Mem. Differential Equations Math. Phys. 25(2002), 153–155

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ON THE STRUCTURE OF THE SET OF SOLUTIONS OF THE WEIGHTED CAUCHY PROBLEM FOR HIGH ORDER EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

(Reported on November 26, 2001)

In the present paper, on the basis of the results obtained in [1,2,3] we study the structure of the set of solutions of the weighted initial problem

$$u^{(n)}(t) = f(u)(t), \tag{1}$$

$$\lim_{t\to a} \frac{u^{(k)}(t)}{h^{(k)}(t)} = 0 \quad (k = 0, \dots, n-1), \tag{2}$$

where $f \in C^{n-1}([a,b]; R^m) \to L_{\text{loc}}(]a,b]; \mathbb{R}^m)$ is a continuous Volterra operator and $h : [a,b] \to [0,+\infty[$ is an $(n-1)$-times continuously differentiable function such that

$$h^{(k)}(a) = 0 \quad (k = 0, \dots, n-2), \quad h^{(n-1)}(t) > 0 \ \text{ for } \ a < t \le b.$$

The problem for the case $n = 1$ has been investigated in [1]. Therefore below we will assume that $n \ge 2$.

Throughout the paper, the use will be made of the following notation.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0; +\infty[$;

$\mathbb{R}^m$ is the space of m-dimensional vectors $x = (x_i)_{i=1}^m$ with real components x_i ($i = 1, \dots, m$) and the norm $\|x\| = \sum\limits_{i=1}^m |x_i|$;

$\mathbb{R}^m_\rho = \{x \in \mathbb{R}^m : \ \|x\| \le \rho\}$.

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$, then $\text{sgn}(x) = (\text{sgn}\, x_i)_{i=1}^m$.

$x \cdot y$ is the scalar product of the vectors x and $y \in \mathbb{R}^m$;

$C^{n-1}([a,b]; \mathbb{R}^m)$ is the space of $(n-1)$-times continuously differentiable vector functions $x : [a,b] \to \mathbb{R}^m$ with the norm

$$\|x\|_{C^{n-1}} = \max\bigg\{\sum_{k=1}^{n-1} \|x^{(k-1)}(t)\| : \ a \le t \le b\bigg\};$$

$C^{n-1}_h([a,b]; \mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b]; \mathbb{R}^m)$ such that

$$\sup\bigg\{\frac{\|u^{(k)}(t)\|}{h^{(k)}(t)} \ : \ a < t \le b\bigg\} < +\infty \quad (k = 0, \dots, n-1);$$

$C^{n-1}_{h,\rho}([a,b]; \mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b]; \mathbb{R}^m)$ satisfying the equalities

$$|u^{(k)}(t)| \le \rho h^{(k)}(t) \ \text{ for } \ a < t \le b \ \ (k = 0, \dots, n-1);$$

If $x :]a,b] \to \mathbb{R}^m$ is a bounded function and $a \le s < t \le b$, then

$$\nu(x)(s,t) = \sup\Big\{\|x(\xi)\| : \ s < \xi < t\Big\};$$

2000 *Mathematics Subject Classification.* 34K05.

Key words and phrases. High order evolution singular functional differential equation, weighted Cauchy problem, Volterra operator, continuable solution, Kneser's property.

$L_{\text{loc}}(]a,b];\mathbb{R}^m)$ is the space of vector functions $x :]a,b] \to \mathbb{R}^m$ which are summable on each segment from $]a,b]$, with the topology of mean convergence on each segment from $]a,b]$.

Definition 1. $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\text{loc}}(]a,b];\mathbb{R}^m)$ is called a Volterra operator if the equality $f(x)(t) = f(y)(t)$ holds almost everywhere on $]a,t_0[$ for any $t_0 \in]a,b]$ and any vector functions x and $y \in C^{n-1}([a,b];\mathbb{R}^m)$ satisfying the condition $x(t) = y(t)$ for $a \le t \le t_0$.

Definition 2. We say that the operator $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\text{loc}}(]a,b];\mathbb{R}^m)$ satisfies the local Carathéodory conditions if it is continuous and there exists a nondecreasing with respect to the second argument function $\gamma :]a,b] \times [0,+\infty[\to [0,+\infty[$ such that $\gamma(\cdot,\rho) \in L_{\text{loc}}(]a,b];\mathbb{R})$ for any $\rho \in]0,+\infty[$, and the equality

$$\big\|f(x)(t)\big\| \le \gamma\big(t, \|x\|_{C^{n-1}}\big)$$

is fulfilled for any $x \in C^{n-1}([a,b];\mathbb{R}^m)$ almost everywhere on $]a,b[$.

Definition 3. If $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\text{loc}}(]a,b];\mathbb{R}^m)$ is a Volterra operator and $b_0 \in]a,b]$, then:

(i) for any $u \in C^{n-1}([a,b_0];\mathbb{R}^m)$ by $f(u)$ is understood the vector function given by the equality $f(u)(t) = f(\overline{u})(t)$ for $a \le t \le b_0$, where

$$\overline{u}(t) = \begin{cases} u(t) & \text{for } a \le t \le b_0 \\ \displaystyle\sum_{k=1}^{n} \frac{(t-b_0)^{k-1}}{(k-1)!}\, u^{(k-1)}(b_0) & \text{for } b_0 < t \le b \end{cases};$$

(ii) the function $u \in C^{n-1}([a,b_0];\mathbb{R}^m)$ is called a solution of the equation (1) on the segment $[a,b_0]$ if $u^{(n-1)}$ is absolutely continuous on each segment contained in $]a,b_0]$, and $u^{(n)}(t) = f(u)(t)$ almost everywhere on $]a,b_0[$;

(iii) a solution u of the equation (1) satisfying on the segment $[a,b_0]$ the initial conditions (2) is called a solution of the problem (1), (2) on the segment $[a,b_0]$.

Definition 4. A solution u of the equation (1) defined on a segment $[a,b_0] \subset [a,b[$ (on a semi-open interval $[a,b_0[\subset [a,b[$) is called continuable if for some $b_1 \in]b_0,b]$ ($b_1 \in [b_0,b]$) the equation (1) has on the segment $[a,b_1]$ a solution v satisfying $u(t) = v(t)$ for $a \le t \le b_0$. A solution u is, otherwise, called noncontinuable.

By $I^*(f;h)$ we denote the set of those $b^* \in]a,b]$ for which the domain of definition of every noncontinuable solution of the problem contains the segment $[a,b^*]$.

Definition 5. We say that the equation (1) has Kneser's property if $I^*(f;h) \ne \varnothing$, and for every $b^* \in I^*(f;h)$ the set of restrictions of noncontinuable solutions on $[a,b^*]$ is compact and connected in the topology of the space $C^{n-1}([a,b^*];\mathbb{R}^m)$.

Theorem. *Let there exist a positive number ρ and summable functions $p_k : [a,b] \to [0,+\infty[$ $(k = 0,\dots,n-1)$ and $q : [a,b] \to [0,+\infty[$ such that*

$$\limsup_{t\to a}\left(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\int_a^t p_k(s)\,ds\right) < 1,$$

$$\lim_{t\to a}\left(\frac{1}{h^{(n-1)}(t)}\int_a^t q(s)\,ds\right) = 0$$

and for any $u \in C^{n-1}_{h,\rho}([a,b];\mathbb{R}^m)$ *the inequality*

$$f(u)(t)\cdot \operatorname{sgn}\Big(u^{(n-1)}(t)\Big) \le \sum_{k=0}^{n-1} p_k(t)\nu\Big(\frac{u^{(k)}}{h^{(k)}}\Big)(a,t)+q(t)$$

is fulfilled almost everywhere on $]a,b[$. *Then the problem* (1), (2) *has Kneser's property.*

A particular case of equation (1) is the vector delay differential equation with

$$u^{(n)}(t) =$$

$$= f_0\bigg(t, u\Big(\tau_{10}(t)\Big),\dots,u^{(n-1)}\Big(\tau_{1\,n-1}(t)\Big),\dots,u\Big(\tau_{\ell\,0}(t)\Big),\dots,u^{(n-1)}\Big(\tau_{\ell\,n-1}(t)\Big)\bigg), \quad (3)$$

where $f_0 :]a,b]\times\mathbb{R}^{\ell mn}\to\mathbb{R}^m$ satisfies the local Carathéodory conditions, and τ_{ik} : $[a,b]\to[a,b]$ are measurable functions such that $\tau_{ik}(t)\le t$ for $a\le t\le b$ ($i=1\dots,\ell$; $k=0\dots,n-1$).

From the above theorem we arrive at the following

Corollary. *Let* $\tau_{\ell\,n-1}(t)\equiv t$ *and let there exist a positive number* ρ, *summable functions* $p_{ik} : [a,b]\to[0,+\infty[$ ($i=1,\dots,\ell$; $k=0,\dots,n-1$) *and* $q : [a,b]\to[0,+\infty[$ *such that the equality*

$$f_0\bigg(t, h\Big(\tau_{10}(t)\Big)x_{10},\dots,h^{(n-1)}\Big(\tau_{1\,n-1}(t)\Big)x_{1\,n-1},\dots,h\Big(\tau_{\ell\,0}(t)\Big)x_{\ell\,0},\dots,$$

$$h^{(n-1)}\Big(\tau_{\ell\,n-1}(t)\Big)x_{\ell\,n-1}\bigg)\cdot\operatorname{sgn}(x_{\ell\,n-1}) \le \sum_{k=0}^{n-1}\sum_{i=1}^{\ell} p_{ik}(t)\|x_{ik}\| + q(t)$$

is fulfilled on $]a,b]\times\mathbb{R}^{\ell mn}_{\rho}$. *Moreover, let*

$$\limsup_{t\to a}\bigg(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\sum_{i=1}^{\ell}\int_a^t p_k(s)\,ds\bigg) < 1,$$

$$\lim_{t\to a}\bigg(\frac{1}{h^{(n-1)}(t)}\int_a^t q(s)\,ds\bigg) = 0.$$

Then the problem (3), (2) *has Kneser's property.*

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