation* (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.3), (7.33) *hold for* $0 < \alpha < \overline{\alpha}$. *Besides, if*

$$c > \max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0, n-1]\},$$

then equation (0.1) *has property* .

Let $c < 0$ *and* $0 < \alpha < \overline{\alpha}$. *Then for equation* (7.35) *to have property* *it is necessary and sufficient that*

$$c < -\max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0, n-1]\}.$$

′ *Let* $F \in V(\tau)$ *and conditions* (0.3), (7.23)–(7.25), (7.27) *be fulfilled. Besides, if for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, *where* $l+n$ *is even, there exists* $\varepsilon \in]0,1[$ *such that* (7.36) *holds, then equation* (0.1) *has property* .

Proof. It suffices to note that the conditions of Theorem 7.2′ are fulfilled with φ and τ defined by (7.28). ■

Let $F \in V(\tau)$ *and conditions* (0.3), (7.29), (7.30), (7.32) *be fulfilled. Besides, if for any* $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, *where* $l+n$ *is even, there exists* $\varepsilon \in]0,1[$ *such that* (7.37) *holds, then equation* (0.1) *has property* .

Let $F \in V(\tau)$, *conditions* (0.2), (7.23)–(7.25), (7.27) *be fulfilled and*

$$\sigma_i(t) \le t \quad \textit{for} \quad t \in R_+ \quad (i = 1, \dots, m). \tag{7.38}$$

Besides, if for any $\lambda \in [n-2, n-1]$ *there exists* $\varepsilon \in]0,1]$ *such that* (7.26_{n-1}) *holds, then equation* (0.1) *has property* .

Proof. Assume the contrary, that equation (0.1) has no property . In that case by (0.2) and (7.23) the inequality

$$u^{(n)}(t) \operatorname{sign} u(t) + \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)| d_s r_i(s,t) \le 0 \tag{7.39}$$

has no property .

Let $u : [t_0, +\infty[\to R$ be a nonoscillatory proper solution of (7.39). Then by Lemma 1.1 there exists $l \in \{0, \dots, n-1\}$ ($l+n$ is odd) such that (2.14_l) is fulfilled. If it is assumed that n is odd and $l = 0$, then (0.4) will hold by virtue of (7.27) because following our assumption inequality (7.39) has no property . Assume that $l \in \{1, \dots, n-1\}$. By (7.25) and (7.27) it is clear that

$$\int^{+\infty} t^{n-l} |u^{(n)}(t)| dt = +\infty \quad \text{for all} \quad l \in \{1, \dots, n-1\}.$$

Therefore by Lemma 1.3

$$|u(t)|/t^l \downarrow \quad \text{for} \quad t \uparrow .$$

Hence on the interval $[t_1, +\infty[$, where t_1 is sufficiently large, the function u is a solution of the inequality

$$u^{(n)}(t) \operatorname{sign} u(t) + \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^l d_s r_i(s,t) \frac{|u(\sigma_i(t))|}{(\sigma_i(t))^l} \leq 0.$$

Like in the case of Lemma 5.2 we can show that on the interval $[t_2, +\infty[$ the last inequality has an oscillatory proper solution u_1 satisfying (2.14_{n-1}) where $t_2 \geq t_1$ is sufficiently large. On the other hand, by Lemma 1.3 we have

$$\frac{u_1(t)}{t^l} \uparrow +\infty \quad \text{for} \quad t \uparrow +\infty.$$ [10]

Therefore on the interval $[t_2, +\infty[$ the function u_1 is a solution of inequality (7.39) satisfying (2.14_{n-1}). But Theorem 7.3 and condition (7.26_{n-1}) imply that (7.39) has no solution satisfying (2.14_{n-1}). The obtained contradiction proves the theorem. ■

If $F \in V(\tau)$, conditions (0.2), (7.29), (7.30), (7.32) *are fulfilled with $\lambda \in [n-2, n-1]$ and there exists $\varepsilon \in]0,1]$ such that condition* (7.31_{n-1}) *holds, then equation* (0.1) *has property* .

If $F \in V(\tau)$ and conditions (0.2), (7.33) *are fulfilled with $0 < \alpha < \overline{\alpha} \leq 1$, then for equation* (0.1) *to have property it is sufficient that*

$$c > \max\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1) \times \\ \times (\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [n-2, n-1]\}. \tag{7.40}$$

Let $c > 0$, $0 < \alpha < \overline{\alpha} \leq 1$. Then condition (7.40) *is necessary and sufficient for equation* (7.35) *to have property* .

Let $F \in V(\tau)$, condition (0.2) *be fulfilled and for some* $t_0 \in R_+$

$$|F(u)(t)| \geq \frac{1}{t^n} \sum_{i=1}^{m} c_i |u(\alpha_i t)| \quad \text{for} \ t \in [t_0, +\infty[, \ \ u \in M_{t_0}, \tag{7.41}$$

[10] It is assumed that $l < n-2$ because otherwise $l = n-1$, i.e. (7.39) has a solution satisfying (2.14_{n-1}).

where $0 < \alpha_i \leq 1$ *and* $c_i > 0$ $(i = 1, \dots, m)$*. Then the condition*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > -\lambda(\lambda-1)\cdots(\lambda-n+1) \;\; for \;\; \lambda \in [n-2, n-1]$$

is sufficient for equation (0.1) *to have property* .

[12]. *Let* $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *and*

$$\varliminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2} p(s)ds > \max\{-\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [n-2, n-1]\}.$$

Then equation (6.8) *has property* .

If $F \in V(\tau)$ *and conditions* (0.3), (7.23)–(7.25), (7.27), (7.38) *are fulfilled, then for equation* (0.1) *to have property it is sufficient in the case of an even* n (*of an odd* n) *that for any* $\lambda \in [n-1, n-2[$ (*for any* $\lambda \in [0,1] \cup [n-3, n-2]$) *there exist* $\varepsilon \in]0,1[$ *such that conditions* (7.26_{n-2}) ((7.26_1) *and* (7.26_{n-2})) *be fulfilled.*

Proof. Let us assume the contrary, i.e. that equation (0.1) has no property . Then by (0.3) and (7.23) the inequality

$$u^{(n)}(t) \operatorname{sign} u(t) \geq \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)| d_s r_i(s,t) \tag{7.42}$$

has no property . Let $u \in [t_0, +\infty[\to R$ be a nonoscillatory proper solution of inequality (7.42). Following Lemma 1.1, there exists $l \in \{0, \dots, n\}$ ($l + n$ is even) such that (2.14_l) is fulfilled. If we assume that $l = n$ (n is even and $l = 0$), then (0.5) ((0.4)) will be fulfilled by virtue of (7.25) and (7.27). Thus, since by our assumption inequality (7.42) has no property , we conclude that $l \in \{1, \dots, n-2\}$. If $l \in \{2, \dots, n-2\}$, then by a reasoning similar to that used in considering Theorem 7.5 we prove that by Corollary 5.1 inequality (7.42) has an oscillatory proper solution satisfying (2.14_{n-2}). This means that for an even n (for an odd n) inequality (7.42) has a solution satisfying (2.14_{n-2}) ((2.14_1) or (2.14_{n-2})).

However by Theorem 7.4 and (7.26_1), (7.26_{n-2}) the differential inequality (7.42) has no proper solution satisfying (2.14_1), (2.14_{n-2}). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$ *and conditions* (0.3), (7.29), (7.30), (7.38) *be fulfilled. Besides, if in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [n-2, n-1]$ (*for any* $\lambda \in [0,1[\cup [n-2, n-1[$) *there exists* $\varepsilon \in]0,1[$ *such that condition* (7.31_{n-2}) ((7.31_1) *and* (7.31_{n-2})) *holds, then equation* (0.1) *has property* .

Let $F \in V(\tau)$ and conditions (0.3), (7.33) *be fulfilled, where $0 < \alpha < \overline{\alpha} \le 1$ and $c \in]0,+\infty[$. Moreover, if*

$$c > \max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)\times$$
$$\times(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0,1] \cup [n-3, n-2]\},$$

then equation (0.1) *has property* .

Let $c < 0$ and $0 < \alpha < \overline{\alpha} \le 1$. Then for equation (7.35) *to have property it is necessary and sufficient that*

$$c < -\max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)\times$$
$$\times(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0,1] \cup [n-3, n-2]\}.$$

′ *If $F \in V(\tau)$ and conditions* (0.3), (7.23)–(7.25), (7.27), (7.38) *are fulfilled, then for equation* (0.1) *to have property it is sufficient in the case of an even n (in the case of an odd n) that for any $\lambda \in [n-3, n-2]$ ($\lambda \in [0,1[\cup[n-3, n-2[$) there exist $\varepsilon \in]0,1[$ such that* (7.36) *be satisfied.*

Proof. This theorem is proved like Theorem 7.6, if we replace Theorem 7.4 by Theorem 7.4′. ■

If $F \in V(\tau)$, and conditions (0.3), (7.29), (7.30), (7.32), (7.38) *are fulfilled, then for equation* (0.1) *to have property it is sufficient in the case of an even n (in the case of an odd n) that for any $\lambda \in [n-3, n-2[$ (for any $\lambda \in [0,1[\cup[n-3, n-2]$) there exist $\varepsilon \in]0,1[$ such that condition* (7.37) *hold.*

Let $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_-)$ and

$$\varliminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}|p(s)|ds > \max\big\{\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0,1]\cup[n-3, n-2]\big\}.$$

Then equation (6.8) *has property* .

In the case of an even n Corollary 2 from [12] is a particular case of the above corollary.

Let $F \in V(\tau)$ and conditions (0.3), (7.41) *be fulfilled, where $c_i > 0$ and $0 < \alpha_i \le 1$ $(i = 1, \dots, m)$. Besides, if*

$$\sum_{i=1}^{m} c_i\alpha_i^{\lambda} > \lambda(\lambda-1)\cdots(\lambda-n+1) \ \ \textit{for} \ \ \lambda \in [0,1] \cup [n-3, n-2],$$

then equation (0.1) *has property* .

If $F \in V(\tau)$, *conditions* (0.2), (7.23), (7.24), (7.27) *are fulfilled,*

$$\tau_i(t) \geq t \ \ for \ \ t \in R_+ \ \ (i = 1, \dots, m) \tag{7.43}$$

and in the case of an even n (*in the case of an odd* n) *for any* $\lambda \in [0, 1[$ (*for any* $\lambda \in [1, 2[\cup[n-2, n-1[$) *there exists* $\varepsilon \in]0, 1[$ *such that condition* (7.26_1) ((7.26_2) *and* (7.26_{n-1})) *holds, then equation* (0.1) *has property* .

Proof. Assume the contrary, i.e. that (0.1) has no property . Then by (0.2) and (7.23) inequality (7.39) has no property .

Assuming $u : [t_0, +\infty[\to R$ to be an oscillatory proper solution of inequality (7.39), by Lemma 1.1 there exists $l \in \{0, \dots, n-1\}$ ($l + n$ is odd) such that (2.14_l) is satisfied. If n is odd and $l = 0$, then (0.4) is fulfilled. Since (7.39) has no property , we conclude that $l \in \{1, \dots, n-1\}$. Similarly to Lemma 5.4 we show that for an even n (for an odd n) inequality (7.39) has a proper solution of form (2.14_1) ((2.14_2) or (2.14_{n-1})).

On the other hand, on account of Theorem 7.3 and (7.26_1) ((7.26_2) and (7.26_{n-1})) inequality (7.39) has no proper solution satisfying (7.14_1) ((7.14_2), (7.14_{n-1})). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$, *conditions* (0.2), (7.29) (7.32), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [0, 1[$ (*for any* $\lambda \in [1, 2[\cup[n-2, n-1[$) *there exist* $\varepsilon \in]0, 1[$ *such that condition* (7.31_1) ((7.31_2) *and* (7.31_{n-1})) *holds. Then equation* (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.2), (7.33) *be fulfilled, where* $1 \leq \alpha < \overline{\alpha}$ *and* $c > 0$. *Moreover, if for an even* n (*for an odd* n)

$$c > \max\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1) \times \times(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [0, 1]\} \tag{7.44}$$

$$\Big(c > \max\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1) \times \times(\overline{\alpha}^{\,\lambda+1} - \alpha^{\lambda+1})^{-1} : \lambda \in [1, 2] \cup [n-2, n-1]\}\Big), \tag{7.45}$$

then equation (0.1) *has property* .

Let $c > 0$ *and* $1 \leq \alpha < \overline{\alpha}$. *Then in the case of an even* n (*in the case of an odd* n) *condition* (7.44) ((7.45)) *is necessary and sufficient for equation* (7.35) *to have property* .

′. *Let* $F \in V(\tau)$, *conditions* (0.2), (7.23)–(7.25), (7.27), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [0, 1[$ (*for any* $\lambda \in [1, 2[\cup[n-2, n-1[$) *there exist* $\varepsilon \in]0, 1[$ *such that* (7.36) *holds. Then equation* (0.1) *has property* .

Proof. This theorem is proved like Theorem 7.7, if we replace Theorem 7.3 by Theorem 7.3′. ■

Let $F \in V(\tau)$, *conditions* (0.2), (7.29), (7.32), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [0,1[$ (*for any* $\lambda \in [1,2[\cup[n-2,n-1[$) *there exist* $\varepsilon \in]0,1[$ *such that* (7.37) *holds. Then equation* (0.1) *has property* .

Let $F \in V(\tau)$, *conditions* (0.3), (7.23), (7.24), (7.27), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [1,2[$ (*for any* $\lambda \in [0,1[$) *there exist* $\varepsilon \in]0,1[$ *such that* (7.26_2) $((7.26_1))$ *holds. Then equation* (0.1) *has property* .

Proof. Assume the contrary, i.e. that (0.1) has no property B. Then by (0.3) and (7.23) inequality (7.42) has no property .

Assume $u : [t_0,+\infty[\to R$ to be a nonoscillatory proper solution of inequality (7.42). Then by Lemma 1.1 there exists $l \in \{0,\dots,n\}$ ($l+n$ is even) such that (2.14_l) holds. If $l=n$ (n is even and $l=0$), then (0.5) ((0.4)) is satisfied. Therefore, since inequality (7.42) has no property by virtue of our assumption, we conclude that $l \in \{1,\dots,n-2\}$. Similarly to Lemma 5.5 it can be shown that for an even n (for an odd n) inequality (7.42) has a proper solution satisfying (2.14_2) $((2.14_1))$.

On the other hand, according to Theorem 7.4 and (7.26_2) $((7.26_1))$ for an even n (for an odd n) inequality (7.42) has no solution satisfying (7.14_2) $((7.14_1))$. The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$, *conditions* (0.3), (7.29), (7.32), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [1,2[$ (*for any* $\lambda \in [0,1[$) *there exist* $\varepsilon \in]0,1[$ *such that* (7.31_2) $((7.31_1))$ *holds. Then equation* (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.3), (7.33) *be fulfilled, where* $1 \le \alpha < \overline{\alpha}$ *and* $c > 0$. *Moreover, if*

$$c > \max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)(\overline{\alpha}^{\lambda+1}-\alpha^{\lambda+1})^{-1} : \lambda \in [0,2]\},$$

then equation (0.1) *has property* .

Let $c < 0$ *and* $1 \le \alpha < \overline{\alpha}$. *Then for equation* (7.35) *to have property* *it is necessary and sufficient that*

$$c < -\max\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)(\overline{\alpha}^{\,\lambda+1}-\alpha^{\lambda+1})^{-1} : \lambda \in [0,2]\}.$$

′ *Let* $F \in V(\tau)$, *conditions* (0.3), (7.23), (7.24), (7.27), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [1,2[$ (*for any* $\lambda \in [0,1[$) *there exist* $\varepsilon \in]0,1[$ *such that* (7.36) *holds. Then equation* (0.1) *has property* .

Proof. This theorem is proved like Theorem 7.8, if we use Theorem 7.3′ instead of Theorem 7.3. ■

Let $F \in V(\tau)$, *conditions* (0.3), (7.29), (7.32), (7.43) *be fulfilled and in the case of an even* n (*in the case of an odd* n) *for any* $\lambda \in [1, 2[$ (*for any* $\lambda \in [0, 1[$) *there exist* $\varepsilon \in]0, 1[$ *such that condition* (7.37) *holds. Then equation* (0.1) *has property* .

Let $F \in V(\tau)$ *and conditions* (0.3), (7.41) *be fulfilled, where* $\alpha_i \geq 1$ *and* $c_i > 0$ $(i = 1, \dots, m)$. *Then the condition*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > \lambda(\lambda-1)\cdots(\lambda-n+1) \quad \textit{for} \quad \lambda \in [0, 2]$$

is sufficient for equation (0.1) *to have property* .

The corollaries formulated in Subsection 7.2 are exact, which is testified to by the validity of the following

Let $F \in V(\tau)$, *condition* (0.2) ((0.3)) *be fulfilled and*

$$|F(u)(t)| \leq \varphi(u)(t) \quad \textit{for} \quad t \in \mathbb{R}_+, \quad u \in C(\mathbb{R}_+; \mathbb{R}_+), \tag{7.46}$$

where $\varphi : C(\mathbb{R}_+; \mathbb{R}_+) \to L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *is a continuous mapping,*

$$\begin{gathered} \varphi(x)(t) \geq \varphi(y)(t) \quad \textit{if} \quad x, y \in C(\mathbb{R}_+; \mathbb{R}_+) \quad \textit{and} \\ x(s) \geq y(s) \quad \textit{for} \quad s \geq \tau(t). \end{gathered} \tag{7.47}$$

Moreover, if for some $t_0 \in \mathbb{R}_+$, $l \in \{1, \dots, n-1\}$ *and* $\lambda \in [l-1, l[$, [11] *where* $l + n$ *is odd* ($l + n$ *is even*) *we have*

$$t^{l-\lambda} \int_t^{+\infty} s^{n-l-1} \varphi(\theta_\lambda)(s) ds \leq \prod_{i=0; i \neq l}^{n-1} |\lambda - i| \quad \textit{for} \quad t \geq t_0, \tag{7.48}$$

where $\theta_\lambda(t) = t^\lambda$, *then equation* (0.1) *has a proper solution satisfying* (2.14_l).

Proof. Let U be the set of functions $u \in C([t_0, +\infty[; \mathbb{R})$ satisfying the condition

$$u(t) = c_l \quad \text{for} \quad t \in [t_0, t_*], \quad c(t-t_*)^{l-1} \leq u(t) \leq t^\lambda \quad \text{for} \quad t > t_*, \tag{7.49}$$

where $t_* = \max\{s : \tau_*(s) \leq t_0\}$, $\tau_*(t) = \min\{t, \tau(t)\}$, $c \in \left[0, \frac{1}{l!} \prod_{i=0}^{l-2} |\lambda - i|\right]$ and $c_l = c$ for $l = 1$, $c_l = 0$ for $l > 1$. Define the operator $T : U \to C([t_0, +\infty[; \mathbb{R})$ by

$$T(u)(t) = \begin{cases} c(t-t_*)^{l-1} + \frac{(-1)^{n+l+1}}{(l-1)!(n-l-1)!} \int_{t_*}^t (t-s)^{l-1} \int_s^{+\infty} (\xi - s)^{n-l-1} \times \\ \qquad \times F(u)(\xi) d\xi \, ds \quad \text{for} \quad t > t_*, \\ c_l \qquad \text{for} \quad t \in [t_0, t_*]. \end{cases} \tag{7.50}$$

[11] When $\lambda = l - 1$ we have $\varphi(\theta_\lambda)(t) = 0$ almost everywhere on the interval $[t_0, +\infty[$. Since in that case the validity of the theorem is obvious, it will be assumed below that $\lambda \in]l-1, l[$.

Show that $TU \subset U$. By (0.2) ((0.3)) and (7.49) and the assumption that $l+n$ is odd ($l+n$ is even) we find that if $u \in U$, then $T(u)(t) \geq c(t-t_*)^{l-1}$ for $t \geq t_*$. Show that if $u \in U$, then $T(u)(t) \leq t^\lambda$ for $t \geq t_*$.

Consider at first the case $l = n-1$. By virtue of (7.46)–(7.49) from (7.50) we obtain

$$T(u)(t) \leq c(t-t_*)^{n-2} + \frac{\prod_{i=0}^{n-2}(\lambda - i)}{(n-2)!} \int_{t_*}^{t} (t-s)^{n-2} s^{\lambda-(n-1)} ds \leq$$

$$\leq c(t-t_*)^{n-2} - \frac{\prod_{i=0}^{n-3}(\lambda - i)}{(n-2)!} t_*^{\lambda+2-n}(t-t_*)^{n-2} + t^\lambda \leq t^\lambda \ \text{ for } \ t > t_*.$$ [12]

Let now $l < n-1$. Then by (7.46)–(7.49) from (7.50) we have

$$T(u)(t) \leq c(t-t_*)^{l-1} - \frac{1}{(l-1)!(n-l-1)!} \int_{t_*}^{t} (t-s)^{l-1} \times$$

$$\times \int_{s}^{+\infty} \Big(\frac{\xi - s}{\xi}\Big)^{n-l-1} d \int_{\xi}^{+\infty} \xi_1^{n-l-1} \varphi(\theta_\lambda)(\xi_1) d\xi_1 \ ds = c(t-t_*)^{l-1} +$$

$$+ \frac{1}{(l-1)!(n-l-1)!} \int_{t_*}^{t} (t-s)^{l-1} \int_{s}^{+\infty} \Big(\Big(\frac{\xi - s}{\xi}\Big)^{n-l-1}\Big)' \times$$

$$\times \int_{\xi}^{+\infty} \xi_1^{n-l-1} \varphi(\theta_\lambda)(\xi_1) d\xi_1 \ d\xi \ ds \leq c(t-t_*)^{l-1} +$$

$$+ \frac{\prod_{i=0; i \neq l}^{n-1} |\lambda - i|}{(l-1)!(n-l-1)!} \int_{t_*}^{t} (t-s)^{l-1} \int_{s}^{+\infty} \xi^{\lambda - l} \Big(\Big(\frac{\xi - s}{\xi}\Big)^{n-l-1}\Big)' d\xi \ ds =$$

$$= c(t-t_*)^{l-1} + \frac{\prod_{i=0}^{n-1} |\lambda - i|}{(l-1)!(n-l-1)!} \int_{t_*}^{t} (t-s)^{l-1} \times$$

$$\times \int_{s}^{+\infty} (\xi - s)^{n-l-1} \xi^{\lambda - n} d\xi \ ds = c(t-t_*)^{l-1} + \frac{\prod_{i=0}^{l-1} |\lambda - i|}{(l-1)!} \times$$

$$\times \int_{t_*}^{t} (t-s)^{l-1} s^{\lambda - l} ds \leq c(t-t_*)^{l-1} - \frac{\prod_{i=0}^{l-2} |\lambda - i|}{(l-1)!} (t-t_*)^{l-1} \times$$

$$\times t_*^{\lambda - l + 1} + t^\lambda \leq t^\lambda \ \text{ for } \ t > t_*.$$

Therefore $TU \subset U$.

On the other hand, it is obvious that U is a closed bounded convex set. We easily ascertain that the operator T is continuous and TU is an equicontinuous set on every finite segment of the interval $[t_0, +\infty[$. Therefore by Lemma 2.1 there exists $u \in U$ such that $Tu = u$. It is easy to see that the function u is a proper solution of equation (0.1) satisfying (2.14_l). ■

′. *Let conditions* (0.2), (7.46), (7.47) ((0.3), (7.46), (7.47)) *be fulfilled and assume that for some* $t_0 \in \mathbb{R}_+$, $l \in \{1, \dots, n-1\}$ *and*

[12] Without loss of generality it can be assumed here that $t_* \geq 1$.

$\lambda \in [l-1, l[$, where $l+n$ is odd ($l+n$ is even) we have

$$t\int_t^{+\infty} \xi^{n-2-\lambda}\varphi(\theta_\lambda)(\xi)d\xi \le \prod_{i=0}^{n-1}|\lambda - i| \ \ \textit{for} \ \ t \ge t_0, \tag{7.51}$$

with $\theta_\lambda(t) = t^\lambda$. Then equation (0.1) *has a proper solution satisfying* (2.14_l).
Proof. To prove the theorem it suffices to show that (7.51) implies (7.48). By (7.51) we have

$$t^{l-\lambda}\int_t^{+\infty} \xi^{n-l-1}\varphi(\theta_\lambda)(\xi)d\xi = -t^{l-\lambda}\int_t^{+\infty}\xi^{\lambda-l+1} \times$$
$$\times d\int_\xi^{+\infty} s^{n-2-\lambda}\varphi(\theta_\lambda)(s)ds = t\int_t^{+\infty}\xi^{n-2-\lambda}\varphi(\theta_\lambda)(\xi)d\xi +$$
$$+(\lambda+1-l)t^{l-\lambda}\int_t^{+\infty}\xi^{\lambda-l}\int_\xi^{+\infty} s^{n-2-\lambda}\varphi(\theta_\lambda)(s)ds\, d\xi \le$$
$$\le \prod_{i=0}^{n-1}|\lambda-i|\Big(1+\frac{\lambda+1-l}{l-\lambda}\Big) = \prod_{i=0; i\ne l}^{n-1}|\lambda - i|. \quad \blacksquare$$

§ 8. Some auxiliary statements

A proper solution $u : [t_0, +\infty[\to \mathbb{R}$ of equation (0.1) is called Kneser-type if there exists $t_1 \in [t_0, +\infty[$ such that

$$(-1)^i u^{(i)}(t) u(t) > 0 \quad \text{for} \quad t \geq t_1 \ \ (i = 0, \dots, n-1). \tag{8.1}$$

Let

$$p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad \delta \in C(\mathbb{R}_+; \mathbb{R}_+), \tag{8.2}$$

$$\delta(t) \leq t \quad \textit{for} \quad t \in \mathbb{R}_+, \quad \lim_{t \to +\infty} \delta(t) = +\infty \tag{8.3}$$

and δ *BE nondecreasing. Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of the equation*

$$u^{(n)}(t) + (-1)^{n+1} p(t) u(\delta(t)) = 0 \tag{8.4}$$

satisfying (8.1). *Then for any* $k \in \{0, \dots, n-1\}$ *we have*

$$\rho_k(t) |u(\delta(t))| \leq (n-k-1)!(n-1)! u_k(t) \ \ \textit{for} \ \ t \geq t_*, \tag{8.5}$$

where

$$t_* = \eta_\delta(t_1), \ \ \eta_\delta(t) = \max\{s : \delta(s) \leq t\}, \tag{8.6}$$

$$u_k(t) = \sum_{i=k}^{n-1} \frac{t^{i-k} |u^{(i)}(t)|}{(i-k)!}, \tag{8.7}$$

$$\rho_k(t) = \max\{\psi_k(t, s, \tau) : \tau \in [t, \eta_\delta(t)], \ s \in [\delta(\tau), t]\} \tag{8.8$_k$}$$

and

$$\psi_k(t, s, \tau) = \int_s^t \xi^{n-k-1} p(\xi) d\xi \int_t^\tau \xi^{n-k-1} p(\xi) d\xi \ \times$$
$$\times \Big(s^{k+1-n} (s - \delta(\tau))^{n-1} + \frac{1}{(n-1)!} \int_{\delta(\tau)}^s (\xi - \delta(\xi))^{n-1} \ \times$$
$$\times p(\xi) (\xi - \delta(\xi))^{n-1} \xi^{k+1-n} d\xi \Big). \tag{8.9$_k$}$$

Proof. By (8.4) and (8.1)

$$u_k(s) \geq \frac{1}{(n-k-1)!} \int_s^t \xi^{n-k-1} p(\xi) |u(\delta(\xi))| d\xi \ \ \text{for} \ \ t_* \leq s \leq t, \tag{8.10}$$

where t_* is defined by (8.6) and u_k – by (8.7).

Let $t \in [t_*, +\infty[$ and (s_0, τ_0) be the point of maximum of the function $\psi_k(t, \cdot, \cdot)$ on $[\delta(\tau_0), t] \times [t, \eta_\delta(t)]$. Then by (8.10)

$$u_k(s_0) \geq \frac{1}{(n-k-1)!} \int_{s_0}^{t} s^{n-k-1} p(s) |u(\delta(s))| ds \geq$$
$$\geq \frac{1}{(n-k-1)!} \int_{s_0}^{t} s^{n-k-1} p(s) ds |u(\delta(t))|, \tag{8.11}$$
$$u_k(t) \geq \frac{1}{(n-k-1)!} \int_{t}^{\tau_0} s^{n-k-1} p(s) |u(\delta(s))| ds \geq$$
$$\geq \frac{1}{(n-k-1)!} \int_{t}^{\tau_0} s^{n-k-1} p(s) ds |u(\delta(\tau_0))|. \tag{8.12}$$

On the other hand, since $u_k(t)$ is nonincreasing, from (8.2) and (1.60_n) we obtain

$$|u(\delta(\tau_0))| \geq \frac{(n-k-1)!}{(n-1)!} \Big[s_0^{k+1-n} (s_0 - \delta(\tau_0))^{n-1} +$$
$$+ \frac{1}{(n-1)!} \int_{\delta(\tau_0)}^{s_0} (\xi - \delta(\tau_0))^{n-1} \times$$
$$\times p(\xi)(\xi - \delta(\xi))^{n-1} \xi^{k+1-n} d\xi \Big] u_k(s_0)$$

whence by (8.11) and (8.12) it follows the validity of (8.5), where ρ_k is defined by (8.8_k) and (8.9_k). ■

Let (8.2) *and* (8.3) *be fulfilled and*

$$\varliminf_{t \to +\infty} \int_{\delta(t)}^{t} p(s) ds > 0, \tag{8.13}$$
$$\text{vrai sup}\{p(t) : t \in \mathbb{R}_+\} < +\infty, \tag{8.14}$$

where the function δ *is nondecreasing. Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.4) *satisfying* (8.1). *Then*

$$\varlimsup_{t \to +\infty} \frac{|u(\delta(t))|}{|u^{(n-1)}(t)|} < +\infty. \tag{8.15}$$

Proof. According to Lemma 8.1, it suffices to show that

$$\varliminf_{t \to +\infty} \rho_{n-1}(t) > 0, \tag{8.16}$$

where the function ρ_{n-1} is defined by (8.8_{n-1}) and (8.9_{n-1}).

In view of (8.13) there exist $c > 0$ and $t_2 > t_1$ such that

$$\int_{\delta(t)}^{t} p(s) ds \geq c \text{ for } t \geq t_2. \tag{8.17}$$

Let $t \in [t_2, +\infty[$. Then by (8.17) there exist $t^* \in]t, \eta_\delta(t)]$, $\overline{t} \in]t, t^*[$ and $\underline{t} \in]\delta(t^*), t[$ such that [13]

$$\int_{\delta(t^*)}^{\underline{t}} p(s)ds \geq \frac{c}{4}, \quad \int_{\overline{t}}^{t} p(s)ds \geq \frac{c}{4}, \quad \int_{t}^{\overline{t}} p(s)ds \geq \frac{c}{4}. \tag{8.18}$$

According to (8.8_{n-1}) and (8.9_{n-1}) it is clear that

$$\rho_{n-1}(t) \geq \int_{\underline{t}}^{t} p(s)ds \int_{t}^{\overline{t}} p(s)ds(\underline{t} - \delta(\overline{t}))^{n-1}. \tag{8.19}$$

On the other hand, by (8.14) from (8.17) we have $\underline{t} - \delta(\overline{t}) \geq \underline{t} - \delta(t^*) \geq \frac{c}{4r}$, where $r = \text{vrai}\sup\{p(t) : t \in \mathbb{R}_+\}$. Therefore (8.18) and (8.19) imply

$$\rho_{n-1}(t) \geq \frac{c^3}{64r} \quad \text{for} \quad t \geq t_2$$

whence it follows the validity of (8.16). ■

Let (8.2), (8.3) *be fulfilled and for some* $k \in \{0, \dots, n-1\}$

$$\varliminf_{t \to +\infty} \int_{\delta(t)}^{t} s^{n-k-1} p(s)ds > 0, \tag{8.20$_k$}$$

$$\text{vrai}\sup\{t^{n-k} p(t) : t \in \mathbb{R}_+\} < +\infty, \tag{8.21$_k$}$$

where δ *is nondecreasing. Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.4) *satisfying* (8.1). *Then*

$$\varlimsup_{t \to +\infty} \frac{\delta^k(t)|u(\delta(t))|}{u_k(t)} < +\infty, \tag{8.22}$$

where u_k *is defined by* (8.7).

Proof. According to Lemma 8.1 it suffices to show that

$$\varliminf_{t \to +\infty} \rho_k(t)\delta^{-k}(t) > 0. \tag{8.23}$$

By (8.20_k) there exist $c > 0$ and $t_2 > t_1$ such that

$$\int_{\delta(t)}^{t} s^{n-k-1} p(s)ds \geq c \quad \text{for} \quad t \geq t_2. \tag{8.24}$$

Let $t \in [t_2, +\infty[$. By (8.24) there exist $t^* \in]t, \eta_\delta(t)]$, $\overline{t} \in]t, t^*[$ and $\underline{t} \in]\delta(t^*), t[$ such that

$$\int_{\delta(t^*)}^{\underline{t}} s^{n-k-1} p(s)ds \geq \frac{c}{4}, \quad \int_{\underline{t}}^{t} s^{n-k-1} p(s)ds \geq \frac{c}{4},$$
$$\int_{t}^{\overline{t}} s^{n-k-1} p(s)ds \geq \frac{c}{4}. \tag{8.25}$$

[13] the function η_δ is defined by (8.6)

From (8.8_k) and (8.9_k) it follows

$$\rho_k(t) \geq \int_{\underline{t}}^{t} s^{n-k-1} p(s) ds \int_{t}^{\overline{t}} s^{n-k-1} p(s) ds \ \underline{t}^{k+1-n} (\underline{t} - \delta(\overline{t}))^{n-1}. \tag{8.26}$$

On the other hand, (8.21_k) and (8.25) imply $\delta(t^*) \leq \exp\{-\frac{c}{4r_k}\}\underline{t}$ with $r_k = \operatorname{vrai}\sup\{t^{n-k}p(t) : t \in \mathbb{R}_+\}$. Hence, since $\underline{t} \geq \delta(t)$, by (8.26) and (8.25) we have

$$\rho_k(t) \geq \frac{c^2}{16} \delta^k(t) \Big(1 - \exp\Big\{ - \frac{c}{4r_k} \Big\}\Big)^{n-1} \quad \text{for} \ \ t \geq t_2.$$

Therefore (8.23) is valid. ■

Let (8.2) *and* (8.3) *be fulfilled and for some* $t_0 \in \mathbb{R}_+$

$$\int_{\delta(t)}^{t} p(s) ds > 0 \quad \textit{for} \quad t \geq t_0. \tag{8.27}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of the inequality*

$$(-1)^{n+1} u^{(n)}(t) \operatorname{sign} u(t) + p(t) |u(\delta(t))| \leq 0 \tag{8.28}$$

satisfying (8.1). *Then there exists* $t_* \geq t_1$ *such that* (8.4) *has a proper solution* $u_* : [t_*, +\infty[\to]0, +\infty[$ *satisfying*

$$(-1)^i u_*^{(i)}(t) > 0 \ \ \textit{for} \ \ t \geq t_* \ \ (i = 0, \dots, n-1), \tag{8.29}$$

$$|u_*^{(i)}(t)| \leq |u^{(i)}(t)| \ \ \textit{for} \ \ t \geq t_* \ \ (i = 0, \dots, n-1). \tag{8.30}$$

Proof. According to (8.28)

$$|u^{(i)}(t)| \geq \frac{1}{(n-i-1)!} \int_t^{+\infty} (s-t)^{n-i-1} p(s) |u(\delta(s))| ds \tag{8.31}$$
$$\text{for} \quad t \geq t_* \quad (i = 0, \dots, n-1),$$

where $t_* = \eta_\delta(t_1)$ (the function η_δ is defined by (8.6)).

Consider the sequence $\{u_i\}_{i=1}^{+\infty}$ of functions defined by

$$u_1(t) = |u(t)| \quad \text{for} \quad t \geq t_1,$$

$$u_i(t) = \begin{cases} \frac{1}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} p(s) u_{i-1}(\delta(s)) ds \ \text{for} \ t \geq t_*, \\ u_i(t_*) + |u(t)| - |u(t_*)| \ \text{for} \ t \in [t_1, t_*[\ \ (i = 2, 3, \dots). \end{cases}$$

This sequence is obviously decreasing. Its limit u_* is a solution of the integral equation

$$u_*(t) = \begin{cases} \frac{1}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} p(s) u_*(\delta(s)) ds \ \text{for} \ t \geq t_*, \\ u_*(t_*) + |u(t)| - |u(t_*)| \ \text{for} \ t \in [t_1, t_*[. \end{cases} \tag{8.32}$$

on $[t_*, +\infty[$.

Show that

$$u_*(t) > 0 \quad \text{for} \quad t \geq t_*. \tag{8.33}$$

Suppose the contrary. Then there exists $t^* \in [t_*, +\infty[$ such that

$$u_*(t) \equiv 0 \;\text{ for }\; t \geq t^*, \;\; u_*(t) > 0 \;\text{ for }\; t \in [t_1, t^*[. \tag{8.34}$$

Denote by E the set of all $t \in [t^*, +\infty[$ satisfying $\delta(t) = t^*$ and put $t^0 = \inf E$. By (8.27) and (8.34) there exists $t_*^0 \in]t^*, t^0]$ such that

$$\int_{t^*}^{t_*^0} (s - t^*)^{n-1} p(s) u_*(\delta(s)) ds > 0.$$

Therefore (8.32) implies

$$u_*(t^*) \geq \frac{1}{(n-1)} \int_{t^*}^{+\infty} (s - t^*)^{n-1} p(s) u_*(\delta(s)) ds \geq$$
$$\geq \frac{1}{(n-1)!} \int_{t^*}^{t_*^0} (s - t^*)^{n-1} p(s) u_*(\delta(s)) ds > 0.$$

But this contradicts (8.34). The obtained contradiction proves that (8.33) is fulfilled. On the other hand, according to (8.31), (8.32) and (8.33) u_* is a solution of (8.4) satisfying (8.29) and (8.30). ■

Let (8.2) *and* (8.3) *be fulfilled, the function* δ *be nondecreasing and for some* $k \in \{0, \dots, n-1\}$ *and* $t_0 \in \mathbb{R}_+$ *let*

$$\rho_k(t) > 0 \;\; \textit{for} \;\; t \geq t_0. \tag{8.35_k}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then*

$$\varliminf_{t \to +\infty} u_k(t) \exp\left\{(n-1)! \int_{t_1}^{t} p(s) s^{n-k-1} (\rho_k(s))^{-1} ds\right\} > 0, \tag{8.36}$$

where u_k *and* ρ_k *are defined respectively by* (8.7) *and* (8.8_k), (8.9_k).

Proof. According to Lemma 8.4 equation (8.4) has a proper solution $u_* : [t_*, +\infty[\to]0, +\infty[$ satisfying (8.29) and (8.30), where t_* is sufficiently large. By (8.35_k) and Lemma 8.1

$$\frac{u_*(\delta(t))}{u_{*k}(t)} \leq (n-k-1)!(n-1)!(\rho_k(t))^{-1} \;\text{ for }\; t \geq t_2,$$

where $t_2 \geq t_*$ is sufficiently large and

$$u_{*k}(t) = \sum_{i=k}^{n-1} \frac{t^{i-k} |u_*^{(i)}(t)|}{(i-k)!}. \tag{8.37}$$

Therefore from (8.4) we have

$$u_{*k}(t) = u_{*k}(t_2)\exp\Big\{-\frac{1}{(n-k-1)!}\int_{t_2}^{t} s^{n-k-1}p(s)\frac{u_*(\delta(s))}{u_{*k}(s)}ds\Big\} \geq$$
$$\geq u_{*k}(t_2)\exp\Big\{-(n-1)!\int_{t_2}^{t} s^{n-k-1}p(s)(\rho_k(s))^{-1}ds\Big\} \text{ for } t \geq t_2$$

whence, taking into account (8.7), (8.30) and (8.37), we deduce (8.36). ■

Taking into account Lemmas 8.1–8.5, we can easily ascertain the validity of the following corollaries.

Let (8.2) *and* (8.3) *be fulfilled, δ be nondecreasing, for some* $k \in \{0, \dots, n-1\}$ (8.35_k) *hold and*

$$\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\int_{t_0}^{t} p(s)s^{n-k-1}(\rho_k(s))^{-1}ds < +\infty, \tag{8.38}$$

where ρ_k is defined by (8.8_k) *and* (8.9_k). *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then there exists $\lambda > 0$ such that*

$$|u(t)|e^{\lambda t} \to +\infty \quad \textit{for} \quad t \to +\infty. \tag{8.39}$$

′ *Let* (8.2), (8.3), (8.13) *and* (8.14) *be fulfilled with δ nondecreasing. Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then $\lambda > 0$ exists such that* (8.39) *holds.*

Let (8.2) *and* (8.3) *be fulfilled, δ be nondecreasing, for some* $k \in \{0, \dots, n-1\}$ (8.35_k) *hold and*

$$\overline{\lim_{t\to+\infty}}\,\frac{1}{\ln t}\int_{t_0}^{t} p(s)s^{n-k-1}(\rho_k(s))^{-1}ds < +\infty, \tag{8.40}$$

where ρ_k is defined by (8.8_k) *and* (8.9_k). *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then there exists $\lambda > 0$ such that*

$$|u(t)|t^{\lambda} \to +\infty \quad \textit{for} \quad t \to +\infty. \tag{8.41}$$

′ *Let* (8.2) *and* (8.3) *be fulfilled with δ nondecreasing and* (8.20_k) *and* (8.21_k) *hold for some* $k \in \{0, \dots, n-1\}$. *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then there exists $\lambda > 0$ such that* (8.41) *holds.*

Let (8.2) *and* (8.3) *be fulfilled, δ be nondecreasing, for some* $k \in \{0, \dots, n-1\}$ (8.35_k) *hold and for some* $r \in \{2, 3, \dots\}$

$$\overline{\lim_{t\to+\infty}}\,\frac{1}{\ln_r t}\int_{t_0}^{t} s^{n-k-1}p(s)(\rho_k(s))^{-1}ds < +\infty, \tag{8.42_k}$$

where $\ln_1 t = \ln t$, $\ln_i t = \ln \ln_{i-1} t$ $(i = 2, \dots, r)$ *and* ρ_k *is defined by* (8.8_k) *and* (8.9_k). *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then there exists* $\lambda > 0$ *such that*

$$|u(t)| \ln_{r-1}^{\lambda} t \to +\infty \quad \text{for} \quad t \to +\infty. \tag{8.43}$$

′ *Let* (8.2), (8.3), (8.20_k) *and* (8.21_k) *hold for some* $k \in \{0, \dots, n-1\}$ *and for some* $r \in \{2, 3, \dots\}$

$$\varlimsup_{t \to +\infty} \frac{1}{\ln_r t} \int_{t_0}^{t} s^{n-k-1} p(s) \delta^{-k}(s) ds < +\infty. \tag{8.44_k}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (8.28) *satisfying* (8.1). *Then there exists* $\lambda > 0$ *such that* (8.43) *holds.*

§ 9. On the existence of Kneser-type solutions

Let $t_0 \in \mathbb{R}_+$. Denote by $H^{-}_{t_0,\tau}$ the set of all functions $u \in \widetilde{C}^{n-1}_{loc}(\mathbb{R}_+; \mathbb{R})$ satisfying

$$(-1)^i u^{(i)}(t) u(t) > 0 \ \ (i = 0, \dots, n-1),$$
$$(-1)^n u^{(n)}(t) u(t) \geq 0 \ \text{ for } \ t \geq t_*,$$

where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$.

Let $F \in V(\tau)$ *and for some* $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \varphi(|u|)(t) \quad \text{for} \quad u \in H^{-}_{t_0,\tau}, \quad t \geq t_0, \tag{9.1}$$

where $\varphi \in M^+(\sigma)$[14] *and*

$$\sigma \in C(\mathbb{R}_+; \mathbb{R}_+) \ \text{ is nondecreasing, } \ \sigma(t) \leq t \quad \text{for } \ t \in \mathbb{R}_+, \ \ \lim_{t \to +\infty} \sigma(t) = +\infty. \tag{9.2}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a proper solution of* (0.1) *satisfying* (8.1) *and there exist* $\gamma \in \widetilde{C}_{loc}([t_0, +\infty[; \mathbb{R}_+)$, $r_2 > 0$ *and* $r_1 \in [0, r_2[$ *such that*

$$\begin{gathered} \gamma(t) \uparrow +\infty \ \text{ as } \ t \uparrow +\infty, \quad \lim_{t \to +\infty} (\gamma(t))^{r_2} |u(t)| = +\infty, \\ \varliminf_{t \to +\infty} (\gamma(t))^{r_1} |u(t)| = 0, \quad \varlimsup_{t \to +\infty} \frac{\gamma(t)}{\gamma(\sigma(t))} < +\infty. \end{gathered} \tag{9.3}$$

Then

$$\varliminf_{t \to +\infty} (\gamma(t))^{r_2} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \leq (n-1)! c^{r_2 - r_1}, \tag{9.4}$$

[14] The definition of the set see on the page 75

where

$$\theta(t) = (\gamma(t))^{-r_2}, \quad c = \overline{\lim_{t\to+\infty}} \frac{\gamma(t)}{\gamma(\sigma(t))}.$$

Proof. Denote

$$\widetilde{u}(t) = \inf\{(\gamma(s))^{r_2}|u(s)| : s \geq t\}. \tag{9.5}$$

By (9.2), (9.3) and (9.5) we have

$$\widetilde{u}(\sigma(t)) \uparrow +\infty \quad \text{for} \quad t \uparrow +\infty \tag{9.6}$$

and

$$\underline{\lim}_{t\to+\infty} \widetilde{u}(\sigma(t))(\gamma(t))^{r_1-r_2} = 0. \tag{9.7}$$

According to (9.5), (9.7) and Lemma 7.1, there exists a sequence of numbers $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \uparrow +\infty$ as $k \uparrow +\infty$ and

$$\begin{gathered}\widetilde{u}(\sigma(t_k)) = (\gamma(\sigma(t_k))^{r_2}|u(\sigma(t_k))|, \ \ (\gamma(t_k))^{r_1-r_2}\widetilde{u}(\sigma(t_k)) \leq \\ \leq (\gamma(t))^{r_1-r_2}\widetilde{u}(\sigma(t)) \quad \text{for} \quad t_* \leq t \leq t_k,\end{gathered} \tag{9.8}$$

where $t_* > t_1$ is sufficiently large.

On the other hand, taking into account (8.1) and (9.1) from (0.1) we have

$$\begin{gathered}|u(\sigma(t))| \geq \frac{1}{(n-1)!}\int_{\sigma(t)}^{t}(s-\sigma(t))^{n-1}\varphi(|u|)(s)ds + \\ +\frac{1}{(n-1)!}\int_{t}^{+\infty}(s-\sigma(t))^{n-1}\varphi(|u|)(s)ds \ \text{ for } \ t \geq t_*.\end{gathered}$$

Hence by (9.5), (9.8) and the fact that $\varphi \in M^+(\sigma)$ we obtain

$$\begin{gathered}|u(\sigma(t_k))| \geq \frac{1}{(n-1)!}\int_{\sigma(t_k)}^{t_k}(s-\sigma(t_k))^{n-1}\widetilde{u}(\sigma(s))\varphi(\theta)(s)ds + \\ +\frac{1}{(n-1)!}\int_{t_k}^{+\infty}(s-\sigma(t_k))^{n-1}\widetilde{u}(\sigma(s))\varphi(\theta)(s)ds \geq \\ \geq \frac{(\gamma(t_k))^{r_1-r_2}(\gamma(\sigma(t_k)))^{r_2}|u(\sigma(t_k))|}{(n-1)!}\int_{\sigma(t_k)}^{t_k}(s-\sigma(t_k))^{n-1} \times \\ \times(\gamma(s))^{r_2-r_1}\varphi(\theta)(s)ds + \frac{(\gamma(\sigma(t_k)))^{r_2}|u(\sigma(t_k))|}{(n-1)!} \times \\ \times\int_{t_k}^{+\infty}(s-\sigma(t_k))^{n-1}\varphi(\theta)(s)ds = \frac{(\gamma(\sigma(t_k)))^{r_2}|u(\sigma(t_k))|}{(n-1)!}\times \\ \times\Big(-(\gamma(t_k))^{r_1-r_2}\int_{\sigma(t_k)}^{t_k}(\gamma(s))^{r_2-r_1}d\int_{s}^{+\infty}(\xi-\sigma(t_k))^{n-1} \times\end{gathered}$$

$$\times\varphi(\theta)(s)ds+\int_{t_k}^{+\infty}(s-\sigma(t_k))^{n-1}\varphi(\theta)(s)ds\Big)\geq$$
$$\geq\frac{(\gamma(\sigma(t_k)))^{r_2}|u(\sigma(t_k))|}{(n-1)!}\Big(\frac{\gamma(\sigma(t_k))}{\gamma(t_k)}\Big)^{r_2-r_1}\times$$
$$\times\int_{\sigma(t_k)}^{+\infty}(s-\sigma(t_k))^{n-1}\varphi(\theta)(s)ds\quad k=2,3,\dots,\tag{9.9}$$

where $\theta(t)=(\gamma(t))^{-r_2}$.

Suppose that $\varepsilon\in]0,\varepsilon_0[$. Then (9.9) implies

$$(\gamma(\sigma(t_k)))^{r_2}\int_{\sigma(t_k)}^{+\infty}(s-\sigma(t_k))^{n-1}\varphi(\theta)(s)ds\leq(n-1)!(c+\varepsilon)^{r_2-r_1}$$
$$\text{for}\;\; k=k_0,k_0+1,\dots,$$

where $k_0\in\mathbb{N}$ is sufficiently large. Since ε is arbitrary, hence it follows (9.4). ■

Let $F\in V(\tau)$ and (9.1) *and* (9.2) *be fulfilled, where $\varphi\in M^+(\sigma)$ and*

$$\varliminf_{t\to+\infty}(\sigma(t)-t)>-\infty.\tag{9.10}$$

Moreover, let

$$\varphi(|u|)(t)\geq p(t)|u(\delta(t))|\;\;\textit{for}\;\; u\in H^-_{t_0,\tau},\;\; t\geq t_0,\tag{9.11}$$

for some $t_0\in\mathbb{R}_+$ (8.2), (8.3) *be fulfilled with δ nondecreasing and for some $k\in\{0,\dots,n-1\}$* (8.35_k), (8.38) *hold, where ρ_k is defined by* (8.8_k), (8.9_k). *Then the condition*

$$\inf\Big\{\varliminf_{t\to+\infty}e^{\lambda t}\int_t^{+\infty}(s-t)^{n-1}\varphi(\theta)(s)ds:\lambda\in]0,+\infty[\Big\}>(n-1)!,\tag{9.12}$$

where $\theta(t)=e^{-\lambda t}$, is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u:[t_0,+\infty[\;\to\mathbb{R}$ satisfying (8.1). According to (8.35_k), (8.38), (9.1), (9.11) and Corollary 8.1 there exists $\lambda>0$ such that (8.39) is fulfilled.

Denote by Λ the set of all λ satisfying (8.39) and put $\lambda_0=\inf\Lambda$. By (9.12) there exist $t_*>t_1$ and $\varepsilon>0$ such that

$$e^{\lambda t}\int_t^{+\infty}(s-t)^{n-1}\varphi(\theta)(s)ds\geq(n-1)!+\varepsilon\tag{9.13}$$
$$\text{for}\;\; t\geq t_*,\;\;\lambda\in]\lambda_0,\lambda_0+\varepsilon].$$

Choose $\varepsilon_2 \in]0, \varepsilon[$ and $\varepsilon_1 \in [0, \varepsilon_2[$ such that

$$\begin{gathered}\lambda_0 - \varepsilon_1 \geq 0, \quad c^{\varepsilon_2+\varepsilon_1}(n-1)! < (n-1)! + \varepsilon, \\ \varlimsup_{t\to+\infty} e^{(\lambda_0+\varepsilon_2)t}|u(t)| = +\infty, \quad \varliminf_{t\to+\infty} e^{(\lambda_0-\varepsilon_1)t}|u(t)| = 0,\end{gathered} \tag{9.14}$$

where $c = \varlimsup\limits_{t\to+\infty} e^{t-\sigma(t)}$. According to (9.1), (9.10) and (9.14) the conditions of Lemma 9.1 are obviously satisfied with $\gamma(t) = e^t$, $r_2 = \lambda_0 + \varepsilon_2$ and $r_1 = \lambda_0 - \varepsilon_1$. Therefore by (9.14) this lemma implies

$$\varliminf_{t\to+\infty} e^{(\lambda_0+\varepsilon_2)t} \int_t^{+\infty} (s-t)^{n-1}\varphi(\theta)(s)ds \leq c^{\varepsilon_1+\varepsilon_2}(n-1)! < (n-1)! + \varepsilon,$$

where $\theta(t) = e^{-(\lambda_0+\varepsilon_2)t}$. But this contradicts (9.13). ∎

Let $F \in V(\tau)$ and (8.2), (8.3), (9.1), (9.2), (9.10) *and* (9.11) *be fulfilled, where $\varphi \in M^+(\sigma)$. Let, moreover, for some $k \in \{0, \dots, n-1\}$* (8.35$_k$) *and* (8.38) *hold with a nondecreasing δ. Then the condition*

$$\inf\left\{\lambda^{-n}(\operatorname*{vrai\,inf}_{t\geq t_0} e^{\lambda t}\varphi(\theta)(t)) : \lambda \in]0, +\infty[\right\} > 1, \tag{9.15}$$

where $\theta(t) = e^{\lambda t}$ and $t_0 \in \mathbb{R}_+$, is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that (9.15) implies (9.12). ∎

′ *Let $F \in V(\sigma)$ and* (8.2), (8.3), (8.13), (8.14), (9.1), (9.2), (9.10) *and* (9.11) *be fulfilled, where $\varphi \in M^+(\sigma)$ and δ is nondecreasing. Then the condition* (9.12) ((9.15)) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. The assertion of the theorem follows from Corollary 8.1′ and Theorem 9.1 (Corollary 9.1). ∎

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1}F(u)(t)\operatorname{sign} u(t) \geq \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|d_s r_i(s,t) \tag{9.16}$$
$$\text{for} \quad u \in H^-_{t_0,\tau}, \quad t \geq t_0,$$

where

$$\tau_i; \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+), \ \ \tau_i(t) \leq \sigma_i(t) \ \text{for } t \in \mathbb{R}_+, \ \ (i = 1, \dots, m), \tag{9.17}$$

$$\varliminf_{t\to+\infty} (\tau_i(t) - t) > -\infty \quad (i = 1, \dots, m), \tag{9.18}$$

$$r_i(s,t) \text{ are measurable}, \ \ r_i(\cdot, t) \text{ are nondecreasing} \ \ (i = 1, \dots, m). \tag{9.19}$$

Let, moreover, there exist $i_0 \in \{1,\dots,m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$\tau_{i_0}(t) \le \delta(t) \le \min\{t, \sigma_{i_0}(t)\} \tag{9.20}$$

and for some $k \in \{0,\dots,n-1\}$ *and* $t_* \in \mathbb{R}_+$ *let*

$$\rho_k(t) > 0 \quad \textit{for} \quad t \ge t_*, \tag{9.21_k}$$

$$\overline{\lim_{t\to+\infty}}\, \frac{1}{t}\int_{t_*}^{t} s^{n-k-1} p(s)(\rho_k(s))^{-1} ds < +\infty, \tag{9.22_k}$$

where ρ_k *is defined by* (8.8_k), (8.9_k) *and*

$$p(t) = r_{i_0}(\delta(t),t) - r_{i_0}(\tau_{i_0}(t),t). \tag{9.23}$$

Then the condition

$$\inf\Big\{ \underline{\lim_{t\to+\infty}}\, e^{\lambda t}\int_t^{+\infty}(s-t)^{n-1}\sum_{i=1}^{m}\int_{\tau_i(s)}^{\sigma_i(s)} e^{-\lambda\xi} d_\xi \,\times$$
$$\times r_i(\xi,s)ds : \lambda \in]0,+\infty[\Big\} > (n-1)!. \tag{9.24}$$

is sufficient for the (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to show that the operator defined by

$$\varphi(u)(t) = \sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)} u(s) d_s r_i(s,t) \tag{9.25}$$

satisfies the conditions of the Theorem 9.1. Indeed, taking into account (9.20) and (9.25) we see that (9.11) holds with p defined by (9.23). On the other hand, by (9.18) the conditions (9.10) and

$$\varphi(xy)(t) \ge x(\sigma(t))\varphi(y)(t)$$
$$\text{for all } \ x,y \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ x(t)\uparrow +\infty \ \text{ as } \ t\uparrow +\infty$$

are fulfilled with

$$\sigma(t) = \inf\{\min(\tau_i(s) : i = 1,\dots,m) : s \ge t\}. \tag{9.26}$$

Therefore $\varphi \in M^+(\sigma)$, so according to (9.21_k), (9.22_k) and (9.24) the operator defined by (9.25) satisfies all the conditions of Theorem 9.1. ■

$'$ *Let* $F \in V(\tau)$, *the conditions* (9.16)–(9.19) *be fulfilled and let there exist* $i_0 \in \{1,\dots,m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that* (9.20), (8.13) *and* (8.14) *hold, where* p *is defined by* (9.23). *Then the condition* (9.24) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. The assertion of the theorem follows from Corollary 8.1$'$ and Theorem 9.2. ■

Let $F \in V(\tau)$, $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$, $\Delta_{i_0} > 0$ *for some* $i_0 \in \{1, \dots, m\}$ *and for some* $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \sum_{i=1}^{m} c_i \int_{t-\Delta_i}^{t-\overline{\Delta}_i} |u(s)| ds \tag{9.27}$$
$$\text{for} \quad u \in H^-_{t_0,\tau}, \quad t \geq t_0.$$

Then the condition

$$\inf \Big\{ \lambda^{-n-1} \sum_{i=1}^{m} c_i (e^{\lambda \Delta_i} - e^{\lambda \overline{\Delta}_i}) : \lambda \in]0, +\infty[\Big\} > 1 \tag{9.28}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that in view of (9.27) and (9.28) the conditions of Theorem 9.2′ are fulfilled with $\tau_i(t) = t - \Delta_i$, $\sigma_i(t) = t - \overline{\Delta}_i$, $r_i(s,t) = c_i s$ $(i = 1, \dots, m)$. ■

′ *Let* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and there exist* $i_0 \in \{1, \dots, m\}$ *such that* $\Delta_{i_0} > 0$. *Then the condition* (9.28) *is necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} c_i \int_{t-\Delta_i}^{t-\overline{\Delta}_i} u(s) ds = 0 \tag{9.29}$$

not to have a Kneser-type solution.

Proof. The sufficiency follows from the Corollary 9.2. If we assume that (9.29) is violated, then (9.29) obviously has the proper solution $u(t) = e^{\lambda t}$ with $\lambda < 0$. ■

Let $F \in V(\tau)$ *and for some* $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \sum_{i=1}^{m} p_i(t) |u(\delta_i(t))| \tag{9.30}$$
$$\text{for} \quad u \in H^-_{t_0,\tau}, \quad t \geq t_0,$$

where

$$\delta_i \in C(\mathbb{R}_+; \mathbb{R}_+), \quad \lim_{t \to +\infty} \delta_i(t) = +\infty, \qquad p_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+) \ \ (i = 1, \dots, m), \tag{9.31}$$

$$\varliminf_{t \to +\infty} (\delta_i(t) - t) > -\infty \quad (i = 1, \dots, m). \tag{9.32}$$

Let, moreover, $i_0 \in \{1,\dots,m\}$ exist such that $\delta_{i_0}(t) \le t$ and for $t \in \mathbb{R}_+$, for some $k \in \{0,\dots,n-1\}$ the conditions (8.35_k), (8.38) *be fulfilled with $\rho_k(t)$ defined by* (8.8_k), (8.9_k), *$p(t) \equiv p_{i_0}(t)$ and $\delta(t) \equiv \delta_{i_0}(t)$. Then the condition*

$$\inf\Big\{ \varliminf_{t\to+\infty} e^{\lambda t} \int_t^{\infty} (s-t)^{n-1} \sum_{i=1}^{m} p_i(s) e^{-\lambda\delta_i(s)} ds : \\ : \lambda \in]0,+\infty[\Big\} > (n-1)! \tag{9.33}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. According to (9.30) the condition (9.16) is fulfilled with

$$\begin{gathered} \tau_i(t) = \delta_i(t) - 1, \quad \sigma_i(t) = \delta_i(t), \\ r_i(s,t) = p_i(t) e(s - \delta_i(t)) \ \ (i = 1,\dots,m).\text{[15]} \end{gathered} \tag{9.34}$$

Therefore by (9.30)–(9.33) all the conditions of Theorem 9.2 are satisfied. This proves the validity of the theorem. ■

Let $F \in V(\tau)$, the conditions (9.30)–(9.32) *be fulfilled and $i_0 \in \{1,\dots,m\}$ exist such that $\delta_{i_0}(t) \le t$ for $t \in \mathbb{R}_+$. Let, moreover, for some $k \in \{0,\dots,n-1\}$ the conditions* (8.35_k), (8.38) *hold with $\rho_k(t)$ defined by* (8.8_k), (8.9_k) *and $p(t) \equiv p_{i_0}(t)$. Then the condition*

$$\inf\Big\{ \lambda^{-n} \operatorname*{vrai\,inf}_{t\ge t_*} \Big(\sum_{i=1}^{m} p_i(t) e^{\lambda(t-\delta_i(t))} \Big) : \lambda \in]0,+\infty[\Big\} > 1 \tag{9.35}$$

with $t_* \in \mathbb{R}_+$ is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that (9.35) implies (9.33). ■

′ *Let $F \in V(\tau)$, the conditions* (9.30)– (9.32) *be fulfilled and $\delta_i(t) \le t$ for $t \in \mathbb{R}_+$ $(i = 1,\dots,m)$. Let, moreover, $i_0 \in \{1,\dots,m\}$ exist such that $\delta_{i_0}(t)$ is nondecreasing, for some $k \in \{0,\dots,n-1\}$ the conditions* (8.35_k), (8.38) *hold with ρ_k defined by* $(8.8._k)$, (8.9_k) *and $p(t) = p_{i_0}(t)$. Then the condition*

$$\operatorname{vrai\,inf}\Big\{ \sum_{i=1}^{m} p_i(t)(t-\delta_i(t))^n : t \in]t_*,+\infty[\Big\} > \Big(\frac{n}{e}\Big)^n \tag{9.36}$$

with $t_* \in \mathbb{R}_+$ is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. Since $e^x \ge x^n \Big(\frac{e}{n}\Big)^n$ for $x \ge 0$, (9.36) obviously implies (9.35). ■

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[15] everywhere below by $e(t)$ we mean

$$e(t) = \begin{cases} 0 & \text{for} \quad t \in]-\infty, 0[, \\ 1 & \text{for} \quad t \in [0,+\infty[. \end{cases}$$

In view of Corollary 8.1′ Theorem 9.3 (Corollary 9.3) easily implies

′ *Let* $F \in V(\tau)$, *the conditions* (9.30)–(9.32) *be fulfilled and* $i_0 \in \{1, \dots, m\}$ *exist such that* $\delta_{i_0}(t) \le t$ *for* $t \in \mathbb{R}_+$. *Let, moreover,* (8.13), (8.14) *hold with* $\delta_{i_0}(t) \equiv \delta(t)$ *and* $p_{i_0}(t) \equiv p(t)$. *Then the condition* (9.33) ((9.35)) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, $c_i > 0$, $\Delta_i \in \mathbb{R}$ $(i = 1, \dots, m)$, $i_0 \in \{1, \dots, m\}$ *exists such that* $\Delta_{i_0} > 0$ *and for some* $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \ge \sum_{i=1}^{m} c_i |u(t - \Delta_i)|$$
$$\text{for} \quad u \in H^-_{t_0 \tau}, \quad t \ge t_0.$$

Then the condition

$$\inf \Big\{ \lambda^{-n} \sum_{i=1}^{m} c_i e^{\lambda \Delta_i} : \lambda \in]0, +\infty[\Big\} > 1 \tag{9.37}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

′ *Let* $c_i > 0$, $\Delta_i \in \mathbb{R}$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then* (9.37) *is necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} c_i u(t - \Delta_i) = 0$$

not to have a Kneser-type solution.

Let $F \in V(\tau)$, *conditions* (9.1), (9.2), (9.11), (8.2) *and* (8.3) *be fulfilled with* $\varphi \in M^+(\sigma)$ *and*

$$\varliminf_{t \to +\infty} \frac{\sigma(t)}{t} > 0. \tag{9.38}$$

Let, moreover, for some $k \in \{0, \dots, n-1\}$ *conditions* (8.35_k) *and* (8.40) *hold with* $\rho_k(t)$ *defined by* (9.8_k) *and* (9.9_k). *Then the condition*

$$\inf \Big\{ \varliminf_{t \to +\infty} t^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds : \lambda \in]0, +\infty[\Big\} > (n-1)! \tag{9.39}$$

with $\theta(t) = t^{-\lambda}$ *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying (8.1). According to (8.35_k), (8.40), (9.1), (9.11) and Corollary 8.2 there exists $\lambda > 0$ such that (8.41) holds. Denote by Λ the set of all λ satisfying (8.41) and put $\lambda_0 = \inf \Lambda$. By (9.39) there exist $t_* > t_1$ and $\varepsilon > 0$ such that

$$t^\lambda \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \geq (n-1)! + \varepsilon \qquad (9.40)$$
$$\text{for} \quad t \geq t_*, \quad \lambda \in]\lambda_0, \lambda_0 + \varepsilon].$$

Choose $\varepsilon_2 \in]0, \varepsilon[$ and $\varepsilon_1 \in [0, \varepsilon_2[$ such that

$$\begin{gathered} \lambda_0 - \varepsilon_1 \geq 0, \quad c^{\varepsilon_1 + \varepsilon_2}(n-1)! < (n-1)! + \varepsilon, \\ \lim_{t \to +\infty} t^{\lambda_0 + \varepsilon_2} |u(t)| = +\infty, \quad \varliminf_{t \to +\infty} t^{\lambda_0 - \varepsilon_1} |u(t)| = 0, \end{gathered} \qquad (9.41)$$

where $c = \varlimsup\limits_{t \to +\infty} \left(\frac{t}{\sigma(t)} \right)$. By (9.1), (9.38) and (9.41) all the conditions of Lemma 9.1 are fulfilled with $\gamma(t) = t$, $r_2 = \lambda_0 + \varepsilon_2$ and $r_1 = \lambda_0 - \varepsilon_1$. Therefore, taking into account (9.41), this lemma implies

$$\varliminf_{t \to +\infty} t^{\lambda_0 + \varepsilon_2} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \leq (n-1)! c^{\varepsilon_1 + \varepsilon_2} < (n-1)! + \varepsilon,$$

where $\theta(t) = t^{-(\lambda_0 + \varepsilon_2)}$. But this inequality contradicts (9.40). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$ *and conditions* (9.1), (9.2), (9.11), (8.2), (8.3) *and* (9.38) *be fulfilled with* $\varphi \in M^+(\sigma)$. *Let, moreover, for some* $k \in \{0, \dots, n-1\}$ (8.35$_k$) *and* (8.40) *hold with* $\rho_k(t)$ *defined by* (8.8$_k$) *and* (8.9$_k$). *Then the condition*

$$\inf \left\{ \frac{1}{\prod_{i=0}^{n-1}(i+\lambda)} \operatorname*{vrai\,inf}_{t \geq t_0} (t^{n+\lambda} \varphi(\theta)(t)) : \lambda \in]0, +\infty[) \right\} > 1 \qquad (9.42)$$

with $\theta(t) = t^{-\lambda}$ *and* $t_* \in \mathbb{R}_+$ *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that (9.42) implies (9.39). ■

In view of Corollary 8.2$'$ Theorem 9.4 (Corollary 9.5) implies

$'$ *Let* $F \in V(\tau)$ *and conditions* (9.1), (9.2), (9.11), (8.2), (8.3) *and* (9.38) *be fulfilled with* $\varphi \in M^+(\sigma)$. *Let, moreover, for some* $k \in \{0, \dots, n-1\}$ (8.20$_k$) *and* (8.21$_k$) *hold. Then* (9.39) ((9.42)) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, *conditions* (9.16), (9.17) *and* (9.19) *be fulfilled and*

$$\varliminf_{t \to +\infty} \frac{\tau_i(t)}{t} > 0 \quad (i = 1, \dots, m). \qquad (9.43)$$

Let, moreover, there exist $i_0 \in \{1, \dots, m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+; \mathbb{R}_+)$ *such that* (9.20)–(9.22_k) *hold with function* ρ_k (*function* p) *defined by* (8.8_k) *and* (8.9_k) ((9.23)). *Then the condition*

$$\inf\Big\{ \varliminf_{t \to +\infty} t^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{-\lambda} d_{\xi} r_i(\xi, s) ds : \\ : \lambda \in]0, +\infty[\Big\} > (n-1)! \tag{9.44}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that the operator defined by (9.25) satisfies all the conditions of Theorem 9.4. ■

In view of Corollary 8.2′ Theorem 9.5 implies

′ *Let* $F \in V(\tau)$ *and conditions* (9.16), (9.17), (9.19) *and* (9.43) *be fulfilled. Let, moreover, there exist* $i_0 \in \{1, \dots, m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+; \mathbb{R}_+)$ *such that* (8.20_k) *and* (8.27_k) *hold with* p *defined by* (9.23). *Then* (9.44) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, $c_i, \alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$, $\alpha_{i_0} < 1$ *for some* $i_0 \in \{1, \dots, m\}$ *and for some* $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} s^{-n-1} |u(s)| ds \tag{9.45}$$
$$\text{for} \quad u \in H^{-}_{t_0, \tau}, \quad t \geq t_0.$$

Then the condition

$$\inf\Big\{ \frac{1}{\prod_{i=0}^{n} (i+\lambda)} \sum_{i=1}^{m} c_i (\alpha_i^{-n-\lambda} - \overline{\alpha}_i^{\,-n-\lambda}) : \lambda \in]0, +\infty[\Big\} > 1 \tag{9.46}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that by (9.45) and (9.46) all the conditions of Theorem 9.5′ are fulfilled with $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \overline{\alpha}_i t$ and $r_i(s,t) = -\frac{c_i s^{-n}}{n}$ $(i = 1, \dots, m)$. ■

′ *Let* $c_i, \alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$, $(i = 1, \dots, m)$ *and for some* $i_0 \in \{1, \dots, m\}$ *let* $\alpha_{i_0} < 1$. *Then the condition* (9.46) *is necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} s^{-n-1} u(s) ds = 0 \tag{9.47}$$

not to have a Kneser-type solution.

Proof. Sufficiency follows from Corollary 9.6. If we assume that (9.46) is violated, then (9.47) has the solution $u(t) = t^{-\lambda}$ with $\lambda > 0$. ■

Let $F \in V(\tau)$, $c_i, \alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$, $(i = 1, \dots, m)$, *there exist* $i_0 \in \{1, \dots, m\}$ *such that* $\alpha_{i0} < 1$ *and for some* $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \ge t^{-n-1} \sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} |u(s)| ds, \qquad (9.48)$$

$$\textit{for} \quad u \in H^-_{t_0, \tau}, \quad t \ge t_0.$$

Then the condition

$$\inf \Big\{ \frac{1}{\prod_{i=-1}^{n-1} (i + \lambda)} \sum_{i=1}^{m} c_i (\overline{\alpha}_i{}^{1-\lambda} - \alpha_i^{1-\lambda}) : \\ : \lambda \in]0, 1[\cup]1, +\infty[\Big\} > 1 \qquad (9.49)$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. By (9.48) and (9.49) all the conditions of Theorem 9.5′ are fulfilled with $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \overline{\alpha}_i t$ and $r_i(s,t) = c_i t^{-n-1} s$ $(i = 1, \dots, m)$. ■

′ *Let* $c_i, \alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$, $(i = 1, \dots, m)$ *and for some* $i_0 \in \{1, \dots, m\}$ *let* $\alpha_{i_0} < 1$. *Then* (9.49) *is necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} \frac{c_i}{t^{n+1}} \int_{\alpha_i t}^{\overline{\alpha}_i t} u(s) ds = 0$$

not to have a Kneser-type solution.

Let $F \in V(\tau)$, (9.30) *and* (9.31) *be fulfilled and*

$$\varliminf_{t \to +\infty} \frac{\delta_i(t)}{t} > 0 \quad (i = 1, \dots, m). \qquad (9.50)$$

Let, moreover, there exist $i_0 \in \{1, \dots, m\}$ *such that* δ_{i_0} *is nondecreasing,* $\delta_{i_0}(t) \le t$ *for* $t \in \mathbb{R}_+$ *and for some* $k \in \{0, \dots, n-1\}$ (8.35_k), (8.40) *hold with* ρ_k *defined by* (8.8_k), (8.9_k), $p(t) \equiv p_{i_0}(t)$ *and* $\delta(t) \equiv \delta_{i_0}(t)$. *Then the condition*

$$\inf \Big\{ \varliminf_{t \to +\infty} t^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} p_i(s) \delta_i^{-\lambda}(s) ds : \\ : \lambda \in]0, +\infty[\Big\} > (n-1)! \qquad (9.51)$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. By (9.30) the inequality (9.16) holds with $\tau_i(t) = \delta_i(t) - 1$, $\sigma_i(t) = \delta_i(t)$ and $r_i(s,t) = p_i(t)e(s - \delta_i(t))$ $(i = 1,\dots,m)$ (the definition of the function e see on p.104). Therefore according to (8.35_k), (8.40), (9.50) and (9.51) all the conditions of Theorem 9.5 are satisfied. This proves the theorem. ■

According to Corollary 8.2′ Theorem 9.6 easily implies

′ *Let* $F \in V(\tau)$ *and* (9.30), (9.31) *and* (9.50) *be fulfilled. Let, moreover, there exist* $i_0 \in \{1,\dots,m\}$ *such that* δ_{i_0} *is nondecreasing,* $\delta_{i_0}(t) \le t$ *for* $t \in \mathbb{R}_+$ *and for some* $k \in \{0,\dots,n-1\}$ (8.20_k), (8.21_k) *hold with* $p(t) \equiv p_{i_0}(t)$ *and* $\delta(t) \equiv \delta_{i_0}(t)$. *Then* (9.51) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$ *and* (9.30), (9.31) *and* (9.50) *be fulfilled. Let, moreover, there exist* $i_0 \in \{1,\dots,m\}$ *such that* δ_{i_0} *is nondecreasing,* $\delta_{i_0}(t) \le t$ *for* $t \in \mathbb{R}_+$ *and for some* $k \in \{0,\dots,n-1\}$ (8.20_k), (8.21_k) *hold with* $p(t) \equiv p_{i_0}(t)$ *and* $\delta(t) \equiv \delta_{i_0}(t)$. *Then the condition*

$$\inf\Big\{\frac{1}{\prod_{i=0}^{n-1}(i+\lambda)}\ \underset{t\ge t_*}{\text{vrai inf}}\Big(t^{n+\lambda}\sum_{i=1}^m p_i(t)\delta_i^{-\lambda}(t)\Big) : \\ : \lambda \in]0,+\infty[\Big\} > 1, \tag{9.52}$$

where $t_* \in \mathbb{R}_+$, *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that (9.52) implies (9.51). ■

Let $F \in V(\tau)$, $c_i, \alpha_i \in]0,+\infty[$ $(i = 1,\dots,m)$, *there exist* $i_0 \in \{1,\dots,m\}$ *such that* $\alpha_{i_0} < 1$ *and for any* $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1}F(u)(t)\,\text{sign}\,u(t) \ge \sum_{i=1}^m \frac{c_i}{t^n}|u(\alpha_i t)| \\ \text{for} \quad u \in H^-_{t_0,\tau}, \quad t \ge t_0.$$

Then the condition

$$\inf\Big\{\frac{1}{\prod_{i=0}^{n-1}(i+\lambda)}\sum_{i=1}^m c_i\alpha_i^{-\lambda} : \lambda \in]0,+\infty[\Big\} > 1 \tag{9.53}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

′ *Let* $c_i, \alpha_i \in]0,+\infty[$ $(i = 1,\dots,m)$ *and for some* $i_0 \in \{1,\dots,m\}$ *let* $\alpha_{i_0} < 1$. *Then* (9.53) *is necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1}\sum_{i=1}^m \frac{c_i}{t^n}u(\alpha_i t) = 0$$

not to have a Kneser-type solution.

Let $n>1$, $F\in V(\tau)$, (8.2) *and* (8.3) *be fulfilled and for some* $t_0\in\mathbb{R}_+$ *let*

$$(-1)^{n+1}F(u)(t)\operatorname{sign}u(t)\geq p(t)|u(\delta(t))| \qquad (9.54)$$
$$\text{for} \quad u\in H^-_{t_0,\tau}, \quad t\geq t_0,$$

where δ *is nondecreasing. Let, moreover, for some* $k\in\{1,\dots,n-1\}$ *and* $r\in\{2,3,\dots\}$ (8.35_k) *and* (8.42_k) *hold with* ρ_k *defined by* (9.8_k) *and* (8.9_k). *Then* (0.1) *has no Knezer-type solution.*

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u:[t_0,+\infty[\;\to\mathbb{R}$ satisfying (8.1). Then according to Lemma 8.5 there exist $\lambda>0$ and $t_*>t_0$ such that $u_k(t)\geq\ln^{-\lambda}t$ for $t\geq t_*$, where u_k is defined by (8.7). The last inequality easily implies the existence of the numbers $t^*>t_*$ and $c>0$ such that $|u^{(k)}(t)|\geq c\,\ln^{-\lambda}t$ for $t\geq t^*$. Therefore, since $k\in\{1,\dots,n-1\}$, by (8.1) we have

$$+\infty>\int_{t^*}^{+\infty}|u^{(k)}(t)|dt\geq c\int_{t^*}^{+\infty}\ln^{-\lambda}t\,dt=+\infty.$$

The obtained contradiction proves the theorem. ■

Let $F\in V(\tau)$ *and* (8.2), (8.3) *and* (9.54) *be fulfilled. Let, moreover, for some* $k\in\{1,\dots,n-1\}$ *and* $r\in\{2,3,\dots\}$ (8.20_k), (8.21_k) *and* (8.44_k) *hold. Then* (0.1) *has no Kneser-type solution.*

Let $F\in V(\tau)$, *conditions* (8.2), (8.3), (9.1), (9.2) *and* (9.11) *be fulfilled with* $\varphi\in M^+(\sigma)$ *and for some* $r\in\{2,3,\dots\}$ *let*

$$\overline{\lim_{t\to+\infty}}\frac{\ln_{r-1}t}{\ln_{r-1}\sigma(t)}<+\infty. \qquad (9.55)$$

Let, moreover (8.35_0) *and* (8.42_0) *hold with* ρ_0 *defined by* (8.8_0) *and* (8.9_0). *Then the condition*

$$\inf\Big\{\underline{\lim}_{t\to+\infty}\ln^{\lambda}_{r-1}t\int_t^{+\infty}(s-t)^{n-1}\varphi(\theta)(s)ds:\lambda\in]0,k]\Big\}>(n-1)! \qquad (9.56)$$
$$\text{for all} \quad k\in\mathbb{N}$$

with $\theta(t)=(\ln_{r-1}t)^{-\lambda}$ *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u:[t_0,+\infty[\;\to\mathbb{R}$ satisfying (8.1). According to (8.35_0), (8.42_0), (9.1), (9.11) and Corollary 8.3, there exists $\lambda>0$ such that (8.43) holds. Denote by Λ the set of all such λ and put $\lambda_0 = \inf \Lambda$. By (9.56) there exist $\varepsilon > 0$ and $t_* > t_1$ satisfying

$$\ln_{r-1}^{\lambda} t \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \geq (n-1)! + \varepsilon \quad \text{for} \quad t \geq t_*, \quad \lambda \in]\lambda_0, \lambda_0 + \varepsilon]. \tag{9.57}$$

Choose $\varepsilon_2 \in]0, \varepsilon[$ and $\varepsilon_1 \in [0, \varepsilon_2[$ such that

$$c^{\varepsilon_1+\varepsilon_2}(n-1)! < (n-1)! + \varepsilon,$$
$$\lim_{t\to+\infty} (\ln_{r-1} t)^{\lambda_0+\varepsilon_2} |u(t)| = +\infty, \quad \varliminf_{t\to+\infty} (\ln_{r-1} t)^{\lambda-\varepsilon_1} |u(t)| = 0, \tag{9.58}$$

where $c = \varlimsup\limits_{t\to+\infty} \frac{\ln_{r-1} t}{\ln_{r-1} \sigma(t)}$. By (9.1), (9.55) and (9.58) all the conditions of Lemma 9.1 are obviously fulfilled with $\gamma(t) = \ln_{r-1} t$, $r_2 = \lambda_0 + \varepsilon_2$ and $r_1 = \lambda_0 - \varepsilon_1$. Therefore, using this lemma and taking into account the first inequality of (9.58), we obtain

$$\varliminf_{t\to+\infty} (\ln_{r-1} t)^{\lambda_0+\varepsilon_2} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \leq c^{\varepsilon_1+\varepsilon_2}(n-1)! < (n-1)! + \varepsilon,$$

where $\theta(t) = (\ln_{r-1} t)^{-(\lambda_0+\varepsilon_2)}$. But this contradicts (9.57). ■

Let $F \in V(\tau)$, conditions (8.2), (8.3), (9.1), (9.2) and (9.11) be fulfilled and for some $r \in \{2, 3, \dots\}$ (9.55) hold with $\varphi \in M^+(\sigma)$. Let, moreover, (8.35_0) and (8.42_0) hold with ρ_0 defined by (8.8_0) and (8.9_0). Then the condition

$$\inf \left\{ \frac{1}{\lambda} \operatorname*{vrai\,inf}_{t \geq t_k} (t^n \ln_1 t \cdots (\ln_{r-1} t)^{\lambda+1} \varphi(\theta)(t)) : \\ : \lambda \in]0, k] \right\} > (n-1)! \ \text{ for all } \ k \in \mathbb{N} \tag{9.59}$$

with $\theta(t) = (\ln_{r-1} t)^{-\lambda}$ is sufficient for (0.1) not to have a Kneser-type solution.

Proof. It suffices to show that (9.59) implies (9.56). Let $k \in \mathbb{N}$. By (9.59) there exist $t_k \in \mathbb{R}_+$ and $\varepsilon \in]0, 1]$ such that

$$\varphi(\theta)(t) \geq \frac{((n-1)! + \varepsilon)\lambda}{t^n \ln_1 t \cdots (\ln_{r-1} t)^{\lambda+1}} \quad \text{for} \quad t \geq t_k, \quad \lambda \in]0, k].$$

Therefore

$$\int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \geq \lambda((n-1)! + \varepsilon) \int_t^{+\infty} \frac{(s-t)^{n-1} ds}{s^n \ln_1 s \cdots (\ln_{r-1} s)^{\lambda+1}} \quad \text{for} \quad t \geq t_k, \quad \lambda \in]0, k]. \tag{9.60}$$

Choose $x \in]0, +\infty[$ such that

$$\left(\frac{x}{1+x} \right)^{n-1} ((n-1)! + \varepsilon) > (n-1)! + \frac{\varepsilon}{2}.$$

Then in view of (9.60) we obtain

$$(\ln_{r-1} t)^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \geq -((n-1)! + \varepsilon)(\ln_{r-1} t)^{\lambda} \times$$
$$\times \int_{(1+x)t}^{+\infty} \Big(1 - \frac{t}{s}\Big)^{n-1} d(\ln_{r-1} s)^{-\lambda} \geq ((n-1)! + \varepsilon)\Big(\frac{x}{1+x}\Big)^{n-1} \times$$
$$\times \Big(\frac{\ln_{r-1} t}{\ln_{r-1}(1+x)t}\Big)^{\lambda} \ \text{ for } \ t \geq t_k, \quad \lambda \in]0, k].$$

So

$$(\ln_{r-1} t)^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \varphi(\theta)(s) ds \geq (n-1)! + \frac{\varepsilon}{3} \ \text{ for } \ t \geq t_k', \ \lambda \in]0, k],$$

where $t_k' > t_k$ is sufficiently large. But this means that (9.56) is true. ■

According to Corollary 8.3′ from Theorem 9.8 (Corollary 9.11) immediately follows

′ *Let* $F \in V(\tau)$, *conditions* (8.2), (8.3), (9.1), (9.2) *and* (9.11) *be fulfilled and for some* $r \in \{2, 3, \dots\}$ (9.55) *hold with* $\varphi \in M^{+}(\sigma)$. *Let, moreover,* (8.20_0), (8.21_0) *and* (8.42_0) *hold. Then* (9.56), ((9.59)) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, *conditions* (9.16), (9.17) *and* (9.19) *be fulfilled and for some* $r \in \{2, 3, \dots\}$ *let*

$$\varlimsup_{t \to +\infty} \frac{\ln_{r-1} t}{\ln_{r-1} \tau_i(t)} < +\infty \quad (i = 1, \dots, m). \tag{9.61}$$

Let, moreover, there exist $i_0 \in \{1, \dots, m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+; \mathbb{R}_+)$ *such that* (9.20), (8.35_0) *and* (8.42_0) *hold with* ρ_0 (p) *defined by* (8.8_0), (8.9_0) ((9.23)). *Then the condition*

$$\inf\Big\{ \varliminf_{t \to +\infty} (\ln_{r-1} t)^{\lambda} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} \int_{\tau_i(s)}^{\sigma_i(s)} (\ln_{r-1} \xi)^{-\lambda} d_{\xi} r_i(\xi, s) ds :$$
$$: \lambda \in]0, k]\Big\} > (n-1)! \ \ \textit{for all} \ \ k \in \mathbb{N} \tag{9.62}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that the operator defined by (9.25) satisfies all the conditions of Theorem 9.8. ■

Theorem 9.8 and Corollary 8.3′ imply

′ *Let* $F \in V(\tau)$, *conditions* (9.16), (9.17) *and* (9.19) *be fulfilled and for some* $r \in \{2, 3, \dots\}$ (9.61) *hold. Let, moreover, there exist* $i_0 \in \{1, \dots, m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+; \mathbb{R}_+)$ *such that* (9.20), (8.20_0), (8.21_0) *and* (8.44_0) *hold with* p *defined by* (9.23). *Then the condition* (9.62) *is sufficient for* (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, $c_i, \alpha_i, \overline{\alpha}_i \in]0,+\infty[$, $\alpha_i < \overline{\alpha}_i$, $(i=1,\dots,m)$, $\alpha_{i_0} < 1$ *for some* $i_0 \in \{1,\dots,m\}$ *and for some* $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1} F(u)(t)\,\mathrm{sign}\,u(t) \geq \sum_{i=1}^{m} \frac{c_i}{t^n} \int_{t^{\alpha_i}}^{t^{\overline{\alpha}_i}} \frac{|u(s)|}{s \ln^2 s} ds$$
$$\text{for} \quad u \in H^-_{t_0,\tau}, \quad t \geq t_0.$$

Then the condition

$$\inf\Big\{\frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{m} c_i(\alpha_i^{-\lambda-1} - \overline{\alpha}_i{}^{-\lambda-1}) : \lambda \in]0,+\infty[\Big\} > (n-1)! \tag{9.63}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Remark 9.1. Condition (9.66) cannot be replaced by

$$\inf\Big\{\frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{m} c_i(\alpha_i^{-\lambda-1} - \overline{\alpha}_i{}^{-\lambda-1}) : \lambda \in]0,+\infty[\Big\} \geq (n-1)! - \varepsilon,$$

however small ε would be.

Let $F \in V(\tau)$, *conditions* (9.30) *and* (9.31) *be fulfilled and for some* $r \in \{2,3,\dots\}$ *let*

$$\overline{\lim_{t\to+\infty}} \frac{\ln_{r-1} t}{\ln_{r-1} \delta_i(t)} < +\infty \quad (i=1,\dots,m). \tag{9.64}$$

Let, moreover, there exist $i_0 \in \{1,\dots,m\}$ *such that* δ_{i_0} *is nondecreasing,* $\delta_{i_0}(t) \leq t$ *for* $t \in \mathbb{R}_+$ *and* (8.20_0), (8.21_0) *hold with* $\delta = \delta_{i_0}$. *Then the condition*

$$\inf\Big\{\underline{\lim}_{t\to+\infty} (\ln_{r-1} t)^\lambda \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} p_i(s)(\ln_{r-1}\delta_i(s))^{-\lambda} ds :$$
$$: \lambda \in]0,k]\Big\} > (n-1)! \quad \text{for all} \quad k \in \mathbb{N} \tag{9.65}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. According to (9.30) inequality (9.16) is valid with $\sigma_i(t) = \delta_i(t)$, $\tau_i(t) = \delta_i(t) - 1$ and $r_i(s,t) = p_i(t) e(s - \delta_i(t))$ $(i=1,\dots,m)$. Therefore, using (9.64) and (9.65), we can easily show that all the conditions of Theorem 9.9′ are satisfied. This proves the theorem. ■

Let $F \in V(\tau)$, *conditions* (9.30) *and* (9.31) *be fulfilled and for some* $r \in \{2,3,\dots\}$ (9.64) *hold. Let, moreover, there exist* $i_0 \in \{1,\dots,m\}$ *such that* δ_{i_0} *is nondecreasing,* $\delta_{i_0}(t) \leq 0$ *for* $t \in \mathbb{R}_+$ *and* (8.20_0),

(8.21_0), (8.44_0) *hold with* $\delta(t) \equiv \delta_{i_0}(t)$. *Then the condition: for any* $k \in \mathbb{N}$ *there exists* $t_k \in \mathbb{R}_+$ *such that*

$$\inf\Big\{\frac{1}{\lambda}\operatorname*{vrai\,inf}_{t\geq t_k}\Big(t^n \ln_1 t \cdots (\ln_{r-1} t)^{\lambda+1}\sum_{i=1}^m p_i(t)(\ln_{r-1}\delta_i(t))^{-\lambda}\Big) : \\ : \lambda \in]0,k]\Big\} > (n-1)! \tag{9.66}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Proof. It suffices to note that (9.66) implies (9.65). ■

Let $F \in V(\tau)$, $c_i, \alpha_i \in]0,+\infty[$, $\alpha_{i_0} < 1$ *for some* $i_0 \in \{1,\dots,m\}$ *and for some* t_0 $t_0 \in \mathbb{R}_+$ *let*

$$(-1)^{n+1}F(u)(t)\operatorname{sign} u(t) \geq \frac{1}{t^n \ln t}\sum_{i=1}^m c_i|u(t^{\alpha_i})| \\ \textit{for} \quad u \in H^-_{t_0,\tau}, \quad t \geq t_0.$$

Then the condition

$$\inf\Big\{\frac{1}{\lambda}\sum_{i=1}^m c_i\alpha_i^{-\lambda} : \lambda \in]0,+\infty[\Big\} > (n-1)!$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Let $F \in V(\tau)$, *conditions* (8.2), (8.3) *and* (9.54) *be fulfilled and for some* $i \in \{0,\dots,n-1\}$ *and* $k \in \mathbb{N}$

$$\varlimsup_{t\to+\infty}\int_{\delta(t)}^t p(s)(s-\delta(t))^{n-i-1}(\delta(t)-\delta(s))^i \times \\ \times g_k(\delta(s),\delta(t))ds > i!(n-i-1)!, \tag{9.67}$$

where the function δ *is nondecreasing and*

$$g_k(t,s) = \exp\Big\{\frac{1}{(n-1)!}\int_t^s (\xi-t)^{n-1}\psi_k(\xi)p(\xi)d\xi\Big\}, \tag{9.68}$$

$$\psi_1(t) = 1, \;\; \psi_j(t) = \exp\Big\{\frac{1}{(n-1)!}\int_{\delta(t)}^t (\xi-\delta(t))^{n-1} \times \\ \times p(\xi)\psi_{j-1}(\xi)d\xi\Big\} \quad (j = 2,\dots,k). \tag{9.69}$$

Then equation (0.1) *has no Kneser-type solution.*

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying (8.1). Then by (0.1) and (9.30) we have

$$-\frac{1}{(n-1)}\int_{\delta(t)}^{s}\frac{(\xi-\delta(t))^{n-1}|u^{(n)}(\xi)|}{u_0(\xi,\delta(t))}d\xi+\frac{1}{(n-1)}\times$$
$$\times\int_{\delta(t)}^{s}(\xi-\delta(t))^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\xi,\delta(t))}d\xi\le 0 \quad\text{for}\quad t_*\le t\le s,$$

where $t_* > t_1$ is sufficiently large and

$$u_0(t,s)=\sum_{j=0}^{n-1}\frac{(t-s)^j}{j!}|u^{(j)}(t)|.$$

Hence we obtain

$$\ln\frac{|u(\delta(t))|}{u_0(s,\delta(t))}\ge\frac{1}{(n-1)!}\int_{\delta(t)}^{s}(\xi-\delta(t))^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\xi,\delta(t))}d\xi \tag{9.70}$$
$$\text{for}\quad t_*\le\delta(t)\le s\le t.$$

Since $u_0(t,s_1)\le u_0(t,s_2)$ for $t_*\le s_2\le s_1\le t$, (9.70) implies

$$\frac{|u(\delta(t))|}{u_0(s,\delta(t))}\ge\exp\left\{\frac{1}{(n-1)!}\int_{\delta(t)}^{s}(\xi-\delta(t))^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\xi,\delta(t))}d\xi\right\}$$

whence we easily conclude that

$$|u(\delta(t))|\ge\exp\left\{\frac{1}{(n-1)!}\int_{\delta(t)}^{s}(\xi-\delta(t))^{n-1}p(\xi)\times\right.$$
$$\left.\times\psi_k(\xi)d\xi\right\}u_0(s,\delta(t))\quad\text{for}\quad t\in[\eta_{\delta k}(t_*),+\infty[, \tag{9.71}$$

where $\eta_{\delta 1}=\sup\{s:\delta(s)<t\}$, $\eta_{\delta j}=\eta_{\delta 1}(\eta_{\delta\, j-1}(t))$ $(j=2,\dots,k)$ and ψ_k is defined by (9.69).

On the other hand, by (81) and (9.54) from (1.6_{in}) we have

$$|u^{(i)}(\delta(t))|\ge\frac{1}{(n-i-1)!}\int_{\delta(t)}^{s}(s-\delta(t))^{n-i-1}p(s)|u(\delta(s))|ds \tag{9.72}$$
$$\text{for}\quad t\ge\eta_{\delta k}(t_*).$$

Since

$$u_0(\delta(t),\delta(s))\ge\frac{(\delta(t)-\delta(s))^i}{i!}|u^{(i)}(\delta(t))|\quad\text{for}\quad t\ge\eta_{\delta 1}(t_*),$$

from (9.71) and (9.72) it follows

$$\int_{\delta(t)}^{t}(s-\delta(t))^{n-i-1}p(s)(\delta(t)-\delta(s))^i g_k(\delta(s),\delta(t))ds\le$$
$$\le i!(n-i-1)!\quad\text{for}\quad t\ge\eta_{\delta k}(t_*),$$

where the function g_k is defined by (9.68). But this inequality contradicts (9.67). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$ *and* (9.54), (8.2) *and* (8.3) *be fulfilled, where* δ *is nondecreasing. Then the condition*

$$\overline{\lim_{t\to+\infty}} \int_{\delta(t)}^{t} p(s)(s-\delta(t))^{n-1} ds > (n-1)! \tag{9.73}$$

is sufficient for (0.1) *not to have a Kneser-type solution.*

Remark 9.2. Condition (9.73) cannot be replaced by

$$\overline{\lim_{t\to+\infty}} \int_{\delta(t)}^{t} p(s)(s-\delta(t))^{n-1} ds > (n-1)!(1-\varepsilon)^n, \tag{9.74}$$

however small $\varepsilon > 0$ would be.

Indeed, let $\varepsilon \in]0,1[$. By the Stirling formula $n_0 \in \mathbb{N}$ can be found such that

$$\sqrt[n]{n!} < \frac{(1+\varepsilon)n}{e} \quad \text{for} \quad n \geq n_0.$$

Choose $\Delta > 0$ and $c > 0$ such that

$$c\Delta^n = (1-\varepsilon)^n n!.$$

Then since

$$e\Delta \sqrt[n]{p} \leq n \quad \text{for} \quad n \geq n_0,$$

the equation

$$u^{(n)}(t) + (-1)^{n+1} c u(t-\Delta) = 0$$

has a Kneser-type solution. On the other hand, (9.74) holds with $p(t) \equiv c$ and $\delta(t) = t - \Delta$.

Let $F \in V(\tau)$, *conditions* (9.54), (8.2) *and* (8.3) *be fulfilled and*

$$\underline{\lim_{t\to+\infty}} \int_{\delta(t)}^{t} (s-\delta(t))^{n-1} p(s) ds > \frac{(n-1)!}{e}. \tag{9.75}$$

then equation (0.1) *has no Kneser-type solution.*

The proof of this theorem is analogous to that of Theorem 6.2.

Remark 9.3. Suppose that $n \geq 2$. Then (9.75) cannot be replaced by

$$\underline{\lim_{t\to+\infty}} \int_{\delta(t)}^{t} (s-\delta(t))^{n-1} p(s) ds \geq \frac{(n-1)!}{e} - \varepsilon, \tag{9.76}$$

however small $\varepsilon > 0$ would be.

Indeed, let $\varepsilon \in \left]0, \frac{(n-1)!}{e}\right[$. Choose $\lambda \in]0,1[$ such that

$$(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)\Big(1-\lambda\sum_{i=0}^{n-1}\frac{1}{n-i}\Big) > (n-1)! - e\varepsilon. \quad (9.77)$$

Put

$$\alpha = e^{-\frac{1}{\lambda}}, \quad p(t) = \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{et^n}. \quad (9.78)$$

Then the equation

$$u^{(n)}(t) + (-1)^{n+1}p(t)u(\alpha t) = 0$$

has the solution $u(t) = t^{-\lambda}$. On the other hand, according to (9.77) and (9.78) condition (9.76) is fulfilled.

In the case $n = 1$ it holds the following

Let $n = 1$, $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$\begin{gathered} 0 \le F(u)(t)\operatorname{sign} u(t) \le p(t)|u(\delta(t))| \\ \text{for} \quad u \in H_{t_0,\tau}, \quad t \ge t_0, \end{gathered} \quad (9.79)$$

where $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\delta \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\lim_{t\to+\infty}\delta(t) = +\infty$ and

$$\int_{\delta_*(t)}^t p(s)ds \le \frac{1}{e} \quad \text{for} \quad t \ge t_0 \quad (9.80)$$

with $\delta_*(t) \le \min\{t, \delta(t)\}$. Then there exists $t_1 \ge t_0$ such that (0.1) has a proper solution $u : [t_1, +\infty[\to]0, +\infty[$ satisfying

$$\exp\Big\{-e\int_{\delta(t_1)}^t p(s)ds\Big\} \le u(t) \le 1 \quad \text{for} \quad t \ge t_1.\text{[16]} \quad (9.81)$$

Proof. Let $t_1 \ge \max\{s : \delta_*(s) \le t_0\}$ and $U \in C_{loc}([t_0,+\infty[;\mathbb{R})$ be the set of all functions $u : [t_1, +\infty[\to \mathbb{R}$ satisfying

$$\begin{gathered} \exp\Big\{-e\int_{t_1}^t p(s)ds\Big\} \le u(t) \le 1 \quad \text{for} \quad t \ge t_1, \\ u(t) = 1 \ \text{ for } \ t \in [t_0, t_1[, \quad u(\delta(t)) \le eu(t) \ \text{ for } \ t \ge t_1. \end{gathered} \quad (9.82)$$

Define the operator $T : U \to C_{loc}([t_0, +\infty[: \mathbb{R})$ by

$$T(u)(t) = \begin{cases} \exp\Big\{-\int_{t_1}^t \frac{F(u)(s)}{u(s)}ds\Big\} & \text{for } t \ge t_1, \\ 1 & \text{for } t \in [t_0, t_1[. \end{cases} \quad (9.83)$$

[16] For the existence of Kneser-type solutions of higher order differential equations with deviating arguments see [77,88]

By (9.80), (9.82) and (9.83) T satisfies all the conditions of Lemma 2.1. Therefore it has a fixed point which, as it can be easily checked up, is a solution of (0.1) satisfying (9.81). ■

Analogously to Theorem 6.3 can be proved

Let $F \in V(\tau)$, conditions (8.2), (8.3) and (9.54) be fulfilled and

$$\varliminf_{t\to+\infty} \int_{\delta(t)}^{t} (s-\delta(t))^{n-1} p(s)ds = c,$$

where the function δ is nondecreasing and $c \in \left]0, \frac{(n-1)!}{e}\right]$.[17] *Then the fulfillement for some $i \in \{0,\dots,n-1\}$ of the condition*

$$\varlimsup_{t\to+\infty} \int_{\delta(t)}^{t} (s-\delta(t))^{n-i-1}(\delta(t)-\delta(s))^{i} p(s)\exp\Big\{\frac{x_0}{(n-1)!}\times \\ \times \int_{\delta(s)}^{\delta(t)} (\xi-\delta(s))^{n-1}p(\xi)d\xi\Big\}ds > i!(n-i-1)! \tag{9.84}$$

is sufficient for (0.1) not to have a Kneser-type solution, where x_0 is the least root of the equation $x = \exp\left\{\frac{cx}{(n-1)!}\right\}$.

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1}F(u)(t)\operatorname{sign} u(t) \ge \varphi(u)(t)\operatorname{sign} u(t) \ge 0 \tag{9.85}$$
$$\text{for} \quad u \in H^-_{t_0,\tau}, \quad t \ge t_0,$$

where $\varphi \in M(\tau,\sigma)$ and $\sigma(t) \le t$ for $t \in \mathbb{R}_+$. Then the condition

$$\widetilde{\varphi}_0 \in M_1(\tau,\sigma) \tag{9.86}$$

with

$$\widetilde{\varphi}_0(u)(t) = \frac{t^{n-1}}{(n-1)!}\varphi\Big(\Big(\frac{t-\sigma(t)}{t}\Big)^{n-1}u(t)\Big)(t)$$

is sufficient for (0.1) not to have a Kneser-type solution.

Proof. Suppose, on the contrary, that (0.1) has a proper solution $u : [t_0,+\infty[\to \mathbb{R}$ satisfying (8.1). Without loss of the generality we can assume that $u(t) > 0$ for $t \ge t_1$. Then by (8.1) from (1.6_{0n}) and (8.1) we have

$$u(\sigma(t)) \ge \sum_{i=0}^{n-1} \frac{|u^{(i)}(t)|}{i!}(t-\sigma(t))^i \ge \\ \ge \Big(\frac{t-\sigma(t)}{t}\Big)^{n-1}\sum_{i=0}^{n-1}\frac{|u^{(i)}(t)|}{i!}t^i \quad \text{for} \quad t \ge \eta_\tau(t_1),$$

[17] If $c > \frac{(n-1)!}{e}$ then (9.75) holds and (9.84) is unnecessary

and

$$\sum_{i=0}^{n-1} \frac{t^i |u^{(i)}(t)|}{i!} \geq \frac{1}{(n-1)!} \int_t^{+\infty} s^{n-1} |u^{(n)}(s)| ds$$
$$\text{for} \quad t \geq \eta_\tau(t_1),$$

where $\eta_\tau(t) = \sup\{s : \tau(s) < t\}$. Hence taking onto account (0.1), (8.1) and (9.85) we obtain

$$y(t) \geq \int_t^{+\infty} \widetilde{\varphi}(y)(s) ds \quad \text{for} \quad t \geq \eta_\tau(t_1)$$

with $y(t) = \sum_{i=0}^{n-1} \frac{|u^{(i)}(t)|}{i!} t^i > 0$ for $t \geq \eta_\tau(t_1)$. But this contradicts (9.86). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \prod_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\lambda_i} d_s r_i(s,t)$$
$$\text{for} \quad u \in H_{t_0,\tau}^{-}, \quad t \geq t_0,$$

where $\tau_i, \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\tau_i(t) \leq \sigma_i(t) \leq t$, $\lim \tau_i(t) = +\infty$, the functions $r_i(s,t)$ are measurable, $r_i(\cdot, t)$ is nondecreasing, $\lambda_i \in]0,1[$ $(i = 1, \dots, m)$ and $\sum_{i=1}^{m} \lambda_i = \lambda < 1$. Then the condition

$$\int^{+\infty} t^{(n-1)(1-\lambda)} \prod_{i=1}^{m} (t - \sigma_i(t))^{\lambda_i(n-1)} (r_i(\sigma_i(t), t) - r_i(\tau_i(t), t)) dt = +\infty$$

is sufficient for (0.1) not to have a Kneser-type solution.

Analogously to Theorem 9.15 can be proved

′ Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \varphi(t, u(\delta_1(t)), \dots, u(\delta_m(t))) \times$$
$$\times \operatorname{sign} u(t) \geq 0 \quad \text{for} \quad u \in H_{t_0,\tau}^{-}, \quad t \geq t_0,$$

where $\varphi \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R})$, $\delta_i \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\lim_{t \to +\infty} \delta_i(t) = +\infty$, $\delta_i(t) \leq t$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$ and

$$\varphi(t, x_1, \dots, x_m) \operatorname{sign} x_1 \geq \varphi(t, y_1, \dots, y_m) \operatorname{sign} y_1 \geq 0$$
$$\text{for} \ t \in \mathbb{R}_+, \ |x_i| \geq |y_i|, \ x_i y_i > 0 \ (i = 1, \dots, m).$$

Then condition (9.86) with

$$\widetilde{\varphi}(u)(t) = \frac{t^{n-1}}{(n-1)!} \varphi\Big(t, \Big(\frac{t - \delta_1(t)}{t}\Big)^{n-1} u(t), \dots, \Big(\frac{t - \delta_m(t)}{t}\Big)^{n-1} u(t)\Big)$$

is sufficient for (0.1) not to have a Kneser-type solution.

Let $F \in V(\tau)$ and for some $t \in \mathbb{R}_+$

$$(-1)^{n+1} F(u)(t) \operatorname{sign} u(t) \geq \prod_{i=1}^{m} p_i(t) |u(\delta_i(t))|^{\lambda_i}$$

$$\text{for} \quad u \in H^{-}_{t_0,\tau}, \quad t \geq t_0,$$

where $p_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, $\delta_i(t) \leq t$ for $t \in \mathbb{R}_+$, $\lim_{t\to+\infty} \delta_i(t) = +\infty$, $\lambda_i \in]0,1[$ $(i = 1, \dots, m)$ and $\sum_{i=1}^{m} \lambda_i = \lambda < 1$. Then the condition

$$\int^{+\infty} t^{(n-1)(1-\lambda)} \prod_{i=1}^{m} p_i(t)(t - \delta_i(t))^{n-1} dt = +\infty$$

is sufficient for (0.1) not to have a Kneser-type solution.

CHAPTER 4

§ 10. Auxiliary statements

In chapter 4 sufficient conditions are given for equation (0.1) not to have a solution satisfying

$$u^{(i)}(t)u(t) > 0 \quad \text{for} \quad t \geq t_0 \ \ (i = 0, \dots, n-1), \tag{10.1}$$

as well as necessary and sufficient ones.

Suppose that

$$p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \ \ \delta \in C(\mathbb{R}_+; \mathbb{R}_+), \ \ \delta \uparrow, \ \ \delta(t) \geq t \ \ \textit{for} \ \ t \in \mathbb{R}_+ \tag{10.2}$$

and let $u : [t_0, +\infty[\to \mathbb{R}$ be a solution of the differential inequality

$$u^{(n)}(t) \operatorname{sign} u(t) \geq p(t)|u(\delta(t))| \tag{10.3}$$

satisfying (10.1). *Then for any $k \in \{0, \dots, n-1\}$ we have*

$$\rho_k^*(t)|u(\delta_i(t))| \leq (n-1)! t^{n-k-1}|u^{(n-1)}(t)| \quad \textit{for} \quad t \geq \delta(t_0), \tag{10.4}$$

where

$$\rho_k^*(t) = \max\{\psi_k^*(t,s,\tau) : s \in [\eta_\delta^*(t), t], \ \tau \in [t, \delta(s)]\}, \tag{10.5$_k$}$$

$$\psi_k^*(t,s,\tau) = \int_s^t \xi^{n-k-1}p(\xi)d\xi \int_t^\tau \xi^{n-k-1}p(\xi)d\xi \ \tau^{k+1-n}\Big[(\delta(s)-\tau)^{n-1} +$$

$$+ \frac{1}{(n-1)!}\int_\tau^{\delta(s)} (\delta(s)-\xi)^{n-1}(\delta(\xi)-\xi)^{n-1}p(\xi)d\xi\Big] \tag{10.6$_k$}$$

$$\eta_\delta^*(t) = \min\{s : \delta(s) \geq t\}. \tag{10.7}$$

Proof. By (10.1) from (10.3) we have

$$t^{n-k-1}|u^{(n-1)}(t)| \geq \int_{t_0}^t s^{n-k-1}p(s)|u(\delta(s))|ds \quad \text{for} \quad t \geq t_0. \tag{10.8}$$

Let $t \geq \delta(t_0)$ and (s_0, τ_0) be a point of maximum of the function $\psi_k^*(t,\cdot,\cdot)$ on $[\eta_\delta^*(t), t] \times [t, \delta(s_0)]$. Then according to (10.8)

$$t^{n-k-1}|u^{(n-1)}(t)| \geq \int_{s_0}^t \xi^{n-k-1}p(\xi)|u(\delta(\xi))|d\xi \geq$$

$$\geq \int_{s_0}^t \xi^{n-k-1}p(\xi)d\xi|u(\delta(s_0))|, \tag{10.9}$$

$$\tau_0^{n-k-1}|u^{(n-1)}(\tau_0)| \geq \int_t^{\tau_0} \xi^{n-k-1}p(\xi)|u(\delta(\xi))|d\xi \geq$$

$$\geq \int_t^{\tau_0} \xi^{n-k-1}p(\xi)d\xi|u(\delta(t))|. \tag{10.10}$$

On the other hand, (1.6_{0n}) in view of (10.1) implies

$$|u(\delta(s_0))| \geq |u^{(n-1)}(\tau_0)| \frac{1}{(n-1)!} \Big[(\delta(s_0) - \tau_0)^{n-1} + $$
$$+ \frac{1}{(n-1)!} \int_{\tau_0}^{\delta(s_0)} (\delta(s_0) - \xi)^{n-1} p(\xi)(\delta(\xi) - \xi)^{n-1} d\xi \Big].$$

Therefore by (10.9) and (10.10) we obtain

$$t^{n-k-1}|u^{(n-1)}(t)| \geq \frac{1}{(n-1)!} \int_{s_0}^{t} \xi^{n-k-1} p(\xi) d\xi \int_{t}^{\tau_0} \xi^{n-k-1} p(\xi) d\xi \times$$
$$\times \Big[(\delta(s_0) - \tau_0)^{n-1} + \frac{1}{(n-1)!} \int_{\tau_0}^{\delta(s_0)} (\delta(s_0) - \xi)^{n-1} p(\xi)(\delta(\xi) - \xi)^{n-1} d\xi \Big] |u(\delta(t))|$$

whence it follows the validity of (10.4). ■

Let (10.2) *be fulfilled and*

$$\varliminf_{t \to +\infty} \int_{t}^{\delta(t)} p(s) ds > 0, \tag{10.11}$$

$$\text{vrai} \sup \{ p(t) : t \in \mathbb{R}_+ \} < +\infty. \tag{10.12}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.3) *satisfying* (10.1). *Then*

$$\varlimsup_{t \to +\infty} \frac{|u(\delta(t))|}{|u^{(n-1)|}(t)|} < +\infty. \tag{10.13}$$

Proof. According to Lemma 10.1 it suffices to prove that

$$\varliminf_{t \to +\infty} \rho^*_{n-1}(t) > 0 \tag{10.14}$$

with $\rho^*_{n-1}(t)$ defined by (10.5_{n-1}) and (10.6_{n-1}).

By (10.11) there exist $c > 0$ and $t_1 \geq t_0$ such that

$$\int_{t}^{\delta(t)} p(s) ds \geq c \ \text{ for } \ t \geq t_1.$$

Therefore for any $t \geq \delta(t_1)$ there exist $\xi_1 \in [\eta^*_\delta(t), t]$ and $\xi_2 \in [t, \delta(\xi_1)]$ such that

$$\int_{\xi_1}^{t} p(s) ds \geq \frac{c}{4}, \ \int_{t}^{\xi_2} p(s) ds \geq \frac{c}{4}, \ \int_{\xi_2}^{\delta(\xi_1)} p(s) ds \geq \frac{c}{4}. \tag{10.15}$$

According to (10.5_{n-1}) and (10.6_{n-1}) we have

$$\rho^*_{n-1}(t) \geq \int_{\xi_1}^{t} p(s) ds \int_{t}^{\xi_2} p(s) ds (\delta(\xi_1) - \xi_2)^{n-1}. \tag{10.16}$$

On the other hand, in view of (10.12) and (10.15) $\delta(\xi_1) - \xi_2 \geq \frac{c}{4r}$ with $r = \text{vrai}\sup\{p(t) : t \in \mathbb{R}_+\}$. Thus the validity of (10.14) obviously follows from (10.15) and (10.16). ■

Let (10.2) *be fulfilled and for some* $k \in \{0, \dots, n-1\}$

$$\varliminf_{t\to+\infty} \int_t^{\delta(t)} s^{n-k-1} p(s)ds > 0, \tag{10.17_k}$$

$$\text{vrai}\sup\{t^{n-k}p(t) : t \in \mathbb{R}_+\} < +\infty. \tag{10.18_k}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.3) *satisfying* (10.1). *Then*

$$\varlimsup_{t\to+\infty} \frac{t^{2k+1-n}|u(\delta(t))|}{|u^{(n-1)}(t)|} < +\infty. \tag{10.19}$$

Proof. According to Lemma 10.1 it suffices to show that

$$\varliminf_{t\to+\infty} \rho_k^*(t)t^{-k} > 0 \tag{10.20}$$

with ρ_k^* defined by (10.5_k) and (10.6_k).

By (10.17_k) there exist $c > 0$ and $t_1 \geq t_0$ such that

$$\int_t^{\delta(t)} s^{n-k-1}p(s)ds \geq c \;\text{ for }\; t \geq t_1.$$

Therefore for any $t \in [\delta(t_1), +\infty[$ there exist $\xi_1 \in [\eta_\delta^*(t), t]$ and $\xi_2 \in [t, \delta(\xi_1)]$ such that

$$\begin{gathered} \int_{\xi_1}^t s^{n-k-1}p(s)ds \geq \frac{c}{4}, \quad \int_t^{\xi_2} s^{n-k-1}p(s)ds \geq \frac{c}{4}, \\ \int_{\xi_2}^{\delta(\xi_1)} s^{n-k-1}p(s)ds \geq \frac{c}{4}. \end{gathered} \tag{10.21}$$

In view of (10.18) and (10.21) we have $\frac{\delta(\xi_1)}{\xi_2} \geq \exp\left\{\frac{c}{4r}\right\}$ with $r = \text{vrai}\sup\{t^{n-k}p(t) : t \in \mathbb{R}_+\}$, so according to (10.5_k), (10.6_k) and (10.21) we obtain

$$\rho_k^*(t) \geq \frac{c^2}{16} t^k \Big(\exp\big\{\frac{c}{4r}\big\} - 1\Big)^{n-1}$$

whence it follows the validity of (10.20). ■

The following lemma can easily be deduced from Lemma 10.1

Let (10.2) *be fulfilled,* $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *and for some* $k \in \{0, \dots, n-1\}$ *let*

$$\rho_k^*(t) > 0 \quad \textit{for} \quad t \geq t_0 \tag{10.22_k}$$

with ρ_k^* *defined by* (10.5_k) *let* (10.6_k)*. Let, moreover* $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of*

$$p(t)|u(\delta(t))| \leq u^{(n)}(t)\,\mathrm{sign}\, u(t) \leq q(t)|u(\delta(t))| \tag{10.23}$$

satisfying (10.1)*. Then there exists* $\lambda > 0$ *such that*

$$\overline{\lim_{t\to+\infty}} |u^{(n-1)}(t)| \exp\Big\{ -\lambda \int_{t_0}^{t} q(s)(\rho_k^*(s))^{-1} s^{n-k-1} ds \Big\} < +\infty.$$

Let (10.2), (10.11) *and* (10.12) *be fulfilled,* $\sigma \in C(\mathbb{R}_+; \mathbb{R}_+)$ *and*

$$\overline{\lim_{t\to+\infty}} (\sigma(t) - t) < +\infty. \tag{10.24}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.3) *satisfying* (10.1)*. Then*

$$\overline{\lim_{t\to+\infty}} \frac{|u(\sigma(t))|}{|u(\delta(t))|} < +\infty. \tag{10.25}$$

Proof. According to (10.11), (10.12) and (10.24) there exist $c_1, c_2 \in]0, +\infty[$ and $t_1 \geq t_0$ such that

$$c_1 < c_2, \quad t + c_1 \leq \delta(t), \quad \sigma(t) \leq t + c_2 \quad \text{for} \quad t \geq t_1. \tag{10.26}$$

Due to Lemma 10.2

$$\overline{\lim_{t\to+\infty}} \frac{|u(\delta(t))|}{|u^{(n-1)}(t)|} < +\infty,$$

so, taking into account the nondecreasing character of $|u(t)|$, by (10.26) we have

$$\overline{\lim_{t\to+\infty}} \frac{|u(t + c_1)|}{|u^{(n-1)}(t)|} < +\infty. \tag{10.27}$$

On the other hand, equality (1.60_n) along with (10.1) implies

$$|u(t + c_1)| \geq \frac{1}{(n-1)!} \Big| u^{(n-1)}\Big(t + \frac{c_1}{2}\Big) \Big| \Big(\frac{c_1}{2}\Big)^{n-1} \quad \text{for} \quad t \geq t_1. \tag{10.28}$$

Therefore in view of (10.27) we obtain

$$\overline{\lim_{t\to+\infty}} \frac{|u^{(n-1)}\big(t + \frac{c_1}{2}\big)|}{|u^{(n-1)}(t)|} < +\infty. \tag{10.29}$$

Let $k \in \mathbb{N}$ satisfy $k > \frac{2c_2}{c_1}$. Then by (10.26)

$$\frac{|u(\sigma(t))|}{|u(\delta(t))|} \leq \frac{\big|u\big(t + k\frac{c_1}{2}\big)\big|}{|u(t + c_1)|} = \frac{|u\big(t + k\frac{c_1}{2}\big)|}{|u^{(n-1)}\big(t + (k-1)\frac{c_1}{2}\big)|} \times$$

$$\times \frac{|u^{(n-1)}\big(t + \frac{c_1}{2}\big)|}{|u(t + c_1)|} \prod_{i=2}^{k-1} \frac{|u^{(n-1)}(t + ic_1)|}{\big|u^{(n-1)}\big(t + (i-1)\frac{c_1}{2}\big)\big|}$$

whence according to (10.27)–(10.29) it obviously follows the validity of (10.25). ■

Let (10.2), (10.17_0) *and* (10.18_0) *be fulfilled,* $\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$ *and*

$$\overline{\lim_{t\to+\infty}}\,\frac{\sigma(t)}{t} < +\infty. \tag{10.30}$$

Let, moreover, $u : [t_0,+\infty[\to \mathbb{R}$ *be a solution of* (10.3) *satisfying* (10.1). *Then* (10.25) *holds.*

Proof. By (10.17_0), (10.18_0) and (10.30) there exist $c_1, c_2 \in]0,+\infty[$ and $t_1 \geq t_0$ such that

$$1 < c_1 < c_2, \quad c_1 t \leq \delta(t), \quad \sigma(t) \leq c_2 t \quad \text{for } t \geq t_1. \tag{10.31}$$

Equality (1.60_{0n}) along with (10.1) implies

$$|u(c,t)| \geq \frac{1}{(n-1)!}|u^{(n-1)}(c_0 t)|\Big(\frac{c_1-1}{2}\Big)^{n-1} t^{n-1} \quad \text{for } t \geq t_1 \tag{10.32}$$

with $c_0 = \frac{1+c_1}{2}$. On the other hand, according to Lemma 10.3 and (10.31) we have

$$\overline{\lim_{t\to+\infty}}\,\frac{|u(c_1 t)|}{t^{n-1}|u^{(n-1)}(t)|} < +\infty. \tag{10.33}$$

Therefore by (10.32) and (10.33)

$$\overline{\lim_{t\to+\infty}}\,\frac{|u^{(n-1)}(c_0 t)|}{|u^{(n-1)}(t)|} < +\infty. \tag{10.34}$$

Choose $k \in \mathbb{N}$ such that $c_0^k \geq c_2$. Then (10.31) implies

$$\frac{|u(\sigma(t))|}{|u(\delta(t))|} \leq \frac{|u(c_0^k t)|}{|u(c_1 t)|} = \frac{|u(c_0^k t|}{|u^{(n-1)}(c_0^{k-1} t)|} \times \\ \times \frac{|u^{(n-1)}(c_0 t)|}{|u(c_1 t)|} \prod_{i=2}^{k-1} \frac{|u^{(n-1)}(c_0^i t)|}{|u^{(n-1)}(c_0^{i-1} t)|}$$

whence in view of (10.32)–(10.34) it follows the validity of (10.24). ■

Let $F \in V(\tau)$ *and*

$$F(u)(t)\,\mathrm{sign}\,u(t) \leq -\varphi(|u|)(t) \quad \textit{for } u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{10.35}$$

where the mapping $\varphi : C(\mathbb{R}_+;\mathbb{R}_+) \to L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ *is such that*

$$\varphi(x)(t) \geq \varphi(y)(t), \quad \textit{if } x, y \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ x(s) \geq y(s) \ \ \textit{for } s \geq \tau(t) \tag{10.36}$$

and for any $c > 0$

$$\int_t^{+\infty} \varphi(c)(s)ds > 0 \quad for \ t \in \mathbb{R}_+. \tag{10.37}$$

Let, moreover, problem (0.1), (10.1) *has a solution. Then the equation*

$$u^{(n)}(t) = \varphi(u)(t) \tag{10.38}$$

has a solution $u_0 : [t_0, +\infty[\to]0, +\infty[$ *satisfying*

$$u_0^{(i)}(t) > 0 \quad for \ t \geq t_* \quad (i = 0, \dots, n-1) \tag{10.39}$$

with $t_* \geq t_0$ *sufficiently large.*

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be a solution of (0.1) satisfying (10.1). Choose $t_1 > t_0$ such that $t_0 \leq t^0 = \inf\{\tau(t) : t \in [t_1, +\infty[\}$ and consider the sequence of functions $\{u_i(t)\}_{i=1}^{+\infty}$ defined by

$$u_1(t) = |u(t)| \quad \text{for} \quad t \geq t_0,$$

$$u_i(t) = \begin{cases} |u(t_*)| + \frac{1}{(n-1)!}\int_{t_*}^t (t-s)^{n-1}\varphi(u_{i-1})(s)ds & \text{for } t \geq t_*, \\ |u(t)| \quad \text{for} \quad t \in [t_0, t_*] \quad (i = 2, 3, \dots) \end{cases} \tag{10.40}$$

with $t_* = \max\{t^0, t_1\}$. By (10.35) and (10.36) this sequence is obviously decreasing. Denote its limit by $u_0(t)$. According to (10.40) u_0 is a solution of (10.38) on $[t_*, +\infty[$. On the other hand, due to (10.37) condition (10.39) is obviously fulfilled. ■

Taking into acount Lemmas 10.3–10.6 one can easily verify the validity of the following lemmas.

Let (10.2) *be fulfilled,* $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, *for some* $k \in \{0, \dots, n-1\}$ (10.22_k) *hold and*

$$\overline{\lim_{t\to+\infty}} \frac{1}{t}\int_{t_0}^t q(s)s^{n-k-1}(\rho_k^*(s))^{-1}ds < +\infty \tag{10.41$_k$}$$

with ρ_k^* *defined by* (10.5_k) *and* (10.6_k). *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.23) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that*

$$|u(t)|e^{-\lambda t} \to 0 \quad for \quad t \to +\infty. \tag{10.42}$$

Let (10.2), (10.11) *and* (10.12) *be fulfilled,* $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *and*

$$\overline{\lim_{t\to+\infty}} \frac{1}{t}\int_0^t q(s)ds < +\infty. \tag{10.43}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.23) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that* (10.42) *holds.*

Let (10.2) *be fulfilled, for some* $k \in \{0, \dots, n-1\}$ (10.17_k) *and* (10.18_k) *hold,* $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *and*

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_1^t q(s) s^{n-2k-1} ds < +\infty.$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.23) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that* (10.42) *holds.*

Let (10.2), (10.11) *and* (10.12) *be fulfilled,*

$$\begin{gathered} q_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+), \\ \overline{\lim_{t\to+\infty}} (\sigma_i(t) - t) < +\infty \quad (i = 1, \dots, m) \end{gathered} \tag{10.44}$$

and

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_0^t \sum_{i=1}^m q_i(s) ds < +\infty. \tag{10.45}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of*

$$p(t)|u(\delta(t))| \le u^{(n)} \operatorname{sign} u(t) \le \sum_{i=1}^m q_i(t) |u(\sigma_i(t))| \tag{10.46}$$

satisfying (10.1). *Then there exists* $\lambda > 0$ *such that* (10.42) *holds.*

Let (10.2) *be fulfilled, for some* $k \in \{0, \dots, n-1\}$ (10.22_k) *hold and*

$$\overline{\lim_{t\to+\infty}} \frac{1}{\ln t} \int_{t_0}^t q(s) s^{n-k-1} (\rho_k^*(s))^{-1} ds < +\infty \tag{10.47_k}$$

with $\rho_k^*(t)$ *defined by* (10.5_k) *and* (10.6_k). *Let, moreover,* $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.23) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that*

$$|u(t)| t^{-\lambda} \to 0 \quad \textit{for} \quad t \to +\infty. \tag{10.48}$$

Let (10.2), (10.17_0) *and* (10.18_0) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \frac{1}{\ln t} \int_0^t s^{n-1} q(s) ds < +\infty. \tag{10.49}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.23) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that* (10.48) *holds.*

Let (10.2), (10.17_0) *and* (10.18_0) *be fulfilled,*

$$\begin{gathered} q_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+), \\ \overline{\lim_{t\to+\infty}} \frac{\sigma_i(t)}{t} < +\infty \quad (i = 1, \dots, m) \end{gathered} \tag{10.50}$$

and

$$\overline{\lim_{t\to+\infty}}\frac{1}{\ln t}\int_0^t s^{n-1}\sum_{i=1}^m q_i(s)ds < +\infty. \tag{10.51}$$

Let, moreover, $u : [t_0, +\infty[\to \mathbb{R}$ *be a solution of* (10.46) *satisfying* (10.1). *Then there exists* $\lambda > 0$ *such that* (10.48) *holds.*

§ 11. On monotonically increasing solutions

Let

$$\begin{gathered}\sigma;\overline{\sigma} \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ \sigma(t) \le \overline{\sigma}(t) \ \text{ for } \ t \in \mathbb{R}_+, \\ \lim_{t\to+\infty}\sigma(t) = +\infty, \quad \sigma\uparrow .\end{gathered} \tag{11.1}$$

Denote by $M^+(\sigma;\overline{\sigma})$ the set of all continuous maps $\varphi : C(\mathbb{R}_+;\mathbb{R}_+) \to L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ which for any $t \in \mathbb{R}_+$ satisfy

$$\begin{gathered}\varphi(x)(t) \ge \varphi(y)(t) \ \text{ if } \ x,y \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ x(s) \ge y(s) \ \text{ for } \ s \in [\sigma(t),\overline{\sigma}(t)], \\ (xy)(t) \ge x(\sigma(t))\varphi(y)(t) \ \text{ if } \ x,y \in C(\mathbb{R}_+,]0,+\infty[), \ \ x(t)\uparrow+\infty \ \text{ as } \ t\uparrow+\infty, \\ \varphi(xy)(t) \ge x(\overline{\sigma}(t))\varphi(y)(t) \ \text{ if } \ x,y \in C(\mathbb{R}_+,]0,+\infty[), \ \ x(t)\downarrow 0 \ \text{ as } \ t\uparrow+\infty.\end{gathered}$$

Let $\varphi \in M^+(\sigma;\overline{\sigma})$ *and the equation*

$$u^{(n)}(t) = \varphi(u)(t) \tag{11.2}$$

have a solution $u : [t_0, +\infty[\to \mathbb{R}$ *satisfying*

$$u^{(i)}(t) > 0 \quad (i = 0,\dots,n-1) \quad \textit{for } \ t \ge t_*. \tag{11.3}$$

Let, moreover, there exist $\gamma \in C(\mathbb{R}_+;]0,+\infty[)$, $r_2 > 0$ *and* $r_1 \in [0, r_2[$ *such that*

$$\begin{gathered}\gamma(t)\downarrow 0 \ \textit{ for } \ t\uparrow+\infty, \quad \lim_{t\to+\infty}(\gamma(t))^{r_1}u(t) = +\infty, \\ \underline{\lim}_{t\to+\infty}(\gamma(t))^{r_2}u(t) = 0,\end{gathered} \tag{11.4}$$

$$\overline{\lim_{t\to+\infty}}\frac{\gamma(\sigma(t))}{\gamma(\overline{\sigma}(t))} < +\infty, \quad \lim_{t\to+\infty}t^{n-1}(\gamma(t))^{r_2} = 0. \tag{11.5}$$

Then

$$\underline{\lim}_{t\to+\infty}(\gamma(t))^{r_2}\int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds \le c^{r_2-r_1}(n-1)! \tag{11.6}$$

with $\theta(t) = (\gamma(t))^{-r_2}$.

Proof. Put

$$\widetilde{u}(t)=\inf\big\{(\gamma(s))^{r_1}|u(s)|: s\geq t\big\}. \tag{11.7}$$

According to (11.1) and (11.7)

$$\widetilde{u}(\sigma(t))\uparrow+\infty \quad \text{for} \quad t\uparrow+\infty \tag{11.8}$$

and

$$\varliminf_{t\to+\infty}\widetilde{u}(\sigma(t))(\gamma(\sigma(t)))^{r_2-r_1}=0. \tag{11.9}$$

Taking into account (11.7)–(11.9) and Lemma 7.1 we see that there exists a sequence $\{t_k\}_{k=1}^{+\infty}$ such that $t_k\uparrow+\infty$ as $k\uparrow+\infty$ and

$$\begin{aligned}\widetilde{u}(\sigma(t_k))=(\gamma(\sigma(t_k)))^{r_1}u(\sigma(t_k)), \quad (\gamma(\sigma(t_k)))^{r_2-r_1}\widetilde{u}(\sigma(t_k))\leq\\ \leq(\gamma(\sigma(t)))^{r_2-r_1}\widetilde{u}(\sigma(t)) \ \text{ for } \ t_*\leq t\leq t_k, \ \ k=k_0,k_0+1,\end{aligned} \tag{11.10}$$

with $k_0\in\mathbb{N}$ and $t_*>t_0$ sufficiently large.

On the other hand, in view of (11.3) from (11.2) we obtain

$$u(\sigma(t))\geq\frac{1}{(n-1)!}\int_{t_*}^{\sigma(t)}(\sigma(t)-s)^{n-1}\varphi(u)(s)ds \quad \text{for} \quad t\geq t_1,$$

with $t_1>t_*$ sufficiently large. Hence by (11.7), (11.10) and the fact that $\varphi\in M^+(\sigma;\overline{\sigma})$ we have

$$\begin{aligned}u(\sigma(t_k))\geq\frac{1}{(n-1)!}\int_{t_*}^{\sigma(t_k)}(\sigma(t_k)-s)^{n-1}\widetilde{u}(\sigma(s))\varphi(\theta_1)(s)ds\geq\\ \geq\frac{(\gamma(\sigma(t_k)))^{r_2}u(\sigma(t_k))}{(n-1)!}\int_{t_*}^{\sigma(t_k)}(\sigma(t_k)-s)^{n-1}\times\\ \times(\gamma(\sigma(s)))^{r_1-r_2}\varphi(\theta_1)(s)ds \quad (k=k_1,k+1,\dots)\end{aligned} \tag{11.11}$$

with $\theta_1(t)=(\gamma(t))^{-r_1}$ and $t_{k_1}\geq t_1$.

Take any $\varepsilon\in]0,1[$ and choose $t^*>t_*$ such that

$$\frac{\gamma(\sigma(t))}{\gamma(\overline{\sigma}(t))}\leq c+\varepsilon \quad \text{for} \quad t\geq t^*.$$

Then since $\varphi\in M^+(\sigma;\overline{\sigma})$, from (11.11) we obtain

$$\begin{aligned}(n-1)!\geq\varlimsup_{k\to+\infty}(\gamma(\sigma(t_k)))^{r_2}\int_{t_*}^{\sigma(t_k)}(\sigma(t_k)-s)^{n-1}(\gamma(\sigma(s)))^{r_1-r_2}\varphi(\theta_1)(s)ds\geq\\ \geq\varlimsup_{k\to+\infty}(\gamma(\sigma(t_k)))^{r_2}\int_{t^*}^{\sigma(t_k)}(\sigma(t_k)-s)^{n-1}\Big(\frac{\gamma(\overline{\sigma}(s))}{\gamma(\sigma(s))}\Big)^{r_2-r_1}\varphi(\theta)(s)ds\geq\\ \geq(c+\varepsilon)^{r_1-r_2}\varlimsup_{k\to+\infty}(\gamma(\sigma(t_k)))^{r_2}\int_{t^*}^{\sigma(t_k)}(\sigma(t_k)-s)^{n-1}\varphi(\theta)(s)ds,\end{aligned}$$

where $\theta(t) = (\gamma(t))^{-r_2}$. Hence, taking onto account the second condition of (11.5),

$$\varliminf_{t\to+\infty} (\gamma(t))^{r_2} \int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds \le \varlimsup_{k\to+\infty} (\gamma(\sigma(t_k)))^{r_2} \times$$
$$\times \int_{t^*}^{\sigma(t_k)} (\sigma(t_k)-s)^{n-1}\varphi(\theta)(s)ds \le (c+\varepsilon)^{r_2-r_1}(n-1)!.$$

Since ε is arbitrary, the last inequality implies (11.6). ■

Let $t_0 \in \mathbb{R}_+$. Denote by $H^+_{t_0,\tau}$ the set of all functions $u \in \widetilde{C}^{n-1}_{loc}(\mathbb{R}_+;\mathbb{R})$ satifying

$$u^{(i)}(t)u(t) > 0 \ \ (i = 0,\dots,n-1), \ \ u^{(n)}(t)u(t) \ge 0 \ \text{ for } \ t \ge t_*,$$

where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \ge t\}$.

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$F(u)(t)\operatorname{sign} u(t) \le -\varphi(|u|)(t) \ \ \textit{for} \ \ u \in H^+_{t_0,\tau}, \ \ t \ge t_0, \tag{11.12}$$
$$p(t)|u(\delta(t)|) \le \varphi(|u|)(t) \le q(t)|u(\delta(t))| \textit{for} \ \ u \in H^+_{t_0,\tau}, \ \ t \ge t_0, \tag{11.13}$$

where $\varphi \in M^+(\sigma;\overline{\sigma})$,

$$\varlimsup_{t\to+\infty} (\overline{\sigma}(t) - \sigma(t)) < +\infty, \tag{11.14}$$
$$p,q \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+), \ \ \delta \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ \delta\uparrow, \ \ \delta(t) \ge t \ \textit{for}\ t \in \mathbb{R}_+, \tag{11.15}$$
$$\int^{+\infty} \delta(t)^{n-1}p(t)dt = +\infty. \tag{11.16}$$

Let, moreover, for some $k \in \{0,\dots,n-1\}$ (10.22_k) and (10.41_k) be fulfilled with $\rho^*_k(t)$ defined by (10.5_k) and (10.6_k). Then the condition

$$\inf\Big\{ \varliminf_{t\to+\infty} e^{-\lambda t}\int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds : \lambda \in]0,+\infty[\Big\} > (n-1)! \tag{11.17}$$

with $\theta(t) = e^{\lambda t}$ is sufficient for problem (0.1), (10.1) to have no solution.

Proof. Suppose, on the contrary, that (0.1) has a solution $u_0 : [t_0,+\infty[\to \mathbb{R}$ satisfying (10.1). According to (11.12) and Lemma 10.7 (11.2) has a solution $u : [t_*,+\infty[\to]0,+\infty[$ satisfying (11.3). By (10.22_k), (10.41_k), (11.13), (11.15) and Lemma 10.3 there exists $\lambda > 0$ such that $\lim\limits_{t\to+\infty} e^{-\lambda t}u(t) = 0$.

Denote by Λ the set of all λ satisfying $\lim\limits_{t\to+\infty} e^{-\lambda t}u(t) = +\infty$ and put $\lambda_0 = \sup\Lambda$. In view of (11.14)–(11.16) it is obvious that $0 \in \Lambda$ and $\lambda_0 < +\infty$. By (11.17) there exist $\varepsilon > 0$ and $t_1 \in \mathbb{R}_+$ such that

$$e^{-\lambda t}\int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds \ge (n-1)! + \varepsilon \tag{11.18}$$
$$\text{for} \ \ t \ge t_1, \ \ \lambda \in]\lambda_0, \lambda_0 + \varepsilon].$$

Choose $\varepsilon_2 \in]0, \varepsilon[$ and $\varepsilon_1 \in [0, \varepsilon_1[$ such that

$$\begin{gathered}\lambda_0 - \varepsilon_1 \geq 0, \quad c^{\varepsilon_2+\varepsilon_1}(n-1)! < (n-1)! + \varepsilon, \\ \lim_{t\to+\infty} e^{-(\lambda_0-\varepsilon_1)t} u(t) = +\infty, \quad \underline{\lim}_{t\to+\infty} e^{-(\lambda_0+\varepsilon_2)t} u(t) = 0,\end{gathered} \tag{11.19}$$

where $c = \overline{\lim}_{t\to+\infty} e^{\overline{\sigma}(t)-\sigma(t)}$. According to (11.19) u is a solution of (11.2) satisfying all the conditions of Lemma 11.1 with $\gamma(t) = e^{-t}$, $r_1 = \lambda_0 - \varepsilon_1$ and $r_2 = \lambda_0 + \varepsilon_2$. Therefore

$$\underline{\lim}_{t\to+\infty} e^{-(\lambda_0+\varepsilon_2)t} \int_0^t (t-s)^{n-1} \varphi(\theta)(s) ds \leq c^{\varepsilon_2+\varepsilon_1}(n-1)! < (n-1)! + \varepsilon$$

with $\theta(t) = e^{(\lambda_0+\varepsilon_2)t}$. But this contradicts (11.18). The obtained contradiction proves the theorem. ■

′ *Let* $F \in V(\tau)$ *and conditions* (11.12)–(11.15), (10.11), (10.12) *and* (10.43) *be fulfilled, where* $\varphi \in M^+(\sigma; \overline{\sigma})$. *Then* (11.17) *with* $\theta(t) = e^{\lambda t}$ *is sufficient for problem* (0.1), (10.1) *to have no solution.*

Proof. The validity of the theorem follows from Theorem 11.1 and Corollary 10.2. ■

Let $F \in V(\tau)$, *conditions* (11.12) *and* (11.14) *be fulfilled with* $\varphi \in M^+(\sigma; \overline{\sigma})$, *for some* $t_0 \in \mathbb{R}_+$

$$p(t)|u(\delta(t)|) \leq \varphi(|u|)(t) \leq \sum_{i=1}^m q_i(t)|u(\sigma_i(t))| \tag{11.20}$$
$$\text{for} \quad u \in H^+_{t_0,\tau}, \quad t \geq t_0$$

and (10.2), (10.11), (10.12), (10.44) *and* (10.45) *hold. Then* (11.1) *is sufficient for problem* (0.1), (10.1) *to have no solution.*

Proof. The validity of the theorem follows from Theorem 11.1 and Lemma 10.9. ■

Let $F \in V(\tau)$ *and for some* $t_0 \in \mathbb{R}_+$

$$F(u)(t) \operatorname{sign} u(t) \leq -\sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)| d_s r_i(s;t) \tag{11.21}$$
$$\text{for} \quad u \in H^+_{t_0,\tau}, \quad t \geq t_0,$$

where

$$\begin{gathered} r_i(s,t) \text{ are measurable, } r_i(\cdot,t) \text{ are nondecreasing,} \\ \tau_i, \sigma_i \in C(\mathbb{R}_+;\mathbb{R}_+), \ \ \tau_i(t) \leq \sigma_i(t) \text{ for } t \in \mathbb{R}_+, \\ \lim_{t\to+\infty} \tau_i(t) = +\infty \ \ (i = 1, \dots, m), \end{gathered} \tag{11.22}$$

$$\overline{\lim}_{t\to+\infty} (\overline{\sigma}(t) - \sigma(t)) < +\infty, \quad \overline{\lim}_{t\to+\infty} (\overline{\sigma}(t) - t) < +\infty \tag{11.23}$$

with $\overline{\sigma}(t)=\max\{\sigma_i(t):i=1,\dots,m\}$, $\sigma(t)=\inf\{\min(\tau_i(s):i=1,\dots,m):s\ge t\}$. *Let, moreover, there exist* $i_0\in\{1,\dots,m\}$ *and a nondecreasing function* $\delta\in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$\sigma_{i_0}(t)>t,\ \ \max\{t,\tau_{i_0}(t)\}\le\delta(t)\le\sigma_{i_0}(t)\ \ \textit{for}\ \ t\in\mathbb{R}_+, \tag{11.24}$$

$$\varliminf_{t\to+\infty}\int_t^{\delta(t)}\big[r_{i_0}(\sigma_{i_0}(s),s)-r_{i_0}(\delta(s),s)\big]ds>0, \tag{11.25}$$

$$\text{vrai sup}\Big\{\big[r_{i_0}(\sigma_{i_0}(t),t)-r_{i_0}(\delta(t),t)\big]:t\in\mathbb{R}_+\Big\}<+\infty, \tag{11.26}$$

$$\varlimsup_{t\to+\infty}\frac{1}{t}\int_0^t\sum_{i=1}^m\big[r_i(\sigma_i(s),s)-r_i(\tau_i(s),s)\big]ds<+\infty. \tag{11.27}$$

Then the condition

$$\inf\Big\{\varliminf_{t\to+\infty}e^{-\lambda t}\int_0^t(t-s)^{n-1}\times$$
$$\times\sum_{i=1}^m\int_{\tau_i(s)}^{\sigma_i(s)}e^{\lambda\xi}d_\xi r_i(\xi,s)ds:\lambda\in]0,+\infty[\Big\}>(n-1)! \tag{11.28}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. It suffices to show that the conditions of Theorem 11.2 are satisfied with

$$\varphi(u)(t)=\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)}u(s)d_sr_i(s,t). \tag{11.29}$$

It is obvious that $\varphi\in M^+(\sigma;\overline{\sigma})$ and (11.14) holds. On the other hand, by (11.24)–(11.27) conditions (11.20), (10.11), (10.12), (10.44) and (10.45) are fulfilled with $p(t)=r_{i_0}(\sigma_{i_0}(t),t)-r_{i_0}(\delta(t),t)$ and $q_i(t)=r_i(\sigma_i(t),t)-r_i(\tau_i(t),t)$ $(i=1,\dots,n)$. Moreover, (11.28) implies (11.17). Therefore the operator defined by (11.29) satisfies all the conditions of Theorem 11.2. ■

Let $F\in V(\tau)$, *conditions* (11.21)–(11.27) *be fulfilled and for some* $t_0\in\mathbb{R}_+$

$$\inf\Big\{\lambda^{-n}\,\underset{t\ge t_0}{\text{vrai inf}}\Big(\sum_{i=1}^m\int_{\tau_i(t)}^{\sigma_i(t)}e^{\lambda(\xi-t)}d_\xi r_i(\xi,t)\Big):\lambda\in]0,+\infty[\Big\}>1. \tag{11.30}$$

Then problem (0.1), (10.1) *has no solution.*

Proof. It suffices to note that (11.30) implies (11.28). ■

Let (11.20)–(11.27) *be fulfilled* , $\sigma_i(t)\ge t$ *for* $t\in\mathbb{R}_+$ $(i=1,\dots,m)$ *and for some* $t_0\in\mathbb{R}_+$

$$\underset{t\ge t_0}{\text{vrai inf}}\Big(\sum_{i=1}^m\int_{\tau_i^*(s)}^{\sigma_i(s)}(\xi-t)^nd_\xi r_i(\xi,t)\Big)>\Big(\frac{n}{e}\Big)^n \tag{11.31}$$

with $\tau_i^*(t) = \max\{t; \tau_i(t)\}$ $(i = 1, \dots, m)$. *Then problem* (0.1), (10.1) *has no solution.*

Proof. It suffices to note that since $e^x \geq x^n(\frac{e}{n})^n$ for $x \geq 0$, (11.31) implies (11.30). ■

Let $F \in V(\tau)$ *and for some* $t_0 \in \mathbb{R}_+$

$$F(u)(t)\,\mathrm{sign}\,u(t) \leq -\sum_{i=1}^{m} c_i \int_{t-\Delta_i}^{t-\overline{\Delta}_i} |u(s)|ds \ \ \textit{for} \ \ u \in H^+_{t_0,\tau}, \ \ t \geq t_0$$

with $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\Delta_i > \overline{\Delta}_i$ $(i = 1, \dots, m)$. *Let, moreover,* $\overline{\Delta}_{i_0} < 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then the condition*

$$\inf\Big\{\lambda^{-n-1}\sum_{i=1}^{m} c_i(e^{-\lambda\overline{\Delta}_i} - e^{-\lambda\Delta_i}) : \lambda \in]0, +\infty[\Big\} > 1 \tag{11.32}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

′ *Let* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\Delta_i > \overline{\Delta}_i$ $(i = 1, \dots, m)$ *and* $\overline{\Delta}_{i_0} < 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then* (11.32) *is necessary and sufficient for the equation*

$$u^{(n)}(t) = \sum_{i=1}^{m} c_i \int_{t-\Delta_i}^{t-\overline{\Delta}_i} u(s)ds \tag{11.33}$$

to have no solution satisfying (10.1).

Proof. Sufficiency follows from Corollary 11.3. If we suppose that (10.32) is violated, then (11.33) has the solution $u(t) = e^{\lambda t}$ with $\lambda > 0$. ■

Let $F \in V(\tau)$ *and for some* $t_0 \in \mathbb{R}_+$

$$F(u)(t)\,\mathrm{sign}\,u(t) \leq -\sum_{i=1}^{m} p_i(t)|u(\delta_i(t))| \ \ \textit{for} \ \ u \in H^+_{t_0,\tau}, \ \ t \geq t_0 \tag{11.34}$$

with

$$\begin{gathered} p_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad \delta_i \in C(\mathbb{R}_+; \mathbb{R}_+), \\ \lim_{t\to+\infty} \delta_i(t) = +\infty \ \ (i = 1, \dots, m), \end{gathered} \tag{11.35}$$

$$\overline{\lim_{t\to+\infty}}(\delta_i(t) - t) < +\infty \quad (i = 1, \dots, m). \tag{11.36}$$

Let, moreover, there exist $i_0 \in \{1, \dots, m\}$ *such that* $\delta_{i_0}(t)$ *is nondecreasing,* $\delta_{i_0}(t) > t$ *for* $t \in \mathbb{R}_+$ *and*

$$\underline{\lim}_{t\to+\infty} \int_t^{\delta_{i_0}(t)} p_{i_0}(s)ds > 0, \quad \mathrm{vrai}\sup\{p_{i_0}(t) : t \in \mathbb{R}_+\} < +\infty. \tag{11.37}$$

Then the condition

$$\inf\Big\{\varlimsup_{t\to+\infty} e^{-\lambda t}\int_0^t (t-s)^{n-1}\times$$

$$\times\sum_{i=1}^m p_i(s)e^{\lambda\delta_i(s)}ds : \lambda\in]0,+\infty[\Big\} > (n-1)! \qquad (11.38)$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. It suffices to note that by (11.34)–(11.38) all the coditions of Theorem 11.3 are satisfied with $\tau_i(t) = \delta_i(t)-1$, $\sigma_i(t) = \delta_i(t)$, $r_i(s,t) = p_i(t)e(s-\delta_i(t))$ $(i=1,\dots,m)$ (the definition of the function $e(t)$ see on page 104). ■

Let (11.34)–(11.37) *be fulfilled and for some* $t_0\in\mathbb{R}_+$

$$\inf\Big\{\lambda^{-n}\operatorname*{vrai\,inf}_{t\ge t_0}\sum_{i=1}^m p_i(t)e^{\lambda(\delta_i(t)-t)} : \lambda\in]0,+\infty[\Big\} > 1. \qquad (11.39)$$

Then problem (0.1), (10.1) *has no solution.*

Proof. It suffices to note that (11.39) implies (11.38). ■

Let $\delta_i(t)\ge t$ *for* $t\in\mathbb{R}_+$ $(i=1,\dots,m)$, (11.34)–(11.37) *be fulfilled and for some* $t_0\in\mathbb{R}_+$

$$\operatorname*{vrai\,inf}_{t\ge t_0}\Big\{\sum_{i=1}^m p_i(t)(\delta_i(t)-t)^n\Big\} > \Big(\frac{n}{e}\Big)^n.$$

Then problem (0.1), (10.1) *has no solution.*

Let $F\in V(\tau)$ *and for some* $t_0\in\mathbb{R}_+$

$$F(u)(t)\operatorname{sign}u(t)\le -\sum_{i=1}^m c_i|u(t-\Delta_i)| \ \ \textit{for}\ \ u\in H^+_{t_0,\tau},\ \ t\ge t_0,$$

where $c_i>0$, $\Delta_i\in\mathbb{R}$ $(i=1,\dots,m)$ *and* $\Delta_{i_0}<0$ *for some* $i_0\in\{1,\dots,m\}$. *Then the condition*

$$\inf\Big\{\lambda^{-n}\sum_{i=1}^m c_ie^{-\lambda\Delta_i} : \lambda\in]0,+\infty[\Big\} > 1 \qquad (11.40)$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Let $c_i>0$, $\Delta_i\in\mathbb{R}$ $(i=1,\dots,m)$ *and* $\Delta_{i_0}<0$ *for some* $i_0\in\{1,\dots,m\}$. *Then* (11.40) *is necessary and sufficient for the equation*

$$u^{(n)}(t) = \sum_{i=1}^m c_iu(t-\Delta_i)$$

to have no solution satisfying (10.1).

Let $F \in V(\tau)$, *conditions* (11.12), (11.13), (11.15) *and* (11.16) *be fulfilled with* $\varphi \in M^+(\sigma;\overline{\sigma})$ *and*

$$\lim_{t\to+\infty} \frac{\overline{\sigma}(t)}{\sigma(t)} < +\infty. \tag{11.41}$$

Let, moreover, for some $k \in \{0,\dots,n-1\}$ (11.22_k), (11.47_k) *hold with* $\rho_k^*(t)$ *defined by* (11.5_k) *and* (11.6_k). *Then the condition*

$$\inf\Big\{ \varliminf_{t\to+\infty} t^{-\lambda} \int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds : \lambda \in [n-1,+\infty[\Big\} > (n-1)!, \tag{11.42}$$

with $\theta(t) = t^\lambda$ *is sufficient for problem* (0.1), (10.1) *to have no solution.*

Proof. Suppose, on the contrary, that (0.1) has a solution $u_0 : [t_0,+\infty[\to \mathbb{R}$ satisfying (10.1). According to (11.12) and Lemma 10.7 (11.2) has a solution $u : [t_*,+\infty[\to]0,+\infty[$ satisfying (11.3). By (10.22_k), (10.47_k), (11.13) , (11.15) and Lemma 10.10 there exists $\lambda > 0$ such that $\lim\limits_{t\to+\infty} t^{-\lambda}u(t) = 0$.

Denote by Λ the set af all λ satisfying $\lim\limits_{t\to+\infty} t^{-\lambda}u(t) = +\infty$ and put $\lambda_0 = \sup\Lambda$. In view of (11.16) it is obvious that $n-1 \in \Lambda$ and $\lambda_0 \in [n-1,+\infty[$. By (11.42) there exist $\varepsilon > 0$ and $t_1 \in \mathbb{R}_+$ such that

$$t^{-\lambda} \int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds \geq (n-1)! + \varepsilon \qquad \text{for } t \geq t_1,\ \lambda \in]\lambda_0, \lambda_0+\varepsilon[. \tag{11.43}$$

Choose $\varepsilon_2 \in]0,\varepsilon[$ and $\varepsilon_1 \in [0,\varepsilon_2[$ such that

$$\begin{gathered} \lambda_0 - \varepsilon_1 \geq 0, \quad c^{\varepsilon_2+\varepsilon_1}(n-1)! < (n-1)! + \varepsilon, \\ \lim_{t\to+\infty} t^{-(\lambda_0+\varepsilon_1)}u(t) = +\infty, \quad \varliminf_{t\to+\infty} t^{-(\lambda_0+\varepsilon_2)}u(t) = 0 \end{gathered} \tag{11.44}$$

with $c = \overline{\lim\limits_{t\to+\infty}} \frac{\overline{\sigma}(t)}{\sigma(t)}$. According to (11.44) u is a solution of (11.2) satisfying all the conditions of Lemma 11.1 with $\gamma(t) = t^{-1}$, $r_1 = \lambda_0 - \varepsilon_1$ and $r_2 = \lambda_0 + \varepsilon_2$. Therefore we have

$$\varliminf_{t\to+\infty} t^{-(\lambda_0+\varepsilon_2)} \int_0^t (t-s)^{n-1}\varphi(\theta)(s)ds < c^{\varepsilon_2+\varepsilon_1}(n-1)! < (n-1)! + \varepsilon$$

with $\theta(t) = t^{\lambda_0+\varepsilon_2}$. But this contradicts (11.43). The obtained contradiction proves the theorem. ■

′ *Let* $F \in V(\tau)$ *and* (11.12), (11.13), (11.15), (11.16), (11.41), (10.17_0) *and* (10.18_0) *be fulfilled with* $\varphi \in M^+(\sigma;\overline{\sigma})$. *Then* (11.42) *is sufficient for problem* (0.1), (10.1) *to have no solution.*

Proof. The validity of the theorem follows from Theorem 11.5 and Corollary 10.3. ■

Let $F \in V(\tau)$ *and* (11.12), (11.21), (11.41), (10.17$_0$), (10.18$_0$), (10.50) *and* (10.51) *be fulfilled with* $\varphi \in M^+(\sigma;\overline{\sigma})$. *Then* (11.42) *is sufficient for problem* (0.1), (10.1) *to have no solution.*

Proof. The validity of the theorem follows from Theorem 11.5 and Lemma 10.11. ■

Let $F \in V(\tau)$, *conditions* (11.21) *and* (11.22) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \frac{\overline{\sigma}(t)}{\sigma(t)} < +\infty, \quad \overline{\lim_{t\to+\infty}} \frac{\overline{\sigma}(t)}{t} < +\infty, \tag{11.45}$$

where $\overline{\sigma}(t) = \max\{\sigma_i(t) : i = 1,\dots,m\}$, $\sigma(t) = \inf\{\min(\tau_i(s) : i = 1,\dots,m) : s \geq t\}$,

$$\overline{\lim_{t\to+\infty}} \frac{1}{\ln t} \int_0^t s^{n-1} \sum_{i=1}^m \big[r_i(\sigma_i(s),s) - r_i(\tau_i(s),s)\big] ds < +\infty. \tag{11.46}$$

Let, moreover, there exist $i_0 \in \{1,\dots,m\}$ *and a nondecreasing function* $\delta \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that* (11.24) *holds,*

$$\underline{\lim}_{t\to+\infty} \int_t^{\delta(t)} s^{n-1}\big[r_{i_0}(\sigma_{i_0}(s),s) - r_{i_0}(\delta(s),s)\big] ds > 0 \tag{11.47}$$

and

$$\text{vrai sup}\,\big\{t^n\big[r_{i_0}(\sigma_{i_0}(t),t) - r_{i_0}(\delta(t),t)\big] : t \in \mathbb{R}_+\big\} < +\infty. \tag{11.48}$$

Then the condition

$$\inf\bigg\{\underline{\lim}_{t\to+\infty} t^{-\lambda} \int_0^t (t-s)^{n-1} \sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^\lambda d_\xi r_i(\xi,s) ds : \\ : \lambda \in]n-1,+\infty[\bigg\} > (n-1)! \tag{11.49}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. As in the proof of Theorem 11.3, we can easily show that the operator defined by (11.29) satisfies all the conditions of Theorem 11.6. This proves the theorem. ■

Let $F \in V(\tau)$, *conditions* (11.22), (11.45)–(11.48) *be fulfilled,* $\varphi \in M^+(\sigma,\overline{\sigma})$ *and for some* $t_0 \in \mathbb{R}_+$

$$\inf\bigg\{\frac{1}{\prod_{i=0}^{n-1}(\lambda-i)} \operatorname*{vrai\,inf}_{t\geq t_0} \Big(t^{n-\lambda} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} \xi^\lambda d_\xi r_i(\xi,t)\Big) : \\ : \lambda \in]n-1,+\infty[\bigg\} > 1. \tag{11.50}$$

Then problem (0.1), (10.1) *has no solution.*

Proof. It suffices to note that (11.50) implies (11.49). ■

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$F(u)(t)\operatorname{sign}u(t) \le -t^{-n-1}\sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} |u(s)|ds \quad for \quad u \in H^{+}_{t_0,\tau}, \quad t \ge t_0$$

with $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,+\infty[$, $\alpha_i < \overline{\alpha_i}$ $(i = 1,\dots,m)$. Let, moreover, $i_0 \in \{1,\dots,m\}$ exist such that $\overline{\alpha}_{i_0} > 1$. Then the condition

$$\inf\left\{\frac{1}{\prod_{i=-1}^{n-1}(\lambda - i)}\sum_{i=1}^{m} c_i(\overline{\alpha}_i^{\ \lambda+1} - \alpha_i^{\lambda+1}) : \lambda \in]n-1,+\infty[\right\} > 1 \quad (11.51)$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

′ Let $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,+\infty[$, $\alpha_i < \overline{\alpha_i}$ $(i = 1,\dots,m)$ and $\overline{\alpha}_{i_0} > 1$ for some $i_0 \in \{1,\dots,m\}$. Then (10.50) *is necessary and sufficient for the equation*

$$u^{(n)} = t^{-n-1}\sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} u(s)ds \qquad (11.52)$$

to have no solution satisfying (10.1).

Proof. Sufficiency follows from Corollary 11.9. If we suppose that (11.50) is violated, then (11.52) has the solution $u(t) = t^{\lambda}$ with $\lambda > n-1$. ■

Let $F \in V(\tau)$, conditions (11.34) *and* (11.35) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \frac{\overline{\sigma}(t)}{\sigma(t)} < +\infty, \quad \overline{\lim_{t\to+\infty}} \frac{\sigma(t)}{t} < +\infty \qquad (11.53)$$

with $\overline{\sigma}(t) = \max\{\delta_i(t) : i = 1,\dots,m\}$, $\sigma(t) = \inf\{\min(\delta_i(s) : i = 1,\dots,m) : s \ge t\}$. Let, moreover, $i_0 \in \{1,\dots,m\}$ exist such that $\delta_{i_0}(t)$ is nondecreasing,

$$\underline{\lim_{t\to+\infty}} \int_{t}^{\delta_{i_0}(t)} s^{n-1} p_{i_0}(s)ds > 0, \qquad (11.54)$$

$$\operatorname{vrai}\sup\{t^n p_{i_0}(t) : t \in \mathbb{R}_+\} < +\infty. \qquad (11.55)$$

Then the condition

$$\inf\left\{\underline{\lim_{t\to+\infty}} t^{-\lambda}\int_0^t (t-s)^{n-1}\sum_{i=1}^{m} p_i(s)\delta_i^{\lambda}(s)ds : \right.$$
$$\left. : \lambda \in]n-1,+\infty[\right\} > (n-1)! \qquad (11.56)$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. It suffices to note that by (11.34), (11.35) and (11.53)-(11.56) all the conditions of Theorem 11.6 are satisfied with $\tau_i(t) = \delta_i(t) - 1$, $\sigma_i(t) = \delta_i(t)$ and $r_i(s,t) = p_i(t)e(s - \delta_i(t))$ (the definition of the function e see on page 104). ■

Let $F \in V(\tau)$, conditions (11.34), (11.35) *and* (11.53)–(11.56) *be fulfilled and for some $t_0 \in \mathbb{R}_+$*

$$\inf\left\{\frac{1}{\prod_{i=0}^{n-1}(\lambda - i)} \operatorname*{vrai\,inf}_{t \geq t_0}\Big(t^{n-\lambda}\sum_{i=1}^{m} p_i(t)\delta_i^{\lambda}(t)\Big) : \right.$$
$$\left. : \lambda \in]n-1, +\infty[\right\} > 1. \tag{11.57}$$

Then problem (0.1), (10.1) *has no solution.*

Proof. It suffices to note that (11.57) implies (11.56). ■

Let $F \in V(\tau)$, for some $t_0 \in \mathbb{R}_+$

$$F(u)(t)\operatorname{sign} u(t) \leq -t^{-n}\sum_{i=1}^{m} c_i|u(\alpha_i t)| \;\; for \;\; u \in H^+_{t_0,\tau}, \;\; t \geq t_0$$

with $c_i > 0$, $\alpha_i > 0$ $(i = 1, \ldots, m)$ and let $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \ldots, m\}$. Then the condition

$$\inf\left\{\frac{1}{\prod_{i=0}^{n-1}(\lambda - i)}\sum_{i=1}^{m} c_i\alpha_i^{\lambda} : \lambda \in]n-1, +\infty[\right\} > 1 \tag{11.58}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

′ *Let $c_i > 0$, $\alpha_i > 0$ $(i = 1, \ldots, m)$ and $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \ldots, m\}$. Then* (11.58) *is necessary and sufficient for the equation*

$$u^{(n)}(t) = t^{-n}\sum_{i=1}^{m} c_i u(\alpha_i t)$$

to have no solution satisfying (10.1).

Let $n > 1$, $F \in V(\tau)$, for any $t_0 \in \mathbb{R}_+$

$$F(u)(t)\operatorname{sign} u(t) \leq -p(t)|u(\delta(t))| \;\; for \;\; u \in H^+_{t_0,\tau}, \;\; t \geq t_0, \tag{11.59}$$

(10.2) *be fulfilled and for some $k \in \{1, \ldots, n-1\}$* (10.17_k), (10.18_k) *hold. Then problem* (0.1), (10.1) *has no solution.*

Proof. Suppose that (0.1) has a solution u satisfying (10.1). Then by (11.58), (11.17_k), (10.18_k) and Lemma 10.3

$$\overline{\lim_{t\to+\infty}} \frac{t^{2k+1-n}|u(\delta(t))|}{|u^{(n-1)}(t)|} < +\infty \tag{11.60}$$

and

$$\underline{\lim_{t\to+\infty}} \frac{\delta(t)}{t} > 1. \tag{11.61}$$

On the other hand, by (10.1) from (1.6_{0n}) we have

$$|u(\delta(t))| \geq \frac{1}{(n-1)!}(\delta(t)-t)^{n-1}|u^{(n-1)}(t)| \ \text{ for } \ t \geq t_1,$$

where $t_1 > t_0$ is sufficiently large. Hence in view of (11.61) we obtain

$$\underline{\lim_{t\to+\infty}} \frac{|u(\delta(t))|t^{1-n}}{|u^{(n-1)}(t)|} > 0.$$

Since $k \geq 1$, the last inequality contradicts (11.60). The obtained contradiction proves the theorem. ■

Analogously can be proved

Let $F \in V(\tau)$, *conditions* (11.59), (10.17_0) *and* (10.18_0) *be fulfilled and*

$$\overline{\lim_{t\to+\infty}} \frac{\delta(t)}{t} = +\infty.$$

Then problem (0.1), (10.1) *has no solution.*

Let $F \in V(\tau)$, *conditions* (10.2) *and* (11.59) *be fulfilled and for some* $k \in \mathbb{N}$ *and* $i \in \{0,\dots,n-1\}$

$$\overline{\lim_{t\to+\infty}} \int_t^{\delta(t)} p(s)(\delta(t)-s)^{n-i-1}(\delta(s)-\delta(t))^i g_k(\delta(s),\delta(t))ds >$$
$$> i!(n-i-1)!, \tag{11.62}$$

where

$$g_k(t,s) = \exp\left\{\frac{1}{(n-1)!}\int_s^t (t-\xi)^{n-1}p(\xi)\psi_k(\xi)d\xi\right\}, \tag{11.63}$$

$$\psi_1(t) = 1, \quad \psi_j(t) = \exp\left\{\frac{1}{(n-1)!}\int_t^{\delta(t)} (\delta(t)-\xi)^{n-1} \times\right.$$
$$\left.\times p(\xi)\psi_{j-1}(\xi)d\xi\right\} \quad (j=2,\dots,k). \tag{11.64}$$

Then problem (0.1), (10.1) *has no solution.*

Proof. Suppose that (0.1) has a solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying (10.1). From (0.1) by (11.59) we have

$$\frac{1}{(n-1)!}\int_s^{\delta(t)} \frac{(\delta(t)-\xi)^{n-1}|u^{(n)}(\xi)|}{u_0(\delta(t),\xi)}d\xi \geq \frac{1}{(n-1)!}\times$$
$$\times\int_s^{\delta(t)}(\delta(t)-\xi)^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\delta(t),\xi)}d\xi \text{ for } t_0\leq s\leq\delta(t), \quad (11.65)$$

where

$$u_0(t,s)=\sum_{j=0}^{n-1}\frac{(t-s)^j}{j!}|u^{(j)}(s)|.$$

Since

$$\frac{du_0(\delta(t),\xi)}{d\xi}=\frac{(\delta(t)-\xi)^{n-1}}{(n-1)!}|u^{(n)}(\xi)|, \ \ u_0(\delta(t),\delta(t))=|u(\delta(t))|,$$

in view of (11.65) we obtain

$$\ln\frac{|u(\delta(t))|}{u_0(\delta(t),s)}\geq\frac{1}{(n-1)!}\int_s^{\delta(t)}(\delta(t)-\xi)^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\delta(t),\xi)}d\xi.$$

As $u_0(\delta(\xi),\xi)\geq u_0(\delta(t),\xi)$ for $t_0\leq t\leq\xi\leq\delta(t)$, the last inequality implies

$$\frac{|u(\delta(t))|}{u_0(\delta(t),s)}\geq\exp\Big\{\frac{1}{(n-1)!}\int_s^{\delta(t)}(\delta(t)-\xi)^{n-1}p(\xi)\frac{|u(\delta(\xi))|}{u_0(\delta(\xi),\xi)}d\xi\Big\}$$
$$\text{for } \ t_0\leq t\leq s\leq\delta(t),$$

whence we easily conclude that

$$|u(\delta(t))|\geq\exp\Big\{\frac{1}{(n-1)!}\int_s^{\delta(t)}(\delta(t)-\xi)^{n-1}p(\xi)\psi_k(\xi)d\xi\Big\}u_0(\delta(t),s)$$
$$\text{for } \ t_0\leq t\leq s\leq\delta(t)$$

with $\psi_k(t)$ defined by (11.64). Therefore according to (10.1) and (11.59) from (0.1) we have

$$|u^{(i)}(\delta(t))|\geq\frac{1}{(n-i-1)!}\int_t^{\delta(t)}(\delta(t)-\xi)^{n-i-1}p(\xi)\times$$
$$\times g_k(\delta(\xi),\delta(t))u_0(\delta(\xi),\delta(t))d\xi \quad (11.66)$$

with $g_k(t,s)$ defined by (11.63). Since

$$u_0(\delta(\xi),\delta(t))\geq\frac{|u^{(i)}(\delta(t))|}{i!}(\delta(\xi)-\delta(t))^i \ \text{ for } \ t_0\leq t\leq\xi,$$

from (11.66) we finally obtain

$$|u^{(i)}(\delta(t))| \geq \frac{1}{i!(n-i-1)!} \int_t^{\delta(t)} (\delta(t)-\xi)^{n-i-1} \times$$
$$\times (\delta(\xi)-\delta(t))^i g_k(\delta(\xi),\delta(t)) d\xi |u^{(i)}(\delta(t))|$$

which contradicts (11.62). The obtained contradiction proves the theorem. ■

Let $F \in V(\tau)$*, conditions* (10.2) *and* (11.59) *be fulfilled and for some* $i \in \{0,\dots,n-1\}$

$$\overline{\lim_{t\to+\infty}} \int_t^{\delta(t)} p(s)(\delta(t)-s)^{n-i-1}(\delta(s)-\delta(t))^i ds > i!(n-i-1)!. \quad (11.67)$$

Then problem (0.1), (10.1) *has no solution.*

Remark 11.1. (11.67) cannot be replaced by

$$\overline{\lim_{t\to+\infty}} \int_t^{\delta(t)} p(s)(\delta(t)-s)^{n-i-1}(\delta(s)-\delta(t))^i ds >$$
$$> i!(n-i-1)!(1-\varepsilon)^n, \quad (11.68)$$

however small $\varepsilon \in]0,1]$ would be.

Indeed, let $\varepsilon \in]0,1]$. Choose $n_0 \in \mathbb{N}$, $\Delta > 0$ and $c > 0$ such that

$$\sqrt[n]{n!} < \frac{(1+\varepsilon)n}{e} \quad \text{for} \quad n \geq n_0, \quad c\Delta^n = (1-\varepsilon)^n n!.$$

Then the equation

$$u^{(n)}(t) = cu(t+\Delta)$$

has the solution $u(t) = e^{\lambda t}$ with $\lambda > 0$. On the other hand, (11.68) holds with $p(t) = c$ and $\delta(t) = t + \Delta$.

Let $F \in V(\tau)$*, conditions* (10.2) *and* (11.59) *be fulfilled and*

$$\underline{\lim_{t\to+\infty}} \int_t^{\delta(t)} p(s)(\delta(t)-s)^{n-1} ds > \frac{(n-1)!}{e}. \quad (11.69)$$

Then problem (0.1), (10.1) *has no solution.*

This theorem can be prooved analogously to Theorem 6.2.

Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$

$$F(u)(t)\,\mathrm{sign}\,u(t) \le -\varphi(u)(t)\,\mathrm{sign}\,u(t) \le 0 \quad \textit{for} \quad u \in H^{+}_{t_0,\tau}, \quad t \ge t_0, \tag{11.70}$$

where $\varphi \in M(\sigma)$, $\sigma(t) \ge t$ for $t \in \mathbb{R}_+$. Then the condition

$$\widetilde{\varphi}_n \in M_2^{\beta}(\sigma) \tag{11.71}$$

with

$$\beta(t) = t, \quad \widetilde{\varphi}_n(u)(t) = \frac{1}{(n-1)!}\varphi((\sigma(t)-t)^{n-1}u(t))(t) \tag{11.72}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. Suppose that (0.1) has a solution $u : [t_0, +\infty[\to \mathbb{R}$ satisfying (10.1). Then from (1.6_{0n}) we have

$$|u(\sigma(t))| \ge \frac{|u^{(n-1)}(t)|}{(n-1)!}(\sigma(t)-t)^{n-1} \quad \text{for} \quad t \ge t_0.$$

Therefore, taking into account the fact that $\varphi \in M(\sigma)$, by (11.70) we obtain

$$x(t) \ge \int_{t_0}^{t} |\widetilde{\varphi}_n(\mathrm{sign}\,u(t_0)x)(s)|ds,$$

where $x(t) = \frac{1}{(n-1)!}|u^{(n-1)}(t)| > 0$ for $t \ge t_0$. But this contradicts (11.71), which proves the validity of the theorem. ■

Analogously can be proved

$'$ *Let $F \in V(\tau)$ and for some $t_0 \in \mathbb{R}_+$*

$$F(u)(t)\,\mathrm{sign}\,u(t) \le -\varphi(t, u(\delta_1(t)), \dots, u(\delta_m(t)))\,\mathrm{sign}\,u(t) \le 0$$
$$\textit{for} \quad u \in H^{+}_{t_0,\tau}, \quad t \ge t_0,$$

where $\delta_i \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\delta_i(t) \ge t$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$, $\varphi \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^m;\mathbb{R})$ and

$$\varphi(t, x_1, \dots, x_m)\,\mathrm{sign}\,x_1 \ge \varphi(t, y_1 \dots, y_m)\,\mathrm{sign}\,y_1 \ge 0$$
$$\textit{for} \;\; t \in \mathbb{R}_+, \;\; x_i x_1 > 0, \;\; x_i y_i > 0, \;\; |x_i| \ge |y_i| \;\; (i = 1, \dots, m).$$

Then condition (11.71) *with*

$$\widetilde{\varphi}_n(u)(t) = \frac{1}{(n-1)!}\varphi(t, (\delta_1(t)-t)^{n-1}u(t), \dots, (\delta_m(t)-t)^{n-1}u(t))$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Let $F \in V(\tau)$, (11.70) *be fulfilled with* $\varphi \in M(\sigma)$, *the function* $\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$ *be nondecreasing and* $\sigma(t) \geq t$ *for* $t \in \mathbb{R}_+$. *Then condition* (11.71) *with*

$$\widetilde{\varphi}_n(u)(t) = \frac{(\sigma(t)-t)^{n-1}}{(n-1)!}|\varphi(u(t))(t)| \tag{11.73}$$

is sufficient for problem (0.1), (10.1) *to have no solution.*

Proof. Suppose that (0.1) has a solution $u : [t_0,+\infty[\to \mathbb{R}$ satisfying (10.1). Then by (10.1), (11.70) and the fact that $\varphi \in M(\sigma)$ we obtain

$$|u(\sigma(t))| \geq \frac{1}{(n-1)!}\int_{t_0}^{\sigma(t)} (\sigma(t)-s)^{n-1}|u^{(n)}(s)|ds \geq$$

$$\geq \frac{1}{(n-1)!}\int_{t_0}^{t} (\sigma(s)-s)^{n-1}\left|\varphi\Big(\operatorname{sign} u(t_0)|u(\sigma(s))|\Big)ds \quad \text{for} \quad t \geq t_0.$$

Therefore

$$x(t) \geq \int_{t_0}^{t} \left|\widetilde{\varphi}_n\big(\operatorname{sign} u(t_0)x\big)(s)\right|ds \quad \text{for} \quad t \geq t_0$$

with $x(t) = |u(\sigma(t))|$. But this contradicts (11.71). The obtained contradiction proves the theorem. ■

CHAPTER 5

In chapter 5 the oscillatory properties which are specific for functional differential equations are studied. These properties have no analogues for ordinary differential equations.

§ 12. Equations with property $\widetilde{\mathbf{A}}$

Everywhere in this section it will be assumed that the inequality

$$(-1)^{n+1} F(u)(t) u(t) \geq 0 \quad \text{for} \quad u \in H_{t_0,\tau}, \quad t \geq t_0 \tag{12.1}$$

holds for some $t_0 \in \mathbb{R}_+$.

Let $F \in V(\tau)$, (12.1) be fulfilled for some $t_0 \in \mathbb{R}_+$ and

$$|F(u)(t)| \geq |\varphi(u)(t)| \quad for \quad u \in H_{t_0,\tau}, \quad t \geq t_0 \tag{12.2}$$

with $\varphi \in M(\tau, \sigma)$, where $\sigma(t) \leq t$ for $t \in \mathbb{R}_+$. Then, if n is odd, the conditions (3.3_{n-1}) and

$$\widetilde{\varphi}_0 \in M_1(\tau, \sigma) \tag{12.3}$$

are sufficient for (0.1) to have property ~ , and if n is even, such are conditions (3.11), (3.3_{n-2}) and (12.3), where $\widetilde{\varphi}_{n-1}$, $\widetilde{\varphi}_{n-2}$ and $\widetilde{\varphi}_0$ are defined, respectively, by (3.12), (3.14) and (9.86).

Proof. Suppose that $u : [t_0, +\infty[\to \mathbb{R}$ is a nonoscillatory proper solution of (0.1). According to Lemma 1.2 and (12.1) $l \in \{0, \dots, n\}$ exists such that l is even and (2.14_l) holds. By Theorems 3.1′, 3.2′ and 9.15 and conditions (3.2_{n-1}), (3.2_{n-2}) and (12.3) we have $l \notin \{0, \dots, n-1\}$. Therefore, n is even and $l = n$, so in view of (3.13) (0.5) is fulfilled. ■

Taking into account Theorems 3.1″, 3.2″ and 9.15, we can analoguosly prove the following

′ *Let $F \in V(\tau)$ and conditions (12.1) and (12.2) be fulfilled with $\varphi \in M(\tau, \sigma)$, where $\sigma(t) \leq t$ for $t \in \mathbb{R}_+$. Then, if n is odd, conditions (3.16), (12.3) are sufficient for (0.1) to have property ~ , while if n is even, such are conditions (3.11), (3.18) and (12.3), where $\widetilde{\varphi}_{n-1}$, $\widetilde{\varphi}_{n-2}$ and $\widetilde{\varphi}_0$ are defined, respectively, by (3.17), (3.18) and (9.15).*

Let $F \in V(\tau)$, (12.1) be fulfilled for some t_0 and

$$|F(u)(t)| \geq \int_{\tau_1(t)}^{\sigma_1(t)} |u(s)|^{\lambda} d_s r(s,t) \quad for \quad u \in H_{t_0,\tau}, \quad t \geq t_0, \tag{12.4}$$

where

$$0<\lambda<1,\ \ \tau_1,\sigma_1\in C(\mathbb{R}_+;\mathbb{R}_+),\ \ \tau_1(t)\le\sigma_1(t)\le t\ \ \textit{for}\ \ t\in\mathbb{R}_+,$$
$$\lim_{t\to+\infty}\tau_1(t)=+\infty,\ \ r(s,t)\ \ \textit{is measurable,}\ \ r(\cdot,t)\ \ \textit{is nondecreasing.} \tag{12.5}$$

Let, moreover,

$$\int^{+\infty}\int_{\tau_1(t)}^{\sigma_1(t)} s^{\lambda(n-1)}d_s r(s,t)dt=+\infty$$

and

$$\int^{+\infty} t^{(n-1)(1-\lambda)}(t-\sigma_1(t))^{\lambda(n-1)}\big(r(\sigma_1(t),t)-r(\tau_1(t),t)\big)dt=+\infty.$$

Then (0.1) *has the property* $\widetilde{\ }$.

′ *Let* $F\in V(\tau)$, *conditions* (12.1) *and* (12.5) *be fulfilled for some* $t_0\in\mathbb{R}_+$ *and*

$$\int_{\tau_1(t)}^{\sigma_1(t)}|u(s)|^\lambda d_s r(s,t)\le|F(u)(t)|\le\delta\int_{\tau_1(t)}^{\sigma_1(t)}|u(s)|^\lambda d_s r(s,t)$$
$$\textit{for}\ \ u\in H_{t_0,\tau},\ \ t\ge t_0,$$

with $0<\lambda<1$ *and* $\delta\in[1,+\infty[$. *Let, moreover,*

$$\varlimsup_{t\to+\infty}\frac{\sigma_1(t)}{\tau_1(t)}<+\infty,\qquad \varliminf_{t\to+\infty}\frac{t^{1-\lambda}(t-\sigma_1(t))^\lambda}{(\tau_1(t))^\lambda}>0.$$

Then the condition

$$\int^{+\infty}\tau_1^{\lambda(n-1)}(t)\big(r(\sigma_1(t),t)-r(\tau_1(t),t)\big)dt=+\infty$$

is necessary and sufficient for (0.1) *to have property* $\widetilde{\ }$.

The validity of Theorems 12.2 and 12.2′ follows from Corollaries 4.1 and 4.2, 9.16.

Let $F\in V(\tau)$, *conditions* (12.1), (8.2) *and* (8.3) *be fulfilled for some* t_0 *and*

$$|F(u)(t)|\ge p(t)|u(\delta(t))|\quad \textit{for}\quad u\in H_{t_0,\tau},\quad t\ge t_0. \tag{12.6}$$

Let, moreover,

$$\int^{+\infty}(\delta(t))^{n-1-\varepsilon}p(t)dt=+\infty \tag{12.7}$$

with $\varepsilon>0$ *and either* (9.73) *or* (9.75) *hold. Then* (0.1) *has property* $\widetilde{\ }$.

Proof. The validity of the theorem follows from Corollaries 3.12 and 9.16 and Theorem 9.12. ■

Let $F \in V(\tau)$*, condition* (12.1) *be fulfilled for some* $t_0 \in \mathbb{R}_+$ *and*

$$|F(u)(t)| \geq \sum_{i=1}^{m} c_i \int_{t-\Delta_i}^{t-\overline{\Delta}_i} |u(s)|ds \ \ for \ \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{12.8}$$

where $c_i > 0$*,* $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$*,* $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$ *for some* $i_0 \in \{1, \dots, m\}$*. Then* (9.28) *is sufficient for* (0.1) *to have the property* $\widetilde{\ }$.

Proof. The validity of the theorem immediately follows from Corollaries 3.1 and 9.2. ■

′ *Let* $c_i > 0$*,* $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$*,* $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$ *for some* $i_0 \in \{1, \dots, m\}$*. Then* (9.28) *is necessary and sufficient for* (9.29) *to have property* $\widetilde{\ }$.

Let $F \in V(\tau)$*, condition* (12.1) *be fulfilled and*

$$|F(u)(t)| \geq \sum_{i=1}^{m} c_i |u(t - \Delta_i)| \ \ for \ \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{12.9}$$

where $c_i > 0$*,* $\Delta_i \in \mathbb{R}$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$ *for some* $i_0 \in \{1, \dots, m\}$*. Then* (9.37) *is sufficient for* (0.1) *to have property* $\widetilde{\ }$.

Proof. The thoerem follows from Corollaries 3.1 and 9.4. ■

Let $F \in V(\tau)$*, condition* (12.1) *be fulfilled and*

$$|F(u)(t)| \geq t^{-1-n} \sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} |u(s)|ds \ \ for \ \ u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{12.10}$$

where $c_i > 0$*,* $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$*,* $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ *and* $\alpha_{i_0} < 1$ *for some* $i_0 \in \{1, \dots, m\}$*. Let, moreover,* (9.49) *hold and*

$$\sum_{i=1}^{m} c_i (\overline{\alpha}_i^{\lambda+1} - \alpha_i^{\lambda+1}) > - \prod_{i=-1}^{n-1} (i - \lambda) \ \ for \ \ \lambda \in [1, n-1].^{18} \tag{12.11}$$

Then (0.1) *has property* $\widetilde{\ }$.

Proof. The theorem follows from Corollary 9.7 and Theorems 7.4 and 7.5. ■

[18] When $n = 1, 2$, (12.11) is unnecessary.

Let $F \in V(\tau)$ and (12.1) and (12.10) be fulfilled with $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,1]$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then the conditions (9.49) and

$$\sum_{i=1}^{m} c_i(\overline{\alpha}_i^{\lambda+1} - \alpha_i^{\lambda+1}) > -\prod_{i=-1}^{n-1}(i-\lambda) \quad \text{for } \lambda \in [n-3, n-1] \qquad (12.12)$$

are sufficient for (0.1) to have property $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,+\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots, m\}$. Then (9.49) and (12.11) are necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} c_i t^{-n-1} \int_{\alpha_i t}^{\overline{\alpha}_i t} u(s)ds = 0 \qquad (12.13)$$

to have property $\widetilde{\ }$.

Proof. The sufficiency follows from Theorem 12.6. If we assume that either (9.49) or (12.11) is violated, then equation (12.13) has the solution $u(t) = t^\lambda$, where either $\lambda < 0$ or $\lambda \in]1, n-1[$. ■

$'$ *Let $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,1]$ and $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then (9.49) and (12.12) are necessary and sufficient for equation (12.13) to have property $\widetilde{\ }$.*

Let $F \in V(\tau)$, condition (12.1) be fulfilled for some $t_0 \in \mathbb{R}_+$ and

$$|F(u)(t)| \geq \sum_{i=1}^{m} c_i \int_{\alpha_i t}^{\overline{\alpha}_i t} s^{-n-1}|u(s)|ds \quad \text{for } u \in H_{t_0,\tau}, \ t \geq t_0, \qquad (12.14)$$

where $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots, m\}$. Then conditions (9.46) and

$$\sum_{i=1}^{m} c_i(\alpha_i^{\lambda-n} - \overline{\alpha}_i^{\lambda-n}) > \prod_{i=0}^{n}(i-\lambda) \quad \text{for } \lambda \in [1, n-1] \qquad (12.15)$$

are sufficient for (0.1) to have property $\widetilde{\ }$.

Proof. The theorem follows from Corollary 9.6 and Theorems 7.4 and 7.5. ■

Let (12.1) and (12.14) be fulfilled with $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0,1]$ and $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then conditions (9.46) and

$$\sum_{i=1}^{m} c_i(\alpha_i^{\lambda-n} - \overline{\alpha}_i^{\lambda-n}) > \prod_{i=0}^{n}(i-\lambda) \quad \text{for } \lambda \in [n-3, n-1] \qquad (12.16)$$

are sufficient for (0.1) to have property $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots, m\}$. Then* (9.46) *and* (12.15) *are necessary and sufficient for* (9.47) *to have property* $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, 1]$ and $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then* (9.46) *and* (12.16) *are necessary and sufficient for* (9.47) *to have property* $\widetilde{\ }$.

Let $F \in V(\tau)$, condition (12.1) *be fulfilled for some $t_0 \in \mathbb{R}_+$ and*

$$|F(u)(t)| \geq t^{-n} \sum_{i=1}^{m} c_i |u(\alpha_i t)| \quad \text{for } u \in H_{t_0,\tau}, \ \ t \geq t_0, \tag{12.17}$$

where $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots, m\}$. Then conditions (9.53) *and*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > \prod_{i=0}^{n-1} (i - \lambda) \quad \text{for } \lambda \in [1, n-1] \tag{12.18}$$

are sufficient for (0.1) *to have property* $\widetilde{\ }$.

Proof. The theorem follows from Corollaries 7.1, 7.5 and 9.9. ∎

Let (12.1) *and* (12.17) *be fulfilled, where $c_i > 0$, $\alpha_i \in]0, 1]$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots .m\}$. Then conditions* (9.53) *and*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > \prod_{i=0}^{n-1} (i - \lambda) \quad \text{for } \lambda \in [n-3, n-1] \tag{12.19}$$

are sufficient for (0.1) *to have property* $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 1$ for some $i_0 \in \{1, \dots, m\}$. Then* (9.53) *and* (12.18) *are necessary and sufficient for the equation*

$$u^{(n)}(t) + (-1)^{n+1} t^{-n} \sum_{i=1}^{m} c_i u(\alpha_i t) = 0 \tag{12.20}$$

to have property $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i \in]0, 1[$ $(i = 1, \dots, m)$ and $\alpha_{i_0} < 0$ for some $i_0 \in \{1, \dots, m\}$. Then* (9.53) *and* (12.19) *are necessary and sufficient for* (12.20) *to have property* $\widetilde{\ }$.

§ 13. Equations with property $\widetilde{\mathbf{B}}$

Let $F \in V(\tau)$ and (0.3) and (12.2) be fulfilled, where

$$\varphi \in M(\sigma), \ \ \sigma \ \text{is nondecreasing}, \ \sigma(t) \geq t \ \text{for} \ t \in \mathbb{R}_+. \tag{13.1}$$

Let, moreover, (3.3_{n-2}) and (11.71) hold, where $\widetilde{\varphi}_n$ is defined by either (11.72) or (11.73) and

$$\widetilde{\varphi}_{n-2}(u)(t) = \frac{t}{(n-1)!}\,\varphi(t^{n-2}u(t))(t). \tag{13.2}$$

Then (0.1) has property $\widetilde{\ }$.

Proof. Suppose that $u : [t_0, +\infty[\to \mathbb{R}$ is a proper nonoscillatory solution of (0.1). By Lemma 1.1 there exists $l \in \{0, \dots, n\}$ such that $l + n$ is even and (2.14_l) holds. According to (13.2) and Theorem 3.2′ $l \notin \{1, \dots, n-1\}$. On the other hand, in view of (11.72)–(11.74) and Theorems 11.12 and 11.13 $l \neq n$. Therefore $l = 0$ which is possible only if n is even, so using (3.3_{n-2}) we can easily show that (0.4) is fulfilled. ■

Taking into account Theorems 3.2″, 11.12 and 11.13, we can analogously prove the following

′ *Let $F \in V(\tau)$ and (0.3), (12.2) and (13.1) be fulfilled with $\varphi \in M(\sigma, \overline{\sigma})$. Let, moreover, (3.18) and (11.71) hold where $\widetilde{\varphi}_n$ is defined by either (11.72) or (11.73) and*

$$\widetilde{\varphi}_{n-2}(u)(t) = \frac{t}{(n-1)!}\,\varphi(\psi_t(u))(t),$$
$$\psi_t(u)(s) = [\overline{\sigma}(t)]^{2-n} t^{n-2} u(t) s^{n-2} \ \ \text{for} \ \ s \in \mathbb{R}_+.$$

Then (0.1) has property $\widetilde{\ }$.

Let $F \in V(\tau)$ and (0.3), (12.2) and (13.1) be fulfilled. Let, moreover, (11.71) and (3.7_1) hold where $\beta(t) = t$, $\widetilde{\varphi}_n$ is defined by either (11.72) or (11.73) and

$$\widetilde{\varphi}_1(u)(t) = \frac{t^{n-1}}{(n-1)!}\,\varphi(u(t))(t).$$

Then (0.1) has property $\widetilde{\ }$.

′ *Let $F \in V(\tau)$ and (0.3), (12.2) and (13.1) be fulfilled with $\varphi \in M(\sigma, \overline{\sigma})$. Let, moreover, (3.22) and (11.71) hold where $\beta(t) = t$, $\widetilde{\varphi}_n$ is defined by either (11.72) or (11.73) and*

$$\widetilde{\varphi}_1(u)(t) = \frac{t^{n-1}}{(n-1)!}\,\varphi(\psi_t(u))(t), \ \ \psi_t(u)(s) = [\overline{\sigma}(t)]^{1-n} u(t) s^{n-1}.$$

Then equation (0.1) *has property* $\widetilde{\ }$.

The validity of Theorem 13.2 (Theorem 13.2′) follows from Theorems 3.3′, 11.12 and 11.13 (Theorems 3.5, 11.12 and 11.13).

Let $F \in V(\tau)$ *and* (0.3), (10.2) *and* (12.6) *be fulfilled. Then* (12.7) *and either* (11.68) *or* (11.70) *are sufficient for* (0.1) *to have property* $\widetilde{\ }$.

Proof. The theorem follows from Corollaries 3.12, 11.12 and Theorem 11.11. ■

Let $F \in V(\tau)$ *and* (0.3), (12.8) *be fulfilled, wh where* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and* $\overline{\Delta}_{i_0} < 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then* (11.32) *is sufficient for* (0.1) *to have property* $\widetilde{\ }$.

Proof. The theorem follows from Corollaries 3.1 and 11.3. ■

′ *Let* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\Delta_i > \overline{\Delta}_i$ $(i = 1, \dots, m)$ *and* $\overline{\Delta}_{i_0} < 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then* (11.32) *is necessary and sufficient for* (11.33) *to have property* $\widetilde{\ }$.

Let $F \in V(\tau)$ *and* (0.3), (12.9) *be fulfilled, where* $c_i > 0$, $\Delta_i \in \mathbb{R}$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} < 0$ *for some* $i_0 \in \{1, \dots, m\}$. *Then* (11.40) *is sufficient for* (0.1) *to have property* $\widetilde{\ }$.

The validity of Theorem 13.5 follows from Corollaries 3.1 and 11.6.

Let $F \in V(\tau)$ *and* (0.3) *and* (12.10) *be fulfilled where* $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ *and* $\overline{\alpha}_{i_0} > 1$ *for some* $i_0 \in \{1, \dots, m\}$. *Then the condition*

$$\sum_{i=1}^{m} c_i(\overline{\alpha}_i^{\lambda+1} - \alpha_i^{\lambda+1}) > \prod_{i=-1}^{n-1} (\lambda - i) \;\; \text{for} \;\; \lambda \in \mathbb{R}_+ \tag{13.3}$$

is sufficient for (0.1) *to have property* $\widetilde{\ }$.

The validity of Theorem 13.6 follows from Theorem 7.4 and Corollary 11.9.

′ *Let* $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ *and* $\overline{\alpha}_{i_0} > 1$ *for some* $i_0 \in \{1, \dots .m\}$. *Then* (13.3) *is necessary and sufficient for* (11.51) *to have property* $\widetilde{\ }$.

Let $F \in V(\tau)$ and (0.3), (12.10) *be fulfilled, where $c_i > 0$ and $1 \le \alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then the condition*

$$\sum_{i=1}^{m} c_i(\overline{\alpha}_i^{\lambda+1} - \alpha_i^{\lambda+1}) > \prod_{i=-1}^{n-1} (\lambda - i) \quad for \quad \lambda \in [0,2] \cup [n-1, +\infty[\tag{13.4}$$

is sufficient for (0.1) *to have property* $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $1 \le \alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$. Then* (13.4) *is necessary and sufficient for* (11.51) *to have property* $\widetilde{\ }$.

Let $F \in V(\tau)$ and (0.3) *and* (12.17) *be fulfilled where, $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ and $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \dots, m\}$. Then the condition*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > \prod_{i=0}^{n-1} (\lambda - i) \quad for \quad \lambda \in \mathbb{R}_+ \tag{13.5}$$

is sufficient for (0.1) *to have property* $\widetilde{\ }$.

The validity of Theorem 13.7 follows from Corollaries 7.5 and 11.11.

$'$ *Let $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ and $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \dots, m\}$. Then* (13.5) *is necessary and sufficient for the equation*

$$u^{(n)}(t) = t^{-n} \sum_{i=1}^{m} c_i u(\alpha_i t) \tag{13.6}$$

to have property $\widetilde{\ }$.

Let $F \in V(\tau)$ and (0.3), (12.17) *be fulfilled, where $c_i > 0$, $\alpha_i \ge 1$ $(i = 1, \dots, m)$ and $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \dots, m\}$. Then the condition*

$$\sum_{i=1}^{m} c_i \alpha_i^{\lambda} > \prod_{i=0}^{n-1} (\lambda - i) \quad for \quad \lambda \in [0,2] \cup [n-1, +\infty[\tag{13.7}$$

is sufficient for (0.1) *to have property* $\widetilde{\ }$.

$'$ *Let $c_i > 0$, $\alpha_i \ge 1$ $(i = 1, \dots, m)$ and $\alpha_{i_0} > 1$ for some $i_0 \in \{1, \dots, m\}$. Then* (13.7) *is necessary and sufficient for* (13.6) *to have property* $\widetilde{\ }$.

§ 14. Oscillatory equations

Sufficient conditions for oscillation of any proper solution of (0.1) in the case of even n were given in §§3,4,6,7, and in the case of odd n in §12. If n is odd and (0.3) holds, the analogous question was studied in §13. In this section we establish sufficient conditions for any proper solution of (0.1) to be oscillatory in the case when n is even and (0.3) is fulfilled. Throughout this section, without mentioning it specially, it will be assumed that n is even.

Let $F \in V(\tau)$ *and* (0.3) *and* (12.8) *be fulfilled, where* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$, $\overline{\Delta}_{i_1} < 0$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (9.28) *and* (11.32) *are sufficient for every proper solution of* (0.1) *to be oscillatory.*

The validity of the theorem follows from Corollaries 3.1, 9.2, and 11.3.

′ *Let* $c_i > 0$, $\Delta_i, \overline{\Delta}_i \in \mathbb{R}$, $\overline{\Delta}_i < \Delta_i$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$, $\overline{\Delta}_{i_1} < 0$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (9.38) *and* (11.32) *are necessary and sufficient for every proper solution of equation* (11.33) *to be oscillatory.*

Proof. Sufficiency follows from Theorem 14.1. If we assume that (9.32) ((11.32)) is violated, then (11.33) has the solution $u(t) = e^{\lambda t}$ with $\lambda < 0$ $(\lambda > 0)$. ■

Let $F \in V(\tau)$ *and* (0.3) *and* (12.9) *be fulfilled, where* $m \geq 2$, $c_i > 0$, $\Delta_i \in \mathbb{R}$ $(i = 1, \dots, m)$ *and* $\Delta_{i_0} > 0$, $\Delta_{i_1} < 0$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (9.37) *and* (11.40) *are sufficient for every proper solution of* (0.1) *to be oscillatory.*

The validity of the theorem follows from Corollaries 3.1, 9.4 and 11.6.

Let $F \in V(\tau)$ *and* (0.3) *and* (12.10) *be fulfilled where* $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ *and* $\alpha_{i_0} < 1$, $\overline{\alpha}_{i_1} > 1$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (9.49) *and* (13.3) *are sufficient for every proper solution of* (0.1) *to be oscillatory.*

The validity if the theorem follows from Corollary 9.7 and Theorem 13.6.

′ *Let* $c_i > 0$, $\alpha_i, \overline{\alpha}_i \in]0, +\infty[$, $\alpha_i < \overline{\alpha}_i$ $(i = 1, \dots, m)$ *and* $\alpha_{i_0} < 1$, $\overline{\alpha}_{i_1} > 1$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (9.49) *and* (13.3) *are necessary and sufficient for every proper solution of* (11.51) *to be oscillatory.*

Let $F \in V(\tau)$ *and* (0.3) *and* (12.17) *be fulfillled where* $m \geq 2$, $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ *and* $\alpha_{i_0} < 1$, $\alpha_{i_1} > 1$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (13.5) *and* (9.53) *are sufficient for every proper solution of* (0.1) *to be oscillatory.*

The validity of the theorem follows from Corollary 9.9 and Theorem 13.7.

$'$ *Let* $m \geq 2$, $c_i > 0$, $\alpha_i > 0$ $(i = 1, \dots, m)$ *and* $\alpha_{i_0} < 1$, $\alpha_{i_1} > 1$ *for some* $i_0, i_1 \in \{1, \dots, m\}$. *Then* (13.5) *and* (9.53) *are necessary and sufficient for every proper solution of* (13.6) *to be oscillatory.*

In this subsection we consider the equation

$$u^{(n)}(t) = \int_{\tau_1(t)}^{\sigma_1(t)} |u(s)|^{\lambda_1} \operatorname{sign} u(s) d_s r_1(s,t) + \\ + \int_{\tau_2(t)}^{\sigma_2(t)} |u(s)|^{\lambda_2} \operatorname{sign} u(s) d_s r_2(s,t), \tag{14.1}$$

where $\lambda_i > 0$, $\tau_i, \sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\tau_i(t) \leq \sigma_i(t)$ for $t \in \mathbb{R}_+$, $r_i(s,t)$ are measurable and $r_i(\cdot, t)$ are nondecreasing $(i = 1, 2)$.

The results of the previous sections enable us to obtain sufficient conditions for every proper solution of (14.1) to be oscillatory in the case when n is even. To illustrate this fact several theorems are given below.

Let $\lambda_1 = 1$, $\lambda_2 < 1$ *and there exist* $t_0 \in \mathbb{R}_+$ *and nondecreasing functions* $\delta_i \in C(\mathbb{R}_+; \mathbb{R}_+)$ $(i = 1, 2)$ *such that*

$$\tau_1(t) \leq \delta_1(t) < t, \quad t < \delta_2(t) \leq \sigma_1(t) \quad \textit{for} \quad t \geq t_0, \tag{14.2}$$

$$\varliminf_{t \to +\infty} \int_{\delta_1(t)}^{t} p_1(s) s^{n-1} ds > 0, \quad \varliminf_{t \to +\infty} \int_{t}^{\delta_2(t)} p_2(s) s^{n-1} ds > 0 \\ \operatorname{vrai} \sup\{t^n p_i(t) : t \in \mathbb{R}_+\} < +\infty \quad (i = 1, 2), \tag{14.3}$$

where

$$p_1(t) = r_1(\delta_1(t), t) - r_1(\tau_1(t), t), \quad p_2(t) = r_1(\sigma_1(t), t) - r_1(\delta_2(t), t). \tag{14.4}$$

Let, moreover,

$$\inf\Big\{ \varliminf_{t \to +\infty} t^{\lambda} \int_{t}^{+\infty} (s-t)^{n-1} \int_{\tau_1(s)}^{\sigma_1(s)} \xi^{-\lambda} d_\xi r_1(\xi, s) ds : \\ : \lambda \in]0, +\infty[\Big\} > (n-1)!, \tag{14.5}$$

$$\inf\Big\{ \varliminf_{t \to +\infty} t^{-\lambda} \int_{0}^{t} (t-s)^{n-1} \int_{\tau_1(s)}^{\sigma_1(s)} \xi^{\lambda} d_\xi r_1(\xi, s) ds : \\ : \lambda \in]n-1, +\infty[\Big\} > (n-1)! \tag{14.6}$$

and

$$\varlimsup_{t \to +\infty} \frac{\sigma_2(t)}{t} < +\infty, \quad \int^{+\infty} t \int_{\tau_2(t)}^{\sigma_2(t)} s^{\lambda_2(n-1)} d_s r_2(s,t) dt = +\infty. \tag{14.7}$$

Then every proper solution of (14.1) *is oscillatory.*

Let $\lambda_1 = 1$, $\lambda_2 < 1$ and (14.7) *be fulfilled. Let, moreover, there exist $t_0 \in \mathbb{R}_+$ and nondecreasing functions $\delta_i \in C(\mathbb{R}_+;\mathbb{R}_+)$ $(i = 1, 2)$ such that* (14.2) *holds and*

$$\begin{aligned} &\varliminf_{t\to+\infty} \int_{\delta_1(t)}^{t} (s-\delta_1(t))^{n-1} p_1(s)ds > \frac{(n-1)!}{e}, \\ &\varliminf_{t\to+\infty} \int_{t}^{\delta_2(t)} (\delta_2(t)-s)^{n-1} p_2(s)ds > \frac{(n-1)!}{e} \end{aligned} \tag{14.8}$$

with p_i $(i = 1, 2)$ defined by (14.4). *Then every proper solution of* (14.1) *is oscillatory.*

Let $\lambda_1 = 1$, $\lambda_2 > 1$, either (14.2)–(14.6) *or* (14.8) *be fulfilled and*

$$\int^{+\infty} [\beta(t)]^{n-1} [\sigma_2(t)]^{(2-n)\lambda_2} \int_{\tau_2(t)}^{\sigma_2(t)} s^{\lambda_2(n-2)} d_s r_2(s,t)dt = +\infty \tag{14.9}$$

with $\beta(t) = \min\{t, \sigma_2(t)\}$. Then every proper solution of (14.1) *is oscillatory.*

Let $\lambda_1 < 1$, $\lambda_2 > 1$, $\sigma_1(t) \le t$, $\tau_2(t) \ge t$ for $t \in \mathbb{R}_+$ and

$$\int^{+\infty} t^{\lambda_1(n-1)} (t-\sigma_1(t))^{\lambda_1(n-1)} [r_1(\sigma_1(t),t) - r_1(\tau_1(t),t)]dt = +\infty,$$
$$\int^{+\infty} (\tau_2(t)-t)^{\lambda_2(n-1)} [r_2(\sigma_2(t),t) - r_2(\tau_2(t),t)]dt = +\infty.$$

Let, moreover, (14.9) *hold. Then every proper solution of* (14.1) *is oscillatory.*

§ 15. Existence of an oscillatory solution.

Suppose that $t_0 > 0$ and $F \in V(\tau;\sigma)$ with

$$\tau(t) \le \sigma(t) < t \quad \text{for} \quad t \ge t_0. \tag{15.1}$$

Consider the following Cauchy problem for (0.1)

$$u(t) = \varphi(t) \ \text{ for } \ t \in [\tau_0, t_0], \ \ u^{(i)}(t_0) = c_i \ \ (i = 1, \dots, n-1), \tag{15.2}$$

where $\tau_0 = \inf\{\tau(t) : t \in [t_0, +\infty[\}$, $\varphi \in C([\tau_0, t_0];\mathbb{R})$, and $c_i \in \mathbb{R}$ $(i = 1, \dots, n-1)$.

Let for any $t \geq t_0$

$$F(u)(t) \neq 0 \ \ \textit{for} \ \ u \in C(\mathbb{R}_+;\mathbb{R}) \ \ \textit{satisfying,} \\ u(s) \geq 0, \ \ u(s) \not\equiv 0 \ \ (u(s) \leq 0, \ \ u(s) \not\equiv 0) \ \ \textit{for} \ \ s \in [\tau(t), \sigma(t)]. \tag{15.3}$$

Let, moreover, $\varphi(t) \not\equiv 0$ *for* $t \in [\tau_0, t_0]$. *Then problem* (0.1), (15.2) *has a unique proper solution.*

Proof. According to (15.1) problem (0.1), (15.2) has a unique solution $u_0 : [t_0, +\infty[\to \mathbb{R}$. Suppose that u_0 is not proper. Then we can find $t_1 \in [t_0, +\infty[$ and $t_2, t_3 \in [t_1, +\infty[$ such that $t_2 < t_3$, $u_0(t) \equiv 0$ for $t \in [t_1, +\infty[$, $u_0(s) \not\equiv 0$ for $s \in [\tau(t), \sigma(t)]$ for any $t \in [t_2, t_3]$ and either $u_0(t) \geq 0$ or $u_0(t) \leq 0$ for $t \in [\tau_1, t_1]$ where $\tau_1 = \min\{\tau(t) : t \in [t_2, t_3]\}$. Therefore by (15.3) from (0.1) we get

$$0 = u_0^{(n-1)}(t_2) - u_o^{(n-1)}(t_3) = \int_{t_2}^{t_3} F(u_0)(s)ds \neq 0.$$

The obtained contradiction proves the lemma. ∎

Suppose that $F \in V(\tau;\sigma)$ where τ, σ satisfy (15.1). For (0.1) consider the following Cauchy problem

$$u(t) = \varphi(t) \ \text{ for } \ t \in [\tau_0, t_0], \ \ \varphi(t_0) = 0, \\ u^{(i)}(t_0) = 0 \ \ (i = 1, \dots, n-2), \ \ u^{(n-1)}(t_0) = \gamma. \tag{15.4}$$

By (15.1) it is obvious that problem (0.1), (15.4) has a unique solution $u(\cdot, \gamma)$ which depends continuously on the parameter γ.

Define the sets Γ_1 and Γ_2 as follows:

$\widetilde{\gamma} \in \Gamma_1$ iff the inequality $\gamma \geq \widetilde{\gamma}$ implies

$$\lim_{t\to+\infty} u^{(i)}(t;\gamma) = +\infty \ \ (i = 0, \dots, n-2), \ \ \lim_{t\to+\infty} u^{(n-1)}(t;\gamma) > 0. \tag{15.5}$$

$\widetilde{\gamma} \in \Gamma_2$ iff the inequality $\gamma \leq \widetilde{\gamma}$ implies

$$\lim_{t\to+\infty} u^{(i)}(t;\gamma) = -\infty \ \ (i = 0, \dots, n-2), \ \ \lim_{t\to+\infty} u^{(n-1)}(t;\gamma) < 0.$$

Suppose that $t_1 \in [t_0, \eta_\tau(t_0)]$, where $\eta_\tau(t) = \sup\{s : \tau(t) < t\}$. Define the sets $E_i(t_1)$ $(i = 1, 2, 3)$ as follows:

$$E_1(t_1) = \{t : t \in [t_0, t_1], \ \tau(t) < t_0, \ \sigma(t) > t_0\}, \\ E_2(t_1) = \{t : t \in [t_0, t_1], \ \sigma(t) \leq t_0\}, \\ E_3(t_1) = \{t : t \in [t_0, t_1], \ \sigma(t) \geq t_0\}.$$

Let $F \in V(\tau;\sigma)$, *conditions* (0.3) *and* (15.1) *be fulfilled and for any* $t_1 \in [t_0, \eta_\tau(t_0)]$ *and* $u \in C(\mathbb{R}_+;\mathbb{R})$ *satisfying* $u(t) = \varphi(t)$ *for* $t \in [\tau_0, t_0]$, $u(t) \neq 0$ *for* $t \in]t_0, t_1]$ *we have*

$$\int_{E_1(t_1)} F(u)(s)ds \leq \int_{E_1(\eta_\tau(t_0))} |F(\theta)(s)|ds, \tag{15.6}$$

where

$$\theta(t) = \begin{cases} 0 & \textit{for} \quad t \in]t_0, +\infty], \\ \varphi(t) & \textit{for} \quad t \in [\tau_0, t_0[. \end{cases}$$

Then the sets Γ_1 *and* Γ_2 *are nonempty and*

$$\gamma_2 \le \gamma_1, \quad \gamma_1, \gamma_2 \notin \Gamma_1 \cup \Gamma_2,$$

where $\gamma_2 = \sup \Gamma_2$, $\gamma_1 = \inf \Gamma_1$.

Proof. Denote

$$\gamma_0 = \int_{E_2(\eta_\tau(t_0))} |F(\varphi)(s)|ds + \int_{E_1(\eta_\tau(t_0))} |F(\theta)(s)|ds. \tag{15.7}$$

Show that if $\gamma > \gamma_0$ then $\gamma \in \Gamma_1$. First prove that

$$u^{(i)}(t;\gamma) > 0 \ \text{ for } \ t \in]t_0, \eta_\tau(t_0)] \ \ (i = 0, \dots, n-1). \tag{15.8}$$

Indeed, otherwise we can find $t_1 \in]t_0, \eta_\tau(t_0)]$ such that

$$u^{(i)}(t;\gamma) > 0 \ \text{ for } \ t \in]t_0, t_1[\ \ (i = 0, \dots, n-1), \ \ u^{(n-1)}(t_1;\gamma) = 0.$$

Then (0.3), (15.6) and (15.7) imply

$$u^{(n-1)}(t_1;\gamma) = \gamma - \int_{t_0}^{t_1} F(u)(s)ds = \gamma -$$

$$- \int_{E_1(t_1)} F(u)(s)ds - \int_{E_2(t_1)} F(u)(s)ds - \int_{E_3(t_1)} F(u)(s)ds \ge$$

$$\ge \gamma - \int_{E_2(\eta_\sigma(t_0))} |F(\varphi)(s)|ds - \int_{E_1(\eta_\sigma(t_0))} |F(\theta)(s)|ds > 0.$$

The obtained contradiction shows that (15.8) is true. Therefore by (0.3) $u(t;\gamma)$ satisfies (15.5), hence $\gamma \in \Gamma_1$.

Analogously it can be shown that if $\gamma < -\gamma_0$, then $\gamma \in \Gamma_2$. The nonemptiness of Γ_1 and Γ_2 and the fact that $\gamma_2 \le \gamma_1$ are thus proved.

Now prove that $\gamma_1 \notin \Gamma_1$. Indeed, otherwise we can find $t_{\gamma_1} \in \mathbb{R}_+$ such that

$$u^{(i)}(t;\gamma_1) > 0 \ \text{ for } \ t \in [t_{\gamma_1}, +\infty[\ \ (i = 0, \dots, n-1).$$

Therefore there exists $\varepsilon > 0$ exists such that if $\gamma \in [\gamma_1 - \varepsilon, \gamma_1]$, then

$$u^{(i)}(t;\gamma) > 0 \ \text{ for } \ t \in [t_{\gamma_1}, \eta_\tau(t_{\gamma_1})] \ \ (i = 0, \dots, n-1).$$

By (0.3) this means that $u(t;\gamma)$ satisfies (15.5) whence it follows that $\gamma_1 - \varepsilon \in \Gamma_1$. But this contradicts the definition of γ_1, so $\gamma_1 \notin \Gamma_1$. Analogously we can show that $\gamma_1, \gamma_2 \notin \Gamma_1 \cup \Gamma_2$. ■

Let n be even, $F \in V(\tau,\sigma)$, conditions (0.2), (15.1) *and* (15.3) *be fulfilled and* (0.1) *have property . Then* (0.1) *has an oscillatory solution.*

Proof. By Lemma 15.1 (0.1) has a proper solution. Since n is even and the equation has the property , this solution has to be oscillatory. ■

Let n be odd, conditions (0.3), (15.1) *and* (15.3) *be fulfilled and suppose that for some $t_0 > 0$ and $\varphi \in C([\tau_0, t_0]; \mathbb{R})$, satisfying $\varphi(t) \not\equiv 0$ for $t \in [\tau_0, t_0]$ and $\varphi(t_0) = 0$ inequality* (15.6) *holds. Let, moreover,* (0.1) *have property . Then it has an oscillatory solution as well as solutions satisfying* (0.5).

Proof. The existence of proper solutions satisfying (0.5) follows from Lemma 15.2 and the definition of property . Besides, Lemmas 15.1 and 15.2 imply the existence of a proper solution not satisfying (0.5). Since n is odd and (0.1) has property , this solution has to be oscillatory. ■

Let $F \in V(\tau;\sigma)$, conditions (12.1), (15.1) *and* (15.3) *be fulfilled and suppose that for some $t_0 > 0$ and $\varphi \in C([\tau_0, t_0]; \mathbb{R})$, satisfying $\varphi(t) \not\equiv 0$ for $t \in [\tau_0, t_0]$ and $\varphi(t_0) = 0$ inequality* (15.6) *holds. Let, moreover, equation* (0.1) *have property $\widetilde{\ }$. Then* (0.1) *has an oscillatory solution. Moreover, if n is even, along with the oscillatory solution it has a proper solutions satisfying* (0.5).

Proof. By Lemma 15.1 (0.1) has a proper solution. Since (0.1) has property $\widetilde{\ }$, this solution has to be oscillatory when n is odd. Suppose now that n is even. Then according to Lemma 15.2 (0.1) has proper solutions satisfying (0.5). Besides, by Lemmas 15.1 and 15.2 this equation has a proper solution not satisfying (0.5). Since (0.1) has property $\widetilde{\ }$ and n is even, this solution has to be oscillatory. ■

§ 16. On a Singular Boundary Value Problem

In this section we shall establish the sufficient conditions for the unique solvability and oscillation of solutions of the following boundary value problem:

$$u''(t) = \sum_{i=1}^{m} p_i(t)u(\delta_i(t)), \tag{16.1}$$

$$u(t) = \varphi(t) \text{ for } t \in [\tau_0, 0], \quad \varliminf_{t\to+\infty} |u(t)| < +\infty, \tag{16.2}$$

where

$$\begin{gathered} p_i \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+), \quad \delta_i \in C(\mathbb{R}_+, \mathbb{R}), \quad \delta_i(t) \le t \\ \text{for } t \in \mathbb{R}_+, \quad \lim_{t\to+\infty} \delta_i(t) = +\infty \quad (i = 1, \dots, m), \\ \varphi \in C([\tau_0, 0]; \mathbb{R}), \quad \tau_0 = \min\{(\inf \delta_i(t) : t \in \mathbb{R}_+) : i = 1, \dots, m\}. \end{gathered} \tag{16.3}$$

If

$$\varlimsup_{t\to+\infty} \int_t^{\eta(t)} (s-t) \sum_{i=1}^{m} p_i(s)ds < +\infty, \tag{16.4}$$

where

$$\begin{gathered} \eta(t) = \max\{\eta_{\delta_i}(t) : i = 1, \dots, m\}, \\ \eta_{\delta_i}(t) = \sup\{s : \delta_i(s) < t\} \quad (i = 1, \dots, m), \end{gathered} \tag{16.5}$$

then problem (16.1), (16.2) *has a unique solution.*

To the theorem we need some auxiliary assertions.

For any $\gamma \in \mathbb{R}$ denote by $u(\cdot;\gamma)$ the solution of (16.1) satisfying

$$u(t) = \varphi(t) \text{ for } t \in [\tau_0, 0], \quad u'(0) = \gamma.$$

Define the sets $\Gamma_1, \Gamma_2 \subset \mathbb{R}$ as follows:

$$\begin{gathered} \gamma \in \Gamma_1 \Leftrightarrow \lim_{t\to+\infty} u(t;\gamma) = +\infty, \quad \lim_{t\to+\infty} u'(t;\gamma) > 0, \\ \gamma \in \Gamma_2 \Leftrightarrow \lim_{t\to+\infty} u(t;\gamma) = -\infty, \quad \lim_{t\to+\infty} u'(t;\gamma) < 0. \end{gathered}$$

Let conditions (16.3) *be fulfilled. Then* Γ_1 *and* Γ_2 *are nonempty and*

$$\gamma_2 \le \gamma_1, \quad \gamma \in [\gamma_2, \gamma_1] \Rightarrow \gamma \notin \Gamma_1 \cup \Gamma_2,$$

where $\gamma_1 = \inf \Gamma_1$, $\gamma_2 = \sup \Gamma_2$. *Moreover, if* $\gamma_2 < \gamma_1$, *then problem* (16.1), (16.2) *has an infinite set of solutions and each of them is oscillatory and unbounded.*

Proof. The nonemptiness of Γ_1 and Γ_2, as well as the inequality $\gamma_2 \leq \gamma_1$ can be proved similarly to Lemma 15.2. If $\gamma_2 < \gamma_1$, then for any $\gamma \in [\gamma_2, \gamma_1]$ we readily find that $u(\cdot;\gamma)$ is oscillatory and unbounded and $\mathbb{R} = \Gamma_1 \cup \Gamma_2 \cup [\gamma_2, \gamma_1]$. ■

Let conditions (16.3) *be fulfilled,* $\gamma_2 < \gamma_1$ *and* $\gamma \in [\gamma_2, \gamma_1]$. *Then there exists* $\gamma^* \in]\gamma_1, +\infty[$ *such that*

$$u(t;\gamma^*) > |u(t;\gamma)|, \ \ u'(t;\gamma^*) > |u'(t;\gamma)|, \ \ \textit{for} \ \ t \geq 1. \tag{16.6}$$

Proof. Introduce the notation

$$c_0 = \max\{|\varphi(t)| : t \in [\tau_0, 0]\} + \max\{|u(t;\gamma)| + |u'(t;\gamma)| : t \in [0, \eta(1)]\},$$

$$\gamma^* = 2c_0 + c_0 \int_0^{\eta(1)} \sum_{i=1}^m p_i(s)ds.$$

In the first place show that

$$u'(t;\gamma^*) \geq 2c_0 \ \text{ for } \ 0 \leq t \leq \eta(1). \tag{16.7}$$

Assume the contrary. Then for some $t_0 \in [0, \eta(1)]$ we have

$$u'(t;\gamma^*) > 0 \ \text{ for } \ 0 \leq t \leq t_0,$$
$$u(t;\gamma^*) > -c_0 \ \text{ for } \ 0 \leq t \leq t_0, \ \text{ and } \ u'(t_0;\gamma^*) < 2c_0$$

which is impossible because (16.1) implies

$$u'(t_0;\gamma^*) = \gamma^* + \int_0^{t_0} \sum_{i=1}^m p_i(s) u(\delta_i(s);\gamma^*)ds \geq \gamma^* - c_0 \int_0^{\eta(1)} \sum_{i=1}^m p_i(s)ds = 2c_0.$$

Therefore (16.7) is valid.

By (16.7) we have

$$u'(\eta(1);\gamma^*) > |u'(\eta(1);\gamma)|, \ \ u(t;\gamma^*) \geq 2c_0 t + \varphi(0) > |u(t;\gamma)|$$
$$\text{for} \quad 1 \leq t \leq \eta(1).$$

Keeping in mind that $p_i(t) \geq 0$ for $t \in \mathbb{R}_+$ $(i = 1, \ldots, m)$, from the latter inequality we obtain (16.6). ■

Proof of Theorem 16.1. Due to Lemma 16.1 it suffices to show that $\gamma_2 = \gamma_1$. Assume on the contrary that $\gamma_2 < \gamma_1$ and take $\gamma \in]\gamma_2, \gamma_1[$. By Lemma 16.2 there exists $\gamma^* \in]\gamma_1, +\infty[$ such that (16.6) holds. We have to show that for any sufficiently large t

$$u(t;\gamma^*) \geq \Big(1 + \frac{1}{\rho}\Big)|u(t;\gamma)|, \tag{16.8}$$

where $\rho \in]0, +\infty[$ satisfies

$$\rho \geq \int_t^{\eta(t)} (s - t) \sum_{i=1}^m p_i(s)ds \ \text{ for } \ t \in \mathbb{R}_+. \tag{16.9}$$

$u(\cdot;\gamma)$ is oscillatory by Lemma 16.1. Let $t_0 \in [\eta(1), +\infty[$ be a zero of $u(\cdot;\gamma)$ and $s_0 \in]t_0, +\infty[$ be any point which is not a zero of $u(\cdot;\gamma)$. For the sake of definiteness assume that $u(s_0;\gamma) > 0$. Then we can choose $s_1 \in [t_0, s_0[$, $s_2 \in]s_0, +\infty[$ and $\widetilde{s} \in]s_1, s_2[$ such that

$$\begin{gathered} u(s_1;\gamma) = u(s_2;\gamma) = u'(\widetilde{s};\gamma) = 0, \\ u(\widetilde{s};\gamma) \geq u(s;\gamma) > 0 \ \text{ for } \ s_1 < s < s_2. \end{gathered} \tag{16.10}$$

It is easy to see that $\widetilde{s} < \eta(s_1)$. Moreover, using (16.6) and (16.9), from the equality

$$u(\widetilde{s};\gamma) = -\int_{s_1}^{\widetilde{s}} (s - s_1) \sum_{i=1}^{m} p_i(s) u(\delta_i(s);\gamma) ds$$

we obtain

$$u(\widetilde{s};\gamma) = -\Big(\sum_{i=1}^{m} \int_{E_i'} (s - s_1) p_i(s) u(\delta_i(s);\gamma) ds +$$

$$+ \sum_{i=1}^{m} \int_{E_i''} (s - s_1) p_i(s) u(\delta_i(s);\gamma) ds \Big) \leq - \sum_{i=1}^{m} \int_{E_i''} (s - s_1) p_i(s) u(\delta_i(s);\gamma) ds \leq$$

$$\leq u(s_1;\gamma^*) \int_{s_1}^{\eta(s_1)} (s - s_1) \sum_{i=1}^{m} p_i(s) ds \leq \rho u(s_1;\gamma^*),$$

where $E_i' = \{s \in [s_1, \widetilde{s}] : \delta_i(s) \geq s_1\}$, $E_i'' = \{s \in [s_1, \widetilde{s}] : \delta_i(s) < s_1\}$ $(i = 1, \dots, m)$. Now by (16.6) and (16.10) we have

$$u(s_0;\gamma^*) = u(s_1;\gamma^*) + u'(s_1;\gamma^*)(s_0 - s_1) + \int_{s_1}^{s_0} (s_0 - s) \times$$

$$\times \sum_{i=1}^{m} p_i(s) u(\delta_i(s);\gamma^*) ds \geq \frac{1}{\rho} u(\widetilde{s};\gamma) +$$

$$+ u'(s_1;\gamma)(s_0 - s_1) + \int_{s_1}^{s_0} (s_0 - s) \sum_{i=1}^{m} p_i(s) u(\delta_i(s);\gamma) ds =$$

$$= \frac{1}{\rho} u(\widetilde{s};\gamma) + u(s_0;\gamma) \geq \Big(1 + \frac{1}{\rho}\Big) u(s_0;\gamma).$$

Since s_0 has been chosen arbitrarily we hence conclude that (16.8) is valid on $[t_0, +\infty[$.

Let $t_1 \in [\eta(t_0), +\infty[$ and $u'(t_1;\gamma) = 0$. By (16.6) and (16.8)

$$u'(t;\gamma^*) = u'(t_1;\gamma^*) + \int_{t_1}^{t} \sum_{i=1}^{m} p_i(s) u(\delta_i(s);\gamma^*) ds > \Big(1 + \frac{1}{\rho}\Big) \times$$

$$\times \int_{t_1}^{t} \sum_{i=1}^{m} p_i(s) |u(\delta_i(s);\gamma)| ds \geq \Big(1 + \frac{1}{\rho}\Big) |u'(t;\gamma)| \ \text{ for } \ t \geq t_1. \tag{16.11}$$

Using (16.8) and (16.1) we can show by induction that there exists a sequence of points $t_i \in \mathbb{R}_+$ $(i = 1, 2, \dots)$ such that for any $i \in \mathbb{N}$ we have the equalities

$$u(t;\gamma^*) \geq \Big(1+\frac{1}{\rho}\Big)^i |u(t;\gamma)|, \quad u'(t;\gamma^*) \geq \Big(1+\frac{1}{\rho}\Big)^i |u'(t;\gamma)|$$
$$\text{for} \quad t \geq t_i, \quad (i = 1, 2, \dots).$$

Therefore

$$\lim_{t\to+\infty} \frac{|u(t;\gamma)|}{u(t;\gamma^*)} = 0. \tag{16.12}$$

Put $\gamma_0 = (\gamma + \gamma_2)/2$ and $c = -(\gamma - \gamma_0)/(\gamma^* - \gamma)$. Clearly, $u(t;\gamma_0) = (1-c)u(t;\gamma) + cu(t;\gamma_*)$. Hence by (16.12) we have

$$\lim_{t\to+\infty} \frac{u(t;\gamma_0)}{u(t;\gamma^*)} = c < 0$$

which is impossible since $\gamma_0 \in]\gamma_2, \gamma_1[$ and by Lemma 16.1 $u(\cdot;\gamma_0)$ is an oscillatory solution. The obtained contradiction proves the theorem. ■

Let $p \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\delta \in C(\mathbb{R}_+;\mathbb{R})$, $\delta(t) \leq t$ *for* $t \in \mathbb{R}_+$, $\lim_{t\to+\infty} \delta(t) = +\infty$,

$$\text{vrai}\sup\{p(t) : t \in \mathbb{R}_+\} < +\infty, \quad \sup\{t - \delta(t) : t \in \mathbb{R}_+\} < +\infty$$

and

$$\int^{+\infty} p(t)dt = +\infty.$$

Then the boundary value problem

$$u''(t) = p(t)u(\delta(t)), \quad u(t) = \varphi(t) \;\; \textit{for} \;\; t \in [\tau_0, 0], \quad \varliminf_{t\to+\infty} |u(t)| < +\infty,$$

where $\tau_0 = \inf\{\delta(t) : t \in \mathbb{R}_+\}$ *and* $\varphi \in C([\tau_0, 0];\mathbb{R})$, *has a unique solution.*

Corollary 16.1 was proved in [102] under the additional restriction $\int^{+\infty} p(t)dt = +\infty$ which, as Theorem 16.1 shows, is quite unnecessary.

In addition to (16.3) *and* (16.4), *let the condtion*

$$\int^{+\infty} t \sum_{i=1}^m p_i(t)dt = +\infty. \tag{16.13}$$

be also fulfilled. Then problem (16.1), (16.2) *has a unique solution* $u : \mathbb{R}_+ \to \mathbb{R}$ *satisfying*

$$\varliminf_{t\to+\infty} |u(t)| = 0. \tag{16.14}$$

Proof. The existence and uniqueness of a solution are provided by Theorem 16.1. Show that (16.14) is fulfilled. Assume the contrary. Then there exist $c \in]0, +\infty[$ and $t_0 \in \mathbb{R}_+$ such that

$$|u(t)| \geq c, \quad u(t)u'(t) \leq 0 \quad \text{for} \quad t \geq t_0.$$

Multiplying both sides of (16.1) by $(t - t_1) \operatorname{sign} u(t_1)$, where $t_1 = \eta(t_0)$, and integrating, we obtain

$$|u(t_1)| = (t - t_1)|u'(t_1)| + |u(t)| + \int_{t_1}^{t} (s - t_1) \sum_{i=1}^{m} p_i(s)|u(\delta_i(s))|ds \quad \text{for} \quad t \geq t_1$$

whence it follows that

$$\int_{t_1}^{+\infty} (s - t_1) \sum_{i=1}^{m} p_i(s)ds \leq \frac{|u(t_1)|}{c}$$

which contradicts (16.13). The obtained contradiction proves the validity of (16.14). ■

Let

$$\begin{gathered} p_i \in L_{loc}(\mathbb{R}_+;]0, +\infty[) \ \ (i = 1, \dots, m), \ \ \varphi(t) \not\equiv 0 \\ \textit{for} \ \ t \in [\tau_0, 0], \ \ \delta_i(t) < t \ \ \textit{for} \ \ t \in \mathbb{R}_+ \\ (i = 1, \dots, m), \quad \varliminf_{t \to +\infty} (\delta_i(t) - t) > -\infty, \\ \operatorname{vrai} \sup\{p_i(t) : t \in \mathbb{R}_+\} < +\infty \quad (i = 1, \dots, m). \end{gathered} \tag{16.15}$$

Then problem (16.1)*,* (16.2) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing and*

$$\varliminf_{t \to +\infty} \int_{\delta_i(t)}^{t} p_i(s)ds > 0 \quad (i = 1, \dots, m), \tag{16.16}$$

$$\inf\Big\{ \varliminf_{t \to +\infty} e^{\lambda t} \int_{t}^{+\infty} (s - t) \sum_{i=1}^{m} p_i(s) e^{-\lambda \delta_i(s)} ds : \lambda \in]0, +\infty[\Big\} > 1, \tag{16.17}$$

then this solution is oscillatory.

Proof. By virtue of (16.15), Lemma 15.1 and Theorem 16.1 problem (16.1), (16.2) has a unique proper solution $u : \mathbb{R}_+ \to \mathbb{R}$. Thus to complete the proof it suffices to show that u is oscillatory. Assume the contrary. Then there exists $t_0 \in \mathbb{R}_+$ such that

$$(-1)^i u^{(i)}(t)u(t) > 0 \ \ \text{for} \ \ t \geq t_0 \ \ (i = 0, 1). \tag{16.18}$$

On the other hand, by Theorem 9.3′ and (16.15)–(16.17) equation (16.1) has no solution satisfying (16.18). The obtained contradiction proves the theorem. ■

Let (16.15) *be fulfilled. Then problem* (16.1), (16.2) *has a unique proper solution. Moreover, if* (16.16) *holds and for some* $t_0 \in \mathbb{R}_+$ *we have*

$$\text{vrai inf}\Big\{\sum_{i=1}^{m} p_i(t)(t-\delta_i(t))^2 : t \in [t_0, +\infty[\Big\} > \Big(\frac{2}{e}\Big)^2, \qquad (16.19)$$

then this solution is ocillatory.

Proof. It suffices to note that (16.19) implies (16.17). ■

′ *Let* $p_i, \Delta_i \in]0,+\infty[$ $(i=1,\dots,m)$, $\varphi \in C([-\Delta,0];\mathbb{R})$ *and* $\varphi(t) \not\equiv 0$ *on* $[-\Delta, 0]$, *where* $\Delta = \max\{\Delta_i : i=1,\dots,m\}$. *Then the problem*

$$u''(t) = \sum_{i=1}^{m} p_i u(t-\Delta_i), \qquad (16.20)$$

$$u(t) = \varphi(t) \ \ for \ \ -\Delta \le t \le 0, \quad \varliminf_{t\to+\infty} |u(t)| < +\infty, \qquad (16.21)$$

has a unique proper solution. Moreover, if

$$\inf\Big\{\frac{1}{\lambda^2}\sum_{i=1}^{m} p_i e^{\lambda\Delta_i} : \lambda \in]0,+\infty[\Big\} > 1,$$

then this solution is oscillatory.

Let

$$p_i \in L_{loc}(\mathbb{R}_+;]0,+\infty[) \ \ (i=1,\dots,m), \ \ \varphi(t) \not\equiv 0 \ \ for \ \ \tau_0 \le t \le 0,$$

$$\delta_i(t) < t \quad for \quad t \in \mathbb{R}_+, \quad \text{vrai sup}\{t^2 p_i(t) : t \in \mathbb{R}_+\} < +\infty$$

$$\varliminf_{t\to+\infty} \frac{\delta_i(t)}{t} > 0 \quad (i=1,\dots,m). \qquad (16.22)$$

Then problem (16.1), (16.2) *has a unique proper solution. Moreover, if* δ_i $(i=1,\dots,m)$ *are nondecreasing and*

$$\varliminf_{t\to+\infty} \int_{\delta_i(t)}^{t} s p_i(s) ds > 0 \quad (i=1,\dots,m), \qquad (16.23)$$

$$\inf\Big\{\varliminf_{t\to+\infty} t^{\lambda} \int_t^{+\infty} (s-t) \sum_{i=1}^{m} p_i(s)\delta_i^{-\lambda}(s) ds : \lambda \in]0,+\infty[\Big\} > 1, \ (16.24)$$

then this solution is oscillatory.

Proof. Following Lemma 15.1, Theorem 16.1 and (16.22), problem (16.1), (16.22) has a unique proper solution $u : \mathbb{R}_+ \to \mathbb{R}$. If we assume that u is nonoscillatory, then there exists $t_0 \in \mathbb{R}_+$ such that (16.18) is fulfilled. But by Corollary 9.9 and (16.22)–(16.24) equation (16.1) has no solution satisfying (16.18). The obtained contradiction proves the theorem. ■

Let (16.22) *be fulfilled. Then problem* (16.1), (16.2) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing,* (16.23) *holds and for some* $t_0 \in \mathbb{R}_+$ *we have*

$$\inf\left\{\frac{1}{\lambda(\lambda+1)}\operatorname*{vrai\,inf}_{t\geq t_0} t^{2+\lambda}\sum_{i=1}^{m} p_i(t)\delta_i^{-\lambda}(t) : \lambda \in]0,+\infty[\right\} > 1, \quad (16.25)$$

then this solution is oscillatory.

′ *Let* $p_i \in]0,+\infty[$ *and* $\alpha_i \in]0,1[$ $(i = 1, \dots, m)$. *Then the problem*

$$u''(t) = \sum_{i=1}^{m} \frac{p_i}{(t+1)^2} u(\alpha_i t), \quad u(0) = c_0 \neq 0 \quad \varliminf_{t\to+\infty} |u(t)| < +\infty,$$

has a unique proper solution. Moreover, if

$$\inf\left\{\frac{1}{\lambda(\lambda+1)}\sum_{i=1}^{m} p_i\alpha_i^{-\lambda} : \lambda \in]0,+\infty[\right\} > 1,$$

then this solution is oscillatory.

Let

$$\begin{gathered} p_i \in L_{loc}(\mathbb{R}_+;]0,+\infty[), \quad \delta_i(t) < t \quad \text{for} \quad t \in \mathbb{R}_+, \\ \varliminf_{t\to+\infty} \frac{\ln\delta_i(t)}{\ln t} > 0 \quad (i = 1, \dots, m), \\ \varphi(t) \not\equiv 0 \quad \text{for} \quad \tau_0 \leq t \leq 0, \\ \operatorname{vrai\,sup}\{t^2 \ln t\, p_i(t) : t \in [1,+\infty[\} < +\infty \quad (i = 1, \dots, m). \end{gathered} \quad (16.26)$$

Then problem (16.1), (16.2) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing,* (16.23) *is fulfilled and for any* $k \in \mathbb{N}$ *we have*

$$\begin{gathered} \inf\left\{\varliminf_{t\to+\infty} (\ln t)^{\lambda}\int_t^{+\infty}(s-t)\times\right. \\ \left.\times\sum_{i=1}^{m} p_i(s)(\ln\delta_i(s))^{-\lambda} : \lambda \in]0,k]\right\} > 1, \end{gathered} \quad (16.27)$$

then this solution is oscillatory.

Proof. By Lemma 15.1, Theorem 16.1 and (16.26) problem (16.1), (16.2) has a unique proper solution. If we assume that u is nonoscillatory, then there exists $t_0 \in \mathbb{R}_+$ such that (16.18) is valid which is impossible by Theorem 9.10′, (61.23), (16.26) and (16.27). ■

Let (16.26) *be fulfilled. Then problem* (16.1), (16.2) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing functions,* (16.23) *holds and for any* $k \in \mathbb{N}$ *there exists* $t_k \in [1, +\infty[$ *such that*

$$\inf\left\{\frac{1}{\lambda}\operatorname*{vrai\,inf}_{t \geq t_k} t^2(\ln t)^{\lambda+1}\sum_{i=1}^{m} p_i(t)(\ln\delta_i(t))^{-\lambda} : \lambda \in]0, k]\right\} > 1, \quad (16.28)$$

then this solution is oscillatory.

Proof. It suffices to note that (6.28) implies (6.27). ■

′ *Let* $p_i \in]0, +\infty[$ *and* $\alpha_i \in]0, 1[$ $(i = 1, \dots, m)$*. Then the problem*

$$u''(t) = \sum_{i=1}^{m}\frac{p_i}{(t+1)^2\ln(t+2)}u(t^{\alpha_i}), \quad u(0) = c_0 \neq 0, \quad \varlimsup_{t\to+\infty}|u(t)| < +\infty$$

has a unique proper solution. Moreover, if

$$\inf\left\{\frac{1}{\lambda}\sum_{i=1}^{m} p_i\alpha_i^{-\lambda} : \lambda \in]0, +\infty[\right\} > 1,$$

then this solution is oscillatory.

§ 17. Existence of Bounded Solutions

In this section we shall establish the sufficient conditions for the existence of bounded solutions of the equation

$$u''(t) = f(t, u(\delta_1(t)), \dots, u(\delta_m(t))) \quad (17.1)$$

where $f \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R})$ and

$$\delta_i(t) < t \text{ for } t \in \mathbb{R}_+, \quad \lim_{t\to+\infty}\delta_i(t) = +\infty \quad (i = 1, \dots, m). \quad (17.2)$$

Let

$$0 \leq f(t, x_1, \dots, x_m)\operatorname{sign} x_1 \leq p(t)\prod_{i=0}^{m}|x_i|^{\lambda_i} \quad (17.3)$$
$$\text{for } t \in \mathbb{R}_+, \quad (x_1, \dots, x_m) \in \mathbb{R}^m,$$

where $\lambda_i \in]0, 1[$ $(i = 1, \dots, m)$, $\sum_{i=1}^{m}\lambda_i < 1$ *and* $p \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+)$*. Moreover, if*

$$\varlimsup_{t\to+\infty}\int_t^{\eta_1(t)}(s-t)p(s)ds < +\infty, \quad (17.4)$$

where $\eta_1(t) = \sup\{s : \delta_1(s) < t\}$*, then every oscillatory solution of* (17.1) *is bounded.*

Proof. Let $u : [t_0, +\infty[\to \mathbb{R}$ be an oscillatory solution of (17.1). By (17.3)

$$u''(t) \operatorname{sign} u(\delta_1(t)) \geq 0 \quad \text{for} \quad t \geq t_* \tag{17.5}$$

and

$$|u''(t)| \leq p(t) \prod_{i=1}^{m} |u(\delta_i(t))|^{\lambda_i} \quad \text{for} \quad t \geq t_*, \tag{17.6}$$

where $t_* \in [t_0, +\infty[$ is sufficiently large.

There are two possibilities: either $\overline{\lim\limits_{t\to+\infty}} |u(t)| \leq |u(t^*)|$ for some $t^* \in [t_*, +\infty[$ or $\overline{\lim\limits_{t\to+\infty}} |u(t)| > |u(t)|$ for $t \geq t_*$. In the first case the validity of the theorem is obvious.

Consider the second case. We find that there exist the increasing sequences $\{t_k\}_{k=1}^{+\infty}$ and $\{\widetilde{t}_k\}_{k=1}^{+\infty}$ tending to $+\infty$ and satisfying

$$\begin{gathered} t_k < \widetilde{t}_k, \quad u(t_k) = u'(\widetilde{t}_k) = 0, \quad u(t) \neq 0, \\ \text{for} \quad t_k < t \leq \widetilde{t}_k \quad (k = 1, 2, \dots) \end{gathered} \tag{17.7}$$

and

$$|u(t)| \leq |u(\widetilde{t}_k)| \quad \text{for} \quad t_* \leq t \leq \widetilde{t}_k \quad (k = 1, 2, \dots). \tag{17.8}$$

Show that

$$\eta_1(t_k) \geq \widetilde{t}_k \quad (k = 1, 2, \dots). \tag{17.9}$$

Indeed, assuming that $\eta_1(t_k) < \widetilde{t}_k$ for some $k \in \mathbb{N}$, by (17.5) and (17.7) we obtain

$$u'(t)u(t) \geq 0, \quad |u(t)| \geq |u(\widetilde{t}_k)| \quad \text{for} \quad t \geq \widetilde{t}_k$$

which is impossible because u is oscillatory.

Inequalities (17.6), (17.8) and (17.9) together with the equality

$$u(\widetilde{t}_k) = -\int_{t_k}^{\widetilde{t}_k} (s - t_k) u''(s) ds$$

imply

$$\begin{aligned} |u(\widetilde{t}_k)| &\leq \int_{t_k}^{\widetilde{t}_k} (s - t_k) p(s) \prod_{i=1}^{m} |u(\delta_i(s))|^{\lambda_i} ds \leq |u(\widetilde{t}_k)|^{\lambda} \int_{t_k}^{\widetilde{t}_k} (s - t_k) p(s) ds \leq \\ &\leq |u(\widetilde{t}_k)|^{\lambda} \int_{t_k}^{\eta_1(t_k)} (s - t_k) p(s) ds \quad (k = k_0, k_0 + 1, \dots), \end{aligned}$$

where $t_{k_0} \geq \eta(t_*)$. Therefore

$$|u(\widetilde{t}_k)|^{1-\lambda} \leq \int_{t_k}^{\eta_1(t_k)} (s - t_k) p(s) ds \quad (k = k_0, k_0 + 1, \dots)$$

with $\lambda = \sum_{i=1}^m \lambda_i < 1$. Hence by (17.4) and (17.8) we conclude that u is bounded. ■

For equation (17.1) consider the initial value problem

$$u(t) = \varphi(t) \ \text{ for } \ \tau_0 \le t \le 0, \ \ u'(0) = \gamma \tag{17.10}$$

and the boundary value problem

$$u(t) = \varphi(t) \ \text{ for } \ \tau_0 \le t \le 0, \ \ \overline{\lim_{t\to+\infty}} |u(t)| < +\infty, \tag{17.11}$$

where $\varphi \in C([\tau_0, 0]; \mathbb{R})$, $\varphi(0) = 0$, $\gamma \in \mathbb{R}$, $\tau_0 = \min\{\inf_{t\ge 0} \delta_i(t) : i = 1, \dots, m\}$.

By (17.2) it is clear that for any $\gamma \in \mathbb{R}$ problem (17.1), (17.10) has a unique solution $u(\cdot;\gamma) : \mathbb{R}_+ \to \mathbb{R}$ depending continuously on the parameter γ. By Lemma 15.2 there exists $\gamma \in \mathbb{R}$ such that $u(\cdot;\gamma)$ either is oscillatory or satisfies

$$u(t,\gamma)u'(t,\gamma) \le 0 \ \text{ for } \ t \ge t_\gamma,$$

where $t_\gamma \in \mathbb{R}_+$ is sufficiently large.

Let (17.3) *and* (17.4) *be fulfilled. Then problem* (17.1), (17.11) *is solvable. Moreover, if* $\varphi(t) \not\equiv 0$ *on* $[\tau_0, 0]$ *and*

$$\begin{gathered} f(t, x_1, \dots, x_m)\operatorname{sign} x_1 > 0 \quad \textit{for} \quad x_1 \ne 0, \\ t \in \mathbb{R}_+, \quad (x_1, \dots, x_m) \in \mathbb{R}^m, \end{gathered} \tag{17.12}$$

then every solution of this problem is proper.

Proof. The first part of the asserion of the theorem follows from Theorem 17.1. As to the second part, its proof can be obtained from Lemma 15.1 due to (17.2) and (17.12). ■

Let (17.2)–(17.4) *and* (17.12) *be fulfilled,* $\lambda_i \in]0, 1[$ $(i = 1, \dots, m)$, $\sum_{i=1}^m \lambda_i < 1$, $p \in L_{loc}(\mathbb{R}_+;]0, +\infty[)$, $\varphi(t) \not\equiv 0$ *on* $[\tau_0, 0]$ *and* $\varphi(0) = 0$. *Then problem* (17.1), (17.11) *has a proper solution. Moreover, if*

$$\begin{gathered} f(t, x_1, \dots, x_m)\operatorname{sign} x_1 \ge q(t) \prod_{i=1}^m |x_i|^{\lambda_i} \ \ \textit{for} \ \ t \in \mathbb{R}_+, \\ (x_1, \dots, x_m) \in \mathbb{R}^m \end{gathered} \tag{17.13}$$

and

$$\int_0^{+\infty} t^{1-\lambda} q(t) \prod_{i=1}^m [t - \delta_i(t)]^{\lambda_i} dt = +\infty, \tag{17.14}$$

where $q \in L_{loc}(\mathbb{R}_+;]0, +\infty[)$, *then every solution of this problem is oscillatory.*

Proof. The fact that problem (17.1), (17.11) has a proper solution $u : \mathbb{R}_+ \to \mathbb{R}$ follows from theorem 17.2. To complete the proof it suffices to show that u is oscillatory. Indeed, otherwise there would exist $t_0 \in \mathbb{R}_+$ such that (16.18) be satisfied which would be impossible by (17.13), (17.14) and Theorem 9.15′. ■

Let $p, \Delta \in]0, +\infty[$, $\lambda \in]0, 1[$. Then the equation

$$u''(t) = p|u(t-\Delta)|^\lambda \operatorname{sign} u(t-\Delta)$$

has a proper bounded solution. Moreover, every oscillatory solution of this equation is bounded.

Let $\delta \in C(\mathbb{R}_+; \mathbb{R})$, $\delta(t) < t$ for $t \in \mathbb{R}_+$, $\lambda \in]0, 1[$, $\alpha \in]0, 1]$, $p \in]0, +\infty[$ and

$$\overline{\lim_{t\to+\infty}} \frac{\delta(t)}{t} < 1, \quad \underline{\lim_{t\to+\infty}} \frac{\delta(t)}{t^\alpha} > 0.$$

Then the equation

$$u''(t) = \frac{p}{(t+1)^2 \ln(t+2)} |u(\delta(t))|^\lambda \operatorname{sign} u(\delta(t))$$

has a proper bounded solution. Moreover, every oscillatory solution of this equation is bounded.

Similarly to Theorem 17.1 one can prove

Let conditions (16.3) *be fulfilled and for any sufficiently large t there hold*

$$\int_t^{\eta(t)} (s-t) \sum_{i=1}^m p_i(s) ds < 1, \tag{17.15}$$

where η is defined by (16.5). *Then every oscillatory solution of* (16.1) *is bounded.*

Let (16.15) *and* (17.15) *be fulfilled. Then problem* (16.1), (17.11) *has a unique proper solution. Moreover, if δ_i $(i = 1, \dots, m)$ are nondereasing,* (16.16) *and* (17.16) *hold and*

$$\underline{\lim_{t\to+\infty}} (\delta_i(t) - t) > -\infty, \quad \text{vrai}\sup\{p_i(t) : t \in \mathbb{R}_+\} < +\infty \tag{17.16}$$
$$(i = 1, \dots, m),$$

then this solution is oscillatory.

Proof. The fact that problem (16.1), (17.1) has a unique proper solution $u : \mathbb{R}_+ \to \mathbb{R}$ follows from Lemma 16.1 and Theorems 16.3 and 17.4. As to the second part of the assertion of the theorem, as while proving Theorem 16.3, the oscillation of u is obtained by Theorem 9.3′, (16.16), (16.17) and (17.6). ■

Let conditions (16.15) *and* (17.15) *be fulfilled. Then problem* (16.1), (17.11) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondereasing and* (16.16), (16.19) *and* (17.16) *hold, then this solution is oscillatory.*

Let $\Delta, p \in]0, +\infty[$ *and*

$$\frac{2}{e^2} < \frac{p\Delta^2}{2} < \frac{19 + 2\sqrt{34}}{18}.$$

Then the problem

$$u''(t) = pu(t - \Delta), \ \ u(t) = \varphi(t) \ \ for \ \ t \in [-\Delta, 0], \ \ \lim_{t \to +\infty} u(t) = 0$$

where $\varphi \in C([-\Delta, 0]; \mathbb{R})$ *and* $\varphi(t) \not\equiv 0$ *on* $[-\Delta, 0]$, *has a unique solution and this solution is oscillatory.*

Proof. The uniqueness and oscillation of a solution satisfying $\overline{\lim\limits_{t \to +\infty}} |u(t)| < +\infty$ are obtained by Theorem 17.5. The fact that this solution tends to zero as $t \to +\infty$ follows from Theorem 77 in [86]. ■

Let (16.15) *and* (17.15) *be fulfilled. Then problem* (16.1), (17.11) *has a unique solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing and* (16.22)–(16.24) *hold, then this solution is oscillatory.*

Proof. The first part of the above assertion follows from Theorem 17.5. The second part is proved by Corollary 9.9. using (16.22)–(16.24). ■

Let (16.15) *and* (17.15) *be fulfilled so that problem* (16.1), (17.11) *has a unique solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing and* (16.22), (16.23), (16.25) *hold, then this solution is oscillatory.*

Let $p \in]0, +\infty[$, $\alpha \in]0, 1[$ *and*

$$p(|\ln \alpha| + \alpha - 1) < 1, \tag{17.17}$$

$$p(\sqrt{4 + \ln^2 \alpha} - 2)e^{\frac{\sqrt{4+\ln^2 \alpha} - |\ln \alpha|}{2}} > \frac{1}{e}. \tag{17.18}$$

Then the problem

$$u''(t) = \frac{p}{(t+1)^2} u(\alpha t), \ \ u(0) = c_0 \neq 0, \ \ \overline{\lim_{t \to +\infty}} |u(t)| < +\infty$$

has a unique solution and this solution is oscillatory.

Remark 17.1. One can easily verify that there really exist $p \in]0, +\infty[$ and $\alpha \in]0, 1[$ satisfying both (17.17) and (17.18).

Let (16.15) *and* (17.15) *be fulfilled. Then problem* (16.1), (17.11) *has a unique proper solution. Moreover, if* δ_i $(i = 1, \dots, m)$ *are nondecreasing functions and* (16.23), (16.26) *and* 16.27 *hold, then this solition is oscillatory.*

Proof. The existence and uniqueness follow from Theorem 17.5. As to the oscillation, it is proved by Theorem 9.10′ using (16.23), (6.26) and (16.27). ■

Let (16.15), (17.15) *be fulfilled. Then problem* (16.1), (17.11) *has a unique proper solution. Moreover, this solution is oscillatory provided that* (16.23), (16.26) *and* (16.28) *hold.*

Let $p \in]0, +\infty[$, $\alpha \in]0, 1[$ *and*

$$\frac{1}{e} < p|\ln \alpha| < 1.$$

Then the problem

$$u''(t) = \frac{p}{(t+1)^2 \ln(t+2)} u(t^\alpha),$$
$$u(0) = c_0 \neq 0, \quad \overline{\lim_{t \to +\infty}} |u(t)| < +\infty$$

has a unique solution and this solution is oscillatory.

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