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OSCILLATION CRITERIA FOR SUBLINEAR AND SUPERLINEAR SECOND ORDER DIFFERENTIAL INCLUSIONS

Abstract. Oscillatory criteria are presented for second order differential inclusions. The results are new also in the single valued case.

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1. Introduction

In [1, 2] we initiated the study of nonoscillatory solutions to the differential inclusion

$$(a(t)\, y'(t))' \in F(t, y(t)) \text{ for a.e. } t \geq t_0 \geq 0. \tag{1.1}$$

However to our knowledge, no oscillatory results are available in the literature for differential inclusions. This paper begins this study. As an added bonus the results of this paper are new even in the single values case i.e. in particular some of the results in [3–7] are extended and improved.

In this paper by a solution y to (1.1) we mean a $y \in C[t_0, \infty)$ with $a\, y' \in C[t_0, \infty)$ and $(a\, y')' \in L^1_{\text{loc}}[t_0, \infty)$. We assume throughout that (1.1) possesses such solutions. Recall a nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

2. Differential Inclusions

In this section a variety of oscillation results will be presented for the differential inclusion

$$(a(t)\, y'(t))' \in F(t, y(t)) \quad \text{for a.e.} \quad t \geq t_0 \geq 0; \tag{2.1}$$

the function a is single valued and $F : [t_0, \infty) \times \mathbf{R} \to 2^{\mathbf{R}}$ is a multifunction (here $2^{\mathbf{R}}$ denotes the family of nonempty subsets of $\mathbf{R}$).

Remark 2.1. The usual standard notation in inclusion theory is used here e.g. $|F(t, u)| = \sup\{|v| :\ v \in F(t, u)\}$ and $F(t, u) > 0$ means $w > 0$ for each $w \in F(t, u)$.

The first few results in this section discuss the case when F has a particular sign. Both sublinear and superlinear results will be presented. Our first result is a theorem of superlinear type.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$a \in C([t_0, \infty), \mathbf{R}^+) \ \ (\textit{here } \mathbf{R}^+ = (0, \infty)), \tag{2.2}$$

$$\begin{cases} F(t, x) < 0 \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (0, \infty) \ \ \textit{and} \\ F(t, x) > 0 \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (-\infty, 0), \end{cases} \tag{2.3}$$

$$\int_{t_0}^{\infty} \frac{du}{a(u)} = \infty, \tag{2.4}$$

$$\begin{cases} \exists\, q : [t_0, \infty) \to (0, \infty) \ \ \textit{with} \ \ q \in L^1_{\text{loc}}[t_0, \infty), \ \psi : \mathbf{R} \to \mathbf{R} \\ \textit{continuous and nondecreasing with} \ \ x\, \psi(x) > 0 \ \ \textit{for} \ \ x \neq 0, \\ \textit{and with} \ \ |F(t, x)| \geq q(t)\, \psi(x) \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (0, \infty) \\ \textit{and} \ \ |F(t, x)| \geq -\, q(t)\, \psi(x) \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (-\infty, 0), \end{cases} \tag{2.5}$$

$$\int^{\infty} \frac{du}{\psi(u)} < \infty \quad \textit{and} \quad \int_{-\infty} \frac{du}{[-\,\psi(u)]} < \infty \tag{2.6}$$

and

$$\int_{t_0}^{\infty} q(u) \int_{t_0}^{u} \frac{ds}{a(s)}\, du \;=\; \infty. \tag{2.7}$$

Then equation (2.1) *is oscillatory.*

Proof. Let y be a nonoscillatory solution of (2.1). Suppose first that $y(t) > 0$ for $t \ge t_0$. We first show

$$y'(t) > 0 \quad \text{for} \quad t > t_0. \tag{2.8}$$

To see this first suppose there exists $\mu > t_0$ with $y'(\mu) < 0$. Let

$$\tau(t) = (a(t)\, y'(t))' \;\; \text{with} \;\; \tau(t) \in F(t, y(t)) \;\; \text{and} \;\; \tau \in L^1_{\text{loc}}[t_0, \infty). \tag{2.9}$$

From (2.3) we have $(a(t)\, y'(t))' \le 0$ for a.e. $t \ge t_0$ and so

$$a(t)\, y'(t) \le a(\mu)\, y'(\mu) \quad \text{for} \quad t > \mu.$$

Now an integration from μ to t $(t > \mu)$ yields

$$y(t) \le y(\mu) + a(\mu)\, y'(\mu) \int_{\mu}^{t} \frac{du}{a(u)}.$$

From (2.4) we have immediately that

$$y(\mu) + a(\mu)\, y'(\mu) \int_{\mu}^{t} \frac{du}{a(u)} \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction. Thus $y'(t) \ge 0$ for $t > t_0$. Next assume there exists $\mu > t_0$ with $y'(\mu) = 0$. Then (2.3) implies $(a(t)\, y'(t))' < 0$ for a.e. $t \ge t_0$, so $a(t)\, y'(t) < 0$ for $t > \mu$, a contradiction. Thus (2.8) is true.

Fix $x > t_0$ and integrate (2.9) from s $(t_0 < s < x)$ to x to obtain

$$y'(s) = \frac{a(x)}{a(s)}\, y'(x) + \frac{1}{a(s)} \int_s^x [-\tau(u)]\, du \ge \frac{1}{a(s)} \int_s^x [-\tau(u)]\, du.$$

This together with (2.3) and (2.5) gives

$$y'(s) \ge \frac{1}{a(s)} \int_s^x q(u)\, \psi(y(u))\, du \quad \text{for} \quad s \in (t_0, x).$$

Divide by $\psi(y(s))$ and integrate from t_0 to x to obtain

$$\int_{t_0}^{x} \frac{y'(s)}{\psi(y(s))}\, ds \ge \int_{t_0}^{x} \int_{s}^{x} \frac{q(u)}{a(s)} \frac{\psi(y(u))}{\psi(y(s))}\, du\, ds.$$

That is

$$\int_{y(t_0)}^{y(x)} \frac{du}{\psi(u)} \ge \int_{t_0}^{x} \int_{t_0}^{u} \frac{q(u)}{a(s)} \frac{\psi(y(u))}{\psi(y(s))}\, ds\, du.$$

From (2.8) and the fact that ψ is nondecreasing we have that

$$\int_{y(t_0)}^{y(x)} \frac{du}{\psi(u)} \ge \int_{t_0}^{x} q(u) \int_{t_0}^{u} \frac{ds}{a(s)}\, du.$$

As a result we have

$$\infty = \int_{t_0}^{\infty} q(u) \int_{t_0}^{u} \frac{ds}{a(s)}\, du \le \int_{y(t_0)}^{\infty} \frac{du}{\psi(u)} < \infty,$$

a contradiction.

Next suppose $y(t) > 0$ for $t \ge t_0$. As in the first part, it is easy to check that

$$y'(t) < 0 \quad \text{for} \quad t > t_0. \tag{2.10}$$

Fix $x > t_0$ and integrate (2.9) from s $(t_0 < s < x)$ to x to obtain

$$\begin{aligned} -y'(s) =& \frac{a(x)}{a(s)}\,[-y'(x)] + \frac{1}{a(s)} \int_s^x \tau(u)\, du \ge \frac{1}{a(s)} \int_s^x \tau(u)\, du \\ \ge& -\frac{1}{a(s)} \int_s^x q(u)\, \psi(y(u))\, du. \end{aligned}$$

Divide by $-\psi(y(s))$ (note $\psi(x) < 0$ for $x < 0$) and integrate from t_0 to x to obtain

$$\int_{y(t_0)}^{y(x)} \frac{du}{\psi(u)} \ge \int_{t_0}^{x} \int_{t_0}^{u} \frac{q(u)}{a(s)} \frac{\psi(y(u))}{\psi(y(s))}\, ds\, du.$$

Now (2.10), ψ nondecreasing and $\psi(x) < 0$ for $x < 0$ implies

$$\int_{y(x)}^{y(t_0)} \frac{du}{[-\psi(u)]} \ge \int_{t_0}^{x} q(u) \int_{t_0}^{u} \frac{ds}{a(s)}\, du,$$

and we again obtain a contradiction by letting $x \to \infty$. □

Remark 2.2. In Theorem 2.1, if (2.4) is not assumed, then (2.1) has no nonoscillatory solutions y which satisfy $y(t)\, y'(t) > 0$ for $t > t_0$.

Our next result is a theorem of sublinear type.

Theorem 2.2. *Suppose* (2.2) *holds and assume the following conditions are satisfied:*

$$\begin{cases} F(t,x) > 0 \ \ \textit{for} \ \ (t,x) \in [t_0,\infty) \times (0,\infty) \ \ \textit{and} \\ F(t,x) < 0 \ \ \textit{for} \ \ (t,x) \in [t_0,\infty) \times (-\infty,0), \end{cases} \tag{2.11}$$

$$\begin{cases} \exists\, q : [t_0,\infty) \to (0,\infty) \ \ \textit{with} \ \ q \in L^1_{\text{loc}}[t_0,\infty), \ \psi : \mathbf{R} \to \mathbf{R} \\ \textit{continuous and nonincreasing with} \ \ x\,\psi(x) > 0 \ \ \textit{for} \ \ x \ne 0, \\ \textit{and with} \ \ |F(t,x)| \ge q(t)\,\psi(x) \ \ \textit{for} \ \ (t,x) \in [t_0,\infty) \times (0,\infty) \\ \textit{and} \ \ |F(t,x)| \ge -q(t)\,\psi(x) \ \ \textit{for} \ \ (t,x) \in [t_0,\infty) \times (-\infty,0), \end{cases} \tag{2.12}$$

$$\int_0 \frac{du}{\psi(u)} < \infty \quad \textit{and} \quad \int^0 \frac{du}{[-\psi(u)]} < \infty \tag{2.13}$$

and

$$\int_{t_1}^{\infty} q(u) \int_{t_1}^{u} \frac{ds}{a(s)}\, du = \infty \ \ \textit{for any} \ \ t_1 \ge t_0. \tag{2.14}$$

Then (2.1) *has no nonoscillatory solution* y *with* $y(t)\, y'(t) < 0$ *eventually.*

Proof. Let y be a nonoscillatory solution of (2.1). Suppose first that $y(t) > 0$ for $t \geq t_0$, and assume $y'(t) < 0$ for $t \geq t_1 \geq t_0$. Let

$$\tau(t) = (a(t)\, y'(t))' \quad \text{with} \quad \tau(t) \in F(t, y(t)) \quad \text{and} \quad \tau \in L^1_{\text{loc}}[t_0, \infty). \tag{2.15}$$

Fix $x > t_1$ and integrate (2.15) from s $(t_1 < s < x)$ to x to obtain

$$-\, y'(s) = \frac{a(x)}{a(s)}\, [-\, y'(x)] + \frac{1}{a(s)} \int_s^x \tau(u)\, du \geq \frac{1}{a(s)} \int_s^x \tau(u)\, du,$$

and this together with (2.12) gives

$$-\, y'(s) \geq \frac{1}{a(s)} \int_s^x q(u)\, \psi(y(u))\, du \quad \text{for} \quad s \in (t_1, x).$$

Divide by $\psi(y(s))$ and integrate from t_1 to x to obtain (see the ideas in Theorem 2.1)

$$\int_{y(x)}^{y(t_1)} \frac{du}{\psi(u)} \geq \int_{t_1}^x \int_{t_1}^u \frac{q(u)}{a(s)}\, \frac{\psi(y(u))}{\psi(y(s))}\, ds\, du.$$

Now $y'(t) < 0$ for $t \geq t_1$, and the fact that ψ is nonincreasing yields

$$\int_{y(x)}^{y(t_1)} \frac{du}{\psi(u)} \geq \int_{t_1}^x q(u) \int_{t_1}^u \frac{ds}{a(s)}\, du.$$

That is

$$\int_{t_1}^x q(u) \int_{t_1}^u \frac{ds}{a(s)}\, du \leq \int_0^{y(t_1)} \frac{du}{\psi(u)}.$$

Let $x \to \infty$ to get

$$\infty = \int_{t_1}^\infty q(u) \int_{t_1}^u \frac{ds}{a(s)}\, du \leq \int_0^{y(t_1)} \frac{du}{\psi(u)} < \infty,$$

a contradiction.

Now suppose $y(t) < 0$ for $t \geq t_0$ and $y'(t) > 0$ for $t \geq t_1 \geq t_0$. Fix $x > t_1$ and notice for $s \in (t_1, x)$ that

$$\begin{aligned} y'(s) =& \frac{a(x)}{a(s)}\, [y'(x)] + \frac{1}{a(s)} \int_s^x [-\, \tau(u)]\, du \geq \frac{1}{a(s)} \int_s^x [-\, \tau(u)]\, du \\ \geq& -\, \frac{1}{a(s)} \int_s^x q(u)\, \psi(y(u))\, du. \end{aligned}$$

Divide by $-\,\psi(y(s))$ (note $\psi(x) < 0$ for $x < 0$) and integrate from t_1 to x to obtain

$$\int_{y(x)}^{y(t_1)} \frac{du}{\psi(u)} \geq \int_{t_1}^x \int_{t_1}^u \frac{q(u)}{a(s)}\, \frac{\psi(y(u))}{\psi(y(s))}\, ds\, du.$$

Now $y' > 0$, ψ nonincreasing and $\psi(x) < 0$ for $x < 0$ implies

$$\int_{y(t_1)}^{y(x)} \frac{du}{[-\, \psi(u)]} \geq \int_{t_1}^x q(u) \int_{t_1}^u \frac{ds}{a(s)}\, du.$$

Thus

$$\int_{t_1}^{x} q(u) \int_{t_1}^{u} \frac{ds}{a(s)}\, du \le \int_{y(t_1)}^{0} \frac{du}{[-\psi(u)]},$$

and we again obtain a contradiction by letting $x \to \infty$. □

In Theorem 2.2 if we assume (2.4) then we have the following result.

Theorem 2.3. *Suppose* (2.2), (2.4) *and* (2.11)–(2.14) *hold. Then every bounded solution of* (2.1) *is oscillatory.*

Proof. Let y be a bounded nonoscillatory solution of (2.1), and without loss of generality assume $y(t) > 0$ for $t \ge t_0$. We claim

$$y'(t) < 0 \quad \text{for} \quad t > t_0. \tag{2.16}$$

To see this suppose there exists $\mu > t_0$ with $y'(\mu) > 0$. Then $(a(t)\, y'(t))' \ge 0$ for a.e. $t \ge t_0$, so $y'(t) \ge \frac{a(\mu)}{a(t)}\, y'(\mu)$ for $t > \mu$. Thus

$$y(t) \ge y(\mu) + a(\mu)\, y'(\mu) \int_{\mu}^{t} \frac{du}{a(u)} \to \infty \quad \text{as} \quad t \to \infty.$$

This contradicts the fact that y is bounded. As a result $y'(t) \le 0$ for $t > t_0$. Next assume there exists $\mu > t_0$ with $y'(\mu) = 0$. Then $(a(t)\, y'(t))' > 0$ for a.e. $t \ge t_0$ together with $y'(\mu) = 0$ implies $a(t)\, y'(t) > 0$ for $t > \mu$, a contradiction. Thus (2.16) holds. Consequently $y(t)\, y'(t) < 0$ for $t > t_0$, which contradicts Theorem 2.2. □

Next we present two results where F does not satisfy a sign change.

Theorem 2.4. *Suppose* (2.2) *and* (2.4) *hold and assume the following conditions are satisfied:*

$$\begin{cases} \exists\, q : [t_0, \infty) \to \mathbf{R} \ \ \textit{with} \ \ q \in L^1_{\text{loc}}[t_0, \infty), \ \psi : \mathbf{R} \to \mathbf{R} \\ \textit{continuous with} \ \ x\, \psi(x) > 0 \ \ \textit{for} \ \ x \ne 0, \ \ \textit{and with} \\ F(t, x) \le -\, q(t)\, \psi(x) \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (0, \infty) \ \ \textit{and} \\ F(t, x) \ge -\, q(t)\, \psi(x) \ \ \textit{for} \ \ (t, x) \in [t_0, \infty) \times (-\infty, 0), \end{cases} \tag{2.17}$$

$$\int^{\infty} q(x)\, dx \;=\; \infty \tag{2.18}$$

and

$$\psi'(x) \ge 0 \ \ \textit{for} \ \ x \ne 0. \tag{2.19}$$

Then equation (2.1) *is oscillatory.*

Proof. Let y be a nonoscillatory solution of (2.1) with $y(t) > 0$ for $t \ge t_0$. Let

$$w(t) \;=\; \frac{a(t)\, y'(t)}{\psi(y(t))} \quad \text{for} \quad t \ge t_0. \tag{2.20}$$

Also let

$$\tau(t) = (a(t)\, y'(t))' \ \ \text{with} \ \ \tau(t) \in F(t, y(t)) \ \ \text{and} \ \ \tau \in L^1_{\text{loc}}[t_0, \infty). \tag{2.21}$$

Notice for $t > t_0$ that

$$w'(t) = \frac{(a(t)\,y'(t))'}{\psi(y(t))} - \frac{\psi'(y(t))\,w^2(t)}{a(t)} \le \frac{\tau(t)}{\psi(y(t))} \le -\,q(t). \tag{2.22}$$

Integrate (2.22) from t_0 to t $(t \ge t_0)$ to obtain

$$w(t) \le w(t_0) - \int_{t_0}^{t} q(s)\,ds. \tag{2.23}$$

Now (2.18) and (2.23) guarantee that there exists $t_1 \ge t_0$ with $w(t) < 0$ for $t \ge t_1$. That is $y'(t) < 0$ for $t \ge t_1$. Also (2.18) guarantees that there exists $t_2 \ge t_1$ with $\int_{t_1}^{t_2} q(s)\,ds = 0$ and $\int_{t_1}^{t} q(s)\,ds > 0$ for $t > t_2$. Integrate (2.21) from t_2 to t $(t > t_2)$ to obtain

$$\begin{aligned} a(t)\,y'(t) &= a(t_2)\,y'(t_2) + \int_{t_2}^{t} \tau(s)\,ds \le a(t_2)\,y'(t_2) - \int_{t_2}^{t} q(s)\,\psi(y(s))\,ds \\ &= a(t_2)\,y'(t_2) - \psi(y(t)) \int_{t_2}^{t} q(s)\,ds + \int_{t_2}^{t} y'(s)\psi'(y(s)) \left(\int_{t_2}^{s} q(u)\,du \right) ds \\ &\le a(t_2)\,y'(t_2). \end{aligned}$$

Thus

$$y'(t) \le \frac{a(t_2)\,y'(t_2)}{a(t)} \quad \text{for} \quad t \ge t_2,$$

so

$$y(t) \le y(t_2) + a(t_2)\,y'(t_2) \int_{t_2}^{t} \frac{ds}{a(s)} \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction.

Next suppose $y(t) < 0$ for $t \ge t_0$ and let w be as in (2.20) and τ as in (2.21). Notice for $t > t_0$ that

$$w'(t) \le \frac{\tau(t)}{\psi(y(t))} \le -\,q(t), \tag{2.24}$$

since $\psi(x) < 0$ for $x < 0$. Integrate (2.24) from t_0 to t $(t \ge t_0)$ to obtain

$$w(t) \le w(t_0) - \int_{t_0}^{t} q(s)\,ds.$$

Now there exists $t_1 \ge t_0$ with $w(t) < 0$ for $t \ge t_1$, and so $y'(t) > 0$ for $t \ge t_1$ since $\psi(x) < 0$ for $x < 0$. Again choose $t_2 \ge t_1$ with $\int_{t_1}^{t} q(s)\,ds > 0$ for $t > t_2$. Integrate (2.21) from t_2 to t $(t > t_2)$ to obtain

$$\begin{aligned} a(t)\,y'(t) &= a(t_2)\,y'(t_2) + \int_{t_2}^{t} \tau(s)\,ds \ge a(t_2)\,y'(t_2) - \int_{t_2}^{t} q(s)\,\psi(y(s))\,ds \\ &= a(t_2)\,y'(t_2) - \psi(y(t)) \int_{t_2}^{t} q(s)\,ds + \int_{t_2}^{t} y'(s)\,\psi'(y(s)) \left(\int_{t_2}^{s} q(u)\,du \right) ds \\ &\ge a(t_2)\,y'(t_2), \end{aligned}$$

and so

$$y(t) \geq y(t_2) + a(t_2)\, y'(t_2) \int_{t_2}^{t} \frac{ds}{a(s)} \to \infty \quad \text{as} \quad t \to \infty,$$

a contradiction. □

Remark 2.3. It is possible to remove condition (2.4) in Theorem 2.4 provided we assume (2.13) and

$$\int^{\infty} \frac{1}{a(s)} \int_{t_0}^{s} q(u)\, du\, ds = \infty. \tag{2.25}$$

To see this let y be a nonoscillatory solution of (2.1) and without loss of generality assume $y(t) > 0$ for $t \geq t_0$. Then (2.23) holds and we may choose $t_1 \geq t_0$ with $y'(t) < 0$ for $t \geq t_1$ and we may also choose $t_2 \geq t_1$ with $\int_{t_0}^{t} q(s)\, ds \geq 2\, w(t_0)$ for $t \geq t_2$, so

$$w(t) \leq -\frac{1}{2} \int_{t_0}^{t} q(s)\, ds \quad \text{for} \quad t \geq t_2.$$

That is

$$\frac{y'(t)}{\psi(y(t))} \leq -\frac{1}{2\, a(t)} \int_{t_0}^{t} q(s)\, ds \quad \text{for} \quad t \geq t_2,$$

so integration from t_2 to t $(t \geq t_2)$ yields

$$\int_{y(t_2)}^{y(t)} \frac{du}{\psi(u)} \leq -\int_{t_2}^{t} \frac{1}{2\, a(s)} \int_{t_0}^{s} q(u)\, du\, ds.$$

Thus for $t \geq t_2$ we have

$$\frac{1}{2} \int_{t_2}^{t} \frac{1}{a(s)} \int_{t_0}^{s} q(u)\, du\, ds \leq \int_{y(t)}^{y(t_2)} \frac{du}{\psi(u)} \leq \int_{0}^{y(t_2)} \frac{du}{\psi(u)},$$

and let $t \to \infty$ to get a contradiction.

It is possible to remove condition (2.18), provided extra conditions are added, as we will see in our next result.

Theorem 2.5. *Suppose* (2.2), (2.4), (2.17) *and* (2.19) *hold and in addition assume the following conditions are satisfied:*

$$\int_{t_0}^{\infty} q(s)\, ds < \infty, \tag{2.26}$$

$$\liminf_{t \to \infty} \int_{T}^{t} q(s)\, ds > 0 \quad \textit{for large} \ \ T, \tag{2.27}$$

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} q(u)\, du\, ds = \infty \tag{2.28}$$

and

$$\int^{\infty} \frac{du}{\psi(u)} < \infty \quad \textit{and} \quad \int_{-\infty} \frac{du}{[-\psi(u)]} < \infty \tag{2.29}$$

Then equation (2.1) *is oscillatory.*

Proof. Let y be a nonoscillatory solution of (2.1) with $y(t) > 0$ for $t \geq t_0$. Let w be as in (2.20), and as in Theorem 2.4 we have

$$w(t) \leq -q(t) \quad \text{for} \quad t \geq t_0. \tag{2.30}$$

Also let

$$\tau(t) = (a(t)\, y'(t))' \quad \text{with} \quad \tau(t) \in F(t, y(t)) \quad \text{and} \quad \tau \in L^1_{\text{loc}}[t_0, \infty). \tag{2.31}$$

There are three cases to consider, either $y'(t) \geq 0$ for $t \geq t_0$, $y'(t) \leq 0$ for $t \geq t_0$, or y' oscillates.

Case (i). $y'(t) \leq 0$ for $t \geq t_0$.

From (2.27) there exists $t_1 \geq t_0$ and $t_2 \geq t_1$ with $\int_{t_1}^t q(x)\, dx > 0$ for $t \geq t_2$. Also from (2.30) we have

$$\int_{t_1}^t q(x)\, dx \leq w(t_1) - w(t) \quad \text{for} \quad t \geq t_2.$$

If there exists $\mu > t_2$ with $y'(\mu) = 0$ then

$$0 \; < \; \int_{t_1}^{\mu} q(x)\, dx \leq w(t_1) \leq 0,$$

a contradiction. Thus $y'(t) < 0$ for $t > t_2$. Integrate (2.31) from t_2 to t $(t > t_2)$ to obtain (as in Theorem 2.4)

$$y'(t) \leq \frac{a(t_2)\, y'(t_2)}{a(t)},$$

and so

$$y(t) \leq y(t_2) + a(t_2)\, y'(t_2) \int_{t_2}^t \frac{ds}{a(s)} \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction.

Case (ii). $y'(t) \geq 0$ for $t \geq t_0$.

Now from (2.30) for $s \geq t \geq t_0$ we have

$$\int_t^s q(x)\, dx \leq w(t) - w(s) \leq w(t).$$

As a result (letting $s \to \infty$) we have

$$\int_t^{\infty} q(x)\, dx \leq \frac{a(t)\, y'(t)}{\psi(y(t))} \quad \text{for} \quad t \geq t_0.$$

Divide by a and integrate from t_0 to t $(t < t_0)$ to obtain

$$\int_{t_0}^t \frac{1}{a(s)} \int_s^{\infty} q(x)\, dx\, ds \leq \int_{y(t_0)}^{y(t)} \frac{du}{\psi(u)} \leq \int_{y(t_0)}^{\infty} \frac{du}{\psi(u)}.$$

Thus

$$\infty = \int_{t_0}^{\infty} \frac{1}{a(s)} \int_s^{\infty} q(x)\, dx\, ds \leq \int_{y(t_0)}^{\infty} \frac{du}{\psi(u)} < \infty,$$

a contradiction.

Case (iii). y' oscillates.

Then there exists a sequence $\{T_n\}_1^\infty$ with $\lim_{n\to\infty} T_n = \infty$ and $y'(T_n) < 0$. Choose N large enough so that

$$\liminf_{t\to\infty} \int_{T_N}^t q(s)\,ds > 0.$$

Now integrate (2.30) from T_n to t $(t > T_N)$ to obtain

$$\frac{a(t)\,y'(t)}{\psi(y(t))} \le \frac{a(T_N)\,y'(T_N)}{\psi(y(T_N))} - \int_{T_N}^t q(s)\,ds,$$

so

$$\limsup_{t\to\infty} \frac{a(t)\,y'(t)}{\psi(y(t))} \le \frac{a(T_N)\,y'(T_N)}{\psi(y(T_N))} + \limsup_{t\to\infty}\left(-\int_{T_N}^t q(s)\,ds\right) < 0.$$

This contradicts the fact that y' oscillates.

Next suppose $y(t) < 0$ for $t \ge t_0$ and let w be as in (2.20) (so (2.30) holds, see Theorem 2.4) and τ be as in (2.31). The same three cases need to be considered here.

Case (i). $y'(t) \le 0$ for $t \ge t_0$.

Now from (2.30) for $s \ge t \ge t_0$ we have

$$\int_t^s q(x)\,dx \le w(t) - w(s) \le w(t),$$

since $y' \le 0$ and $\psi(x) < 0$ for $x > 0$. Thus

$$\int_t^\infty q(x)\,dx \le \frac{a(t)\,y'(t)}{\psi(y(t))} \quad \text{for} \quad t \ge t_0,$$

so divide by a, integrate from t_0 to t $(t < t_0)$, and let $t \to \infty$ to obtain

$$\infty = \int_{t_0}^\infty \frac{1}{a(s)} \int_s^\infty q(x)\,dx\,ds \le \int_{-\infty}^{y(t_0)} \frac{du}{\psi(u)} < \infty,$$

a contradiction.

Case (ii). $y'(t) \ge 0$ for $t \ge t_0$.

Now there exists $t_1 \ge t_0$ and $t_2 \ge t_1$ with $\int_{t_1}^t q(x)\,dx > 0$ for $t \ge t_2$. Also from (2.30) we have

$$\int_{t_1}^t q(x)\,dx \le w(t_1) - w(t) \quad \text{for} \quad t \ge t_2.$$

If there exists $\mu > t_2$ with $y'(\mu) = 0$ then

$$0 < \int_{t_1}^\mu q(x)\,dx \le w(t_1) \le 0,$$

since $y' \ge 0$ and $\psi(x) < 0$ for $x < 0$. Thus $y'(t) > 0$ for $t > t_2$. Integrate (2.31) from t_2 to t $(t > t_2)$ to obtain (as in Theorem 2.4)

$$y'(t) \ge \frac{a(t_2)\,y'(t_2)}{a(t)},$$

and so

$$y(t) \geq y(t_2) + a(t_2)\, y'(t_2) \int_{t_2}^{t} \frac{ds}{a(s)} \to \infty \quad \text{as} \quad t \to \infty,$$

a contradiction.

Case (iii). y' oscillates.

Then there exists a sequence $\{T_n\}_1^\infty$ with $\lim_{n\to\infty} T_n = \infty$ and $y'(T_n) > 0$. Choose N large enough so that

$$\liminf_{t\to\infty} \int_{T_N}^{t} q(s)\, ds \,>\, 0.$$

Integrate (2.30) from T_n to t $(t > T_N)$, and take $\limsup' s$ to obtain

$$\limsup_{t\to\infty} \frac{a(t)\, y'(t)}{\psi(y(t))} \leq \frac{a(T_N)\, y'(T_N)}{\psi(y(T_N))} + \limsup_{t\to\infty} \left(- \int_{T_N}^{t} q(s)\, ds \right) \,<\, 0,$$

a contradiction. □

Remark 2.4. It is easy to see that (2.27) can be replaced in Theorem 2.5 by

$$\liminf_{t\to\infty} \int_{T}^{t} q(s)\, ds \geq 0 \quad \text{for large} \quad T.$$

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