Mem. Differential Equations Math. Phys. 28(2003), 141–146

T. Kiguradze

ON INITIAL–BOUNDARY VALUE PROBLEMS FOR DEGENERATE LINEAR HYPERBOLIC SYSTEMS

(Reported on September 23, 2002)

Let $l,\ m$ and $n \geq 2$ be natural numbers, $0 < b < +\infty$ and $I \subset \mathbb{R}$ be a compact interval containing zero. In the rectangle $\Omega = I \times (0,b)$ consider the hyperbolic system

$$\gamma_n(y)u^{(m,n)} = \sum_{k=0}^{n-1} \gamma_k(y)P_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} \gamma_k(y)P_{jk}(x,y)u^{(j,k)} + q(x,y) \tag{1}$$

with the initial–boundary conditions

$$\begin{aligned} &u^{(j,0)}(0,y) = \varphi_j(y)\ (j = 0,\dots,m-1),\\ &h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k(x)\ (k = 1,\dots,n), \end{aligned} \tag{2}$$

where

$$u^{(j,k)}(x,y) = \frac{\partial^{j+k}u(x,y)}{\partial x^j \partial y^k} \quad (j = 0,\dots,m;\ k = 0,\dots,n),$$

$$\gamma_k(y) = y^k(b-y)^k \quad (k = 0,\dots,n-1), \qquad \gamma_n(y) = \gamma_{n-1}(y).$$

Everywhere below it will be assumed that $P_{jk} : \Omega \to \mathbb{R}^{l\times l}$ $(j = 0,\dots,m;\ k = 0,\dots,n;\ j+k < m+n)$ are continuous and bounded matrix functions, $q : \Omega \to \mathbb{R}^l$ and $\psi_k : I \to \mathbb{R}^l$ $(k = 1,\dots,n)$ are continuous and bounded vector functions, $\varphi_j : (0,b) \to \mathbb{R}^l$ $(j = 0,\dots,m-1)$ are n–times continuously differentiable vector functions such that

$$\sup\big\{\gamma_k(y)\|\varphi_j^{(k)}(y)\| : 0 < y < b\big\} < +\infty \quad (j = 0,\dots,m-1;\ k = 0,\dots,n),$$

and $h_k : C([0,b];\mathbb{R}^l) \to C(I;\mathbb{R}^l)$ $(k = 1,\dots,n)$ are bounded linear operators.

System (1) degenerates along the intervals $y = 0$ and $y = b$. These degeneration is removable only when P_{jk} and q admit the representation

$$P_{jk}(x,y) = \frac{\gamma_n(y)}{\gamma_k(y)}\widetilde{P}_{jk}(x,y) \quad (j = 0,\dots,m;\ k = 1,\dots,n), \quad q(x,y) = \gamma_n(y)\widetilde{q}(x,y),$$

i.e., when system (1) has the form

$$u^{(m,n)} = \sum_{k=0}^{n-1} \widetilde{P}_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} \widetilde{P}_{jk}(x,y)u^{(j,k)} + \widetilde{q}(x,y),$$

where $\widetilde{P}_{jk} : \overline{\Omega} \to \mathbb{R}^{l\times l}$ $(j = 0,\dots,m;\ k = 0,\dots,n;\ j+k < m+n)$ and $\widetilde{q} : \overline{\Omega} \to \mathbb{R}^l$ are continuous matrix and vector functions. In this case the criterion of well–posedness of problem (1),(2) is established in [3]. However, in the case, where degeneration is not removable (e.g., when $\limsup_{y\to 0}\|P_{jk}(x,y)\| > 0$, or $\limsup_{y\to b}\|P_{jk}(x,y)\| > 0$ for

2000 *Mathematics Subject Classification.* 35L35.
Key words and phrases. Linear degenerate hyperbolic system.

some $x \in I$ and $j \in \{0, \dots, m\}$) the question of well–posedness of problem (1),(2) was opened. The results formulated below concern this case namely.

Throughout the paper we will use the following notation:

$\mathbb{R} = (-\infty, +\infty)$;

$\mathbb{R}^l$ is the space of column–vectors $z = (z_i)_{i=1}^l$ with the real components $z_i \in \mathbb{R}$ $(i = 1, \dots, l)$ and the norm $\|z\| = \max\{|z_1|, \dots, |z_l|\}$;

$\mathbb{R}^{l \times l}$ is the space of $l \times l$ matrices $Z = (z_{ik})_{i,k=1}^l$ with the components $z_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, l)$ and the norm

$$\|Z\| = \max\Big\{ \sum_{k=1}^{l} |z_{ik}| : i = 1, \dots, l \Big\};$$

If $z = (z_i)_{i=1}^l \in \mathbb{R}^l$ and $Z = (z_{ik})_{i,k=1}^l \in \mathbb{R}^{l \times l}$, then

$$|z| = (|z_i|)_{i=1}^l, \quad |Z| = (|z_{ik}|)_{i,k=1}^l;$$

Z^{-1} is the matrix reciprocal to a nonsingular matrix $Z \in \mathbb{R}^{l \times l}$;

$r(Z)$ is the spectral radius of a matrix $Z \in \mathbb{R}^{l \times l}$;

$C(I; \mathbb{R}^l)$ and $C(\Omega; \mathbb{R}^l)$ are the spaces of continuous and bounded vector functions $\varphi : I \to \mathbb{R}^l$ and $z : \Omega \to \mathbb{R}^l$ with the norms

$$\|\varphi\|_{C(I;\mathbb{R}^l)} = \sup\{\|\varphi(s)\| : s \in I\}, \qquad \|z\|_{C(\Omega;\mathbb{R}^l)} = \sup\{\|z(x,y)\| : (x,y) \in \Omega\};$$

$S^n((0,b); \mathbb{R}^l)$ is the space of n–times continuously differentiable functions $\varphi : (0,b) \to \mathbb{R}^l$ such that

$$\|\varphi\|_{S^n((0,b);\mathbb{R}^l)} = \sup\Big\{ \sum_{k=0}^{n} \gamma_k(y) \|\varphi^{(k)}(y)\| : 0 < y < b \Big\} < +\infty;$$

$S^{m,n}(\Omega; \mathbb{R}^l)$ is the space of functions $u : \Omega \to \mathbb{R}^l$ having the continuous partial derivatives $u^{(j,k)}$ $(j = 0, \dots, m;\ k = 0, \dots, n)$ such that

$$\|u\|_{S^{m,n}(\Omega;\mathbb{R}^l)} = \sup\Big\{ \sum_{j=0}^{m} \sum_{k=0}^{n} \gamma_k(y) \|u^{(j,k)}(x,y)\| : (x,y) \in \Omega \Big\} < +\infty;$$

If $\varphi \in S^n((0,b); \mathbb{R}^l)$ and $u \in S^{m,n}(\Omega; \mathbb{R}^l)$, then there exist the limits

$$\lim_{y \to 0} \varphi(y), \quad \lim_{y \to b} \varphi(y), \quad \lim_{y \to 0} u^{(j,0)}(x,y), \quad \lim_{y \to b} u^{(j,0)}(x,y) \quad (j = 0, \dots, m),$$

which are denoted by $\varphi(0),\ \varphi(b),\ u^{(j,0)}(x,0),\ u^{(j,0)}(x,b)$ $(j = 0, \dots, m)$.

By a solution of problem (1),(2) we understand a vector function $u \in S^{m,n}(\Omega; \mathbb{R}^l)$ satisfying system (1) and conditions (2) in Ω.

Definition. Problem (1),(2) is called *well–posed* if it is uniquely solvable for arbitrary $q \in C(\Omega; \mathbb{R}^l)$, $\varphi_j \in S^n((0,b); \mathbb{R}^l)$ $(j = 0, \dots, m-1)$, $\psi_k \in C(I; \mathbb{R}^l)$ $(k = 1, \dots, n)$ and for an arbitrary interval $J \subset I$ containing zero the restriction of a solution of this problem on $J \times (0,b)$ admits the estimate

$$\|u\|_{S^{m,n}(J\times(0,b);\mathbb{R}^l)} \le \rho\Big(\sum_{j=0}^{m-1} \|\varphi_j\|_{S^n((0,b);\mathbb{R}^l)} + \sum_{k=1}^{n} \|\psi_k\|_{C(J;\mathbb{R}^l)} + \|q\|_{C(J\times(0,b);\mathbb{R}^l)} \Big), \tag{3}$$

where ρ is a positive constant independent of q, φ_j, ψ_k $(j = 0, \dots, m-1;\ k = 1, \dots, n)$ and J.

For an arbitrarily fixed $x \in I$ in the interval $(0, b)$ consider the system of ordinary differential equations

$$\gamma_n(y)\frac{d^n v}{dy^n} = \sum_{k=0}^{n-1} \gamma_k(y) P_{mk}(x,y)\frac{d^k v}{dy^k} \tag{4}$$

with the homogeneous boundary conditions

$$h_k(v)(x) = 0 \quad (k = 1, \dots, n). \tag{5}$$

We will seek for a solution of problem (4),(5) in the class of vector functions $z : [0, b] \to \mathbb{R}^l$ continuous on $[0, b]$ and n–times continuously differentiable in $(0, b)$.

Theorem. *Problem* (1), (2) *is well–posed if and only if for any* $x \in I$ *problem* (3), (4) *has only a trivial solution.*

To prove the Theorem we need to give two auxiliary propositions. The first of them concerns continuity with respect to x of a solution of the problem

$$\gamma_n(y)\frac{d^n v}{dy^n} = \sum_{k=0}^{n-1} \gamma_k(y) P_{mk}(x,y)\frac{d^k v}{dy^k} + q_0(y), \tag{6}$$

$$h_k(v)(x) = c_k \quad (k = 1, \dots, n). \tag{7}$$

Lemma 1. *Let for any* $x \in I$ *problem* (4), (5) *have only a trivial solution. Then for an arbitrary* $q_0 \in C((0, b); \mathbb{R}^l)$, $c_k \in \mathbb{R}^l$ $(k = 1, \dots, n)$ *and* $x \in I$ *problem* (6), (7) *has a unique solution* $v(x, \cdot)$ *which is continuous with respect to* x. *Moreover, the vector functions* $v^{(0,k)} : \Omega \to \mathbb{R}^l$ $(k = 0, \dots, n-1)$ *are continuous and there exists a positive constant* ρ_0, *independent of* q_0 *and* c_k $(k = 1, \dots, n)$, *such that the inequality*

$$\sum_{k=0}^{n} \gamma_k(y)\|v^{(0,k)}(x,y)\| \le \rho_0\Big(\sum_{k=1}^{n}\|c_k\| + \|q_0\|_{C((0,b);\mathbb{R}^l)}\Big)$$

holds in Ω.

This lemma follows from Theorem 1.1 from [2].

The following lemma concerns the operator equation

$$u(x, y) = g(u)(x, y) + f(x, y), \tag{8}$$

where $g : S^{m,n}(\Omega; \mathbb{R}^l) \to S^{m,n}(\Omega; \mathbb{R}^l)$ is a linear bounded operator and $f \in S^{m,n}(\Omega; \mathbb{R}^l)$.

For an arbitrary $i \in \{0, \dots, m\}$ and $z \in C^{m,n}(\Omega; \mathbb{R}^l)$ set

$$\|z^{(i,0)}(x,\cdot)\|_{S^n((0,b);\mathbb{R}^l)} = \sup\Big\{\sum_{k=0}^{n}\gamma_k(y)\|z^{(i,k)}(x,y)\| : 0 < y < b\Big\}.$$

Lemma 2. *Let there exist a positive number* ρ_1 *such that for any* $z \in S^{m,n}(\Omega; \mathbb{R}^l)$ *the inequality*

$$\sum_{i=0}^{m}\Big\|\frac{\partial^i g(z)(x,\cdot)}{\partial x^i}\Big\|_{S^n((0,b);\mathbb{R}^l)} \le \rho_1\sum_{i=0}^{m}\Big|\int_0^x \big\|z^{(i,0)}(\xi,\cdot)\big\|_{S^n((0,b);\mathbb{R}^l)}\,d\xi\Big| \tag{9}$$

holds in I. *Then equation* (8) *has a unique solution* u *in the space* $S^{m,n}(\Omega; \mathbb{R}^l)$ *and*

$$\sum_{i=0}^{m}\big\|u^{(i,0)}(x,\cdot)\big\|_{S^n((0,b);\mathbb{R}^l)} \le \rho_1\exp(|x|)\sum_{i=0}^{m}\big\|f^{(i,0)}(x,\cdot)\big\|_{S^n((0,b);\mathbb{R}^l)} \quad \textit{for} \quad x \in I. \tag{10}$$

This lemma can be proved similarly to Lemma 2.3 from [3].

Proof of the Theorem. We will prove the sufficiency since the necessity can be proved by the method applied in [3] for proving Theorem 1.1.

By Lemma 1, there exists a linear bounded operator

$$g_0 : \mathbb{R}^l \times C((0, b); \mathbb{R}^l) \to S^{0,n}(\Omega; \mathbb{R}^l)$$

such that if $x \in I$, $c_k \in \mathbb{R}^l$ $(k = 1, \dots, n)$ and $q_0 \in C((0,b);\mathbb{R}^l)$, then the vector function $v(x,\cdot) : (0,b) \to \mathbb{R}^l$ is a solution of problem (6),(7) if and only if

$$v(x,y) = g_0(c_1, \dots, c_l, q_0)(x,y) \quad \text{for} \quad 0 < y < b.$$

For an arbitrary $z \in S^{m,n}(\Omega;\mathbb{R}^l)$ set

$$w(z)(x,y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n} \frac{\gamma_k(y)}{(m-j-1)!} P_{jk}(x,y) \int_0^x (x-s)^{m-j-1} z^{(m,k)}(s,y)\, ds, \tag{11}$$

$$g(z)(x,y) = \frac{1}{(m-1)!} \int_0^x (x-s)^{m-1} g_0(0, \dots, 0, w(z)(s,\cdot))(s,y)\, ds. \tag{12}$$

Furthermore, introduce the vector functions

$$w_0(x,y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n} \gamma_k(y) P_{jk}(x,y) \sum_{i=j}^{m-1} \frac{x^{i-j}}{(i-j)!} \varphi_i^{(k)}(y) + q(x,y), \tag{13}$$

$$f(x,y) = \frac{1}{(m-1)!} \int_0^x (x-s)^{m-1} g_0(\psi_1(s), \dots, \psi_l(s), w_0(s,\cdot))(s,y)\, ds. \tag{14}$$

In view of notation (11)–(14) it is not difficult to see that problem (1),(2) is equivalent to equation (8), i.e., every solution of problem (1),(2) is a solution of equation (8) and vice versa. Therefore to prove the theorem it is sufficient to show that in the space $S^{m,n}(\Omega;\mathbb{R}^l)$ equation (8) has a unique solution admitting estimate (3) on every interval $J \subset I$ containing zero, where ρ is a positive constant independent of q, φ_j, ψ_k $(j = 0, \dots, m-1;\ k = 1, \dots, n)$ and J.

By Lemma 1, there exists a positive constant ρ_0 such that for arbitrary $c_j \in \mathbb{R}^l$ $(j = 1, \dots, l)$ and $q_0 \in C((0,b);\mathbb{R}^l)$ the inequality

$$\sum_{k=0}^{n} \gamma_k(y) \Big| \frac{\partial^k}{\partial y^k} g_0(c_1, \dots, c_l, q_0)(x,y) \Big| \le \rho_0 \Big(\sum_{k=1}^{n} \|c_k\| + \|q_0\|_{C((0,b);\mathbb{R}^l)} \Big) \tag{15}$$

holds in the rectangle Ω. According to equalities (11),(13) and boundedness of the matrix functions P_{jk} $(j = 0, \dots, m-1;\ k = 0, \dots, n)$, without loss of generality we may assume that the inequalities

$$\|w(z)(x,\cdot)\|_{C((0,b);\mathbb{R}^l)} \le \rho_0 \Big| \int_0^x \big\|z^{(m,0)}(s,\cdot)\big\|_{S^n((0,b);\mathbb{R}^l)}\, ds \Big|, \tag{16}$$

$$\|w_0(x,\cdot)\|_{C((0,b);\mathbb{R}^l)} \le \rho_0 \Big(\sum_{j=0}^{m-1} \|\varphi_j\|_{S^n((0,b);\mathbb{R}^l)} + \|q(x,\cdot)\|_{C((0,b);\mathbb{R}^l)} \Big) \tag{17}$$

hold on I.

In view of conditions (15) and (16), inequality (9) follows from (12), where

$$\rho_1 = \rho_0^2 \sum_{i=0}^{m} \frac{1}{(m-i)!} |I|^{m-i}$$

and $|I|$ is the length of the interval I. On the other hand, by (15) and (17), it follows from (14) that for an arbitrary interval $J \subset I$ the function f admits the estimate

$$\|f\|_{S^{m,n}(J\times(0,b);\mathbb{R}^l)} \le \rho_1 \Big(\sum_{j=0}^{m-1} \|\varphi_j\|_{S^n((0,b);\mathbb{R}^l)} + \sum_{k=1}^{n} \|\psi_k\|_{C(J;\mathbb{R}^l)} + \|q\|_{C(J\times(0,b);\mathbb{R}^l)} \Big). \tag{18}$$

By Lemma 2, in the space $S^{m,n}(\Omega;\mathbb{R}^l)$ equation (8) has a unique solution admitting estimate (10). However, estimate (3) follows from (10) and (18), where $\rho=\rho_1^2\exp(|I|)$ is a positive constant independent of q, φ_j, ψ_k $(j=0,\dots,m-1;\ k=1,\dots,n)$ and J. $\square$

The initial–boundary conditions

$$u^{(j,0)}(0,y)=\varphi_j(y)\ \ (j=0,\dots,m-1),\quad u^{(m,0)}(x,y_k(x))=\psi_k(x)\ \ (k=1,\dots,n) \tag{19}$$

are the particular case of (2), where $y_k: I\to\mathbb{R}$ $(k=1,\dots,n)$ are continuous functions satisfying the inequalities

$$0\le y_1(x)<y_2(x)\le\cdots<y_n(x)\le b\quad\text{for}\quad x\in I.$$

Let $g(\cdot,\cdot;x):[y_1(x),y_n(x)]\times[y_1(x),y_n(x)]\to\mathbb{R}$ be the Green's function of the differential equation

$$\frac{d^n v}{dy^n}=0$$

with multi–point boundary conditions

$$v(y_k(x))=0\quad (k=1,\dots,n). \tag{20}$$

Then by Lemma 8.5 from [1], we have

$$\mu_{nj}(x)\overset{def}{=}\sup\Big\{\frac{\gamma_j(y)}{\gamma_n(t)}\Big|\frac{\partial^j g(y,t;x)}{\partial y^j}\Big|: y_1(x)<y,\ t<y_n(x),\ y\ne t\Big\}<$$
$$<+\infty\quad\text{for}\quad x\in I\quad (j=0,\dots,n-1). \tag{21}$$

Corollary. *If*

$$r\Big(\sum_{k=0}^{n-1}\mu_{nk}(x)\int\limits_{y_1(x)}^{y_n(x)}|P_{mk}(x,t)|\,dt\Big)<1\quad\textit{for}\quad x\in I, \tag{22}$$

then problem (1), (19) *is well–posed.*

Proof. Let $v=(v_i)_{i=1}^l$ be a solution of problem (4),(20) for an arbitrarily fixed $x\in I$. By the above proved theorem, to prove the Corollary we need to show that $v(y)\equiv 0$.

It is easy to see

$$w_i=\sup\Big\{\frac{\gamma_k(y)}{\mu_{nk}(x)}|v_l^{(k)}(y)|:0<y<b;\ k=0,\dots,n-1\Big\}<+\infty\quad (i=1,\dots,l).$$

Set $w=(w_i)_{i=1}^l$. Then taking into account (21) from the equalities

$$\gamma_j(y)v^{(j)}(y)=\int\limits_{y_1(x)}^{y_n(x)}\frac{\gamma_j(y)}{\gamma_n(t)}\frac{\partial^j g(y,t;x)}{\partial y^j}\Big(\sum_{k=0}^{n-1}P_{mk}(x,t)\gamma_k(t)v^{(k)}(t)\Big)\,dt\ \ (j=0,\dots,n-1)$$

we find

$$w\le\Big(\sum_{k=0}^{m-1}\mu_{mk}(x)\int\limits_{y_1(x)}^{y_n(x)}|P_{mk}(x,t)|\,dt\Big)w.$$

Hence in view of conditions (22) and nonnegativity of the vector w we get $w=0$. Consequently $v(y)\equiv 0$. $\square$

Remark 1. Condition (22) is nonimprovable in the sense that it cannot be replaced by the condition

$$r\Big(\sum_{k=0}^{n-1}\mu_{nk}(x)\int\limits_{y_1(x)}^{y_n(x)}|P_{mk}(x,t)|\,dt\Big)<1+\varepsilon$$

for arbitrarily small $\varepsilon>0$.

Remark 2. It immediately follows from (21) that

$$\mu_{20}(x) = \frac{1}{y_2(x) - y_1(x)}, \quad \mu_{21}(x) = 1 \quad \text{for} \quad x \in I.$$

Therefore in the case, where $n = 2$ condition (2) coincides with the condition of unique solvability of two–point problem (4),(20) given in [4].

Acknowledgment

This work was supported by INTAS (grant No 00136).

References

1. I. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.
2. I. Kiguradze, B. Půža and I. P. Stavroulakis, On singular boundary value problems for functional differential equations of higher order. *Georgian Math. J.* **8**(2000), No. 4, 791–814.
3. T. Kiguradze and T. Kusano, On well-posedness of initial-boundary value problems for higher order linear hyperbolic equations with two independent variables. *Differentsial'nye Uravneniya* **39**(2003), No. 3.
4. B. Půža and A. Rabbimov, On a weighted boundary value problem for a system of singular functional differential equations. *Mem. Differential Equations Math. Phys.* **21**(2000), 125–130.

Author's address:

Faculty of Physics
I. Javakhishvili Tbilisi State University
3, Chavchavadze Ave., Tbilisi 380028
Georgia
E-mail: tkig@rmi.acnet.ge